Counting rational curves with multiple points and Gromov-Witten invariants of blow-ups

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Abstract

We study Gromov-Witten invariants on the blow-up of $\mathbb{P}^n$ at a point, which is probably the simplest example of a variety whose moduli spaces of stable maps do not have the expected dimension. It is shown that many of these invariants can be interpreted geometrically on $\mathbb{P}^n$ as certain numbers of rational curves having a multiple point of given order at the blown up point. Moreover, all these invariants can actually be calculated, giving enumerative invariants of $\mathbb{P}^n$ which have not been known before.

0 Introduction

Over the last few years, Gromov-Witten invariants have become a very powerful tool in enumerative geometry. Let us briefly recall their definition. If X is an n-dimensional smooth complex projective variety and $\beta \in H_2(X)$ a homology class, then for every $N \geq 3$ one defines a moduli space $\bar{M}_N(X,\beta)$ of genus zero N-pointed stable maps into X, which is a compactification of the space of all maps $f : C \to X$, where C is a smooth N-pointed rational curve (which is allowed to vary) [BM]. If X is a convex variety, i.e. if $h^1(C, f^*T_X) = 0$ for all $f : C \to X$, then this moduli space is a smooth Deligne-Mumford stack of the expected dimension

$$\text{virt dim}\,\bar{M}_N(X,\beta) = -K_X \cdot \beta + n - 3 + N.$$

Now if $\alpha_1, \dots, \alpha_N \in A^*(X)$ are cohomology classes whose codimensions sum up to virt dim $\bar{M}_N(X,\beta)$, one defines the associated Gromov-Witten invariant by

$$\Phi_X(\alpha_1, \dots, \alpha_N \mid \beta) = (p_1^*\alpha_1 \cdots p_N^*\alpha_N) \cdot [\bar{M}_N(X,\beta)] \qquad (*)$$

where $p_i : \bar{M}_N(X,\beta) \to X$ are the obvious evaluation maps and $[\bar{M}_N(X,\beta)]$ denotes the fundamental class of $\bar{M}_N(X,\beta)$. Geometrically, this invariant can be interpreted as the number of rational curves in X of homology class β which pass through generically chosen subvarieties $V_i \subset X$ with $[V_i] = \alpha_i$.

If X is not a convex variety, however, the actual dimension of $\bar{M}_N(X,\beta)$ will in general be greater than the expected one, such that the above definition of the Gromov-Witten invariants is not applicable. In this case, K. Behrend [B] has shown recently that it is possible to define a virtual fundamental class

$$[\bar{M}_N(X,\beta)]^{virt} \in A_{\text{virt dim}\,\bar{M}_N(X,\beta)}(\bar{M}_N(X,\beta))$$

such that, if one uses this class in $(*)$ instead of the usual fundamental class, this defines Gromov-Witten invariants (satisfying the usual axioms [KM]) on an arbitrary smooth complex projective variety X. Of course, in this case there is no longer an obvious geometric interpretation of the invariants.

In this paper, we study this construction in the case where $X = \tilde{\mathbb{P}}^n$ is the blow-up of projective n-space in a point $P \in \mathbb{P}^n$. Let H' be the class of a line in $\mathbb{P}^n$ and E' be the class of a line in the exceptional divisor E. We consider the commutative diagrams

$$\begin{array}{ccc} \bar{M}_N(\tilde{\mathbb{P}}^n, dH'-eE') & \xrightarrow{\phi} & \bar{M}_N(\mathbb{P}^n, dH') \\ \tilde{p}_i \downarrow & & \downarrow p_i \\ \tilde{\mathbb{P}}^n & \xrightarrow{\pi} & \mathbb{P}^n \end{array}$$

and show that, although there are components in $\bar{M}_N(\tilde{\mathbb{P}}^n, dH'-eE')$ whose dimension is too large, these are actually mapped by ϕ to a subspace in $\bar{M}_N(\mathbb{P}^n, dH')$ whose dimension is *smaller* than the expected one, which means that they are irrelevant if one can compute the intersection products on the moduli space $\bar{M}_N(\mathbb{P}^n, dH')$. This is obviously the case if all the classes α_i in the Gromov-Witten invariant are pullbacks of classes on $\mathbb{P}^n$. Hence, in this case it will again be possible to give a geometric interpretation of the invariants.

It is even possible to give an interpretation of the Gromov-Witten invariants on $\tilde{\mathbb{P}}^n$ in terms of curves on $\mathbb{P}^n$: via strict transform, curves of degree d in $\mathbb{P}^n$ which pass through P with total multiplicity e correspond to curves in $\tilde{\mathbb{P}}^n$ of homology class $dH'-eE'$. This will lead to our main result (proposition 4.5):

Let $d > 0$, $e \geq 0$ and $\alpha_1, \dots, \alpha_N \in A^{\geq 1}(\mathbb{P}^n)$ such that

$$\sum_i (\operatorname{codim} \alpha_i - 1) = d\,(n+1) - e\,(n-1) + n - 3.$$

Let $P \in \mathbb{P}^n$ be a point. Choose generic subschemes $V_i \subset \mathbb{P}^n$ with $[V_i] = \alpha_i$ (in a sense that will be made precise).

Then the number of rational curves in $\mathbb{P}^n$ (purely 1-dimensional subvarieties birational to $\mathbb{P}^1$) of degree d which have non-empty intersection with all V_i and which pass through *the point P with total multiplicity e, where each such curve C is counted with multiplicity*

$$\sharp(C \cap V_1) \cdots \sharp(C \cap V_N),$$

is equal to the Gromov-Witten invariant on $\tilde{\mathbb{P}}^n$

$$\Phi_{\tilde{\mathbb{P}}^n}(\pi^*\alpha_1, \ldots, \pi^*\alpha_N \mid dH' - eE').$$

Moreover, it will be shown that all Gromov-Witten invariants of $\mathbb{P}^n$ can be computed using the First Reconstruction Theorem of Kontsevich and Manin [KM] and some initial data that we will calculate.

In the case $n = e = 2$, our results reproduce the numbers of rational curves of degree d in $\mathbb{P}^2$ having a node in P and passing through $3d-3$ further points in the plane, which have already been computed last year by R. Pandharipande [P] using different methods.

I have been informed that L. Göttsche and R. Pandharipande have been working on Gromov-Witten invariants of multiple blow-ups of $\mathbb{P}^2$ and their geometric interpretation.

The paper is organized as follows: In section 1 we will give a correspondence between stable maps to X and embedded curves in X. In section 2 we calculate the cohomology $h^1(C, f^*T_{\tilde{\mathbb{P}}^n})$ for all maps $f : C \to \tilde{\mathbb{P}}^n$ which will enable us in section 3 to prove the statement on the dimension of the image of the map $\phi : \bar{M}_N(\tilde{\mathbb{P}}^n, dH' - eE') \to \bar{M}_N(\mathbb{P}^n, dH')$ that was mentioned above. The proof of our main result on the geometric meaning of the invariants on $\tilde{\mathbb{P}}^n$ will be given in section 4. Finally, in section 5 we show how to calculate the invariants on $\tilde{\mathbb{P}}^n$ and give some examples.

Notations and Conventions. We will always work over the ground field $\mathbb{C}$ of complex numbers. If X is an n-dimensional smooth projective variety, we follow [BM] and let

$$H_2^+(X) = \{\beta \in \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic} X, \mathbb{Z}) \mid \beta(L) \geq 0 \text{ whenever } L \text{ is ample}\}$$

be the semigroup of "positive homology classes" in X. If $N > 3$ and $\beta \in H_2^+(X)$, we denote by $\bar{M}_N(X, \beta)$ the Deligne-Mumford stack of N-pointed stable maps in X of genus zero and homology class β as defined in [BM]. In this paper, the terminology *stable map* will always be used to denote a stable map of genus zero. A stable map $(C, x_1, \ldots, x_N, f)$ will be called *irreducible* if the curve C is, and reducible otherwise.

If $\alpha \in A^c(X)$ is a cycle, we denote the *codimension* of α by $c = \operatorname{codim} \alpha$. If $\alpha_1, \ldots, \alpha_N \in A^*(X)$ are cycles on X whose codimensions sum up to the virtual dimension

$$\text{virt dim}\, \bar{M}_N(X, \beta) = -K_X \cdot \beta + n - 3 + N$$

of $\bar{M}_N(X, \beta)$, then K. Behrend [B] has defined an associated Gromov-Witten invariant which is denoted by

$$\Phi_X(\alpha_1, \ldots, \alpha_N \mid \beta) = (p_1^*\alpha_1 \cdots p_N^*\alpha_N) \cdot [\bar{M}_N(X, \beta)]^{virt}$$

where $p_i : \bar{M}_N(X, \beta) \to X$ are the evaluation maps and $[\bar{M}_N(X, \beta)]^{virt}$ is the virtual fundamental class of the moduli space $\bar{M}_N(X, \beta)$.

1 Stable maps and their images

Gromov-Witten invariants are concerned with stable maps into X. Since our final aim is to make statements about the numbers of rational curves embedded in a variety X, we start by collecting some relations between these two points of view. So let us begin by defining the moduli spaces between which we will find a correspondence later.

Definition 1.1 Let X be a smooth projective variety over $\mathbb{C}$, $0 \neq \beta \in H_2^+(X)$ a fixed homology class, and $N \geq 3$.

(i) A *relevant map* in X is a (genus zero) stable map $(C, x_1, \ldots, x_N, f)$ such that $[f(C)] = f_*[C] = \beta \in H_2^+(X)$, i.e. there is no irreducible component of C on which f is a finite covering. (The map f may, however, contract some irreducible components of C to a point.) The set of all such relevant maps in X modulo isomorphism can be regarded as a substack of $\bar{M}_N(X, \beta)$. It will be denoted by $RM_N(X, \beta)$.

(ii) The set of all irreducible relevant maps (i.e. relevant maps as above whose underlying curve C is irreducible) such that $f^{-1}(f(x_i)) = \{x_i\}$ for all i will be denoted by $RM'_N(X, \beta)$.

(iii) A *relevant curve* in X is defined to be a tuple $(D, y_1, \ldots, y_N)$, where D is a (purely) 1-dimensional, closed, connected subvariety of X with $[D] = \beta$ such that every irreducible component of D is rational (i.e. birational to $\mathbb{P}^1$, not necessarily smooth), and where the y_i are points on D, not necessarily distinct. Let $RC_N(X, \beta)$ be the set of all such relevant curves.

Remark. We call such maps "relevant maps" because we will show later (see proposition 4.4) that, under favourable circumstances, we can arrange that all curves counted by the Gromov-Witten invariants are of this type.

The first property we want to show is that every relevant curve is the image of some relevant map. This and the following results in this section will be set-theoretic, since this is all we need.

Proposition 1.2 *Let X be a smooth projective variety, $0 \neq \beta \in H_2^+(X)$, and $N \geq 3$. Then there is a natural surjective map*

$$\mu : RM_N(X, \beta) \to RC_N(X, \beta)$$

which is given by

$$\mu(C, x_1, \ldots, x_N) = (f(C), f(x_1), \ldots, f(x_N)).$$

Proof. Let $(C, x_1, \ldots, x_N, f)$ be a relevant map in X. We have to show that $(f(C), f(x_1), \ldots, f(x_N))$ is a relevant curve.

- Obviously, $f(C) \subset X$ is a 1-dimensional, closed, connected subvariety of X with $f(x_i) \in f(C)$.
- By definition of a relevant map, $[f(C)] = f_*[C] = \beta$.
- Every irreducible component of $f(C)$ can be written as $f(C_0)$ with C_0 a rational irreducible component of C. Since C_0 is rational, so is $f(C_0)$.

This shows that $(f(C), f(x_1), \ldots, f(x_N))$ is a relevant curve.

To show the surjectivity of the map μ, let $(D, y_1, \ldots, y_N)$ be a relevant curve in X and $D = D_1 \cup \cdots \cup D_m$ its decomposition into irreducible components.

By assumption, each D_i is a rational curve, hence we can find rational maps $f_i : C_i \to D_i$ with $C_i \cong \mathbb{P}^1$ for all i. Of course, the f_i have to be surjective morphisms. Moreover, we have $f_*[C_i] = D_i$, since the f_i are birational.

Now the procedure to glue the $f_i : C_i \to X$ to a map $f : C \to X$ and to choose points $x_1, \ldots, x_N \in C$ such that $(C, x_1, \ldots, x_N, f)$ becomes a relevant map with image $(D, y_1, \ldots, y_N)$ under μ is rather obvious, but nevertheless we will give it in detail.

Since D is connected, we can assume that its components D_i are numbered in such a way that for each $i = 2, \ldots, m$ there is an $a(i) \in \{1, \ldots, i-1\}$ such that there exists a point $y'_i \in D_i \cap D_{a(i)}$.

Now, for every $y \in D$ which is either one of the y_k or one of the y'_i, do the following:

If $y = y'_i$ for exactly one $i = 2, \ldots, m$, but $y \neq y_k$ for all $k = 1, \ldots, N$, then glue C_i and $C_{a(i)}$ together at a point which is mapped to y both by f_i and by $f_{a(i)}$.

If $y = y_k$ for exactly one $k = 1, \ldots, N$, but $y \neq y'_i$ for all $i = 2, \ldots, m$, then choose some point x_k in some C_i with $f_i(x_k) = y$.

In all other cases, let C'_y be a smooth rational curve and let $f'_y : C'_y \to \{y\}$ be the constant map. If $y \neq y'_i$ for all $i = 2, \ldots, m$, glue C'_y to some point in some C_i which is mapped to y. Otherwise, for any component C_i such that $y = y'_i$ or $i = a(j)$ for some j with $y = y'_j$, choose a point on this component which is mapped to y and glue it at this point to some point on C'_y. In both cases, for every k with $y = y_k$ choose a point $x_k \in C'_y$. (All points chosen on C'_y, the x_k as well as the points glued to the C_i, have to be distinct.)

Now let C be the union of all C_i and C'_y, glued together as described above. Let $f : C \to X$ be the map induced by f_i and f'_y. Then, as can be seen from the construction, $(C, x_1, \ldots, x_N, f)$ is a relevant map with image $(D, y_1, \ldots, y_N)$ under μ. Hence, the map μ is surjective. □

The next result will eventually allow us to establish a one-to-one correspondence between certain stable maps and their images.

Lemma 1.3 *Let*

$$\mu' = \mu|_{RM'_N(X,\beta)} : RM'_N(X,\beta) \to RC_N(X,\beta)$$

be the restriction of the map considered in proposition 1.2. Then

(i) the image of μ' is contained in the subset of $RC_N(X,\beta)$ parametrizing irreducible curves.

(ii) μ' is injective.

(iii) The stable maps in $RM'_N(X,\beta)$ have no non-trivial automorphisms.

PROOF. If $(C, x_1, \dots, x_N, f) \in RM'_N(X,\beta)$, then C is irreducible, so $f(C)$ is irreducible, too. This shows (i).

Let $(C, x_1, \dots, x_N, f)$ and $(C', x'_1, \dots, x'_N, f')$ be two relevant maps in $RM'_N(X,\beta)$ having the same image

$$(f(C), f(x_1), \dots, f(x_N)) = (f'(C'), f'(x'_1), \dots, f'(x'_N)).$$

Since $C \cong C' \cong \mathbb{P}^1$ and f, f' are birational, we can consider the composition $\alpha : f^{-1} \circ f'$ which is a also a birational map and therefore an isomorphism between C' and C. Because of the condition $f^{-1}(f(x_i)) = \{x_i\}$, α maps each x'_i to x_i and hence induces an isomorphism between the two relevant maps, so they represent the same element in $RM'_N(X,\beta)$, which proves (ii). Finally, the isomorphism constructed is obviously unique, which shows (iii). □

2 Calculation of the obstruction $h^1(C, f^*T_{\tilde{X}})$

We now start to study the relation between stable maps on a variety X and on its blow-up $\tilde{X}$ in a point $P \in X$. Our main problem will be that $\tilde{X}$ is never a convex variety in the sense of [K], since there are always stable maps $(C, x_1, \dots, x_N, f)$ whose image is contained in the excepional divisor where $h^1(C, f^*T_{\tilde{X}})$ does not vanish (see e.g. lemma 2.2). Hence, we expect the moduli spaces of stable maps into $\tilde{X}$ to have the "wrong" dimension. Indeed, it is easy to give an example, even for curves that do not lie entirely in the exceptional divisor:

EXAMPLE. Let $X = \mathbb{P}^n$ and denote by H' the class of a line in X. We consider curves of degree d, whose moduli space is well known to have the expected dimension, namely

$$\dim \bar{M}_N(X, dH') = \text{virt} \dim \bar{M}_N(X, dH') = d\,(n+1) + n - 3 + N.$$

Now consider curves on the blow-up $\tilde{X}$ having homology class dH'. Of course, the virtual dimension remains the same as above, but now we have new possibilities of realizing curves with this homology class using reducible curves where parts of it are lying in the exceptional divisor E: for example, take the strict transform C_1 of any curve of degree d on X passing e times through the blown up point. This curve has homology class

$dH' - eE'$, where E' denotes the class of a line in $E \cong \mathbb{P}^{n-1}$. Since passing through P gives $n-1$ conditions, the dimension of the space of such curves is (at least)

$$D_1 = d\,(n+1) - e\,(n-1) + n - 3.$$

Now take any curve C_2 in the exceptional divisor $E \cong \mathbb{P}^{n-1}$ of degree e, the space of such curves has dimension $en + n - 4$. To be able to glue it to C_1, we have to require it to pass through one of the points where C_1 meets the exceptional divisor, which gives $n-2$ conditions on C_2. Hence, for the choice of C_2, we have

$$D_2 = en - 2$$

degrees of freedom. Now we can consider reducible curves made up of two components C_1 and C_2 as above, these curves have homology class dH' and hence give, together with N marked points, a subspace of $\bar{M}_N(\tilde{X}, dH')$. But the dimension of this space is (at least)

$$D_1 + D_2 + N = d\,(n+1) + n - 3 + N + e - 2$$

which is bigger than the expected dimension of $\bar{M}_N(\tilde{X}, dH')$ if $e > 2$.

It will be the aim of this section to calculate $h^1(C, f^* T_{\tilde{X}})$ if $f : C \to \tilde{X}$ is a map from a prestable curve of genus zero to $\tilde{X}$. This will tell us more precisely which parts of the moduli spaces have the "correct" dimension.

First we introduce some notation which will be used throughout the rest of the paper when dealing with blow-ups. Let X be a smooth n-dimensional projective variety, $n \geq 2$. We will soon specialize to the case where $X = \mathbb{P}^n$, but for the moment we keep it arbitrary.

Let $\tilde{X}$ be the blow-up of X in a fixed point $P \in X$, such that we have a cartesian diagram

$$\begin{array}{ccc} E & \xrightarrow{\ i\ } & \tilde{X} \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ P & \longrightarrow & X \end{array}$$

By a Mayer-Vietoris type argument (see e.g. [GH], ch. 4.1) one sees that

$$H_2(\tilde{X}) = H_2(X) \oplus \mathbb{Z} \cdot E'$$

where E' is the class of a line in the exceptional divisor $E \cong \mathbb{P}^{n-1}$.

In the case $X = \mathbb{P}^n$ we will denote both the hyperplane class of X as well as its pullback to $\tilde{X}$ by H. The class of a line will be denoted H'.

We now start by proving a lemma which will allow us in favourable cases to reduce the calculation of $h^1(C, f^* T_{\tilde{X}})$ to computations that do not involve $\tilde{X}$ but only X.

Lemma 2.1 *Let C be a smooth curve and $f : C \to \tilde{X}$ a morphism such that $f(C) \not\subset E$. Let $D := f^*(E)$, which is an effective divisor on C. Then there is a commutative diagram of sheaves on C*

$$\begin{array}{ccc} f^*\pi^*T_X(-D) & \longrightarrow & f^*\pi^*T_X \\ & \searrow \quad \nearrow & \\ & f^*T_{\tilde{X}} & \end{array}$$

where all the three morphisms are injective, and all of them are isomorphisms away from the support of D.

PROOF. Since $E = P \times_X \tilde{X}$, we have $i^*\Omega_{\tilde{X}/X} = \Omega_{E/P} = \Omega_E$. As $\Omega_{\tilde{X}/X}$ has support on E, this can be rewritten as $i_*\Omega_E = \Omega_{\tilde{X}/X}$. Hence, there is an exact sequence of sheaves on $\tilde{X}$

$$0 \to \pi^*\Omega_X \to \Omega_{\tilde{X}} \to i_*\Omega_E \to 0.$$

Dualizing, we get

$$0 \to T_{\tilde{X}} \to \pi^*T_X \to \mathcal{E}xt^1(i_*\Omega_E, \mathcal{O}_{\tilde{X}}) \to 0.$$

By duality ([HR], thm. III 6.7), we have

$$\mathcal{E}xt^1(i_*\Omega_E, \mathcal{O}_{\tilde{X}}) = i_*\,\mathcal{E}xt^1(\Omega_E, N_{E/\tilde{X}}) = i_*T_E(-1),$$

therefore we get a morphism $\pi^*T_X \to i_*T_E(-1)$ which we can restrict to E to get a morphism $\pi^*T_X|_E \to i_*T_E(-1)$ fitting into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi^*T_X(-E) & \longrightarrow & \pi^*T_X & \longrightarrow & \pi^*T_X|_E & \longrightarrow & 0 \\ & & & & \| & & \downarrow & & \\ 0 & \longrightarrow & T_{\tilde{X}} & \longrightarrow & \pi^*T_X & \longrightarrow & i_*T_E(-1) & \longrightarrow & 0 \end{array}$$

From this we can deduce the existence of a map $\pi^*T_X(-E) \to T_{\tilde{X}}$ giving a commutative diagram

$$\begin{array}{ccc} \pi^*T_X(-E) & \longrightarrow & \pi^*T_X \\ & \searrow \quad \nearrow & \\ & T_{\tilde{X}} & \end{array}$$

with all three morphisms injective. Finally, apply the functor f^* to get the desired result

$$\begin{array}{ccc} f^*\pi^*T_X(-D) & \longrightarrow & f^*\pi^*T_X \\ & \searrow \quad \nearrow & \\ & f^*T_{\tilde{X}} & \end{array}$$

Since $f(C) \not\subset E$ by assumption, the morphisms are still injective, and certainly they are also isomorphisms away from the support of D. $\square$

We are now ready to compute the relevant cohomology groups in the case where $C = \mathbb{P}^1$ and $X = \mathbb{P}^n$, namely $h^1(C, f^*T_{\tilde{X}}(-\epsilon))$ for $\epsilon \in \{0, 1\}$, where $f^*T_{\tilde{X}}(-\epsilon)$ is to be understood as $(f^*T_{\tilde{X}}) \otimes \mathcal{O}_C(-\epsilon)$.

Lemma 2.2 *Let $C = \mathbb{P}^1$, $X = \mathbb{P}^n$, $f : C \to \tilde{X}$ a morphism, and $\epsilon \in \{0, 1\}$.*

(i) If $f(C) \not\subset E$ or f is a constant map then $h^1(C, f^*T_{\tilde{X}}(-\epsilon)) = 0$.

(ii) If $f(C) \subset E$ and the map $f : C \to E \cong \mathbb{P}^{n-1}$ has degree $e > 0$ then

$$h^1(C, f^*T_{\tilde{X}}(-\epsilon)) = e - 1 + \epsilon.$$

Proof. (i): If f is a constant map, the assertion is trivial, so let us assume that the homology class of the map is $f_*[C] = dH' - eE'$ with $d > 0$, $e \geq 0$. We have $d = f^*H$ and $e = f^*E$, and since $f^*(H - E)$ is an effective divisor on C, it follows that $e \leq d$.
By lemma 2.1, there is a commutative diagram

$$\begin{array}{ccc} f^*\pi^*T_X(-e-\epsilon) & \longrightarrow & f^*\pi^*T_X(-\epsilon) \\ & \searrow \quad \nearrow & \\ & f^*T_{\tilde{X}}(-\epsilon) & \end{array}$$

where, in particular, the map $f^*\pi^*T_X(-e-\epsilon) \to f^*T_{\tilde{X}}(-\epsilon)$ is injective and an isomorphism on a dense open subset of C. Hence we have an exact sequence

$$0 \to f^*\pi^*T_X(-e-\epsilon) \to f^*T_{\tilde{X}}(-\epsilon) \to Q \to 0$$

with some sheaf Q on C which has zero-dimensional support. Therefore, to prove the lemma, it suffices to show that $h^1(C, f^*\pi^*T_X(-e-\epsilon)) = 0$. But this follows easily from the Euler sequence, pulled back to C and twisted by $\mathcal{O}_C(-e-\epsilon)$:

$$0 \to \mathcal{O}_C(-e-\epsilon) \to (n+1)\,\mathcal{O}_C(d-e-\epsilon) \to f^*\pi^*T_X(-e-\epsilon) \to 0,$$

since $d - e - \epsilon \geq -\epsilon \geq -1$.
In particular, we also see that $h^1(C, f^*\pi^*T_X(-\epsilon)) = 0$, which will be needed in the proof of part (ii).
(ii): We consider the normal sequence

$$0 \to T_E \to i^*T_{\tilde{X}} \to N_{E/\tilde{X}} \to 0.$$

As $N_{E/\tilde{X}} = \mathcal{O}_E(-1)$, pulling back to C and twisting by $\mathcal{O}_C(-\epsilon)$ yields

$$0 \to f^*T_E(-\epsilon) \to f^*T_{\tilde{X}}(-\epsilon) \to \mathcal{O}_C(-e-\epsilon) \to 0.$$

By the remark at the end of part (i), applied to $E \cong \mathbb{P}^{n-1}$ instead of $X = \mathbb{P}^n$, we see that $h^1(C, f^*T_E(-\epsilon)) = 0$. Since $h^1(C, \mathcal{O}_C(-e-\epsilon)) = e - 1 + \epsilon$, the result follows. □

Finally, we consider the case where C is a genus 0 prestable curve in the sense of [BM], i.e. a curve with at most ordinary double points as singularities and whose arithmetic genus is zero.

Proposition 2.3 *Let C be a genus 0 prestable curve, $X = \mathbb{P}^n$, and $f : C \to \tilde{X}$ a morphism. Let $e' = e'(C)$ be "the sum of the exceptional degrees of all components of C which are mapped into E", i.e.*

$$e' := \sum_{C'} \{ \; e \mid C' \textit{ is an irreducible component of } C \textit{ such that } f_*[C'] = e \cdot E' \; \}.$$

Then $h^1(C, f^*T_{\tilde{X}}) \leq e'$, with strict inequality holding if $e' > 0$.

PROOF. The proof is by induction on the number of irreducible components of C. If C itself is irreducible, the statement follows immediately from lemma 2.2.

Now let C be reducible, so assume $C = C_0 \cup C'$ where $C' \cong \mathbb{P}^1$, $C_0 \cap C' = \{Q\}$, and where C_0 is a prestable curve for which the induction hypothesis holds. Consider the exact sequences

$$0 \to f^*T_{\tilde{X}} \to f_0^*T_{\tilde{X}} \oplus f'^*T_{\tilde{X}} \xrightarrow{\varphi} f_Q^*T_{\tilde{X}} \to 0$$
$$0 \to f'^*T_{\tilde{X}}(-Q) \to f'^*T_{\tilde{X}} \xrightarrow{\psi} f_Q^*T_{\tilde{X}} \to 0$$

where f_0, f', and f_Q denote the restrictions of f to C_0, C', and Q, respectively.

From these sequences we deduce that

$$\dim \operatorname{coker} H^0(\varphi) = h^1(C, f^*T_{\tilde{X}}) - h^1(C_0, f_0^*T_{\tilde{X}}) - h^1(C', f'^*T_{\tilde{X}})$$
$$\dim \operatorname{coker} H^0(\psi) = h^1(C', f'^*T_{\tilde{X}}(-Q)) - h^1(C', f'^*T_{\tilde{X}}).$$

Since we certainly have $\dim \operatorname{coker} H^0(\varphi) \leq \dim \operatorname{coker} H^0(\psi)$, we can combine these equations into the single inequality

$$h^1(C, f^*T_{\tilde{X}}) \leq h^1(C_0, f_0^*T_{\tilde{X}}) + h^1(C', f'^*T_{\tilde{X}}(-Q)).$$

Now, by the induction hypothesis on f_0, we have $h^1(C_0, f_0^*T_{\tilde{X}}) \leq e'(C_0)$ with strict inequality holding if $e'(C_0) > 0$. On the other hand, we get $h^1(C', f'^*T_{\tilde{X}}(-Q)) \leq e'(C')$ by lemma 2.2. Since $e'(C) = e'(C_0) + e'(C')$, the proposition follows by induction. □

3 The morphism $\phi : \bar{M}_N(\tilde{X}, \beta - eE') \to \bar{M}_N(X, \beta)$

The proposition 2.3 shows us that, at least in the case $X = \mathbb{P}^n$, problems with nonvanishing $h^1(C, f^*T_{\tilde{X}})$ only arise if some irreducible components of C are mapped into the exceptional divisor. Since these components are contracted by the map $\pi : \tilde{X} \to X$, we are led to study the relation between stable maps in the blow-up $\tilde{X}$ and their image in X.

Let $M_\tau(X, \beta)$ denote the substack of $\bar{M}_N(X, \beta)$ of those stable maps $(C, x_1, \ldots, x_N, f)$ where $(C, x_1, \ldots, x_N)$ has a fixed topology which is encoded in the graph τ as introduced in [BM]. We will not need the details of this encoding in this paper, all that will be important for us is that there is a stratification of $\bar{M}_N(X,\beta)$ by the $M_\tau(X,\beta)$ for all possible τ, such that in each stratum we have a fixed structure of the singularities of the curve C, and the marked points lie in fixed components of C. Since we only consider stable maps, there is only a finite number of possible graphs τ for given N and β.

Note that this is *not* the stack $\bar{M}_\tau(X,\beta)$ as defined in [BM].

By the functorial properties of moduli spaces of stable maps [BM], the map $\pi : \tilde{X} \to X$ induces morphisms $\phi : \bar{M}_N(\tilde{X},\beta - eE') \to \bar{M}_N(X,\beta)$ for all e, where $\beta \in H_2^+(X)$ and where we use the decomposition $H_2(\tilde{X}) = H_2(X) \oplus \mathbb{Z} \cdot E'$. We may restrict these maps to the case where the underlying curves have topology τ, so we also get morphisms

$$\phi_\tau : M_\tau(\tilde{X},\beta - eE') \to \bar{M}_N(X,\beta).$$

As we have seen in the example above, the dimension of the stack $\bar{M}_N(\tilde{X},\beta - eE')$ may be larger than its virtual dimension, which is

$$\begin{aligned} \text{virt dim}\,\bar{M}_N(\tilde{X},\beta - eE') &= \chi(C, f^*T_{\tilde{X}}) + N - 3 \\ &= n - K_X \cdot \beta - e\,(n-1) + N - 3 \\ &= \text{virt dim}\,\bar{M}_N(X,\beta) - e\,(n-1). \end{aligned}$$

The aim of this section is to prove the following proposition:

Proposition 3.1 *Let $X = \mathbb{P}^n$ and $\phi : \bar{M}_N(\tilde{X},\beta - eE') \to \bar{M}_N(X,\beta)$ be the morphism as above. Then*

$$\dim \phi(\bar{M}_N(\tilde{X},\beta - eE')) \le \text{virt dim}\,\bar{M}_N(\tilde{X},\beta - eE').$$

Moreover, if $R \subset \bar{M}_N(\tilde{X},\beta - eE')$ denotes the subspace of all reducible stable maps in $\bar{M}_N(\tilde{X},\beta - eE')$ (i.e. the maps $(C,x_1,\dots,x_N,f)$ with C reducible), then

$$\dim \phi(R) < \text{virt dim}\,\bar{M}_N(\tilde{X},\beta - eE').$$

Proof. Since the moduli spaces $M_\tau(\tilde{X},\beta - eE')$ form a stratification of $\bar{M}_N(\tilde{X},\beta - eE')$, it is enough to show the statement for the restricted maps ϕ_τ. Hence we fix a topology τ and associate to it the following numerical invariants:

- Let S be the number of nodes of a curve with topology τ. We divide this number into $S = S_{EE} + S_{XX} + S_{XE}$, where S_{EE} (resp. S_{XX}, S_{XE}) denotes the number of nodes joining two exceptional components of C (resp. two non-exceptional components, or one exceptional with one non-exceptional component). Here and in the following we call an irreducible component of C exceptional if it is mapped by f into the exceptional divisor and it is not contracted by f, and non-exceptional otherwise.

- Let P be the (minimal) number of additional marked points which are necessary to stabilize C. We divide the number P into $P = P_E + P_X$, where P_E (resp. P_X) is the number of marked points that have to be added on exceptional components (resp. non-exceptional components) of C to stabilize C.

Now let $\mathcal{C} = (C, x_1, \dots, x_N, f) \in M_\tau(\tilde{X}, \beta - eE')$ be a stable map of topology τ, and let

$$T_\mathcal{C}\phi_\tau : T_{M_\tau(\tilde{X},\beta-eE'),\mathcal{C}} \to T_{\bar{M}_N(X,\beta),\phi(\mathcal{C})}$$

be the differential of the map ϕ_τ at the point $\mathcal{C}$. As we always work over the ground field of complex numbers, to prove the proposition it suffices to show that

$$\dim \operatorname{im} T_\mathcal{C}\phi_\tau \leq \text{virt dim}\, \bar{M}_N(\tilde{X}, \beta - eE')$$

for all $\mathcal{C}$, and that strict inequality holds if τ is a topology corresponding to reducible stable maps.

The tangent space $T_{M_\tau(\tilde{X},\beta-eE'),\mathcal{C}}$ is given by the hypercohomology group [K]

$$T_{M_\tau(\tilde{X},\beta-eE'),\mathcal{C}} = \mathbb{H}^1(T'_C \to f^*T_{\tilde{X}})$$

where $T'_C = T_C(-x_1 - \dots - x_N)$ and where we put the sheaves T'_C and $f^*T_{\tilde{X}}$ in degrees 0 and 1, respectively. This means that there is an exact sequence

$$0 \to H^0(C, T'_C) \to H^0(C, f^*T_{\tilde{X}}) \to T_{M_\tau(\tilde{X},\beta-eE'),\mathcal{C}} \to H^1(C, T'_C)$$

(note that the first map is injective because $\mathcal{C}$ is a stable map). By Riemann-Roch we get $\chi(C, T'_C) = S + 3 - N$. Moreover, by proposition 2.3 we have

$$\dim H^0(C, f^*T_{\tilde{X}}) \leq \chi(C, f^*T_{\tilde{X}}) + e' \qquad (*)$$

where e' is the "sum of the exceptional degrees of the components of C" as introduced there. It follows that

$$\begin{aligned} \dim T_{M_\tau(\tilde{X},\beta-eE'),\mathcal{C}} &\leq \chi(C, f^*T_{\tilde{X}}) + e' + N - S - 3 \\ &= \text{virt dim}\, \bar{M}_N(\tilde{X}, \beta - eE') + e' - S. \end{aligned}$$

We will now study the map

$$\phi' : H^0(C, f^*T_{\tilde{X}})/H^0(C, T'_C) \to T_{\bar{M}_N(X,\beta),\phi(\mathcal{C})}$$

induced by the composition of the maps $H^0(C, f^*T_{\tilde{X}}) \to T_{M_\tau(\tilde{X},\beta-eE'),\mathcal{C}} \to T_{\bar{M}_N(X,\beta),\phi(\mathcal{C})}$ considered above. To prove the proposition, we will show that

$$\dim \ker \phi' \geq e' - S$$

and that strict inequality holds in certain cases. Obviously, we may assume that $e' - S \geq 0$.

Let C_0 be a maximal connected subscheme of C consisting only of exceptional components of C. Let f_0 be the restriction of f to C_0 and let $Q_1, \ldots, Q_a$ be the nodes of C which join C_0 with the rest of C (they are of type S_{XE}). Now every section of $f_0^* T_E(-Q_1 - \cdots - Q_a)$ can be extended by zero to a section of $f^* T_{\tilde{X}}$ which is mapped to zero by ϕ' since these deformations take place entirely within the exceptional divisor. As $E \cong \mathbb{P}^{n-1}$ is a convex variety, we have

$$h^0(C_0, f_0^* T_E) = \chi(C_0, f_0^* T_E) = n - 1 + n \cdot e'(C_0)$$

and therefore we can estimate the dimension of the space of deformations that we have just found:

$$h^0(C_0, f_0^* T_E(-Q_1 - \cdots - Q_a)) \geq n - 1 + n \cdot e'(C_0) - (n-1)\, a.$$

(The right hand side of this inequality may well be negative, but nevertheless the statement is correct also in this case, of course.)

We will now add up these numbers for all possible C_0, say there are B of them. The sum of the $e'(C_0)$ will then give $e' = e'(C)$, and the sum of the a will give S_{XE}. Note that there is a P_E-dimensional space of infinitesimal automorphisms of C, i.e. a subspace of $H^0(C, T'_C)$, included in the deformations that we have just found, and that these do not give non-trivial elements in the kernel of ϕ'. Therefore we have

$$\begin{aligned} \dim \ker \phi' &\geq B\,(n-1) + ne' - (n-1) S_{XE} - P_E \\ &= (n-1) \cdot (B + e' - S_{XE}) + e' - P_E \\ &\geq B + e' - S_{XE} + e' - P_E \qquad (B + e' - S_{XE} \geq 0 \text{ since } e' \geq S) \\ &= e' - S + (B + e' + S_{EE} - P_E) + S_{XX}. \qquad (+) \end{aligned}$$

Hence, it is certainly sufficient to show that $P_E \leq B + e' + S_{EE}$.

To show this, we look at the exceptional components of C where marked points have to be added to stabilize C. We have to distinguish three cases:

- Components on which two points have to be added, and whose (only) node is of type S_{EE}: those give a contribution of 2 to P_E, but they also give at least 1 to e' and to S_{EE} (and every node of type S_{EE} "belongs" to at most one such component).
- Components on which two points have to be added, and whose (only) node is of type S_{XE}: those give a contribution of 2 to P_E, but they also give at least 1 to e' and to B (since such a component alone is one of the C_0 considered above).
- Components on which only one point has to be added: those give a contribution of 1 to P_E, but they also give at least 1 to e'.

This finishes the proof of the "$\leq$"-part of the lemma.

To show the strict inequality if C is reducible, we again distinguish two cases:

- $e' > 0$: Then, by lemma 2.3, the inequality $(*)$ is already strict.
- $e' = 0$: If C is reducible and $e' = 0$, then we must have $S_{XX} > 0$, hence we get the strict inequality by $(+)$.

This completes the proof. $\square$

4 Geometric interpretation of the Gromov-Witten invariants on $\tilde{X}$

We start this section by proving some moving lemmas which will be needed to show that, in favourable cases, the intersection product on the moduli space which defines the Gromov-Witten invariants can be made transverse.

Lemma 4.1 *Let X be a scheme of finite type and $f : X \to \mathbb{P}^m$ a morphism. Then, for a generic hyperplane $H \subset \mathbb{P}^m$, we have:*

(i) $f^{-1}(H)$ is (empty or) of pure codimension 1 in X.

(ii) If X is smooth then the divisor $f^{-1}(H)$ is a smooth subscheme of X counted with multiplicity one.

PROOF. See [J], cor. 6.11. $\square$

Lemma 4.2 *Let X be a scheme of finite type, Y a smooth, connected, projective scheme, and $f : X \to Y$ a morphism. Let L be a base point free linear system on Y. Then, for generic $D \in L$, we have:*

(i) $f^{-1}(D)$ is (empty or) purely 1–codimensional.

(ii) If X is smooth then the divisor $f^{-1}(D)$ is a smooth subscheme of X counted with multiplicity one.

PROOF. The base point free linear system L on Y gives rise to a morphism $s : Y \to \mathbb{P}^m$ where $m = \dim L$. Composing with f yields a morphism $X \to \mathbb{P}^m$, and the divisors $D \in L$ correspond to the inverse images under s of the hyperplanes in $\mathbb{P}^m$. Hence, the statement follows from lemma 4.1, applied to the map $X \to \mathbb{P}^m$. $\square$

Lemma 4.3 *Let X be a Deligne-Mumford stack of finite type, Y_i smooth, connected, projective schemes, and $f_i : X \to Y_i$ morphisms for $i = 1, \ldots, N$. Let $\alpha_i \in A^{c_i}(Y_i)$*

be cycles of codimensions $c_i \geq 1$ on Y_i that can be written as intersection products of divisors on Y_i

$$\alpha_i = [D'_{i,1}] \cdot \; \cdots \; \cdot [D'_{i,c_i}] \qquad (i = 1, \dots, N)$$

such that the complete linear systems $|D'_{i,j}|$ are base point free (this always applies, for example, in the case $Y_i = \mathbb{P}^n$). Let $c = c_1 + \cdots + c_N$. Then, for almost all $D_{i,j} \in |D'_{i,j}|$, we have:

(i) $V_i := D_{i,1} \cap \cdots \cap D_{i,c_i}$ is smooth of pure codimension c_i in Y_i, and the intersection is transverse. In particular, $[V_i] = \alpha_i$.

(ii) $V := f_1^{-1}(V_1) \cap \cdots \cap f_N^{-1}(V_N)$ is of pure codimension c in X. In particular, if $\dim X < c$ then $V = \emptyset$.

(iii) If $\dim X = c$ and X contains a dense, open, smooth substack U such that each geometric point of U has no nontrivial automorphisms then V consists of exactly $(f_1^* \alpha_1 \cdots f_N^* \alpha_N)[X]$ points of X (which lie in U and are counted with multiplicity one).

Proof. (i) follows immediately by recursive application of lemma 4.1 to the schemes Y_i.

If X is a scheme, then (ii) follows by recursive application of lemma 4.2. If X is a Deligne-Mumford stack, then it has an étale cover $S \to X$ by a scheme S, so (ii) holds for the composed maps $S \to X \to Y_i$. But since the map $S \to X$ is étale, the statement is also true for the maps $X \to Y_i$.

A Deligne-Mumford stack U whose generic geometric point has no nontrivial automorphisms always has a dense open substack U' which is a scheme (see e.g. [V]. To be more precise, U is a functor and hence an algebraic space ([DM], ex. 4.9), but an algebraic space always contains a dense open subset U' which is a scheme ([Kn], p. 25)). Since U' is dense in X and therefore has smaller dimension, applying (ii) to the restrictions $f_i|_{X \backslash U'} : X \backslash U' \to Y_i$ gives that V is contained in the smooth scheme U', hence it suffices to consider the restrictions $f_i|_{U'} : U' \to Y_i$. But since U' is a smooth scheme, we can apply lemma 4.2 (ii) recursively and get the desired result. □

We are now ready to give a geometric interpretation of the Gromov-Witten invariants on $\tilde{\mathbb{P}}^n$ with only non-exceptional classes. First, we show that the invariants can be thought of as certain numbers of stable maps to $\tilde{\mathbb{P}}^n$.

Proposition 4.4 *Let $X = \mathbb{P}^n$, $0 \neq \beta \in H_2^+(X)$ and $e \in \mathbb{Z}$. Let $\alpha_1, \dots, \alpha_N \in A^{\geq 1}(X)$ such that $\sum_i \operatorname{codim} \alpha_i = \operatorname{virt} \dim \bar{M}_N(\tilde{X}, \beta - eE')$. Choose generic subschemes $V_i \subset X$ with $[V_i] = \alpha_i$ in the sense of lemma 4.3, which do not meet the blown up point P, such that the V_i may also be regarded as subschemes of $\tilde{X}$.*

Then the number of stable maps $(C, x_1, \dots, x_N, f) \in \bar{M}_N(\tilde{X}, \beta - eE')$ passing through the V_i (i.e. such that $f(x_i) \in V_i$) is finite and equal to the Gromov-Witten invariant

$$\Phi_{\tilde{X}}(\pi^* \alpha_1, \dots, \pi^* \alpha_N \mid \beta - eE').$$

Moreover, all these stable maps are contained in the space $RM'(\tilde{X}, \beta - eE')$ (see definition 1.1), and they are all counted with multiplicity one in the Gromov-Witten invariant.

Remark. In particular, if $e < 0$ and $\beta \neq 0$ it follows that

$$\Phi_{\tilde{X}}(\pi^*\alpha_1, \ldots, \pi^*\alpha_N \mid \beta - eE') = 0$$

for all $\alpha_1, \ldots, \alpha_N \in A^*(X)$, because there are no irreducible curves in $\tilde{X}$ with homology class $\beta - eE'$ for $e < 0$, $\beta \neq 0$.

Proof. For $i = 1, \ldots, N$ consider the commutative diagram

$$\begin{array}{ccc} \bar{M}_N(\tilde{X}, \beta - eE') & \xrightarrow{\phi} & \bar{M}_N(X, \beta) \\ \tilde{p}_i \downarrow & & \downarrow p_i \\ \tilde{X} & \xrightarrow{\pi} & X \end{array}$$

where p_i, $\tilde{p}_i$ are the evaluation maps at the i-th marked point. By definition [B], the Gromov-Witten invariant mentioned in the proposition is equal to

$$\Phi_{\tilde{X}}(\pi^*\alpha_1, \ldots, \pi^*\alpha_N \mid \beta - eE') = (\tilde{p}_1^*\pi^*\alpha_1 \cdots \tilde{p}_N^*\pi^*\alpha_N) \cdot [\bar{M}_N(\tilde{X}, \beta - eE')]^{virt} \qquad (*)$$

where $[\bar{M}_N(\tilde{X}, \beta - eE')]^{virt}$ denotes the virtual fundamental class. As shown in proposition 2.3, we have $h^1(C, f^*T_{\tilde{X}}) = 0$ whenever C is irreducible. Therefore, on this part of the moduli space, the virtual fundamental class coincides with the usual one (since, by construction of virtual fundamental classes, this can be checked locally). This means that if $I \subset \bar{M}_N(\tilde{X}, \beta - eE')$ is the substack consisting of irreducible maps and $\bar{I}$ its closure, then one can write

$$[\bar{M}_N(\tilde{X}, \beta - eE')]^{virt} = [\bar{I}] + \gamma \qquad \in A_{\text{virt dim}\,\bar{M}_N(\tilde{X}, \beta - eE')}(\bar{M}_N(\tilde{X}, \beta - eE'))$$

where γ is some cycle in $\bar{M}_N(\tilde{X}, \beta - eE')$ whose support is entirely contained in the substack $R \subset \bar{M}_N(\tilde{X}, \beta - eE')$ of reducible stable maps. But, by proposition 3.1, we have $\dim \phi(R) < \text{virt dim}\,\bar{M}_N(\tilde{X}, \beta - eE')$, therefore, by lemma 4.3 (ii) applied to the restrictions $p_i|_{\phi(R)} : \phi(R) \to X$ one concludes that, for generic choice of the V_i, there are no stable maps in $\phi(R)$ passing through the V_i. This means that there are no reducible stable curves in $\bar{M}_N(\tilde{X}, \beta - eE')$ passing through the V_i. In particular, the contribution in $(*)$ coming from the cycle γ vanishes, and we have

$$\Phi_{\tilde{X}}(\pi^*\alpha_1, \ldots, \pi^*\alpha_N \mid \beta - eE') = (\tilde{p}_1^*\pi^*\alpha_1 \cdots \tilde{p}_N^*\pi^*\alpha_N) \cdot [\bar{I}].$$

So we only have to evaluate an intersection product with the usual fundamental class on $\bar{I}$. By lemma 4.3 (ii) this means that the number of stable maps passing through the V_i is finite and we are simply counting the number of such maps, although we do not yet know whether they are counted with multiplicity one.

We do know, however, that all stable maps passing through the V_i are irreducible, which means, for example, that we can restrict ourselves to the case $e \geq 0$ since otherwise there certainly are no such curves. But we can restrict this even further. For example, we can assume that none of these maps is a finite covering map: for each irreducible finite covering map $f : \mathbb{P}^1 \to \tilde{X}$ of degree $a > 1$ and homology class $\beta - eE'$, there is also an irreducible stable map $f' : \mathbb{P}^1 \to \tilde{X}$ of homology class $(\beta - eE')/a$. But the moduli space of maps of homology class $(\beta - eE')/a$ is smaller than that of $\beta - eE'$, since, if $\beta = dH'$ for some $d > 0$,

$$\begin{aligned}
\text{virt dim}\, \bar{M}_N(\tilde{X}, \beta - eE') - \text{virt dim}\, \bar{M}_N(\tilde{X}, (\beta - eE')/a) \\
&= -K_X \cdot \beta - e(n-1) + \frac{1}{a}\,(K_X \cdot \beta + e\,(n-1)) \\
&= (d\,(n+1) - e\,(n-1)) \cdot (1 - \frac{1}{a}) \\
&= [(d-e) \cdot (n+1) + 2e] \cdot (1 - \frac{1}{a}) \\
&> 0 \qquad \text{since } d \geq e \geq 0
\end{aligned}$$

(note that on the space of irreducible curves, the virtual dimension coincides with the actual one). Hence by lemma 4.3 (ii), for generic V_i there are no maps f' as above passing through the V_i, and therefore there are also no finite covering maps $f : \mathbb{P}^1 \to X$.

So, in the terminology of definition 1.1, we are only counting irreducible relevant maps. Let us denote the subspace of irreducible relevant maps by $Z \subset \bar{M}_N(\tilde{X}, \beta - eE')$. Certainly, the locus $RM'_N(\tilde{X}, \beta - eE') \subset Z$ of the maps $(C, x_1, \ldots, x_N, f)$ in Z where $f^{-1}(f(x_i)) = \{x_i\}$ for all i is dense in Z. Since $RM'_N(\tilde{X}, \beta - eE')$ is smooth and, by lemma 1.3, the geometric points in $RM'_N(\tilde{X}, \beta - eE')$ have no non-trivial automorphisms, we can apply lemma 4.3 (iii) to the restricted maps $p_i|_Z : Z \to \tilde{X}$ and the statement of the proposition follows. □

Finally, we use the results of section 1 to reformulate this proposition in terms of embedded curves in $\mathbb{P}^n$ with multiple points, which is our main result.

Proposition 4.5 *Let $d > 0$, $e \geq 0$ and $\alpha_1, \ldots, \alpha_N \in A^{\geq 1}(\mathbb{P}^n)$ such that*

$$\sum_i (\operatorname{codim} \alpha_i - 1) = d\,(n+1) - e\,(n-1) + n - 3.$$

Let $P \in \mathbb{P}^n$ be a point. Choose generic subschemes $V_i \subset \mathbb{P}^n$ with $[V_i] = \alpha_i$ in the sense of lemma 4.3.

Then the number of rational curves in $\mathbb{P}^n$ (purely 1-dimensional subvarieties birational to $\mathbb{P}^1$) of degree d which have non-empty intersection with all V_i and which pass through the point P with total multiplicity e, where each such curve C is counted with multiplicity

$$\sharp(C \cap V_1) \cdots \sharp(C \cap V_N),$$

is equal to the Gromov-Witten invariant on $\tilde{\mathbb{P}}^n$

$$\Phi_{\tilde{\mathbb{P}}^n}(\pi^*\alpha_1, \dots, \pi^*\alpha_N \mid dH' - eE').$$

PROOF. By proposition 4.4, for generic V_i the Gromov-Witten invariant mentioned in the proposition counts maps $(C, x_1, \dots, x_N, f)$ in $RM'_N(\tilde{\mathbb{P}}^n, dH' - eE')$ with multiplicity one which pass through the V_i, and there are no further stable maps in $\bar{M}_N(\tilde{X}, dH' - eE')$ passing through the V_i. By proposition 1.2 and lemma 1.3, this can be reformulated by saying that we are counting irreducible relevant curves in $\tilde{\mathbb{P}}^n$ with multiplicity one which pass through the V_i, i.e. rational curves $C \subset \tilde{\mathbb{P}}^n$ of homology class $dH' - eE'$ together with points $p_i \in C$ such that $p_i \in V_i$ for all i. This is obviously the same as counting rational curves $C \subset \tilde{\mathbb{P}}^n$ meeting all the V_i, and counting them with multiplicity $\sharp(C \cap V_1) \cdots \sharp(C \cap V_N)$.

But, using the strict transform of curves $C \subset \mathbb{P}^n$ under the blow-up $\pi : \tilde{\mathbb{P}}^n \to \mathbb{P}^n$, there is a one-to-one correspondence between rational curves in $\mathbb{P}^n$ of degree d which pass through P with multiplicity e and rational curves in $\tilde{\mathbb{P}}^n$ with homology class $dH' - eE'$. Of course, the property of meeting the subvarieties V_i is not affected by this strict transform, hence the proposition follows. □

5 Calculation of the Gromov-Witten invariants of $\tilde{\mathbb{P}}^n$

To compute the Gromov-Witten invariants of $\tilde{\mathbb{P}}^n$, we recall the First Reconstruction Theorem of Kontsevich and Manin.

Proposition 5.1 *Let X be a smooth projective variety such that the algebraic part of $H^*(X)$ is generated as a ring by divisor classes on X. Then all Gromov-Witten invariants $\Phi_X(\alpha_1, \dots, \alpha_N \mid \beta)$ with $\alpha_i \in A^*(X)$ and $\beta \in H_2^+(X)$ can be reconstructed from those with $N = 3$ and $\alpha_3 \in A^1(X)$.*

PROOF. [KM] theorem 3.1, applied to the Gromov-Witten classes constructed in [B]. An explicit algoritm to compute the invariants is also given in [KM]. □

Now we apply this result to the case of $X = \tilde{\mathbb{P}}^n$. We set $H_i = H^i$, $E_i = -(-E)^i$, and choose

$$B = \{H_0, H = H_1, H_2, \dots, H_n, E = E_1, E_2, \dots, E_{n-1}\}$$

as a basis of $A^*(\tilde{\mathbb{P}}^n)$. In the following, we consider only invariants whose classes are in this basis.

Proposition 5.2 *All Gromov-Witten invariants on $\tilde{\mathbb{P}}^n$ can be computed recursively by proposition 5.1 using the initial data* *(i)* $\Phi_{\tilde{\mathbb{P}}^n}(pt, pt, H \mid H') = 1$*, where* pt *denotes the class of a point,*

(ii) $\Phi_{\tilde{\mathbb{P}}^n}(E_{n-1}, E_{n-1}, E \mid E') = -1$*,*

(iii) $\Phi_{\tilde{\mathbb{P}}^n}(\alpha_1, \alpha_2, \alpha_3 \mid H' - E') = 1$ *if* $\alpha_i \in B$ *and* $\operatorname{codim} \alpha_1 + \operatorname{codim} \alpha_2 + \operatorname{codim} \alpha_3 = n+2$*,*

(iv) $\Phi_{\tilde{\mathbb{P}}^n}(\alpha_1, \alpha_2, \alpha_3 \mid dH' + eE') = 0$ *in all other cases where* $\operatorname{codim} \alpha_3 = 1$ *and* $\alpha_i \in B$*.*

PROOF. Let $\Phi_{\tilde{\mathbb{P}}^n}(\alpha_1, \alpha_2, \alpha_3 \mid dH' + eE')$ be an invariant with $\operatorname{codim} \alpha_3 = 1$. Let $i = \operatorname{codim} \alpha_1$ and $j = \operatorname{codim} \alpha_2$, such that $1 \le i, j \le n$.

Case 1: $d = 0$, $e > 0$. Then the dimension condition reads

$$i + j = e\,(n-1) + n - 1 = (e-1)(n-1) + 2n - 2.$$

Since these curves are contained in the exceptional divisor, the Gromov-Witten invariant is zero if there is a point class (or any other non-exceptional class) among the α_i. Hence we may assume that $i + j \le 2n - 2$. But then it follows that $e = 1$ and $\alpha_1 = \alpha_2 = E_{n-1}$, and we are in case (ii).

To prove (ii), note that for maps $f : C \to E$ of degree 1 into the exceptional divisor, we have $h^1(C, f^* T_{\tilde{\mathbb{P}}^n}) = 0$ by proposition 2.3, hence the corresponding moduli stack is smooth of the expected dimension, and its virtual fundamental class coincides with the usual one. Now consider the invariant

$$\Phi_{\tilde{\mathbb{P}}^n}(H_{n-1} - E_{n-1}, H_{n-1} - E_{n-1}, H - E \mid E').$$

The classes $H_{n-1} - E_{n-1}$ and $H - E$ are represented on $\tilde{\mathbb{P}}^n$ by the strict transform of a line (resp. hyperplane) in $\mathbb{P}^n$ passing through P. These intersect the exceptional divisor transversally in a point (resp. in a hyperplane in E), hence this Gromov-Witten invariant simply counts the number of lines in E through two points in E (and intersecting a hyperplane in E), which is 1.

Note that Gromov-Witten invariants $\Phi_{\tilde{\mathbb{P}}^n}(\alpha_1, \ldots, \alpha_N \mid eE')$ with $e > 0$ vanish if one of the α_i is a non-exceptional class, since one can choose a subvariety $V_i \subset \mathbb{P}^n$ representing α_i which does not pass through the exceptional divisor, such that there are no stable maps of homology class eE' passing through V_i. Hence, by linearity of the Gromov-Witten invariants it follows that

$$\Phi_{\tilde{\mathbb{P}}^n}(E_{n-1}, E_{n-1}, E \mid E') = -\Phi_{\tilde{\mathbb{P}}^n}(H_{n-1} - E_{n-1}, H_{n-1} - E_{n-1}, H - E \mid E') = -1,$$

which proves (ii).

Case 2: $d > 0$. Then we must have

$$0 \le (H - E)(dH' + eE') = d + e$$

and the dimension condition is

$$\begin{aligned} i + j &= d\,(n+1) + e\,(n-1) + n - 1 \\ &= (d+e)(n-1) + 2d + n - 1. \qquad (*) \end{aligned}$$

If $d+e>0$, then we have $i+j \geq 2d+2n-2$, therefore it follows that $d=1$ and $\alpha_1 = \alpha_2 = pt$, which is (i). The statement of (i) follows immediately from proposition 4.5.

Now assume that $e=-d$. Since any curve in $\mathbb{P}^n$ of degree d passing with multiplicity d through a given point is a union of lines, the image of any stable map of homology class $dH' - dE'$ is simply a line passing through the exceptional divisor (note that it is not possible to have a union of some lines since such a curve would not be connected in $\tilde{\mathbb{P}}^n$).

So we consider the moduli space of lines in $\tilde{\mathbb{P}}^n$ intersecting E, which is canonically isomorphic to $E \cong \mathbb{P}^{n-1}$ itself. The condition that such a line meets a generic linear subspace of codimension k in $\mathbb{P}^n$ or E is given by a codimension $k-1$ linear subspace in the moduli space E. Therefore, the moduli space of lines intersecting E and two linear subspaces corresponding to α_1 and α_2 is

$$\dim E - (i-1) - (j-1) = n - i - j + 1. \qquad (+)$$

So to get a non-zero invariant, we must have $i+j \leq n+1$. Therefore from $(*)$ it follows that $d=1$, in which case $(+)$ is zero, such that there is exactly one line satisfying the incidence conditions. Note that by lemma 2.3 the moduli stack of stable maps of homology class $H' - E'$ is smooth of the expected dimension, and we indeed count the number of lines intersecting E and two classes representing α_1 and α_2. This proves (iii).

Finally, we have also shown that (i)–(iii) are the only non-zero invariants on $\mathbb{P}^n$ with $N=3$ and $\operatorname{codim} \alpha_3 = 2$, which proves (iv). □

A few numerical examples of invariants can be found in tables 1 to 3. There are some remarks that can be made about these numbers:

- Let $\alpha_1, \dots, \alpha_k$ be non-exceptional classes, i.e. $\alpha_i \in \pi^* A^*(\mathbb{P}^n)$. If $d>0$ then

  $$\Phi_{\tilde{\mathbb{P}}^n}(\alpha_1, \dots, \alpha_n \mid dH' - E') = \Phi_{\tilde{\mathbb{P}}^n}(\alpha_1, \dots, \alpha_n, pt \mid dH').$$

  This follows, for example, from proposition 4.5, since both sides count the number of rational curves in $\mathbb{P}^n$ of degree d which meet subvarieties representing the α_i and one additional point.

  This is no longer true if one of the α_i is exceptional.

- Similarly, if $\alpha_1, \dots, \alpha_k$ are non-exceptional classes, $d>0$ and $e<0$, then

  $$\Phi_{\tilde{\mathbb{P}}^n}(\alpha_1, \dots, \alpha_n \mid dH' - eE') = 0.$$

  Again, this follows from proposition 4.5 and is no longer true if one of the α_i is exceptional. Indeed, the invariants need not even be positive, so that there can be no geometric interpretation of these invariants as numbers of curves (at least not in an obvious way).

- Again let $\alpha_1, \ldots, \alpha_k$ be non-exceptional classes. If $e = 0$ then
$$\Phi_{\tilde{\mathbb{P}}^n}(\alpha_1, \ldots, \alpha_n \mid dH') = \Phi_{\mathbb{P}^n}(\alpha_1, \ldots, \alpha_n \mid dH').$$
This can also be deduced from proposition 4.5. The Gromov-Witten invariants of $\mathbb{P}^2$ can be found e.g. in [KM] 5.2.1., and those of $\mathbb{P}^3$ and $\mathbb{P}^4$ in [JS].

- The numbers of rational curves of degree d in $\mathbb{P}^2$ passing through $3d - 3$ points in the plane and in addition through the point P with multiplicity two, which we have computed as Gromov-Witten invariants on $\tilde{\mathbb{P}}^2$ (case $e = 2$ in table 1), have already been calculated by R. Pandharipande in [P] by different methods, and indeed the numbers agree.

- Consider the invariants $\Phi_{\tilde{\mathbb{P}}^2}(pt, \ldots, pt \mid dH' - (d-1)E')$ for $d > 1$, where we have $2d$ point classes in the invariant. A curve C of degree d in $\mathbb{P}^2$ passing with multiplicity $d - 1$ through a point P has genus
$$\frac{1}{2}(d-1)(d-2) - \frac{1}{2}(d-1)(d-2) = 0,$$
i.e. it is always a rational curve. Hence the space of degree d rational curves with a $(d-1)$-fold point in P is simply a linear system of the expected dimension. Therefore we have
$$\Phi_{\tilde{\mathbb{P}}^2}(pt, \ldots, pt \mid dH' - (d-1)E') = 1$$
which can also be seen in table 1.

- From the dimension of the moduli space of stable maps of homology class $dH' - eE'$ in $\mathbb{P}^2$, it can be seen that the dimension of the space of degree d rational curves in $\mathbb{P}^2$ which pass through a given point with multiplicity e is $3d - 1 - e$. This can also be understood geometrically by a dimension count as follows: if $C \subset \mathbb{P}^2$ is a curve of degree d having ordinary k_i-fold points in the points P_i and no further singularities, the genus g of the normalization of C is
$$g = \frac{1}{2}(d-1)(d-2) - \sum_i \frac{1}{2} k_i \, (k_i - 1)$$
Hence, if we already have an e-fold point at P, to get a rational curve we need additional
$$\frac{1}{2}(d-1)(d-2) - \frac{1}{2} e \, (e-1)$$
double points. Now consider the linear system of degree d curves in $\mathbb{P}^2$, which has dimension $\frac{1}{2}(d+1)(d+2) - 1$. Having an e-fold point at P gives $\frac{1}{2} e \, (e+1)$ conditions, and each of the above additional double points imposes one more constraint (since the points where these double points occur are not specified). Hence the dimension of the space of curves we considered is
$$\frac{1}{2}(d+1)(d+2) - 1 - \frac{1}{2} e \, (e+1) - \left[\frac{1}{2}(d-1)(d-2) - \frac{1}{2} e \, (e-1) \right] = 3d - 1 - e.$$

	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
$e=0$	1	1	12	620	87304	26312976	14616808192
$e=1$	1	1	12	620	87304	26312976	14616808192
$e=2$	0	0	1	96	18132	6506400	4059366000
$e=3$	0	0	0	1	640	401172	347987200
$e=4$	0	0	0	0	1	3840	7492040
$e=5$	0	0	0	0	0	1	21504
$e=6$	0	0	0	0	0	0	1

Table 1: Some Gromov-Witten invariants $\Phi_{\tilde{\mathbb{P}}^2}(pt,\dots,pt \mid dH'-eE')$, where the number of point classes in the invariant is $3d-1-e$. By proposition 4.5, these are the numbers of degree d rational curves in $\mathbb{P}^2$ meeting $3d-1-e$ generic points in the plane, and in addition passing through the point P with multiplicity e.

	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
$e=0$	1	0	1	4	105	2576	122129	7397760
$e=1$	1	0	1	4	105	2576	122129	7397760
$e=2$	0	0	0	0	12	384	23892	1666128
$e=3$	0	0	0	0	0	0	620	72528
$e=4$	0	0	0	0	0	0	0	0

Table 2: Some Gromov-Witten invariants $\Phi_{\tilde{\mathbb{P}}^3}(pt,\dots,pt \mid dH'-eE')$, where the number of point classes in the invariant is $2d-e$. These are the numbers of rational curves of degree d in $\mathbb{P}^3$ meeting $2d-e$ generic points, and in addition passing through the point P with multiplicity e.

	$d=1$	$d=2$	$d=3$	$d=4$
$e=-3$	2925	4849635	25767926176	362956315020486
$e=-2$	-68	-35832	-89070592	-730861150688
$e=-1$	3	342	382720	1793900214
$e=0$	0	0	-2332	-5810112
$e=1$	1	-3	40	23825
$e=2$	0	0	4	960
$e=3$	0	0	0	45

Table 3: Some Gromov-Witten invariants $\Phi_{\tilde{\mathbb{P}}^3}(E_2,\dots,E_2 \mid dH'-eE')$, where $E_2=-E^2$ and the number of classes in the invariant is $4d-2e$.

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