TROPICAL ALGEBRAIC GEOMETRY

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There are many examples in algebraic geometry in which complicated geometric or algebraic problems can be transformed into purely combinatorial problems. The most prominent example is probably given by toric varieties — a certain class of varieties that can be described purely by combinatorial data, e.g. by giving a convex polytope in an integral lattice. As a consequence, most questions about these varieties can be transformed into combinatorial questions on the defining polytope that are then hopefully easier to solve.

Tropical algebraic geometry is a recent development in the field of algebraic geometry that tries to generalize this idea substantially. Ideally, every construction in algebraic geometry should have a combinatorial counterpart in tropical geometry. One may thus hope to obtain results in algebraic geometry by looking at the tropical (i.e. combinatorial) picture first and then trying to transfer the results back to the original algebro-geometric setting.

The origins of tropical geometry date back about twenty years. One of the pioneers of the theory was Imre Simon [Si], a mathematician and computer scientist from Brazil — which is by the way the only reason for the peculiar name "tropical geometry". Originally, the theory was developed in an applied context of discrete mathematics and optimization, but it has not been part of the mainstream in either of mathematics, computer science or engineering. Only in the last few years have people realized its power for applications in fields such as combinatorics, computational algebra, and algebraic geometry.

This is also why the theory of tropical algebraic geometry is still very much in its beginnings: not even the concept of a variety has been defined yet in tropical geometry in a general and satisfactory way. On the other hand there are already many results in tropical geometry that show the power of these new methods. For example, Mikhalkin has proven recently that tropical geometry can be used to compute the numbers of plane curves of given genus g and degree d through $3d + g - 1$ general points [M1] — a deep result that had been obtained first by Caporaso and Harris about ten years ago by a complicated study of moduli spaces of plane curves [CH].

In this expository article we will for simplicity restrict ourselves mainly to the well-established theory of tropical plane curves. Even in this special case there are several seemingly different approaches to the theory. We will describe these approaches in turn in chapter 1 and discuss possible generalizations at the end. We will then explain in chapter 2 how some well-known results from classical geometry — e.g. the degree-genus formula and Bézout's theorem — can be recovered (and reproven) in the language of tropical geometry. Finally, in chapter 3 we will discuss the most powerful applications of tropical geometry known so far, namely to complex and real enumerative geometry.

1. PLANE TROPICAL CURVES

1.1. **Tropical curves as limits of amoebas.** With classical (complex) algebraic geometry in mind the most straightforward way to tropical geometry is via so-called *amoebas* of algebraic varieties. For a complex plane curve C the idea is simply to restrict it to the open subset $(\mathbb{C}^*)^2$ of the (affine or projective) plane and then to map it to the *real* plane by the map

$$\text{Log}: \quad (\mathbb{C}^*)^2 \to \mathbb{R}^2$$
$$z = (z_1, z_2) \mapsto (x_1, x_2) := (\log|z_1|, \log|z_2|).$$

The resulting subset $A = \text{Log}(C \cap (\mathbb{C}^*)^2)$ of $\mathbb{R}^2$ is called the *amoeba* of the given curve. It is of course a two-dimensional subset of $\mathbb{R}^2$ since complex curves are real two-dimensional. The following picture shows three examples (where we set $e := \exp(1)$):

(a) $C = \{z; z_1 + z_2 = 1\}$ (b) $C = \{z; e^3 z_1 + e^2 z_2 = 1\}$ (c) $C =$ a generic conic
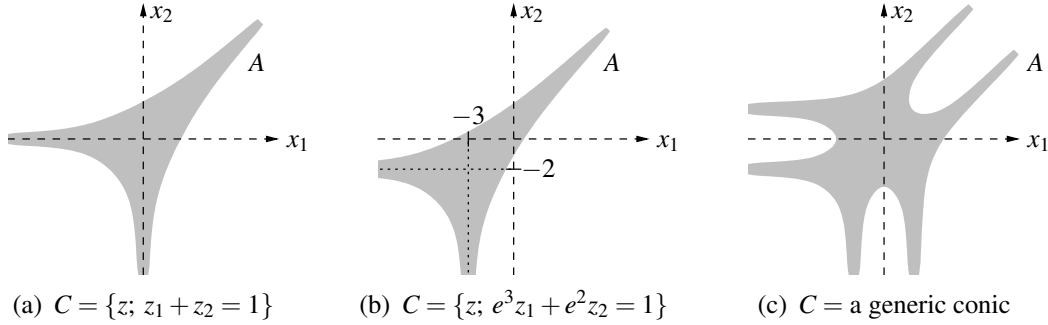
FIGURE 1. Three amoebas of plane curves

In fact, the shape of these pictures (that also explains the name "amoeba") can easily be explained. In case (a) for example the curve C contains exactly one point whose z_1-coordinate is zero, namely $(0,1)$. As $\log 0 = -\infty$ a small neighborhood of this point is mapped by Log to the "tentacle" of the amoeba A pointing to the left. In the same way a neighborhood of $(1,0) \in C$ leads to the tentacle pointing down, and points of the form $(z, 1-z)$ with $|z| \to \infty$ to the tentacle pointing to the upper right.

In case (b) the multiplicative change in the variable simply leads to an (additive) shift of the amoeba. In (c) a generic conic, i.e. a curve given by a general polynomial of degree 2, has two points each where it meets the coordinate axes, leading to two tentacles in each of the three directions. In the same way one could consider curves of an arbitrary degree d that would give us amoebas with d tentacles in each direction.

To make these amoebas into combinatorial objects the idea is simply to shrink them to "zero width". So instead of the map Log above let us consider the maps

$$\text{Log}_t : \quad (\mathbb{C}^*)^2 \to \mathbb{R}^2$$

$$(z_1, z_2) \mapsto (-\log_t |z_1|, -\log_t |z_2|) = \left(-\frac{\log |z_1|}{\log t}, -\frac{\log |z_2|}{\log t} \right)$$

for small $t \in \mathbb{R}$ and study the limit of the amoebas $\text{Log}_t(C \cap (\mathbb{C}^*)^2)$ as t tends to zero. As Log_t differs from Log only by a rescaling of the two axes the result for the curve in figure 1 (a) is the graph Γ shown in the following picture on the left. We call Γ the *tropical curve* determined by C.

(a) $C = \{z; z_1 + z_2 = 1\}$ (b) $C_t = \{z; t^{-3} z_1 + t^{-2} z_2 = 1\}$ (c) C_t : a family of conics
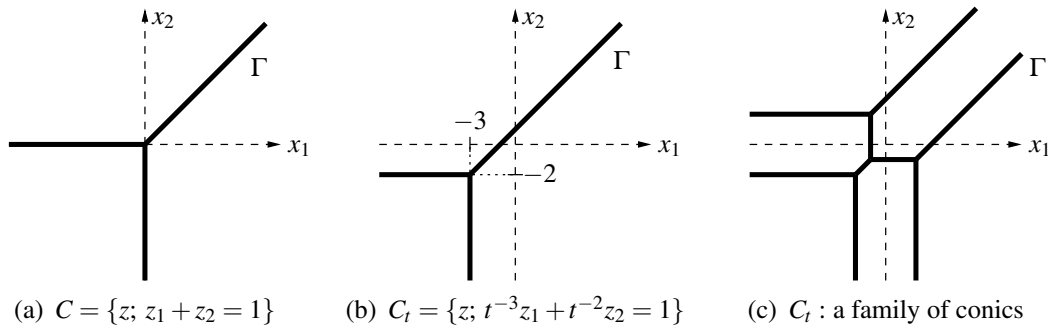
FIGURE 2. The tropical curves corresponding to the amoebas in figure 1

If we did the same thing with the curve (b) the result would of course be that we not only shrink the amoeba to zero width, but also move its "vertex" to the origin, leading to the same tropical curve as in (a). To avoid this we consider not only one curve $C = \{z; \; e^3 z_1 + e^2 z_2 = 1\}$ but the *family* of curves $C_t = \{z; \; t^{-3} z_1 + t^{-2} z_2 = 1\}$ for small $t \in \mathbb{R}$. This family has the property that C_t passes through $(0, t^2)$ and $(t^3, 0)$ for all t, and hence all $\mathrm{Log}_t(C_t \cap (\mathbb{C}^*)^2)$ have their horizontal and vertical tentacles at $z_2 = -2$ and $z_1 = -3$, respectively. So if we now take the limit as $t \to 0$ we shrink the width of the amoeba to zero but keep its position in the plane: we get the shifted tropical curve as in figure 2 (b). We call this the tropical curve determined by the family (C_t).

For (c) we can proceed in the same way: using a suitable family of conics we can shrink the width of the amoeba A to zero while keeping the position of its tentacles fixed. The resulting tropical curve Γ, i.e. the limit of $\mathrm{Log}_t(C_t \cap (\mathbb{C}^*)^2)$, may e.g. look like in figure 2 (c). We will see at the end of section 1.4 however that this is not the only type of graph that we can obtain by a family of conics in this way.

Summarizing we can say informally that a tropical curve should be a subset of $\mathbb{R}^2$ obtained as the limit (in a certain sense) of the amoebas $\mathrm{Log}_t(C_t \cap (\mathbb{C}^*)^2)$, where (C_t) is a suitable family of plane algebraic curves. They are all piecewise linear graphs with certain properties that we will study later.

1.2. **Tropical curves via varieties over the field of Puiseux series.** Of course this method of constructing (or even defining) tropical curves is very cumbersome as it always involves a limiting process over a whole family of complex curves. There is an elegant way to hide this limiting process by replacing the ground field $\mathbb{C}$ by the field K of so-called *Puiseux series*, i.e. the field of formal power series $a = \sum_{q \in \mathbb{Q}} a_q t^q$ in a variable t such that the subset of $\mathbb{Q}$ of all q with $a_q \neq 0$ is bounded below and has a finite set of denominators. For such an $a \in K$ with $a \neq 0$ the infimum of all q with $a_q \neq 0$ is actually a minimum; it is called the *valuation* of a and denoted $\mathrm{val}\, a$.

Using this construction we can say for example that our family (b) in section 1.1

$$C = \{z \in K^2; \; t^{-3} z_1 + t^{-2} z_2 = 1\} \subset K^2$$

now defines *one single curve* in the affine plane over this new field K. How do we now perform the limit $t \to 0$ in this set-up? For an element $a = \sum_{q \in \mathbb{Q}} a_q t^q \in K$ only the term with the smallest exponent, i.e. $a_{\mathrm{val}\, a} t^{\mathrm{val}\, a}$, will be relevant in this limit. So applying the map $\log_t$ we get for small t

$$\log_t |a| \approx \log_t |a_{\mathrm{val}\, a} t^{\mathrm{val}\, a}| = \mathrm{val}\, a + \log_t |a_{\mathrm{val}\, a}| \approx \mathrm{val}\, a.$$

In our new picture the operations of applying the map Log_t and taking the limit for $t \to 0$ therefore correspond to the map

$$\mathrm{Val} : (K^*)^2 \to \mathbb{R}^2$$
$$(z_1, z_2) \mapsto (x_1, x_2) := (-\mathrm{val}\, z_1, -\mathrm{val}\, z_2).$$

Using this observation we can now give our first rigorous definition of plane tropical curves:

Definition A. *A plane tropical curve is a subset of $\mathbb{R}^2$ of the form $\mathrm{Val}(C \cap (K^*)^2)$, where C is a plane algebraic curve in K^2. (Strictly speaking we should take the closure of $\mathrm{Val}(C \cap (K^*)^2)$ in $\mathbb{R}^2$ since the image of the valuation map Val is by definition contained in $\mathbb{Q}^2$.)*

Note that this definition is now purely algebraic and does not involve any limit taking processes. As K is an algebraically closed field of characteristic zero (in fact it is the algebraic closure of the field of Laurent series in t) the theory of algebraic geometry of plane curves over K is largely identical to that of algebraic curves over $\mathbb{C}$.

As an example let us consider again case (b) of section 1.1, i.e. the curve $C \subset K^2$ given by the equation $t^{-3} z_1 + t^{-2} z_2 = 1$. If $(z_1, z_2) \in C \cap (K^*)^2$ then $\mathrm{Val}(z_1, z_2)$ can give three different kinds of results:

- If $\mathrm{val}\, z_1 > 3$ then the valuation of $z_2 = t^2 - t^{-1} z_1$ is 2 since all exponents of t in $t^{-1} z_1$ are bigger than 2. Hence these points map precisely to the left edge of the tropical curve in figure 2 (b) under Val.
- In the same way we get the bottom edge of this tropical curve if $\mathrm{val}\, z_2 > 2$.

- If $\operatorname{val} z_1 \leq 3$ and $\operatorname{val} z_2 \leq 2$ then the equation $t^{-3}z_1 + t^{-2}z_2 = 1$ shows that the leading terms of $t^{-3}z_1$ and $t^{-2}z_2$ must have the same valuation, i.e. that $\operatorname{val} z_1 = \operatorname{val} z_2 + 1$. This leads to the upper right edge of the tropical curve in figure 2 (b).

So we recover our old result, i.e. the tropical curve drawn in figure 2 (b).

One special case is worth mentioning: if the curve $C \subset K^2$ is given by an equation whose coefficients lie in $\mathbb{C}$ (i.e. are "independent of t") then for any point $(z_1(t), z_2(t)) \in C$ the points $(z_1(t^q), z_2(t^q))$ for $q \in \mathbb{Q}$ are obviously in C as well. As replacing t by t^q for some $q > 0$ simply multiplies the valuation with q we conclude that the tropical variety associated to C in this case is a *cone* (i.e. a union of half-rays starting at the origin) — as it was the case e.g. in figure 2 (a).

1.3. **Tropical curves as varieties over the max-plus semiring.** We now want to study definition A in more detail. Let $C \subset K^2$ be a plane algebraic curve given by the polynomial equation

$$C = \left\{ (z_1, z_2) \in K^2; \ f(z_1, z_2) := \sum_{i,j \in \mathbb{N}} a_{ij} z_1^i z_2^j = 0 \right\}$$

for some $a_{ij} \in K$ of which only finitely many are non-zero. Note that the valuation of a summand of $f(z_1, z_2)$ is

$$\operatorname{val}(a_{ij} z_1^i z_2^j) = \operatorname{val} a_{ij} + i \operatorname{val} z_1 + j \operatorname{val} z_2.$$

Now if (z_1, z_2) is a point of C then all these summands add up to zero. In particular, the lowest valuation of these summands must occur at least twice since otherwise the corresponding terms in the sum could not cancel. For the corresponding point $(x_1, x_2) = \operatorname{Val}(z_1, z_2) = (-\operatorname{val} z_1, -\operatorname{val} z_2)$ of the tropical curve this obviously means that in the expression

$$g(x_1, x_2) := \max\{ix_1 + jx_2 - \operatorname{val} a_{ij}; \ (i,j) \in \mathbb{N}^2 \text{ with } a_{ij} \neq 0\} \qquad (*)$$

the maximum is taken on at least twice. It follows that the tropical curve determined by C is contained in the "corner locus" of this convex piecewise linear function g, i.e. in the locus where this function is not differentiable. In fact, Kapranov's theorem states that the converse inclusion holds as well, i.e. that the tropical curve determined by C is precisely this corner locus (see e.g. [K], [Sh]).

As an example let us consider again the curve $C = \{(z_1, z_2); \ t^{-3}z_1 + t^{-2}z_2 - 1 = 0\} \subset K^2$ that we discussed in the previous section. The corresponding convex piecewise linear function is

$$g(x_1, x_2) = \max\{x_1 + 3, x_2 + 2, 0\}$$

The following picture shows how the corner locus of this function gives us back the tropical curve of figure 2 (b).
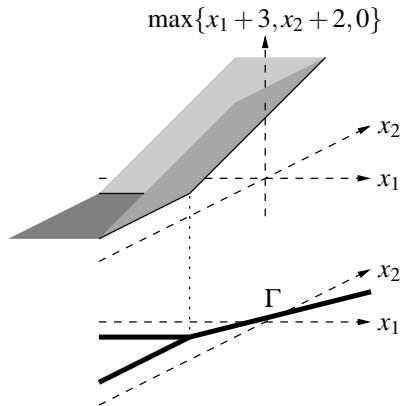

FIGURE 3. A tropical curve as the corner locus of a convex piecewise linear function

These convex piecewise linear functions are often written in a different way in order to resemble the notation of the original polynomial: for two real numbers x, y we define "tropical addition" and "tropical multiplication" simply by

$$x \oplus y := \max\{x, y\} \qquad \text{and} \qquad x \odot y := x + y.$$

The real numbers together with these two operations form a semiring, i.e. they satisfy all properties of a ring except for the existence of additive neutral and inverse elements. Sometimes an element $-\infty$ is formally added to the real numbers to serve as a neutral element, but there is certainly no way to construct inverse elements as this would require equations of the form $\max\{-\infty, x\} = -\infty$ to be solvable.

Using this notation we can write our convex piecewise linear function $(*)$ above as

$$g(x_1, x_2) = \bigoplus_{i,j} (-\operatorname{val} a_{ij}) \odot x_1^{\odot i} \odot x_2^{\odot j}.$$

We call this expression the *tropicalization* of the original polynomial f. It can be considered as a "tropical polynomial", i.e. as a polynomial in the tropical semiring. For example, the tropicalization of the polynomial $t^{-3} z_1 + t^{-2} z_2 - 1$ is just

$$3 \odot x_1 \oplus 2 \odot x_2 \oplus 0 = \max\{x_1 + 3, x_2 + 2, 0\}.$$

(Note that the addition of 0 is not superfluous here since 0 is not a neutral element for tropical addition!)

We can therefore now give an alternative definition of plane tropical curves that does not involve the somewhat complicated field of Puiseux series any more:

Definition B. *A plane tropical curve is a subset of $\mathbb{R}^2$ that is the corner locus of a tropical polynomial, i.e. of a polynomial in the tropical semiring $(\mathbb{R}, \oplus, \odot) = (\mathbb{R}, \max, +)$.*

Again there is a special case that is completely analogous to the one mentioned at the end of section 1.2: if the tropical polynomial g is the maximum of linear functions without constant terms (e.g. because it is the tropicalization of a polynomial with coefficients that do not depend on t) then the corner locus of g is a cone. If this is not the case and g is the maximum of many affine functions then its corner locus will in general be a complicated piecewise linear graph in the plane as e.g. in figure 2 (c).

1.4. **Tropical curves as balanced graphs.** Our definition B now allows us to give an easy and entirely geometric characterization of plane tropical curves. We have already seen that a tropical curve Γ is a graph in $\mathbb{R}^2$ whose edges are line segments. Let us consider Γ locally around a vertex $V \in \Gamma$. For simplicity we shift coordinates so that V is the origin in $\mathbb{R}^2$ and thus Γ becomes a cone locally around V. We have seen already at the end of section 1.3 that Γ is then locally the corner locus of a tropical polynomial of the form

$$g(x_1, x_2) = \bigoplus_i x_1^{\odot a_1^{(i)}} \odot x_2^{\odot a_2^{(i)}} = \max\{a_1^{(i)} x_1 + a_2^{(i)} x_2; \ i = 1, \ldots, n\}.$$

for some $a^{(i)} = (a_1^{(i)}, a_2^{(i)}) \in \mathbb{N}^2$. Let Δ be the convex hull of the points $a^{(i)}$, as indicated in the following picture on the left:
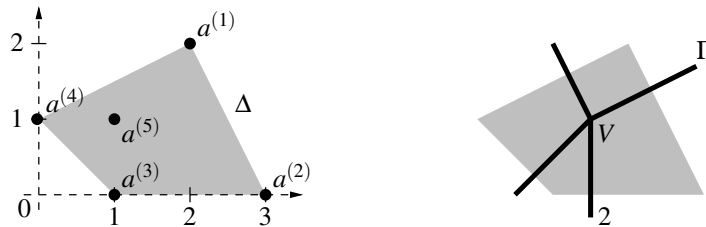

FIGURE 4. A local picture of a plane tropical curve

First of all we claim that any point $a^{(i)}$ that is not a vertex of Δ is irrelevant for the tropical curve Γ. In fact, it is impossible for such an $a^{(i)}$ (as e.g. $a^{(5)}$ in the example above) that the expression $a_1^{(i)}x_1 + a_2^{(i)}x_2$ is strictly bigger than all the other $a_1^{(j)}x_1 + a_2^{(j)}x_2$ for some $x_1, x_2 \in \mathbb{R}$. Hence g and therefore also its corner locus remain the same if we drop this term. In particular we see that — unlike in classical algebraic geometry — *there is no hope for a one-to-one correspondence between tropical curves and tropical polynomials (up to scalars)*.

It is now easy to see that the corner locus of g consists precisely of those points where

$$g(x_1, x_2) = a_1^{(i)}x_1 + a_2^{(i)}x_2 = a_1^{(j)}x_1 + a_2^{(j)}x_2$$

for two adjacent vertices $a^{(i)}$ and $a^{(j)}$ of Δ. An easy computation shows that for fixed i and j this is precisely the half-ray starting from the origin and pointing in the direction of the outward normal of the edge joining $a^{(i)}$ and $a^{(j)}$. So as shown in figure 4 on the right the tropical curve Γ is simply the union of all these outward normal lines locally around V. In particular all edges of Γ have rational slopes.

There is one more important condition on the edges of Γ around V that follows from this observation. If $a^{(1)}, \ldots, a^{(n)}$ are the vertices of Δ in clockwise direction then an outward normal vector of the edge joining $a^{(i)}$ and $a^{(i+1)}$ (where we set $a^{(n+1)} := a^{(1)}$) is $v^{(i)} := (a_2^{(i)} - a_2^{(i+1)}, a_1^{(i+1)} - a_1^{(i)})$ for all $i = 1, \ldots, n$. In particular it follows that $\sum_{i=1}^{n} v^{(i)} = 0$. This fact is usually expressed as follows: we write the vectors $v^{(i)}$ as $v^{(i)} = w^{(i)} \cdot u^{(i)}$ where $u^{(i)}$ is the primitive integral vector in the direction of $v^{(i)}$ and $w^{(i)} \in \mathbb{N}_{>0}$. We call $w^{(i)}$ the *weight* of the corresponding edge of Γ and thus consider Γ to be a weighted graph. Our equation $\sum_i v^{(i)} = 0$ then states that the weighted sum of the primitive integral vectors of the edges around every vertex of Γ is 0. This is usually called the *balancing condition*. For example, in figure 4 the edge of Γ pointing down has weight 2 (since $v^{(2)} = (0, -2) = 2 \cdot (0, -1)$), whereas all other edges have weight 1. In this paper we will usually label the edges with their corresponding weights unless these weights are 1. The balancing condition around the vertex V then reads

$$(2, 1) + 2 \cdot (0, -1) + (-1, -1) + (-1, 2) = (0, 0)$$

in this example.

Together with our observation of section 1.1 that (at least generic) plane algebraic curves of degree d lead to plane tropical curves with d ends each in the directions $(-1, 0)$, $(0, -1)$, and $(1, 1)$, we arrive at the following somewhat longer but purely geometric definition of plane tropical curves:

Definition C. *A plane tropical curve of degree d is a weighted graph Γ in $\mathbb{R}^2$ such that*

(a) *every edge of Γ is a line segment with rational slope;*
(b) *Γ has d ends each in the directions $(-1, 0)$, $(0, -1)$, and $(1, 1)$ (where an end of weight w counts w times);*
(c) *at every vertex V of Γ the balancing condition holds: the weighted sum of the primitive integral vectors of the edges around V is zero.*

Strictly speaking we have only explained above why a plane tropical curve in the sense of definition B gives rise to a curve in the sense of definition C. One can show that the converse holds as well; a proof can e.g. be found in [M1] or [Sp] chapter 5.

With this definition it has now become a combinatorial problem to find all types of plane tropical curves of a given degree. In fact, the construction given above globalizes well: assume that Γ is the tropical curve given as the corner locus of the tropical polynomial

$$g(x_1, x_2) = \max\{a_1^{(i)}x_1 + a_2^{(i)}x_2 + b^{(i)}; \, i = 1, \ldots, n\}.$$

If g is the tropicalization of a polynomial of degree d then the $a^{(i)}$ are all integer points in the triangle $\Delta_d := \{(a_1, a_2); \, a_1 \geq 0, a_2 \geq 0, a_1 + a_2 \leq d\}$. Consider two terms $i, j \in \{1, \ldots, n\}$ with $a^{(i)} \neq a^{(j)}$. If there is a point $(x_1, x_2) \in \mathbb{R}^2$ such that

$$g(x_1, x_2) = a_1^{(i)}x_1 + a_2^{(i)}x_2 + b^{(i)} = a_1^{(j)}x_1 + a_2^{(j)}x_2 + b^{(j)}$$

then we draw a straight line in Δ_d through the points $a^{(i)}$ and $a^{(j)}$. This way we obtain a subdivision of Δ_d whose edges correspond to the edges of Γ and whose 2-dimensional cells correspond to the vertices of Γ. This subdivision is usually called the *Newton subdivision* corresponding to Γ. So to find all types of plane tropical curves of degree d one has to list all subdivisions of Δ_d and check which of them are induced by a tropical curve as above. As the simplest example there is only the trivial subdivision of Δ_1, leading to the only type of plane tropical curve of degree 1:

FIGURE 5. Tropical lines are "dual" to Δ_1

The following picture shows all non-degenerated cases in degree 2, where by "non-degenerate" we mean that the subdivision of Δ_2 is maximal.

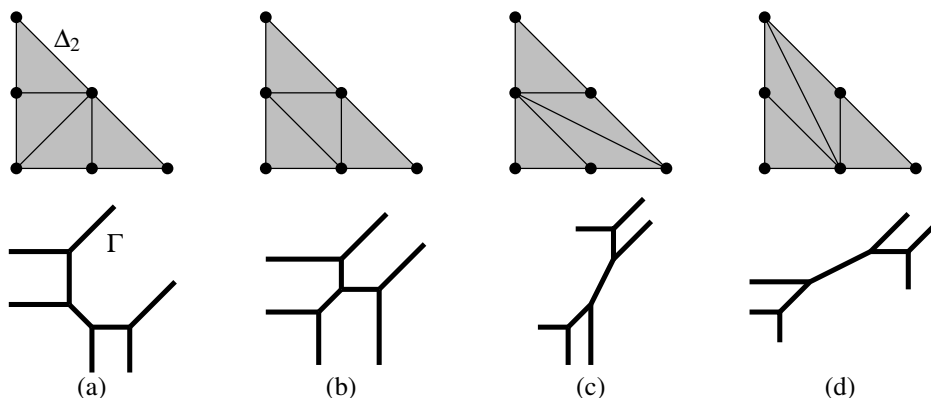(a)   (b)   (c)   (d)

FIGURE 6. The four types of (non-degenerated) tropical plane conics

In general, the problem of finding an algorithm to generate all such subdivisions has already been studied extensively in geometric combinatorics [IMTI, R].

Note however that not every subdivision gives rise to a type of tropical curves. First of all it is obvious by our constructions above that we need subdivisions into *convex* polytopes in order to have a tropical curve corresponding to it. But this condition is not sufficient: assume for example that we have the following (local) picture Δ somewhere in the Newton subdivision of a plane tropical curve:
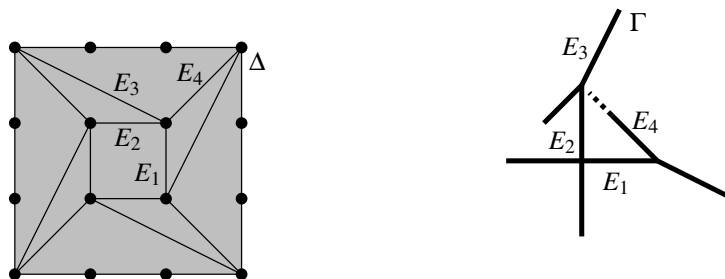
FIGURE 7. A subdivision that is not induced by a tropical curve

In the tropical curve Γ that would correspond to this subdivision the edge E_4 would have to meet E_3 and not E_2 at the dotted end, so E_2 must be longer than E_1. But the same argument can be used cyclically to conclude that each edge around the central vertex must be longer than the previous one. As this is not possible we conclude that there cannot be a tropical curve corresponding to this subdivision of Δ. In fact, the subdivisions corresponding to tropical curves are precisely the ones that are usually called the *regular polyhedral subdivisions*, i.e. the ones that can be written as the corner locus of a piecewise linear convex function similarly to figure 3.

It should also be stressed that the subdivision of Δ_d determines only the combinatorial type of the tropical curve and not the curve itself. For example, each of the types in figure 6 describes a (real) 5-dimensional family of plane tropical conics since the lengths of the bounded edges (3 parameters) as well as the position in the plane (2 parameters) can vary arbitrarily. Note that this agrees nicely with the classical picture: conics in the complex plane vary in a 5-dimensional family as well (corresponding to the 6 coefficients of a quadratic equation modulo a common scalar).

1.5. **Generalizations.** At the end of this chapter let us briefly describe how the theory of plane tropical curves given above can be generalized.

First of all it is quite obvious to note that essentially the same constructions and the same theory can be carried through for curves that are not necessarily in the plane but in any *toric surface*, i.e. in any surface X with a $\mathbb{C}^*$-action that contains $(\mathbb{C}^*)^2$ as a dense open subset. Definition A remains unchanged in this case; the resulting tropical curves will still be graphs in $\mathbb{R}^2$. Definition B only has to be modified as to allow Laurent polynomials compatible with the chosen homology class of the curves; and in definition C the only change is in the directions of the ends of Γ. In fact, a toric surface X together with a positive homology class corresponds exactly to a convex integral polytope Δ, and the types of tropical curves coming from this homology class are precisely those determined by subdivisions of Δ as explained at the end of section 1.4.

In the same way one can also consider general hypersurfaces instead of plane curves. Except for the existence of more variables there are no changes in definitions A and B, and there is an analogous version of the balancing condition and definition C too. The resulting tropical hypersurfaces are weighted polyhedral complexes in a real vector space. It is also true in this case that the combinatorial types of hypersurfaces correspond to subdivisions of a higher-dimensional polytope.

The theory becomes more difficult however in the case of varieties of higher codimension, e.g. space curves. It is probably agreed upon that definition A would be the "correct" one also in this case, i.e. that tropical varieties are by definition the images of classical varieties over the field of Puiseux series under the valuation map. This definition is however hard to work with in practice — it would be much more convenient to think of tropical hypersurfaces as in definition B and of general tropical varieties as intersections of such tropical hypersurfaces. Unfortunately, if $X \subset K^n$ is a variety given by some polynomial equations $f_1 = \cdots = f_r = 0$ it is (in contrast to the codimension-1 case) in general *not* true that the tropical variety corresponding to X (i.e. the image of X under the valuation map) is simply given by the intersection of the corner loci of the tropicalizations of $f_1, \ldots, f_r$. In fact, it is not even true in general that the intersection of tropical hypersurfaces is a tropical variety at all: if we intersect e.g. a tropical line as in figure 2 (b) with the same line shifted a bit to the left then the result is a single half-ray — which is not a tropical variety. It has been shown however that the tropical variety corresponding to X can always be written as an intersection of the corner loci of the tropicalizations of (finitely many) suitably chosen generators of the ideal $(f_1, \ldots, f_r)$ defining X [BJSST]. Moreover, there exists an implemented algorithm to perform these computations explicitly [BJSST, J] so that — from an algorithmic point of view — "every variety can be tropicalized". However, the necessary calculations rely on Gröbner basis techniques and thus become complicated very soon (more precisely, most questions regarding the computation of tropical varieties are NP-hard in the language of complexity analysis [Th]).

As for definition C the conditions listed there can be adapted to make sense in higher codimensions as well, so one could try to use definition C and say that e.g. space curves are simply balanced graphs in $\mathbb{R}^3$. Unfortunately it turns out that this definition would not be equivalent to definition A. While it is true that every tropical space curve in the sense of definition A gives rise to a balanced graph in $\mathbb{R}^3$ in the sense of definition C the converse does not hold in general [Sp]. There is no known general criterion yet to decide exactly when a balanced graph in $\mathbb{R}^3$ can be obtained as the image of a curve in K^3 under the valuation map, although a sufficient criterion is given in [Sp].

Finally one should note that even the most generally applicable definition A depends on a given embedding of the original variety over K in an affine or projective space (or a toric variety). There is no (known) way to associate a tropical variety to a given abstract variety over K. In fact, there is not even a good theory yet of what an abstract tropical variety should be. There is some recent work of Mikhalkin however that tries to build up a theory of tropical geometry completely in parallel to algebraic geometry, replacing the ground field by the semiring $(\mathbb{R}, \oplus, \odot)$ [M2].

2. TROPICAL VERSIONS OF CLASSICAL THEOREMS

As tropical curves are simply images of classical curves by definition A we can hope to find tropical — and thus combinatorial — versions of many results known from classical geometry. We will list a few important and interesting ones in this chapter.

2.1. **Tropical factorization.** Let us start with a very simple statement: in classical geometry it is obvious that for two polynomials $f_1, f_2 \in K[z_1, z_2]$ the plane curve defined by the equation $(f_1 \cdot f_2)(z) = 0$ is simply the union of the two curves with the equations $f_1(z) = 0$ and $f_2(z) = 0$.

It is easy to see that an analogous statement holds in the tropical semiring as well: if g_1, g_2 are two tropical polynomials, in particular convex piecewise linear functions, then the corner locus of $g_1 \odot g_2 = g_1 + g_2$ is simply the union of the corner loci of g_1 and g_2. In particular, the union of two plane tropical curves of degrees d_1 and d_2 is always a plane tropical curve of degree $d_1 + d_2$.

We can therefore consider the "tropical factorization problem" both in a geometric and an algebraic version. In the geometric language (i.e. using definition C) we would start with a weighted balanced graph in the plane and ask whether this graph is a union of two weighted subgraphs that are themselves balanced. In the algebraic language (i.e. using definition B) we would start with a tropical polynomial and ask whether it can be written as a (tropical) product of two polynomials of smaller degrees.

Note however that these two problems are not entirely equivalent since we have seen already in section 1.4 that there is no one-to-one correspondence between tropical polynomials and tropical curves. As an easy example consider the tropical polynomial $g(x_1, x_2) = x_1 \oplus x_2 \oplus 0$ whose corner locus is the curve in figure 2 (a). If we now consider the tropical square of this polynomial

$$g(x_1, x_2) \odot g(x_1, x_2) = x_1^{\odot 2} \oplus x_2^{\odot 2} \oplus 0 \oplus (x_1 \odot x_2) \oplus x_1 \oplus x_2$$
$$= \max\{2x_1, 2x_2, 0, x_1 + x_2, x_1, x_2\}$$

then the tropical curve determined by this polynomial is still the same as before (but with weight 2). But as piecewise linear maps the function $g(x_1, x_2) \odot g(x_1, x_2)$ is the same as

$$\max\{2x_1, 2x_2, 0\} = x_1^{\odot 2} \oplus x_2^{\odot 2} \oplus 0,$$

and this tropical polynomial *cannot* be written as a product of two linear tropical polynomials (to prove this just note that there are no additive inverses in the tropical semiring, so no additive cancellations are possible anywhere in the expansion of the product).

Geometrically it is in principle easy to decide (although maybe complicated combinatorially if the degree of the curve is large) whether a given balanced graph is the union of two smaller ones. Algebraically, it has been shown for tropical polynomials in one variable that any such polynomial can be replaced by another one defining the same piecewise linear function that can then be written as a tropical

product of linear factors (see [SS] section 2). For polynomials in more than one variable not much is known however. The factorization of a tropical polynomial into irreducible polynomials is in general not unique, and there is no algorithm known to determine whether a given polynomial is irreducible resp. to compute a (or all) possible decomposition into irreducible factors (see [SS] section 2). Results in this direction would be very interesting since it has been shown that a solution to the tropical factorization problem would also be useful to compute factorizations of ordinary polynomials more efficiently [GL].

2.2. **The degree-genus formula.** If C is a smooth complex plane projective curve of degree d then it is well-known that its genus (i.e. the "number of holes" in the real surface C) is given by the so-called degree-genus formula $g = \frac{1}{2}(d-1)(d-2)$. If C is not smooth then there are several slightly different ways to define its genus, but for any of these definitions the genus will be at most the above number $\frac{1}{2}(d-1)(d-2)$.

Let us study the same questions in tropical geometry. If $\Gamma \subset \mathbb{R}^2$ is a plane tropical curve then the most natural way to define its genus is simply to let it be the number of loops in the graph Γ, i.e. its first Betti number $g = \dim H_1(\Gamma, \mathbb{R})$.

How is this genus related to the degree of Γ? To see this let us denote the set of vertices and bounded edges of Γ by Γ_0 and Γ_1, respectively. Moreover, for a vertex $V \in \Gamma_0$ we define its *valence* $\mathrm{val}\, V$ to be the number of (bounded or unbounded) edges adjacent to V. As Γ has $3d$ unbounded and $|\Gamma_1|$ bounded edges it follows that

$$3d + 2|\Gamma_1| = \sum_{V \in \Gamma_0} \mathrm{val}\, V.$$

Since the genus of Γ can be computed as $1 + |\Gamma_1| - |\Gamma_0|$ we conclude that

$$g = 1 + \frac{1}{2} \sum_{V \in \Gamma_0} \mathrm{val}\, V - \frac{3}{2}d - |\Gamma_0| = \frac{1}{2}(d-1)(d-2) - \underbrace{\left(\frac{1}{2}d^2 - \sum_{V \in \Gamma_0} \frac{1}{2}(\mathrm{val}\, V - 2) \right)}_{(*)}.$$

Recall from section 1.4 that every vertex $V \in \Gamma_0$ corresponds to a convex polygon with $\mathrm{val}\, V$ vertices in the Newton subdivision of Δ_d corresponding to Γ. As such a polygon has area at least $\frac{1}{2}(\mathrm{val}\, V - 2)$ and the total area of Δ_d is $\frac{1}{2}d^2$ it follows that the expression $(*)$ above is always non-negative and thus the genus of Γ is always at most $\frac{1}{2}(d-1)(d-2)$, as in the classical case. Equality holds if and only if all polygons in the Newton subdivision have minimal area for its number of vertices.

There is nothing like a general "tropical singularity theory" yet, but usually one says that a plane tropical curve Γ is *smooth* if its Newton subdivision is maximal (i.e. consists of d^2 triangles of area $\frac{1}{2}$ each), or equivalently if every vertex of Γ has valence 3, all weights of the edges are 1, and the primitive integral vectors along the edges adjacent to any vertex generate the lattice $\mathbb{Z}^2$. With this definition it follows from our computations above that *the genus of a smooth plane tropical curve of degree d is $\frac{1}{2}(d-1)(d-2)$*, just as in classical geometry.

The following picture shows three examples of plane cubics: the curve (a) is smooth (and hence of genus 1), (b) is not smooth but still of genus 1 (since the Newton subdivision contains only a parallelogram of area 1 and triangles of area $\frac{1}{2}$), and (c) has genus 0 (since the Newton subdivision contains triangles of area greater than $\frac{1}{2}$).
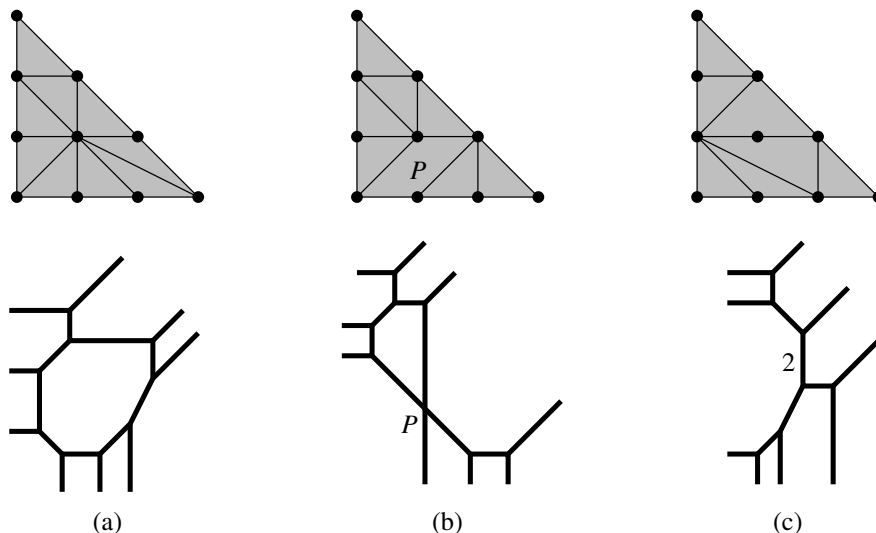
FIGURE 8. Three plane tropical cubics

Let us consider case (b) in more detail. Obviously the parallelogram P in the Newton subdivision gives rise to a point in the tropical curve (that we also denoted by P) where two straight edges intersect. We can therefore think of Γ as the planar image of a graph of genus 0 that has a "crossing" at P. This corresponds exactly to a normal crossing singularity in the classical case, i.e. to a complex curve C with a point $P \in C$ where two smooth branches meet transversely. In fact, for a plane tropical curve whose Newton subdivision contains only of triangles and parallelograms one sometimes subtracts the number of parallelograms from the genus defined above for that reason (so that e.g. the genus of the curve (b) above would then be 0).

2.3. **Bézout's theorem.** Let C_1 and C_2 be two distinct smooth plane projective complex curves of degrees d_1 and d_2, respectively. Bézout's theorem states that the intersection $C_1 \cap C_2$ then consists of at most $d_1 \cdot d_2$ points, and that in fact equality holds if the intersection points are counted with the correct multiplicity, namely with the local intersection multiplicities of C_1 and C_2.

We have seen in section 1.5 already that there is a slight problem if we try to find an analogous statement in tropical geometry: it is not even true that the intersection of two distinct (smooth) plane tropical curves is always finite — they might as well share some common line segments.

Let us ignore this problem for a moment however and assume that we have two smooth tropical curves Γ_1 and Γ_2 of degrees d_1 and d_2 respectively that intersect in finitely many points, and that none of these intersection points is a vertex of either curve. In this case it is in fact easy to find a tropical Bézout theorem: by section 2.1 the union $\Gamma_1 \cup \Gamma_2$ is a plane tropical curve of degree $d_1 + d_2$ and hence corresponds to a Newton subdivision of $\Delta_{d_1+d_2}$. The vertices of $\Gamma_1 \cup \Gamma_2$ are of two types:

- the 3-valent vertices of Γ_1 and Γ_2 are of course also present in $\Gamma_1 \cup \Gamma_2$. As $\Gamma_1 \cup \Gamma_2$ looks locally the same as Γ_1 resp. Γ_2 around such a vertex the triangles in the Newton subdivisions for Γ_1 and Γ_2 can also be found in the subdivision for $\Gamma_1 \cup \Gamma_2$.
- every intersection point in $\Gamma_1 \cap \Gamma_2$ gives rise to a 4-valent vertex of $\Gamma_1 \cup \Gamma_2$ where two straight lines meet. As in the end of section 2.2 this gives rise to a parallelogram in the Newton subdivision of $\Gamma_1 \cup \Gamma_2$.

The following picture illustrates this for the case of two conics, one of type (a) and one of type (d) in the notation of figure 6:
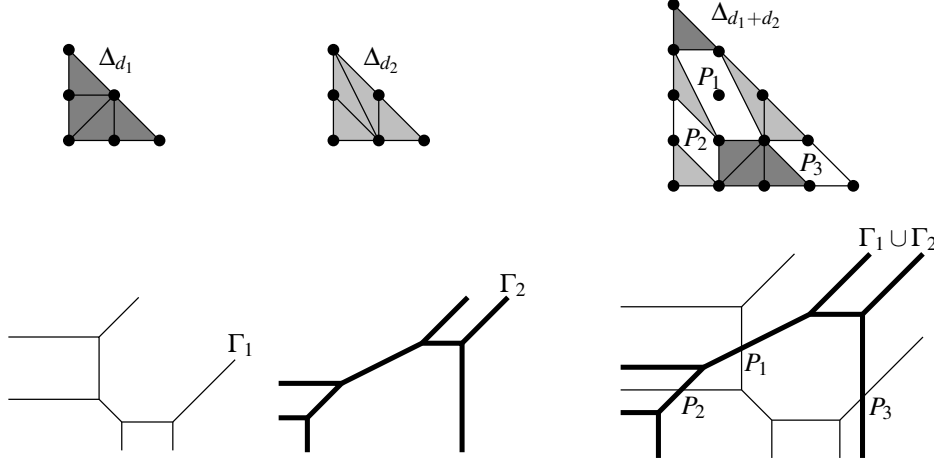
FIGURE 9. The intersection of two smooth plane conics

We conclude that the area covered by the parallelograms corresponding to the points in $\Gamma_1 \cap \Gamma_2$ is

$$\text{area}(\Delta_{d_1+d_2}) - \text{area}\,\Delta_{d_1} - \text{area}\,\Delta_{d_2} = \frac{1}{2}(d_1+d_2)^2 - \frac{1}{2}d_1^2 - \frac{1}{2}d_2^2 = d_1 d_2.$$

So if we define the intersection multiplicity of Γ_1 and Γ_2 in a point $P \in \Gamma_1 \cap \Gamma_2$ to be the area of the parallelogram corresponding to P in the Newton subdivision of $\Delta_{d_1+d_2}$ (which is a positive integer) then it follows that the sum of all these intersection multiplicities is $d_1 d_2$. This is the tropical Bézout theorem that has first appeared in the literature in [RST] (albeit with a different proof). For example, in figure 9 there are three intersection points of which P_1 has multiplicity 2 and the other two have multiplicity 1, so the total weighted number is $4 = \deg \Gamma_1 \cdot \deg \Gamma_2$.

It is easy to see that one can also interpret the intersection multiplicity in terms of the graphs Γ_1 and Γ_2 themselves, without using their Newton subdivisions: if w_1, w_2 are the weights and u_1, u_2 the primitive integral vectors along the two edges meeting in a point $P \in \Gamma_1 \cap \Gamma_2$ then the intersection multiplicity of Γ_1 and Γ_2 in P is $w_1 \cdot w_2 \cdot |\det(u_1, u_2)|$.

It is quite straightforward to generalize our tropical Bézout theorem to other cases. For example, it is easily checked that our genericity assumptions (i.e. that the curves are smooth and do not intersect in vertices) are not necessary — with a suitable definition of intersection multiplicity the same theorem holds if only the intersection $\Gamma_1 \cap \Gamma_2$ is finite. Moreover, by replacing the triangles Δ_d by other convex polytopes the very same proof can be used for tropical curves in toric surfaces (see section 1.5), leading to some very basic intersection theory for curves on such surfaces. The above proof can also be iterated to the case of hypersurfaces: if n tropical hypersurfaces in $\mathbb{R}^n$ intersect in finitely many points then the weighted number of such points is the product of the degrees of the hypersurfaces (resp. the mixed volume of the convex polytopes in the case of hypersurfaces in toric varieties). There is however no tropical version yet of Chow groups and a general intersection theory in the sense of Fulton [F].

Much more surprising is the fact that — unlike in classical geometry — one can in fact get a version of Bézout's theorem without *any* condition on the curves Γ_1 and Γ_2, even if they coincide or share common line segments. In this case the strategy is to move one of the curves, say Γ_2, to a nearby curve Γ_2' so that the intersection is finite. As Γ_2' is moved back to Γ_2 it can be shown that the finitely many intersection points in $\Gamma_1 \cap \Gamma_2'$ have well-defined limit points in $\Gamma_1 \cap \Gamma_2$ (see [RST] section 4). This is called the *stable intersection* of Γ_1 and Γ_2. For example, in the following picture the stable intersection of the conic Γ_1 with itself is simply the union of the four vertices $Q_1, \ldots, Q_4$ of the curve, each counted with multiplicity 1.

$$\Gamma_1 \cap \Gamma_2' = \{P_1, P_2, P_3, P_4\}$$
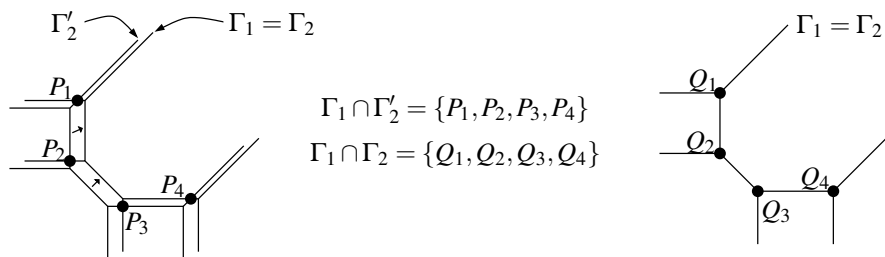$$\Gamma_1 \cap \Gamma_2 = \{Q_1, Q_2, Q_3, Q_4\}$$

FIGURE 10. The stable intersection of a conic with itself

The proof of Bézout's theorem that we have given above is purely combinatorial and does not use the corresponding classical statement. In fact, we have mentioned already that one of the big advantages of tropical geometry is that possibly complicated algebraic or geometric questions can be reduced to entirely combinatorial ones. Sometimes it is also interesting and instructive however to remember that tropical geometry is nothing but an "image of classical geometry (over the field K of Puiseux series) under the valuation map". This way one can try to transfer results from classical geometry directly to tropical geometry. In our case at hand we could proceed as follows: if Γ_1 and Γ_2 are two plane tropical curves of degrees d_1 and d_2 respectively they can be realized as the images under the valuation map of two classical plane curves C_1 and C_2 of these degrees over K. Now if we use the classical Bézout theorem for these curves we can conclude that C_1 and C_2 intersect in $d_1 d_2$ points (counted with the correct multiplicities). Of course the images of these points under the valuation map are intersection points of $\Gamma_1 \cap \Gamma_2$, and in a generic situation these will be the only intersection points of these tropical curves. To make Bézout's theorem hold in the tropical setting we therefore should define the intersection multiplicity of Γ_1 and Γ_2 in a point P to be the sum of the intersection multiplicities of C_1 and C_2 at all points that map to P under the valuation map.

Let us check in a simple example that this agrees with the definition of tropical intersection multiplicity that we have given above. We consider a local situation around an intersection point of Γ_1 and Γ_2. For simplicity let us choose coordinates so that the intersection point is the origin and the two curves have local equations $\Gamma_1 = \{(x_1, x_2);\ x_2 = 0\}$ and $\Gamma_2 = \{(x_1, x_2);\ x_2 = nx_1\}$ for some $n \in \mathbb{N}_{>0}$. We can then choose plane curves C_1 and C_2 mapping to Γ_1 and Γ_2 as in the following picture:

$$\Gamma_1 = \{(x_1, x_2) \in \mathbb{R}^2;\ x_2 = 0\}$$
$$\Gamma_2 = \{(x_1, x_2) \in \mathbb{R}^2;\ x_2 = nx_1\}$$

$$C_1 = \{(z_1, z_2) \in K^2;\ z_2 = 1\}$$
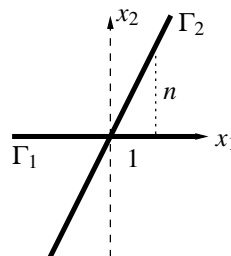$$C_2 = \{(z_1, z_2) \in K^2;\ z_2 = z_1^n\}$$

FIGURE 11. Classical interpretation of the tropical intersection multiplicity

We see that the intersection $C_1 \cap C_2$ consists of n points — corresponding to a choice of n-th root of unity for z_1 — that all map to $(0,0)$ in the tropical picture under the valuation map. So the intersection multiplicity of Γ_1 and Γ_2 should be defined to be n. In fact this agrees with our definition above since the primitive integral vectors of the two tropical curves are $u_1 = (1,0)$ and $u_2(1,n)$, and $|\det(u_1, u_2)| = n$.

2.4. **The group structure of a plane cubic curve.** Again let C be a smooth plane projective complex curve. The group of *divisors* $\operatorname{Div} C$ on C is the free abelian group generated by the points of C, i.e. the points of $\operatorname{Div} C$ are finite formal linear combinations $D = \lambda_1 P_1 + \cdots + \lambda_n P_n$ with $\lambda_i \in \mathbb{Z}$ and $P_i \in C$. For such a divisor we call the number $\lambda_1 + \cdots + \lambda_n \in \mathbb{Z}$ the degree of D. Obviously the subset $\operatorname{Div}^0(C)$ of divisors of degree 0 is a subgroup of $\operatorname{Div} C$.

If C' is any other curve then the intersection $C \cap C'$ consists of finitely many points. Hence we can consider this intersection to be an element of $\operatorname{Div} C$, where we count each point with its intersection multiplicity. This divisor is usually denoted $C \cdot C'$. By Bézout's theorem its degree is $\deg C \cdot \deg C'$.

Two divisors $D_1, D_2 \in \operatorname{Div} C$ are called *equivalent* if there are curves C', C'' of the same degree such that $D_1 - D_2 = C \cdot C' - C \cdot C''$. The group of equivalence classes is usually denoted $\operatorname{Pic}(C)$, or $\operatorname{Pic}^0(C)$ if we restrict to divisors of degree 0.

Now if C has degree 3 it can be shown that after picking a base point $P_0 \in C$ the map $C \to \operatorname{Pic}^0(C), P \mapsto P - P_0$ is a bijection. Hence we can use this map to define a group structure on C, or vice versa the structure of a complex algebraic curve on $\operatorname{Pic}^0(C)$. Alternatively, it follows by the degree-genus formula that C has genus 1, i.e. it is a torus. It can be shown that one can realize this torus in the form $\mathbb{C}/\Lambda$ with a lattice $\Lambda \cong \mathbb{Z}^2$ so that the group structure on $\mathbb{C}/\Lambda$ induced by addition on $\mathbb{C}$ is precisely the group structure defined by the bijection with $\operatorname{Pic}^0(C)$ constructed above.

Which of all these results remain true in the tropical world? Let Γ be a plane tropical curve, and let us start by defining the group of divisors $\operatorname{Div} \Gamma$ in the same way as above, i.e. as the free abelian group generated by the points of Γ. As we have seen already that Bézout's theorem holds in the tropical set-up as well we can also define the group $\operatorname{Pic}^0(\Gamma)$ in the same way as before, i.e. it is the group of divisors on Γ of degree 0 modulo those that can be written as $\Gamma \cdot \Gamma' - \Gamma \cdot \Gamma''$ for some tropical curves Γ' and Γ'' of the same degree.

We can also still define a map $\Gamma \to \operatorname{Pic}^0(\Gamma)$ by sending P to $P - P_0$ after choosing a base point $P_0 \in P$. However, this map is *not* a bijection in the tropical case as can be seen from the following picture:
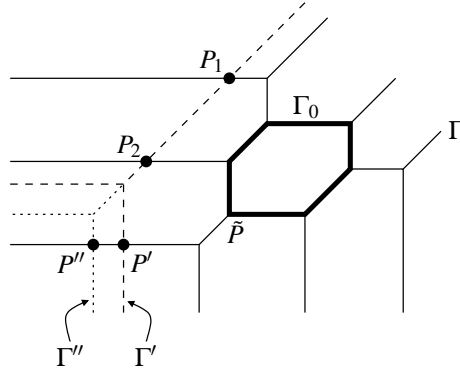

FIGURE 12. The group $\operatorname{Pic}^0(\Gamma)$ of a plane cubic curve Γ

As the two lines Γ' and Γ'' meet the cubic curve Γ in the points P_1, P_2, P' and P_1, P_2, P'', respectively, it follows that P' and P'' are equivalent; in particular $P' - P_0 = P'' - P_0 \in \operatorname{Pic}^0(\Gamma)$ for any choice of base point P_0.

In fact, if Γ_0 is the unique loop in Γ (drawn in bold in the picture above) Vigeland [V] has shown by the same techniques that any point of $\Gamma \backslash \Gamma_0$ is equivalent to the point of Γ_0 "nearest to it", e.g. P' and P'' in figure 12 are both equivalent to $\tilde{P}$. Moreover, he has proven that the map $P \mapsto P - P_0$ in fact gives a bijection of Γ_0 with $\operatorname{Pic}^0(\Gamma)$, so that the loop Γ_0 inherits a group structure from $\operatorname{Pic}^0(\Gamma)$. In fact, this

group structure is just the ordinary group structure of the unit circle S^1 after choosing a suitable bijection of S^1 with Γ_0.

So to a certain extent we can find analogues of the classical results about plane cubics in the tropical world as well. Despite these encouraging results one should note however that there is no good theory of divisors and their equivalence yet on arbitrary tropical curves.

3. TROPICAL TECHNIQUES IN ENUMERATIVE GEOMETRY

In the last chapter we have seen that many classical results from algebraic geometry have a tropical counterpart. However, the main reason why tropical geometry received so much attention recently is that it can be used very successfully to solve even complicated problems in complex and real enumerative geometry. So in the rest of this paper we want to give a brief sketch of the progress that has been made so far in tropical enumerative geometry.

3.1. **Complex enumerative geometry and Gromov-Witten invariants.** If we stick to plane curves the main basic question in complex enumerative geometry is: given $d \geq 1$ and $g \geq 0$, what are the numbers $N_{g,d}$ of curves of genus g and degree d in the complex projective plane that pass through $3d + g - 1$ general given points? (The number $3d + g - 1$ is chosen so that a naive count of dimensions versus conditions leads one to expect a finite non-zero answer.) Except for some special cases the answer to this problem has not been known until the invention of Gromov-Witten theory about ten years ago.

For curves in projective spaces the main objects of study in Gromov-Witten theory are the so-called *moduli spaces of stable maps* $\bar{M}_{g,n}(r,d)$ for $n, r \geq 0$. We will be more specific about the definition of these spaces in section 3.2 — for the moment it suffices to say that they are reasonably well-behaved, compact spaces that parametrize curves of genus g and degree d with n marked points in $\mathbb{P}^r$. Of course there are *evaluation maps* $\mathrm{ev}_i : \bar{M}_{g,n}(r,d) \to \mathbb{P}^r$ for $i = 1, \ldots, n$ that map such an n-pointed curve to the position of its i-th marked point in $\mathbb{P}^r$.

In the case of plane curves mentioned above we now set $n = 3d + g - 1$ and choose n general points $P_1, \ldots, P_n \in \mathbb{P}^2$. The intersection $\mathrm{ev}_1^{-1} P_1 \cap \cdots \cap \mathrm{ev}_n^{-1} P_n$ then obviously corresponds to those plane curves of the given genus and degree that pass through the specified points. As we have chosen n so that the expected dimension of this intersection is 0 it makes sense to *define* the number $N_{g,d}$ to be the (zero-dimensional) intersection product

$$\mathrm{ev}_1^* P_1 \cdot \cdots \cdot \mathrm{ev}_n^* P_n \in \mathbb{Z}$$

on $\bar{M}_{g,n}(r,d)$. Taking this as a definition has the advantage that we do not have to care about whether (or for which collections of points P_i) the number of curves through the P_i actually *is* finite — we get a well-defined number in any case. These numbers are called the Gromov-Witten invariants.

In sections 3.2 and 3.3 we will explain how these numbers can actually be computed — both in Gromov-Witten theory and in tropical geometry. For the moment let us just explain how our problem can be set up in the tropical world. In the same way as at the end of section 2.3 the idea is of course simply to map the whole situation to the real plane by the logarithm resp. valuation map. A plane curve over $\mathbb{C}$ resp. the field of Puiseux series K through some points $P_1, \ldots, P_n$ then simply maps to a plane tropical curve in $\mathbb{R}^2$ through the n image points. It should therefore also be possible to compute the numbers $N_{g,d}$ by counting plane *tropical* curves of the given genus and degree through n given points in the real plane. The following picture shows the simplest example of this statement, namely that also in the tropical world there is always exactly one line through two given (general) points. Note that the relative position of the points in the plane determines on which edges of the tropical line the points lie (i.e. whether we are in case (a), (b), or (c)):
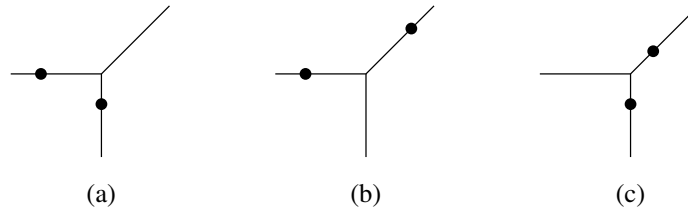
FIGURE 13. There is always one tropical line through two given (general) points

This program has been carried out successfully for all $N_{g,d}$ (and in fact for curves in any toric surface) by Mikhalkin [M1]. One of the main problems when transferring the situation to the tropical world is to determine — in the same way as for Bézout's theorem at the end of section 2.3 — *how many* complex curves through the given points map to the same tropical curve, i.e. with what multiplicity the tropical curves have to be counted. The answer to this question turns out to be surprisingly simple and even independent of the chosen points: let $\Gamma \subset \mathbb{R}^2$ be a tropical curve through the given points. If the points are in general position then all vertices of Γ will have valence 3. For any such vertex V we first define its multiplicity to be $w_1 w_2 |\det(u_1, u_2)|$, where w_1, w_2, w_3 and u_1, u_2, u_3 are the weights and primitive integer vectors along the three edges adjacent to V (the balancing condition ensures that it does not matter which two of the edges we use in the formula). The multiplicity of Γ is then simply the product of the multiplicities of all its vertices. For example, the multiplicity of the following curve is 4 (the multiplicity of both vertices is 2):
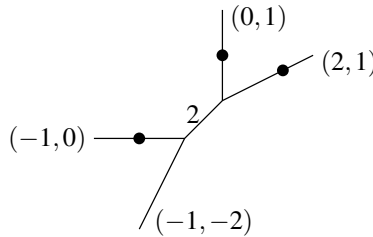

FIGURE 14. A tropical curve whose multiplicity is 4

(Note that this is not a plane curve of some degree since the ends do not point in the right directions for this. We can interpret figure 14 either as a tropical curve in a different toric surface or as a local picture of a plane tropical curve.)

Mikhalkin's "Correspondence Theorem" now states that this is precisely the correct multiplicity for our purposes, i.e. that the numbers $N_{g,d}$ of complex curves through given points $P_1, \ldots, P_n$ are the same as the numbers of tropical curves of the same genus and degree through the images of $P_1, \ldots, P_n$ under the logarithm (resp. valuation) map.

One easy corollary of this statement is worth mentioning: as the numbers of complex curves do not depend on the choice of points P_i (as long as they are in general position) the same must be true in the tropical setting. To see that this is in fact a non-trivial statement let us consider again the tropical curve of figure 14. Because of the balancing condition this curve is fixed in the plane by the directions of the outer edges and the positions of the three marked points in the plane. If we now move the rightmost point down this has the effect of shrinking the bounded edge of the curve (see picture (a) below) until we reach a curve with a 4-valent vertex in (b) (note that the slopes of all edges are fixed):

multiplicity 4
(a)

(b)

multiplicity 3
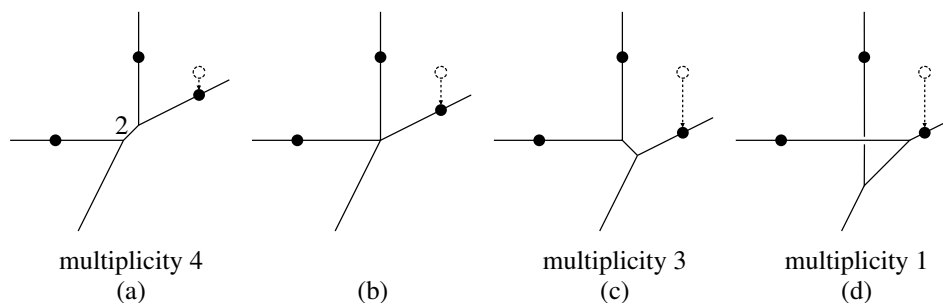(c)

multiplicity 1
(d)

FIGURE 15. The weighted number of tropical curves does not change when moving the points

If we move the marked point down further we see that there are now *two* combinatorially different possibilities (c) and (d) what the curve might look like. These two types have multiplicities 3 and 1 respectively, so their total *weighted* sum is the same as before, in accordance with Mikhalkin's theorem. A purely tropical proof of the statement that the weighted number of tropical curves through some given points does not depend on the choice of points can be found in [GM1].

It would of course be desirable to set up tropical moduli spaces of stable maps that are themselves tropical varieties, establish a tropical intersection theory and use this to explain both the definition of the multiplicity of the curves and the constancy of the numbers $N_{g,d}$ by general principles of this theory — in the same way as in the complex case. So far it is not known how to do this however.

Before we explain in the next sections how the numbers $N_{g,d}$ can be computed in Gromov-Witten theory and how these methods can be transferred to the tropical setting we should mention that Mikhalkin has also found a different way to calculate these numbers tropically by an algorithm that has no analogue in classical geometry [M1]. His idea is to identify the curves in question by their Newton subdivisions and thus translate the problem into one of counting subdivisions of the polytope Δ_d with certain properties. In comparison to the algorithms that we will describe next this method has a very high combinatorial complexity however and is therefore probably not suited very well for actual numerical computations.

3.2. **The tropical WDVV equations.** After having discussed the tropical way to set up enumerative problems let us now focus on how the numbers can be computed. We will first deal with *rational* curves, i.e. with the numbers $N_d := N_{0,d}$ of plane curves of genus 0 that pass through $3d - 1$ points. These numbers have first been computed by Kontsevich about ten years ago. His result was that the numbers N_d are given recursively by the initial value $N_1 = 1$ and the equation

$$N_d = \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right) N_{d_1} N_{d_2}$$

for $d > 1$ (see [KM] claim 5.2.1).

The main tool in deriving this formula is the so-called WDVV equations. To explain the origin of these equations we have to study the moduli spaces of stable maps $\bar{M}_{0,n}(r,d)$ in a little more detail first. One of the key ideas in the construction of these spaces is that it is usually better to parametrize curves in $\mathbb{P}^r$ as maps from an abstract curve to $\mathbb{P}^r$ rather than embedded curves in $\mathbb{P}^r$ (hence the name "stable maps"). More precisely, the points of $\bar{M}_{0,n}(r,d)$ are in bijection to tuples $(C,x_1,\ldots,x_n,f)$ where $x_1,\ldots,x_n$ are distinct smooth points on a rational nodal curve C and $f : C \to \mathbb{P}^r$ is a morphism of degree d (with a stability condition). A special case of this construction is $r = 0$ when there is no map and we simply parametrize certain "stable" nodal rational curves with n marked points. This gives rise to the well-known *moduli spaces of stable curves* $\bar{M}_{0,n}$ that have first been considered by Deligne and Mumford [DM]. The most important example for our purposes is the space $\bar{M}_{0,4}$ of 4-pointed stable rational

curves. It is well-known that this space is isomorphic to $\mathbb{P}^1$. The general point of $\bar{M}_{0,4}$ corresponds to a smooth rational curve with 4 distinct marked points, whereas there are also three special points corresponding to the curves
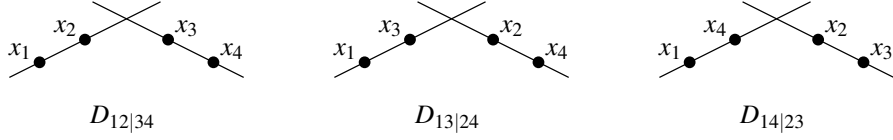

$$D_{12|34} \qquad\qquad D_{13|24} \qquad\qquad D_{14|23}$$

FIGURE 16. The three special points in $\bar{M}_{0,4}$

As $\bar{M}_{0,4}$ is isomorphic to $\mathbb{P}^1$ these three points obviously define the same homology class (resp. the same divisor) on this moduli space.

The important point is now that there are "forgetful maps" $\pi : \bar{M}_{0,n}(r,d) \to \bar{M}_{0,4}$ for all $n \geq 4$ that send a stable map $(C,x_1,\ldots,x_n,f)$ to (the stabilization of) $(C,x_1,\ldots,x_4)$. Pulling back the equality $D_{12|34} = D_{13|24}$ of homology classes on $\bar{M}_{0,4}$ we conclude that $\pi^*D_{12|34}$ and $\pi^*D_{13|24}$ define the same homology class in $\bar{M}_{0,n}(r,d)$. Now $\pi^*D_{12|34}$ (and similarly of course for $\pi^*D_{13|24}$) can be described explicitly as the locus of all *reducible* stable maps with two components such that the marked points x_1,x_2 lie on one component and x_3,x_4 on the other. Note that this space has many irreducible components since the degree and the marked points $x_5,\ldots,x_n$ can be distributed onto the two components in an arbitrary way.

If we now intersect the equation $\pi^*D_{12|34} = \pi^*D_{13|24}$ of codimension-1 cycles with suitable cycles of dimension 1 (that correspond to the condition that the curves in question pass through given subspaces of $\mathbb{P}^r$ at the marked points) we get some equations between certain numbers of reducible curves through given points. But these numbers of reducible curves are just products of the corresponding numbers for their irreducible components, i.e. products of certain numbers of curves of smaller degree. This way one obtains recursion formulas that can be shown to determine all the numbers completely (for more details see e.g. [CK] section 7.4.2); in the case of $\mathbb{P}^2$ we get Kontsevich's formula stated above.

It has been shown very recently that Kontsevich's formula can also be proven in essentially the same way in tropical geometry [GM3]. For a tropical version of the above proof it is very important that we also adapt the "stable map picture" to the tropical setting, i.e. parametrize plane tropical curves as maps from an "abstract tropical curve" to $\mathbb{P}^r$. Here by abstract tropical curve we simply mean a connected graph Γ obtained by glueing closed (not necessarily bounded) real intervals together at their boundary points in such a way that every vertex has valence at least 3. In particular, every bounded edge of such an abstract tropical curve has an intrinsic length. Following an idea of Mikhalkin [M3] the unbounded ends of Γ will be labeled and called the marked points of the curve. The most important example for our applications is of course the "tropical $\bar{M}_{0,4}$" whose points correspond to tree graphs with 4 unbounded ends. There are four possible combinatorial types for this:
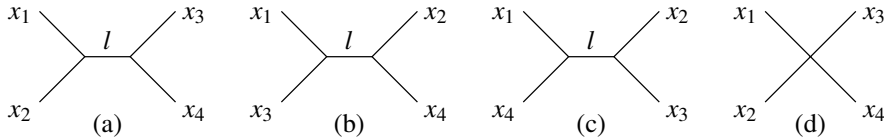

FIGURE 17. The four combinatorial types of curves in the tropical $\bar{M}_{0,4}$

In the types (a) to (c) the bounded edge has an intrinsic length l; so each of these types leads to a stratum of $\bar{M}_{0,4}$ isomorphic to $\mathbb{R}_{>0}$ parametrized by this length. The last type (d) is simply a point in $\bar{M}_{0,4}$ that

can be seen as the boundary point where the other three strata meet. Hence $\bar{M}_{0,4}$ is again a rational tropical curve:
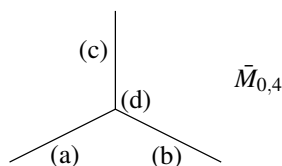

FIGURE 18. $\bar{M}_{0,4}$ is again a rational tropical curve

In analogy to the complex case plane tropical curves are now parametrized as tuples $(\Gamma, x_1, \ldots, x_n, h)$, where Γ is an abstract tropical curve, $x_1, \ldots, x_n$ are distinct unbounded ends of Γ, and $h : \Gamma \to \mathbb{R}^2$ is a piecewise linear map with certain conditions (see [GM3] for details). The most important feature of this definition is that h may be a constant map on some edges of Γ, and is in fact *required* to be a constant map on the unbounded ends $x_1, \ldots, x_n$. For example, the following picture shows a 4-pointed plane tropical conic, i.e. of the tropical analogue of $\bar{M}_{0,4}(2,2)$:
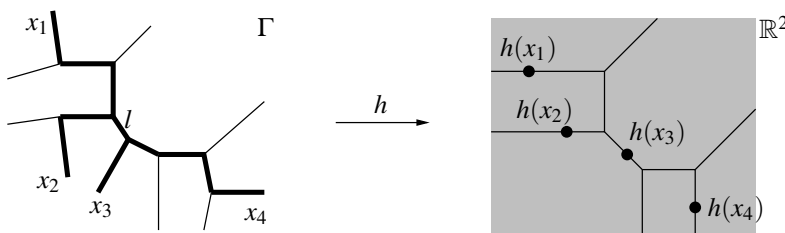

FIGURE 19. A tropical stable map in $\bar{M}_{0,4}(2,2)$

Note that the balancing condition around every (3-valent) vertex adjacent to a marked point (resp. edge) x_i ensures that the other two edges around this vertex form a straight line in $\mathbb{R}^2$.

It is easy to see from this picture already that the tropical moduli spaces $\bar{M}_{0,n}(r,d)$ with $n \geq 4$ admit forgetful maps to $\bar{M}_{0,4}$: given an n-marked plane tropical curve $(\Gamma, x_1, \ldots, x_n, h)$ we simply forget the map h, take the minimal connected subgraph of Γ that contains $x_1, \ldots, x_4$, and "straighten" this graph to obtain an element of $\bar{M}_{0,4}$. In the picture above we simply obtain the "straightened version" of the subgraph drawn in bold, i.e. the element of $\bar{M}_{0,4}$ of type (a) in figure 17 with length parameter l as indicated in the picture.

To obtain the WDVV equations we now simply consider the inverse image under this forgetful map of a point of $\bar{M}_{0,4}$ of type (a) resp. (b) in figure 17 with a very large length parameter l. It can be shown that such very large lengths can occur only if there is a bounded edge (of a very large length) in Γ on which h is constant:
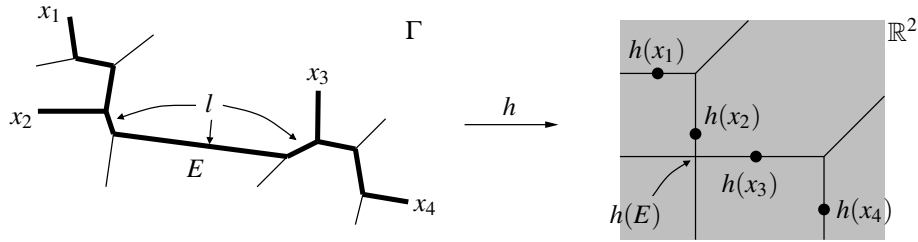
FIGURE 20. A very large length in $\bar{M}_{0,4}$ leads to a tropical stable map with a contracted bounded edge

Again the balancing condition around the contracted bounded edge requires that the image of the tropical curve in $\mathbb{R}^2$ is locally a union of two straight lines. We can therefore consider such curves as being reducible and made up of two tropical curves of smaller degrees (in figure 20 we have a reducible tropical conic that is a union of two tropical lines). The picture is now exactly the same as in the classical case, and in fact the rest of the proof of Kontsevich's formula works in the same way as in Gromov-Witten theory. It is expected that essentially the same proof can be used to reprove the WDVV equations for rational curves in higher-dimensional spaces as well.

This application of tropical geometry shows very well that it should be possible to carry many concepts from classical complex geometry over to the tropical world: moduli spaces of curves and stable maps, morphisms, divisors and divisor classes, intersection multiplicities, and so on. In [GM3] these concepts were introduced only in the specific cases needed for Kontsevich's formula.

3.3. **The tropical Caporaso-Harris formula.** After having discussed rational curves let us now turn to the general numbers $N_{g,d}$ for arbitrary genus g. These numbers have first been computed by Caporaso and Harris [CH]. The idea in their proof is is to define new invariants that count plane curves of given degree and genus having specified local contact orders to a fixed line L and passing in addition through the appropriate number of general points. By specializing one point after the other to lie on L one can then derive recursive relations among these new invariants that finally suffice to compute all the numbers $N_{g,d}$.

Instead of explaining the general formula let us look at an example of what happens in this specialization process, referring to [CH] for details. We consider plane rational cubics having a point of contact order 3 to L at a fixed point $P_1 \in L$ and passing in addition through 5 general points $P_2, \ldots, P_6 \in \mathbb{P}^2$ as in the following picture on the left:
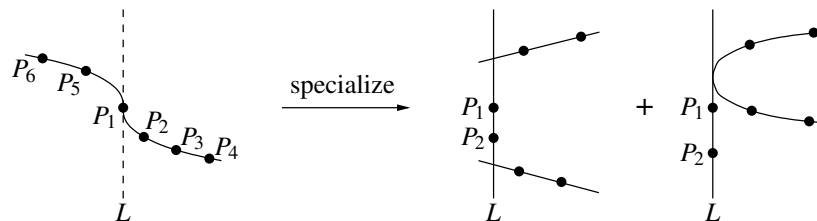

FIGURE 21. Computing the number of cubics with a point of contact order 3 to a line

To compute the number of such curves we specialize P_2 to lie on L. As the cubics intersect L already with multiplicity 3 at P_1 they cannot pass through another point on L unless they become reducible and have L as a component. Hence there are two possibilities after the specialization (see figure 21): the cubics can degenerate into a union of three lines $L \cup L_1 \cup L_2$ where L_1 and L_2 each pass through

two of the points $P_3, \ldots, P_6$, or they can degenerate into $L \cup C$, where C is a conic tangent to L and passing through $P_3, \ldots, P_6$. The initial number of rational cubics with a point of contact order 3 to L at a fixed point and passing through 5 more general points is therefore a sum of two numbers (counted with suitable multiplicities) related to only lines and conics. This is the general idea of Caporaso and Harris how specialization finally reduces the degree of the curves and allows a recursive solution to compute the numbers $N_{g,d}$ as well as all the newly introduced numbers of curves with multiplicity conditions.

In fact, the same constructions can again be made in tropical geometry [GM2]. Intuitively, if we pick our line $L \subset \mathbb{C}^2$ to be the line with z_1-coordinate 0 then its image under the logarithm map is "the vertical line with x_1-coordinate $-\infty$". Hence the process of moving P_2 to the line L in complex geometry now simply corresponds to moving P_2 to the very far left in tropical geometry. Moreover, curves with higher contact orders to L in complex geometry just correspond to tropical curves with unbounded ends of higher weight to the left. So the tropical analogue of the specialization process of figure 21 is
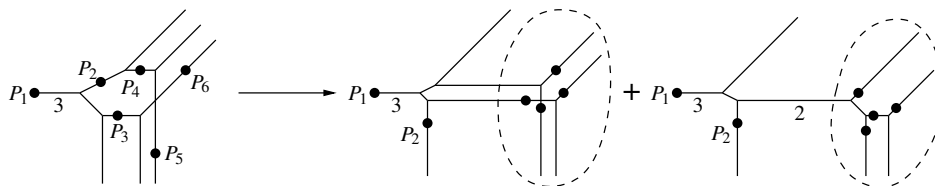

FIGURE 22. The degeneration of figure 21 in the tropical setting

where P_1 is to be considered to lie infinitely far to the left. Note that the curves after moving P_2 to the left are not reducible, but they still "split" into two parts: a left part (through P_2) and a right part (through the remaining points, circled in the picture above). We get the same "degenerations" as in the complex case: one where the right part consists of two lines through two of the points $P_3, \ldots, P_6$ each, and one where it consists of a conic "tangent to a line" (i.e. with an unbounded edge of multiplicity 2 to the left).

Using this idea it has been shown in [GM2] that the Caporaso-Harris formula can also be proven in tropical geometry — in fact with a much simpler proof than in complex geometry since we are only dealing with combinatorial objects and do not have to construct and study complicated moduli spaces in complex geometry.

3.4. **Real enumerative geometry and Welschinger invariants.** Of course, the same questions as in the previous sections can be asked for real instead of for complex curves: given $d \geq 1$, $g \geq 0$, and $3d + g - 1$ points in general position in the real projective plane $\mathbb{P}^2_{\mathbb{R}}$, how many curves of degree d and genus g are there in $\mathbb{P}^2_{\mathbb{R}}$ that intersect all the given points? Note that every such real curve has a complexification by just considering its equation as an equation in the complex rather than the real plane, and by its degree and genus we simply mean the degree resp. genus of its complexification.

As usual in algebraic geometry this real case is much more difficult to handle than the complex case. The first problem is already that the answer to this question will in general depend on the position of the points so that there are no well-defined numbers $N_{g,d}$ as in the complex case. Instead one could ask questions of the following type:

- is there at least one real curve of degree d and genus g through any choice of $3d + g - 1$ given points? Or even better: can we compute a lower bound for the number of real curves through any choice of points?
- Is there a way to assign multiplicities to the real curves through the given points so that the *weighted* sum of these curves is independent of the choice of points?

A few years ago Welschinger has found a solution to the second question for the case of curves of genus 0 [W]. To state his result note first that a general *complex* plane rational curve of degree d has exactly

$\frac{1}{2}(d-1)(d-2)$ nodes, i.e. points where two smooth branches of the curve intersect transversally. If C is now a real rational plane curve then each of the nodes of its complexification is of one of the following three types:

(a) nodes that are in the real plane $\mathbb{P}^2_{\mathbb{R}}$ and where the local equation of the curve is of the form $x^2 - y^2 = 0$, i.e. $(x+y)(x-y) = 0$ for suitable local analytic coordinates. In a local real picture C is simply the union of two smooth curves intersecting transversely.

(b) nodes that are in the real plane $\mathbb{P}^2_{\mathbb{R}}$ and where the local equation of the curve is of the form $x^2 + y^2 = 0$ for suitable local analytic coordinates. In a local real picture such a node leads to an isolated point corresponding to the values $x = y = 0$.

(c) nodes that are not in the real plane $\mathbb{P}^2_{\mathbb{R}}$. These nodes obviously come in pairs as the complex conjugate of such a node is again a node of the complexification. They are not visible in the real curve C.

Welschinger's main theorem is now the following: if we assign to each real plane curve C the multiplicity $(-1)^m$ where m is the number of nodes of type (b) of its complexification then the corresponding weighted sum of all curves through the given points is independent of the choice of points. It is called the *Welschinger invariant W_d*. In particular, W_d gives a lower bound for the actual number of real plane curves through any set of $3d - 1$ general given points, whereas the complex number $N_{0,d}$ is of course an upper bound.

Unfortunately, except for a few special cases Welschinger was not able to actually compute the numbers W_d. So the question whether there always exists a real rational plane curve of degree d through any set of $3d - 1$ general points remained open.

Some time ago however a tropical way has been found to compute the Welschinger invariants W_d and in fact also the actual (non-invariant) numbers of real curves through some configurations of points in the plane [IKS1, M1]. In the same way as in Mikhalkin's Correspondence Theorem the strategy is to identify complex resp. real algebraic curves with tropical curves, and then to count such tropical curves with the proper multiplicities. More precisely, to obtain the Welschinger invariant W_d one has to count rational plane tropical curves of degree d through $3d - 1$ points in the same way as for the computation of $N_{0,d}$ — but one does not count them with the "complex multiplicity" as described in section 3.1, but rather with the multiplicity

$$
\begin{array}{rl}
0 & \text{if the ``complex multiplicity'' is even,} \\
1 & \text{if the ``complex multiplicity'' is congruent to 1 modulo 4,} \\
-1 & \text{if the ``complex multiplicity'' is congruent to 3 modulo 4.}
\end{array}
$$

One can then try to do this count by enumerating the corresponding Newton subdivisions of the polytope Δ_d as in section 1.4. Although the algorithm is combinatorially very complicated (and cannot be explained here in detail) it can be used to prove that the numbers W_d are all positive and hence that there is always at least one real rational plane curve of degree d through any set of $3d - 1$ points in general position. In fact, a more careful study of the algorithm even allows one to prove that the lower and upper bounds W_d resp. $N_{0,d}$ for the numbers of these curves grow approximately with the same speed as d increases [IKS2].

Very recently Itenberg has shown that both the proof of [GM1] that the invariants are independent of the marked points and the proof of [GM2] of the tropical Caporaso-Harris formula (see section 3.3) can be adapted to the Welschinger case. In particular, this yields a "real Caporaso-Harris formula" that gives a fast method to compute the Welschinger invariants. It is not known yet whether there exists an analogue of the WDVV equations (see section 3.2) in the real case.

CONCLUSION

In the last few years tropical algebraic geometry has evolved with a tremendous speed. Its general approach to replace algebro-geometric problems by combinatorial ones often leads to new insights, sometimes to easier proofs of known statements, and occasionally even to new results in algebraic geometry.

Nevertheless tropical geometry is still in its beginnings since even the most basic objects of algebraic geometry — (abstract) varieties and their morphisms — do not have a satisfactory counterpart yet in the tropical world. Consequently, there are many open problems in tropical geometry for the near future, and one could reasonably expect that the solution to these problems gives an entirely new strategy to attack many problems in algebraic geometry. In fact, two such recent new examples in which tropical ideas have already been applied successfully are the study of compactifications of subvarieties of algebraic tori (in particular moduli spaces of rational stable curves) [Te] and low-dimensional topology [Ti].

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