MODULI SPACES OF CURVES IN TROPICAL VARIETIES

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ABSTRACT. We describe a framework to construct tropical moduli spaces of rational stable maps to a smooth tropical hypersurface or curve. These moduli spaces will be tropical cycles of the expected dimension, corresponding to virtual fundamental classes in algebraic geometry. As we focus on the combinatorial aspect, we take the weights on certain basic 0-dimensional local combinatorial curve types as input data, and give a compatibility condition in dimension 1 to ensure that this input data glues to a global well-defined tropical cycle. As an application, we construct such moduli spaces for the case of lines in surfaces, and in a subsequent paper for stable maps to a curve [GMO17].

1. INTRODUCTION

Moduli spaces of stable maps to a smooth projective variety are one of the most important tools in modern enumerative geometry [BM96, FP97]. Intersection theory on these spaces has been used successfully to solve many enumerative problems, such as e. g. determining the numbers of plane curves of fixed genus and degree through given points, or the numbers of rational curves of fixed degree in a general quintic threefold [Kon95, CH98].

In recent times, tropical geometry has also been proven to be very useful for attacking enumerative problems, starting with Mikhalkin's famous Correspondence Theorem that provided the first link between such problems in algebraic and tropical geometry [Mik05]. Accordingly, it is an important goal in tropical enumerative geometry to construct tropical analogues of moduli spaces of stable maps, i. e. tropical cycles whose points parametrize curves with certain properties in a given tropical variety. This has been achieved for rational curves in toric varieties (corresponding to tropical curves in a real vector space) in [Mik06, GKM09], and (tropical) intersection theory on these spaces has been used in many cases to attack and solve enumerative problems from a purely combinatorial point of view.

Of course, it would be very desirable to have such moduli spaces of tropical stable maps also for other target varieties. However, there is currently no general known method to construct such spaces, mainly for the following two reasons:

(a) Already in algebraic geometry, the moduli spaces of stable maps may have bigger dimension than expected from deformation theory. One can solve this problem by introducing virtual fundamental classes, i. e. cycle classes in the moduli spaces which are of the expected dimension and replace the ordinary fundamental classes

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for intersection-theoretic purposes [BF97, Beh97]. These classes can usually be constructed as certain Chern classes of vector bundles. However, there is no corresponding counterpart of this theory in tropical geometry yet.

(b) Tropical curves (in the sense of: metric graphs of the given degree and genus) in the given tropical variety might not be tropicalizations of actual algebraic curves inside the algebraic variety. Consequently, the naive tropical moduli space may not capture the situation from algebraic geometry appropriately, and it might have too big dimension even if the algebraic moduli space does not. This already happens for lines in cubic surfaces: Whereas each smooth algebraic cubic contains exactly 27 lines, there are smooth tropical cubics with infinitely many lines on them [Vig10]. In general, already this question whether a tropical curve is realizable by an algebraic one inside the given ambient variety is an unsolved problem. It is known that the space of realizable tropical curves is a polyhedral set [Yu15], but there is currently no explicit way to describe it.

In this paper, we will therefore study these tropical moduli spaces from an axiomatic and purely combinatorial point of view. Given a smooth tropical hypersurface or curve X, we will describe a set of (essentially 0-dimensional) input data and (essentially 1-dimensional) compatibility conditions that allow to construct from them tropical moduli spaces $\mathscr{M}_0(X, \Sigma)$ of rational curves in X that are cycles of the expected dimension (where the subscript 0 denotes the genus and Σ the degree of the curves). We can therefore consider such cycles as tropical analogues of both the algebraic moduli spaces of stable maps and their virtual fundamental classes.

A central concept for this construction is the *resolution dimension* $\mathrm{rdim}(V)$ of a vertex V of a curve in X. It is an integer determined by the local combinatorial type of the curve and X at V that describes the expected dimension of the moduli space of such local curves when the vertex of the curve is resolved, modulo the local lineality space of X. For example, a 4-valent vertex in the plane $\mathbb{R}^2$ has resolution dimension 1, since it can be resolved to two 3-valent vertices in a 1-dimensional family (modulo translations in $\mathbb{R}^2$). The origin of the curve in the following picture has resolution dimension 0 since this curve piece cannot be resolved or moved in X.
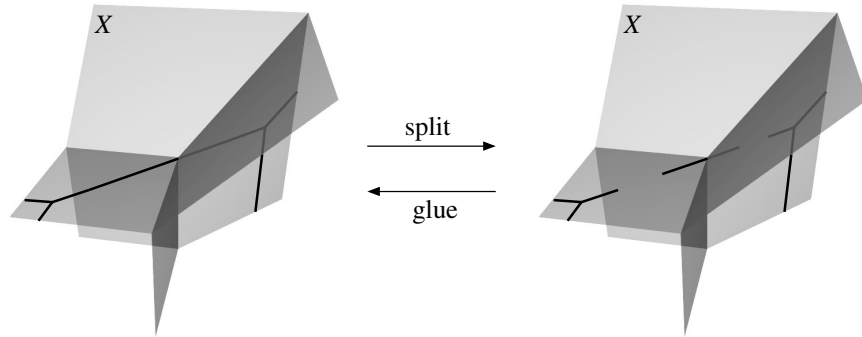
As a first rule, vertices of negative resolution dimension are not allowed in our moduli spaces — in the case of target curves this is known as the Riemann-Hurwitz condition [BBM11, BM16]. To construct the maximal cells of $\mathcal{M}_0(X, \Sigma)$ together with their weights, the general idea is then that splitting and gluing of the curves allows to reduce this question to vertices of resolution dimension 0. For example, the picture above on the left shows a line in a plane in $\mathbb{R}^3$, which can vary in a 2-dimensional moduli space by moving its two vertices along the direction of its bounded edge. We can split the curve in three pieces as in the picture on the right, all of which have resolution dimension 0. If we have weights for the moduli spaces for these three local pieces, we can glue them back together using tropical intersection theory to obtain a moduli space for our original situation on the left. Technically, this means that we consider the curve pieces to have bounded ends, and that we impose the intersection-theoretic condition that corresponding endpoints map to the same point in X by suitable evaluation maps.

In this way, we can make $\mathcal{M}_0(X, \Sigma)$ into a weighted polyhedral complex of the expected dimension by just giving weights for vertices of resolution dimension 0 as input data (we will refer to them as *moduli data* in Definition 3.9). However, as this input data can a priori be arbitrary, we need a certain compatibility condition for the resulting polyhedral complex to be balanced. A central result of our paper is that checking this condition in resolution dimension 1 is enough to ensure that the gluing process then works for all dimensions of the moduli spaces (Corollaries 3.17 and 3.18).

We check this condition for the moduli spaces of lines in surfaces in $\mathbb{R}^3$, leading e. g. to a well-defined 0-dimensional moduli cycle of lines on an arbitrary smooth tropical cubic surface, even if the actual number of such lines is infinite. In a particular example from [Vig10] of such a cubic with infinitely many lines, we verify that this 0-cycle still has degree 27 as expected. In a subsequent paper, we use our methods to obtain well-defined moduli spaces of rational stable maps (of any degree) to an arbitrary target curve [GMO17]. In any case, the initial 0-dimensional input data is obtained by tropicalization from the algebraic situation. For example, for the vertex in the origin in the picture above the weight is just the number of lines in $\mathbb{P}^3$ through $L_1 \cap L_2$ and $L_3 \cap L_4$ for any four general lines $L_1, L_2, L_3, L_4 \subset \mathbb{P}^3$, which is 1.

The organization of this paper is as follows. In Chapter 2 we give the necessary background from tropical geometry. While most of this material is well-known, there are three techniques that go beyond the usual theory: partially open versions of tropical varieties in Section 2.1, quotient maps (and their intersection-theoretic properties) in Section 2.2, and pull-backs of diagonals of smooth varieties in Appendix A. Chapter 3 then describes the gluing process for curves and constructs the tropical moduli spaces from the given input data. Finally, in Chapter 4 we study the case of lines in surfaces.

This paper is based on parts of the Ph. D. thesis of the second author [Och13]. It would not have been possible without extensive computations of examples which enabled us to establish and prove conjectures about polyhedra and their weights in our moduli spaces. For this we used the polymake extension a-tint [GJ00, Ham14] and GAP [GAP16]. We thank an anonymous referee for useful suggestions on the exposition. The work of the first

2. PRELIMINARIES

2.1. **Partially open tropical varieties.** Although most of the spaces occurring in this paper will be tropical varieties, some of our intermediate constructions also involve "partially open" versions of them. In these more general spaces the boundary faces of some polyhedra can be missing, and thus the balancing condition is required to hold at fewer places. The constructions in this introductory chapter are adapted to this setting and thus sometimes slightly more general than usual. However, since all constructions relevant to us are local, the required changes are minimal and straightforward.

For more details on the notions of tropical cycles and fans, see e. g. [GKM09, AR10].

Notation 2.1 (Polyhedra). Let Λ denote a lattice isomorphic to $\mathbb{Z}^N$ for some $N \geq 0$, and let $V := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ be the corresponding real vector space. A *partially open (rational) polyhedron* in V is a subset $\sigma \subset V$ that is the intersection of finitely many open or closed half-spaces $\{x \in V : f(x) < c\}$ resp. $\{x \in V : f(x) \leq c\}$ for $c \in \mathbb{R}$ and f in $\Lambda^{\vee}$, the dual of Λ. We call σ a (closed) polyhedron if it can be written in this way with only closed half-spaces.

The *(relative) interior* σ° of a partially open polyhedron is the topological interior of σ in its affine span. We denote by $V_{\sigma} \subset V$ the linear space which is the shift of the affine span of σ to the origin, and set $\Lambda_{\sigma} := V_{\sigma} \cap \Lambda$. The *dimension* of σ is defined to be the dimension of V_{σ}.

A *face* τ of a partially open polyhedron σ is a non-empty subset of σ that can be obtained by changing some of the non-strict inequalities $f(x) \leq c$ defining σ into equalities. We write this as $\tau \leq \sigma$, or $\tau < \sigma$ if in addition $\tau \neq \sigma$. If $\dim \tau = \dim \sigma - 1$ we call τ a *facet* of σ. In this case we denote by $u_{\sigma/\tau} \in \Lambda_{\sigma}/\Lambda_{\tau}$ the *primitive normal vector* of σ relative to τ, i. e. the unique generator of $\Lambda_{\sigma}/\Lambda_{\tau}$ lying in the half-line of σ in $V_{\sigma}/V_{\tau} \cong \mathbb{R}$.

Definition 2.2 (Polyhedral complexes and tropical varieties). A *partially open polyhedral complex* in V is a collection X of partially open polyhedra in a vector space $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, also called cells, such that

   (a) if $\sigma \in X$ and τ is a face of σ then $\tau \in X$; and
   (b) if $\sigma_1, \sigma_2 \in X$ then $\sigma_1 \cap \sigma_2$ is empty or a face of both σ_1 and σ_2.

It is called pure-dimensional if each inclusion-maximal cell has the same dimension. The *support of X*, denoted by $|X|$, is the union of all $\sigma \in X$ in V.

A *weighted partially open polyhedral complex* is a pair (X, ω_X), where X is a purely k-dimensional partially open polyhedral complex, and ω_X is a map associating a *weight* $\omega_X(\sigma) \in \mathbb{Z}$ to each k-dimensional cell $\sigma \in X$. If there is no risk of confusion we will write ω_X as ω, and (X, ω_X) just as X. A *partially open tropical cycle X* in V is a weighted polyhedral complex such that for each cell τ of dimension $k-1$ the *balancing condition*

$$\sum_{\sigma : \sigma > \tau} \omega(\sigma) \cdot u_{\sigma/\tau} = 0 \quad \in V/V_{\tau}$$

holds. It is called a *partially open tropical variety* if all weights are non-negative. If all polyhedra in X are closed, we omit the attribute "partially open" and speak e. g. of tropical cycles and tropical varieties. A *tropical fan* is a tropical variety all of whose polyhedra are cones.

Often, the exact polyhedral complex structure of our cycles is not important. We call two (partially open) cycles equivalent if they allow a common refinement (where a refinement is required to respect the weights, and every polyhedron of a refinement must be closed in its corresponding cell of the original cycle). By abuse of notation, the corresponding equivalence classes will again be called (partially open) tropical cycles.

A *morphism* $f : X \to Y$ between (partially open) tropical cycles X and Y is a locally affine linear map $f : |X| \to |Y|$, with the linear part induced by a map between the underlying lattices. It is called an *isomorphism* if it has a two-sided inverse (up to refinements) and respects the weights.

Example 2.3 (Linear spaces). Let $V = \mathbb{R}^k$, denote by e_i for $i = 1, \ldots, k$ the negatives of the vectors of the standard basis, and set $e_0 = -e_1 - \cdots - e_k$. For $r < k$ we denote by L_r^k the tropical fan whose simplicial cones are indexed by subsets $I \subset \{0, \ldots, k\}$ with at most r elements and given by the cones generated by all e_i with $i \in I$. The weights of the top-dimensional cones, corresponding to subsets of size r, are all set to 1. This is the tropicalization of a general r-dimensional linear space over the Puiseux series with constant coefficient equations [FS05, Proposition 2.5 and Theorem 4.1].

The following pictures illustrate these spaces, where all displayed cones are thought to be extending to infinity. If instead we interpret the pictures as bounded spaces they represent partially open tropical varieties obtained by intersecting L_r^k with an open bounded polyhedron.

$$L_1^2 \qquad\qquad L_2^3 \qquad\qquad L_1^3$$
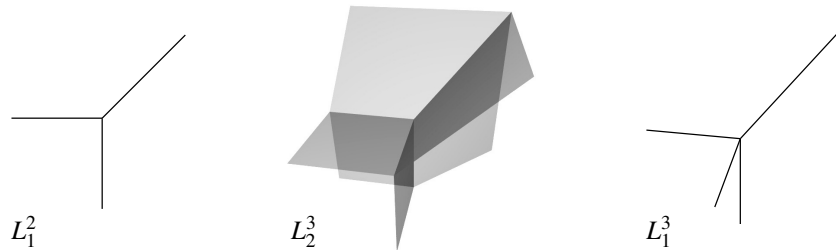
Remark 2.4. In our Definition 2.2 it is allowed that two partially open polyhedra in a complex do not intersect although their closures do. For example, in the pictures above we could replace all polyhedra by their relative interiors. This would give us weighted partially open polyhedral complexes with the same support, and whose face relations and balancing conditions are trivial. However, spaces of this type will not occur in our constructions in this paper — we will always have partially open polyhedral complexes X such that $\sigma \cap \tau = \emptyset$ for given $\sigma, \tau \in X$ implies $\overline{\sigma} \cap \overline{\tau} = \emptyset$.

Definition 2.5 (Smooth varieties)**.** For simplicity, in this paper we will follow [All12] and call a tropical variety X *smooth* if it is locally isomorphic to some $L_r^k \times \mathbb{R}^m$ around 0 at each

point (where $r + m = \dim X$ is fixed, but otherwise k, r, m may depend on the chosen point). This is more special than the usual definition of smoothness which allows any polyhedral complex locally isomorphic to a matroid fan as in Appendix A. We expect that our results would hold in this more general setting as well.
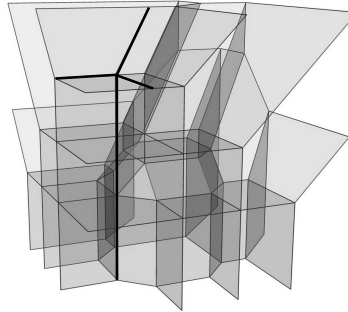
For a smooth variety X, following [Mik06, Section 5.3] the *canonical divisor K_X* of X is defined to be the weighted polyhedral complex given by the codimension-1 skeleton of X, where the weight of a codimension-1 cell of X is the number of adjacent maximal cells minus 2.

Example 2.6 (Smooth curves and hypersurfaces). In this paper, the following two cases of smooth varieties will be of particular importance.

(a) Consider a connected 1-dimensional tropical variety X in $\mathbb{R}^N$ which is a tree — we will refer to such a space as a *rational curve*. The smoothness condition then means that for any vertex of X the primitive integer vectors in the directions of the adjacent edges can be mapped to $e_0, \ldots, e_k$ in $\mathbb{R}^k$ for some $k \leq N$ by a $\mathbb{Z}$-linear map, and that all edges have weight 1. Such a vertex occurs in the canonical divisor K_X with weight $k - 1$.

(b) Consider a hypersurface X in $\mathbb{R}^N$, i.e. a tropical variety given by a *tropical polynomial* in N variables [RGST03]. The coefficients of this polynomial determine a subdivision of its Newton polytope, and the weighted polyhedral complex structure of X is induced by this subdivision. Smoothness then means that this subdivision is unimodular [Mik05, Proposition 3.11], and hence K_X contains each codimension-1 cell of X with weight 1. Most important for us will be the case of a cubic surface in $\mathbb{R}^3$, i.e. of X being dual to a unimodular subdivision of the lattice polytope in $\mathbb{Z}^3$ with vertices $(0,0,0)$, $(3,0,0)$, $(0,3,0)$, and $(0,0,3)$.
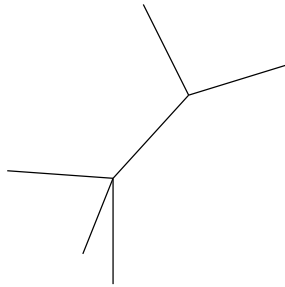
(a) a smooth curve in $\mathbb{R}^3$ (b) a cubic surface in $\mathbb{R}^3$
                                              with a line on it

2.2. Tropical quotients.

In this section we will define quotients of partially open tropical cycles by vector spaces in certain cases.

Definition 2.7 (Lineality space). Let X be a partially open polyhedral complex in a vector space $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, and let $L \subset V$ be a vector subspace defined over $\mathbb{Q}$. We say that L is a

lineality space for X if, for a suitable complex structure of X, for all $\sigma \in X$ and $x \in \sigma$ the intersection $\sigma \cap (x+L)$ is open in $x+L$ and equal to $|X| \cap (x+L)$.

Remark 2.8. If X is a tropical variety, i. e. σ is closed in V for all $\sigma \in X$, then $\sigma \cap (x+L)$ can only be open and non-empty if it is all of $x+L$. So in this case we arrive at the usual notion of lineality space found in the literature (although note that most authors only call a maximal subspace with this property a lineality space).

Lemma 2.9. *Let X be a partially open polyhedral complex in V with a lineality space L. Denote by $q : V \to V/L$ the quotient map, where V/L is considered to have the underlying lattice $\Lambda/(\Lambda \cap L)$. Then for all $\sigma, \tau \in X$ we have:*

(a) *$q(\sigma)$ is a partially open polyhedron of dimension $\dim q(\sigma) = \dim \sigma - \dim L$.*

(b) *If $\tau \leq \sigma$ then $q(\tau) \leq q(\sigma)$.*

(c) *$q(\sigma \cap \tau) = q(\sigma) \cap q(\tau)$.*

(d) *If $q(\sigma) = q(\tau)$ then $\sigma = \tau$.*

(e) *$\Lambda_{q(\sigma)} = \Lambda_\sigma/(\Lambda \cap L)$; and if τ is a facet of σ then $u_{q(\sigma)/q(\tau)} = \overline{u_{\sigma/\tau}}$ with this identification.*

Proof. By induction it suffices to prove the statements for $\dim L = 1$. We choose coordinates $(x,y) \in \mathbb{R}^{\dim V - 1} \times \mathbb{R} \cong V$ such that $L = \{(x,y) : x = 0\}$, and consider x as coordinates on V/L.

(a) In the defining inequalities for σ we may assume that all of them that contain y are strict: if one of the non-strict defining inequalities $f(x,y) \leq c$ containing y was satisfied as an equality at a point $(x_0, y_0) \in \sigma$ it could not be satisfied in a neighborhood of $y_0 \in \mathbb{R}$, contradicting the openness of $\sigma \cap ((x_0, y_0) + L)$. Hence σ can be written as

$$\sigma = q^{-1}(\sigma_0) \cap \{(x,y) : y > f_i(x) + a_i \text{ and } y < g_j(x) + b_j \text{ for all } i,j\} \tag{1}$$

for some linear forms f_i, g_j, constants a_i, b_j, and a partially open polyhedron σ_0 given by the defining inequalities of σ that do not contain y. But then

$$q(\sigma) = \sigma_0 \cap \{x : f_i(x) + a_i < g_j(x) + b_j \text{ for all } i,j\}, \tag{2}$$

since for x satisfying these conditions we can always find $y \in \mathbb{R}$ with $f_i(x) + a_i < y < g_j(x) + b_j$ for all i,j. Hence $q(\sigma)$ is a partially open polyhedron. Moreover, the openness of $\sigma \cap ((x,y) + L)$ means that all non-empty fibers of $q|_\sigma$ have dimension $\dim L$, so that $\dim q(\sigma) = \dim \sigma - \dim L$.

(b) If $\tau \leq \sigma$ is obtained from σ by changing some non-strict inequalities to equalities this means that τ can be written as in (1) for some $\tau_0 \leq \sigma_0$ and with the same f_i, g_j. Then (2) holds for τ and τ_0 as well, and we conclude that $q(\tau)$ is a face of $q(\sigma)$.

(c) The inclusion "$\subset$" is obvious. Conversely, if $x \in q(\sigma) \cap q(\tau)$ then there are $y, y' \in \mathbb{R}$ with $(x,y) \in \sigma$ and $(x,y') \in \tau$. Hence

$$(x,y) \in \sigma \cap ((x,y') + L) \subset |X| \cap ((x,y') + L),$$

which implies $(x, y) \in \tau \cap ((x, y') + L)$ by definition of a lineality space. Hence $(x, y) \in \sigma \cap \tau$, i. e. $x = q(x, y) \in q(\sigma \cap \tau)$.

(d) By (c), the equality $q(\sigma) = q(\tau)$ implies $q(\sigma \cap \tau) = q(\sigma)$. In particular, $\sigma \cap \tau \neq \emptyset$. Hence $\sigma \cap \tau$ is a face of σ, which by the dimension statement of (a) must be of the same dimension as σ. But this means that $\sigma \cap \tau = \sigma$, and by symmetry thus also $\sigma \cap \tau = \tau$.

(e) From the definition of a lineality space it follows that $L \subset V_\sigma$; hence $V_{q(\sigma)} = V_\sigma / L$ and thus $\Lambda_{q(\sigma)} = \Lambda_\sigma / (\Lambda \cap L)$. In particular, for τ a facet of σ we have $\Lambda_\sigma / \Lambda_\tau = \Lambda_{q(\sigma)} / \Lambda_{q(\tau)}$ (with the isomorphism given by taking quotients by $\Lambda \cap L$), from which the statement about the primitive normal vector follows. $\qquad \square$

Corollary 2.10 (Quotients). *Let X be an n-dimensional partially open tropical variety in $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ with a lineality space L, and let $q : V \to V/L$ be the quotient morphism. Then*

$$X/L := \{q(\sigma) : \sigma \in X\}$$

together with the weights $\omega_{X/L}(q(\sigma)) := \omega_X(\sigma)$ is a partially open tropical variety of dimension $n - \dim L$ in the vector space V/L with lattice $\Lambda / (\Lambda \cap L)$. We will also denote it by $q(X)$.

Proof. By Lemma 2.9 (a) we see that X/L is a collection of partially open polyhedra which by (d) are in bijection to the polyhedra in X. The statements (b) and (c) of the lemma now imply that X/L is a partially open polyhedral complex as in Definition 2.2. Moreover, the dimension statement of part (a) of the lemma means that X/L is of pure dimension $n - \dim L$, and that $\omega_{X/L}(q(\sigma)) := \omega_X(\sigma)$ defines a weight function on the top-dimensional cones. Finally, part (e) of the lemma shows that the images of the balancing conditions for X in V give us the balancing conditions for X/L in V/L. $\qquad \square$

Our main examples for this quotient construction are (cycles in) the moduli spaces of tropical curves, which we introduce in Section 2.4.

2.3. **Tropical intersection theory.** We now briefly recall some constructions and results of tropical intersection theory. For a detailed introduction we refer to [AR10, Sha11, Sha13]. Although the theory is only developed for closed tropical cycles there, the extension to the case of partially open tropical cycles stated below follows immediately since all constructions involved are local.

Construction 2.11 (Rational functions and divisors). For a partially open pure-dimensional tropical cycle X, a *(non-zero) rational function on X* is a continuous function $\varphi : |X| \to \mathbb{R}$ that is affine linear on each cell (for a suitable polyhedral complex structure), with linear part given by an element of $\Lambda^\vee$. We can associate to such a rational function φ a *(Weil) divisor*, denoted by $\varphi \cdot X$. It is a partially open tropical subcycle of X (i.e. a partially open cycle whose support is a subset of $|X|$) of codimension 1. Its support is contained in the subset of $|X|$ where φ is not locally affine linear [AR10, Construction 3.3].

Two important examples for rational functions and divisors in this paper are:

(a) On $X = \mathbb{R}^N$, a tropical polynomial φ defines a rational function, and its divisor is just the tropical hypersurface defined by φ.

(b) If X is one-dimensional, the divisor $\varphi \cdot X$ consists of finitely many points. We define its *degree* of to be the sum of the weights of the points in $\varphi \cdot X$. By abuse of notation, we sometimes write this degree as $\varphi \cdot X$ as well.

Multiple intersection products $\varphi_1 \cdot \ \cdots \ \cdot \varphi_r \cdot X$ are commutative by [AR10, Proposition 3.7].

Remark 2.12 (Pull-backs and push-forwards). Rational functions on a (partially open) tropical cycle $Y \subset \Lambda' \otimes_{\mathbb{Z}} \mathbb{R}$ can be *pulled back* along a morphism $f : X \to Y$ to rational functions $f^*\varphi = \varphi \circ f$ on X. Also, we can *push forward* subcycles Z of X to subcycles f_*Z of Y of the same dimension [AR10, Proposition 4.6 and Corollary 7.4], where in the partially open case we will always restrict ourselves to injective maps f so that no problems can arise from two partially open polyhedra with different boundary behavior that are mapped by f to an overlapping image. In any case, by picking a suitable refinement of X we can ensure that the partially open image polyhedra $f(\sigma)$ for $\sigma \in X$ form a partially open polyhedral complex f_*Z. For a top-dimensional cell $\sigma' \in f_*Z$ its weight $\omega_{f_*Z}(\sigma')$ is given by

$$\omega_{f_*Z}(\sigma') := \sum_{\sigma} \omega_X(\sigma) \cdot |\Lambda'_{\sigma'}/f(\Lambda_{\sigma})|,$$

where the sum goes over all top-dimensional cells $\sigma \in Z$ with $f(\sigma) = \sigma'$. Of course, in the partially open case, there will be at most one such σ in each sum since f is assumed to be injective. As expected, push-forwards and pull-backs satisfy the projection formula [AR10, Proposition 4.8 and Corollary 7.7].

Construction 2.13 (Pull-backs along quotient maps). Let X be a partially open tropical variety with a lineality space L, so that there is a quotient variety X/L as in Corollary 2.10 with quotient map $q : X \to X/L$. Moreover, let Z be a partially open subcycle of X/L. Then the collection of polyhedra $q^{-1}(\sigma)$ for $\sigma \in Z$, together with the weight function $\omega(q^{-1}(\sigma)) = \omega_Z(\sigma)$, is a partially open subcycle of X of dimension $\dim Z + \dim L$ (in fact, the balancing conditions follows from Lemma 2.9 (e)). We denote it by q^*Z.

We conclude this short excursion into tropical intersection theory with two compatibility statements between the quotient construction and pull-backs of rational functions resp. push-forward of cycles.

Lemma 2.14. *Let X be a partially open tropical cycle with lineality space L and quotient map $q : X \to X/L$. Then for any rational function φ on X/L we have*

$$(q^*\varphi \cdot X)/L = \varphi \cdot (X/L).$$

Proof. This is obvious from the definitions. $\square$

Lemma 2.15. *Let $f : X \to Y$ be a morphism between partially open tropical cycles. Assume that X and Y have lineality spaces L and L', respectively, and that the linear part of f on*

each cell maps $\Lambda_X \cap L$ isomorphically to $\Lambda_Y \cap L'$. If $g : X/L \to Y/L'$ is the morphism giving a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\;f\;} & Y \\
\Big\downarrow{\scriptstyle q} & & \Big\downarrow{\scriptstyle q'} \\
X/L & \xrightarrow{\;g\;} & Y/L'
\end{array}
$$

then $f_*(X)/L' = g_*(X/L)$.

Proof. Assume that the polyhedral complex structure of all cycles is sufficiently fine to be compatible with all of the morphisms. Let σ be a maximal cell in X, and set $\tau = f(\sigma)$ and $\rho = q'(\tau)$. Applying a suitable translation, we may assume that f (and thus also g) is linear on σ.

If $\dim \rho < \dim \sigma - \dim L$ then σ does not contribute to either cycle in the statement of the lemma. Otherwise, the assumption implies that f maps $\Lambda_X \cap L$, and hence every saturated sublattice of $\Lambda_X \cap L$ such as $\Lambda_\sigma \cap L$, to a saturated sublattice of Λ_Y. Hence, in the inclusion

$$
f(\Lambda_\sigma \cap L) \subset f(\Lambda_\sigma) \cap f(L) = f(\Lambda_\sigma) \cap L' \subset \Lambda_\tau \cap L'
$$

we must have equality since both sides are saturated lattices in Λ_Y of the same rank. By the last equality in this chain it then follows from the weight formulas of Corollary 2.10 and Remark 2.12 that the contribution of σ to the cell ρ in the two cycles of the lemma is

$$
\begin{aligned}
\omega_{f_*(X)/L'}(\rho) &= \omega_X(\sigma) \cdot |\Lambda_\tau / f(\Lambda_\sigma)| \\
&= \omega_X(\sigma) \cdot |(\Lambda_\tau/(\Lambda_\tau \cap L'))/(f(\Lambda_\sigma)/(f(\Lambda_\sigma) \cap L'))| \\
&= \omega_X(\sigma) \cdot |q'(\Lambda_\tau)/q'(f(\Lambda_\sigma))| \\
&= \omega_X(\sigma) \cdot |q'(\Lambda_\tau)/g(q(\Lambda_\sigma))| \\
&= \omega_{g_*(X/L)}(\rho).
\end{aligned}
$$

$\square$

2.4. Tropical moduli spaces of curves.
We will now come to the construction of the moduli spaces $\mathscr{M}_0(\mathbb{R}^N, \Sigma)$ of tropical curves in $\mathbb{R}^N$. Analogously to the algebraic case, elements of these spaces will be given by a rational n-marked abstract tropical curve, together with a map to $\mathbb{R}^N$ whose image is a (not necessarily smooth) tropical curve in $\mathbb{R}^N$ as in Section 2.1, with unbounded directions as determined by Σ.

For later purposes we will also need a version $\mathscr{M}_0'(\mathbb{R}^N, \Sigma)$ of these spaces where the ends x_i have bounded lengths. In contrast to the original spaces $\mathscr{M}_0(\mathbb{R}^N, \Sigma)$ they will only be partially open tropical varieties since there is no limit curve when the length of such a bounded end approaches zero.

Let us start with the discussion of abstract tropical curves.

Construction 2.16 (Abstract curves). A *(rational) abstract tropical curve* [GKM09, Definition 3.2] is a metric tree graph Γ with all vertices of valence at least 3. Unbounded ends (with no vertex there) are allowed and labeled by $x_1, \ldots, x_n$. The tuple $(\Gamma, x_1, \ldots, x_n)$ will

be referred to as an *n-marked abstract tropical curve*. Two such curves are called isomorphic (and will from now on be identified) if there is an isometry between them that respects the markings. The set of all *n*-marked abstract tropical curves (modulo isomorphisms) is denoted $\mathcal{M}_{0,n}$.

For $n \geq 3$ it follows from [SS04, Theorem 3.4], [Mik07, Section 2], or [GKM09, Theorem 3.7] that $\mathcal{M}_{0,n}$ can be given the structure of a tropical fan as follows: Consider the map

$$d : \mathcal{M}_{0,n} \to \mathbb{R}^{\binom{n}{2}}, \quad (\Gamma, x_1, \ldots, x_n) \mapsto (\mathrm{dist}_\Gamma(x_i, x_j))_{i<j}$$

where $\mathrm{dist}_\Gamma(x_i, x_j)$ denotes the distance between the two marked ends x_i and x_j in the metric graph Γ. Moreover, consider the linear map

$$\varphi : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{2}}, \quad (a_i)_i \mapsto (a_i + a_j)_{i,j},$$

let $u_i = \varphi(e_i)$ be the images of the unit vectors, let $U_n := \varphi(\mathbb{R}^n) = \langle u_1, \ldots, u_n \rangle$, and

$$q_n : \mathbb{R}^{\binom{n}{2}} \to Q_n := \mathbb{R}^{\binom{n}{2}}/U_n$$

be the quotient map. For a subset $I \subset [n]$ with $2 \leq |I| \leq n-2$ define v_I to be the image under $q_n \circ d : \mathcal{M}_{0,n} \to Q_n$ of a tree Γ with exactly one bounded edge of length one, the marked leaves x_i with $i \in I$ on one side and the leaves x_i for $i \notin I$ on the other. Let $\Lambda_n := \langle v_I \rangle_{\mathbb{Z}} \subset Q_n$ be the lattice in Q_n generated by the vectors v_I. By [SS04, Theorem 4.2] the map $q_n \circ d : \mathcal{M}_{0,n} \to Q_n$ is injective, and its image is a purely $(n-3)$-dimensional simplicial tropical fan in $Q_n = \Lambda_n \otimes_{\mathbb{Z}} \mathbb{R}$, with all top-dimensional cones having weight 1. In the following we will always consider $\mathcal{M}_{0,n}$ with this structure of a tropical fan.

The cones of $\mathcal{M}_{0,n}$ are labeled by the *combinatorial types* of marked curves, i.e. by the homeomorphism classes of the curves relative to their ends. The vectors v_I generate the rays of $\mathcal{M}_{0,n}$.

Construction 2.17 (Abstract curves with bounded ends). We will now adapt Construction 2.16 to the case when the ends also have bounded lengths. So we say that an *n-marked abstract tropical curve with bounded ends* is a metric tree graph as above with the unbounded ends replaced by bounded intervals, i.e. a metric graph Γ without 2-valent vertices, and the 1-valent vertices labeled by $x_1, \ldots, x_n$. The notions of isomorphisms and combinatorial types carry over from the case with unbounded ends. The set of all *n*-marked curves (modulo isomorphisms) with bounded ends is denoted $\mathcal{M}'_{0,n}$.

To make $\mathcal{M}'_{0,n}$ for $n \geq 3$ into a partially open tropical variety we consider the distance map $d : \mathcal{M}'_{0,n} \to \mathbb{R}^{\binom{n}{2}}$ as above, which in this case however includes the lengths of the bounded ends. Then d is injective: the vectors $u_i = \varphi(e_i) \in Q_n$ in the notation of Construction 2.16 correspond exactly to a change of the length of the bounded edge at x_i — so by Construction 2.16 the image point under $q_n \circ d : \mathcal{M}'_{0,n} \to Q_n$ allows to reconstruct the combinatorial type of the graph as well as the lengths of all edges not adjacent to the markings, whereas the full vector in $\mathbb{R}^{\binom{n}{2}}$ then allows to reconstruct the lengths of the ends as well.

Note that the combinatorial types of these curves with bounded ends are in one-to-one correspondence with the types of curves with unbounded ends, and that d maps $\mathcal{M}'_{0,n}$ to

a partially open polyhedral complex (with cones in bijection to the combinatorial types), which by abuse of notation we will also denote by $\mathscr{M}'_{0,n}$. Taking again the lattice in $\mathbb{R}^{\binom{n}{2}}$ generated by all graphs with integer lengths, and giving all cells of $\mathscr{M}'_{0,n}$ weight 1, we get in fact a partially open tropical variety in $\mathbb{R}^{\binom{n}{2}}$ of dimension $2n-3$. It has lineality space Q_n in the sense of Definition 2.7, and we have $\mathscr{M}'_{0,n}/Q_n = \mathscr{M}_{0,n}$ as in Corollary 2.10.

Construction 2.18 (Parametrized curves in $\mathbb{R}^N$). A *(rational, parametrized) curve in $\mathbb{R}^N$* [GKM09, Definition 4.1] is a tuple $(\Gamma, x_1, \ldots, x_n, h)$, with $(\Gamma, x_1, \ldots, x_n)$ a rational n-marked abstract tropical curve and $h : \Gamma \to \mathbb{R}^N$ a continuous map satisfying:

(a) On each edge e of Γ, with metric coordinate t, the map h is of the form $h(t) = a + t \cdot v$ for some $a \in \mathbb{R}^N$ and $v \in \mathbb{Z}^N$. If $V \in e$ is a vertex and we choose t positive on e, the vector v will be denoted $v(e,V)$ and called the *direction* of e (at V). If e is an end and t pointing in its direction, we write v as $v(e)$.

(b) For every vertex V of Γ we have the *balancing condition*

$$\sum_{e:V \in e} v(e,V) = 0.$$

Two such curves in $\mathbb{R}^N$ are called isomorphic (and will be identified) if there is an isomorphism of the underlying abstract n-marked curves commuting with the maps to $\mathbb{R}^N$.

The *degree* of a curve $(\Gamma, x_1, \ldots, x_n, h)$ in $\mathbb{R}^N$ as above is the n-tuple $\Sigma = (v(x_1), \ldots, v(x_n)) \in (\mathbb{Z}^N)^n$ of directions of its ends. We denote the space of all curves in $\mathbb{R}^N$ of a given degree Σ by $\mathscr{M}_0(\mathbb{R}^N, \Sigma)$.

Note that Σ may contain zero vectors, corresponding to contracted ends that can be thought of as marked points in the algebraic setting (see Construction 2.19).

If $n \geq 3$ and there is at least one end x_i with $v(x_i) = 0$ we can use the bijection

$$\mathscr{M}_0(\mathbb{R}^N, \Sigma) \to \mathscr{M}_{0,n} \times \mathbb{R}^N, \quad (\Gamma, x_1, \ldots, x_n, h) \mapsto ((\Gamma, x_1, \ldots, x_n), h(x_i))$$

(which forgets h except for its image on x_i) to give it the structure of a tropical variety [GKM09, Proposition 4.7].

Construction 2.19 (Evaluation maps). For each i with $v(x_i) = 0$ there is an *evaluation map*

$$\mathrm{ev}_i : \mathscr{M}_0(\mathbb{R}^N, \Sigma) \to \mathbb{R}^N$$

assigning to a tropical curve $(\Gamma, x_1, \ldots, x_n, h)$ the position $h(x_i)$ of its i-th marked end (note that this is well-defined since the marked end x_i is contracted to a point). By [GKM09, Proposition 4.8], these maps are morphisms of tropical fans.

Construction 2.20 (Parametrized curves in $\mathbb{R}^N$ with bounded ends). Constructions 2.17 and 2.18 can obviously be combined to obtain moduli spaces $\mathscr{M}'_0(\mathbb{R}^N, \Sigma)$ of curves in $\mathbb{R}^N$ with bounded ends, as pull-backs under the quotient maps that forget the lengths of the bounded ends. They are partially open tropical varieties and admit evaluation maps to $\mathbb{R}^N$ as in Construction 2.19 at all ends (i. e. not just at the contracted ones).

Remark 2.21 (Parametrized curves in $\mathbb{R}^N$ with few markings). In the following, we will also need the moduli spaces $\mathscr{M}_0(\mathbb{R}^N, \Sigma)$ of parametrized curves $(\Gamma, x_1, \ldots, x_n, h)$ for the special cases when $n < 3$ or there is no x_i with $v(x_i) = 0$, so that the above bijection with $\mathscr{M}_{0,n} \times \mathbb{R}^N$ is not available. In order to overcome this technical problem and still give $\mathscr{M}_0(\mathbb{R}^N, \Sigma)$ the structure of a tropical variety, there are two possibilities:

    (a) One can use *barycentric coordinates*, taking a certain weighted average of the vertices of the curve.

    (b) One can combine evaluation maps at several non-contracted ends, where at each such end the evaluations are only taken modulo the direction of the edge to make them well-defined.

Details on these alternative construction can be found in [Och13, Section 1.2]. In the following, we will just assume that $\mathscr{M}_0(\mathbb{R}^N, \Sigma)$ has the structure of a tropical variety in any case.

Remark 2.22 (Degree of the canonical divisor on curves). As in Example 2.6, let $X \subset \mathbb{R}^N$ be a smooth rational curve or a smooth hypersurface. Consider an *n*-marked curve $(\Gamma, x_1, \ldots, x_n, h) \in \mathscr{M}_0(\mathbb{R}^N, \Sigma)$ with $h(\Gamma) \subset X$, so that h can also be viewed as a morphism from the (abstract) tropical curve Γ to X.

We can then consider K_X as a divisor on X as in Construction 2.11 and compute its pull-back $h^* K_X$ according to Remark 2.12. Its degree (as in Construction 2.11 (b)) depends only on X and Σ, and not on h:

    (a) If X is a rational curve note that any two points in X are rationally equivalent divisors in the sense of [AR10]. Hence the degree of the divisor h^*P for a point $P \in X$ does not depend on P; for a general point P it is just the sum of the weights of the direction vectors for all edges of Γ that map some point to P under h. In particular, taking for P a point far out on an unbounded edge of X, we see that this degree depends only on Σ and not on h. It will be called the *degree* $\deg h$ of h and is the tropical counterpart of the notion of degree of a morphism between smooth curves in algebraic geometry. The degree of $h^* K_X$ is now just $\deg h \cdot \deg K_X$; in particular by the above it depends only on Σ and not on h.

    (b) If X is a hypersurface in $\mathbb{R}^N$ a local computation shows that $K_X = X \cdot X$, and hence by the projection formula the degree of $h^* K_X$ on Γ is the same as the degree of $h_* \Gamma \cdot X$ on $\mathbb{R}^N$. But the 1-cycle $h_* \Gamma$ in $\mathbb{R}^N$ is rationally equivalent to its so-called *recession fan*, i.e. the fan obtained by shrinking all bounded edges to zero length. As this fan is determined by Σ it follows also in this case that the degree of $h^* K_X$ depends only on Σ and not on h.

We will therefore denote the degree of $h^* K_X$ also by $K_X \cdot \Sigma$.

3. GLUING MODULI SPACES

Throughout this section, let $X \subset \mathbb{R}^N$ be a smooth rational curve or a smooth hypersurface as in Example 2.6. Aiming at enumerative applications, we want to construct tropical

analogues of the algebraic moduli spaces of stable maps to a variety, or in other words gen-eralizations of the tropical moduli spaces $\mathscr{M}_0(\mathbb{R}^N, \Sigma)$ of Construction 2.18 to other target spaces than $\mathbb{R}^N$. The naive approach would simply be to use the subset

$$\tilde{\mathscr{M}}_0(X, \Sigma) := \{ (\Gamma, x_1, \ldots, x_n, h) \in \mathscr{M}_0(\mathbb{R}^N, \Sigma) : h(\Gamma) \subset X \}$$

of tropical curves mapping to X. As $K_X \cdot \Sigma$ is independent of the curves in this space by Remark 2.22, the algebro-geometric analogue tells us that we expect $\tilde{\mathscr{M}}_0(X, \Sigma)$ to be of dimension $\dim X + |\Sigma| - 3 - K_X \cdot \Sigma$ (in fact, this independence is the reason why we restrict to the curve and hypersurface cases in this paper). However, just as in the algebraic case, the actual dimension of this space might be bigger, as the following example shows.

Example 3.1. Let $X = L_1^2$ be the tropical line in $\mathbb{R}^2$ as in Example 2.3, and consider degree-2 covers of X, i.e. $\Sigma = (e_0, e_0, e_1, e_1, e_2, e_2)$. The following pictures show combinatorial types of tropical curves in $\tilde{\mathscr{M}}_0(X, \Sigma)$. Edges are labeled with their weight if it is not 1, and their directions in the picture indicate which cell of X they are mapped to. The edge drawn with a dashed line is contracted to a point.
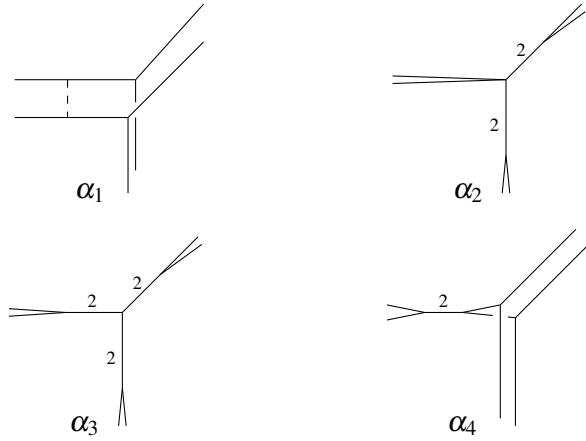
Note that in α_1 and α_4 the lengths of the bounded edges are not independent, since in both cases the two horizontal bounded edges adjacent to the origin must have the same length. Hence the types α_1, α_2, and α_4 are described by 2-dimensional cells in the moduli spaces, whereas α_3 is 3-dimensional (with α_2 as one of its faces). As we expect our tropical moduli space to have dimension 2 (equal to the space of algebraic degree-2 covers of $\mathbb{P}^1$), we see that $\tilde{\mathscr{M}}_0(X, \Sigma)$ has too big dimension.

Our first aim is therefore to define a suitable subset $\mathscr{M}_0(X, \Sigma)$ of $\tilde{\mathscr{M}}_0(X, \Sigma)$ of the expected dimension. We fix the polyhedral complex structure on X to be the unique coarsest one (which exists since X is smooth), and choose the polyhedral complex structure on $\tilde{\mathscr{M}}_0(X, \Sigma)$ as follows.

Notation 3.2 (Curves in X). Let $(\Gamma, x_1, \ldots, x_n, h) \in \tilde{\mathscr{M}}_0(X, \Sigma)$. In the following, all isolated points of $h^{-1}(\sigma)$ for a cell $\sigma \in X$ will be considered as (possibly additional 2-valent) ver-tices of Γ. The interior of every edge of Γ then maps to the relative interior of a unique cell of X, and we include this information in the combinatorial type of a curve in X.

For such a combinatorial type α, we denote the set of all curves in $\tilde{\mathcal{M}}_0(X,\Sigma)$ of this type by $\mathcal{M}(\alpha)$. These are partially open polyhedra, and their closures $\overline{\mathcal{M}(\alpha)}$ give $\tilde{\mathcal{M}}_0(X,\Sigma)$ the structure of a polyhedral complex. We write $\beta \geq \alpha$ for two combinatorial types with $\overline{\mathcal{M}(\beta)} \supset \mathcal{M}(\alpha)$, and say in this case that α is a *face* of β. If in addition $\beta \neq \alpha$ we call β a *resolution* of α.

Notation 3.3 (Local curves). Let $C = (\Gamma, x_1, \ldots, x_n, h) \in \tilde{\mathcal{M}}_0(X,\Sigma)$ be a tropical curve in X, and fix a vertex V of Γ. We can restrict C to the local situation around V and obtain the following data, all written with an index V:

(a) The collection of all vectors $v(e,V)$ for $e \ni V$ as in Construction 2.18 is called the *local degree* Σ_V of the curve at V.

(b) The *star* of X at $h(V)$ will be denoted X_V; it is a shifted tropical fan.

(c) By our assumption on X, we know that X_V is isomorphic to $L_{r_V}^{k_V} \times \mathbb{R}^{m_V}$ for unique k_V, r_V, m_V (where $0 < r_V < k_V$ unless $(k_V, r_V) = (0,0)$). Note that the degree of the canonical divisor as in Remark 2.22 then splits up as

$$K_X \cdot \Sigma = \sum_V K_{X_V} \cdot \Sigma_V,$$

with the sum taken over all vertices of Γ.

(d) Let $C_V \in \tilde{\mathcal{M}}_0(X_V, \Sigma_V)$ be the curve in X_V with one vertex mapping to $h(V)$, and unbounded ends of directions Σ_V. We refer to C_V as a *local curve*. Its combinatorial type α_V will be called the *trivial combinatorial type* of V in $\tilde{\mathcal{M}}_0(X_V, \Sigma_V)$. We will refer to a resolution of α_V also as a *resolution of* V.

Definition 3.4. Let $V \in \Gamma$ be a vertex of a curve $(\Gamma, x_1, \ldots, x_n, h) \in \tilde{\mathcal{M}}_0(X,\Sigma)$. Using Notation 3.3, we define

(a) the *virtual dimension* of V as $\mathrm{vdim}(V) = \mathrm{val}(V) - K_{X_V} \cdot \Sigma_V + \dim X - 3$,

(b) the *resolution dimension* of V as $\mathrm{rdim}(V) = \mathrm{val}(V) - K_{X_V} \cdot \Sigma_V + r_V - 3$,

(c) the *classification number* of V as $c_V = \mathrm{val}(V) + r_V$.

Example 3.5. Let $X = L_3^4$, and consider the combinatorial types $\alpha > \beta$ of curves in X as shown in the picture below. In the type β, the vertex V is mapped to the origin, and the unbounded ends have directions $e_0 + e_1 + e_2$ (with weight 6), and twice e_3 and e_4 (with weight 3). The type α has the same directions of the ends, V_1 and V_2 map to the origin, and consequently V_0 to a positive multiple of $e_0 + e_1 + e_2$.
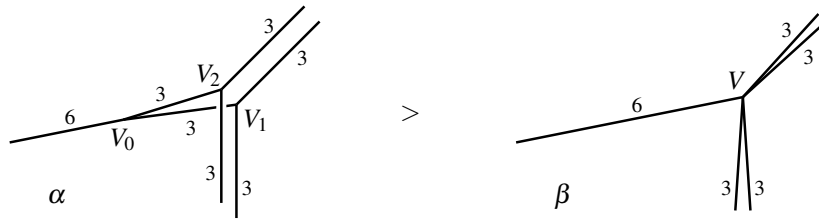
We then have $\operatorname{vdim}(V) = \operatorname{rdim}(V) = -1$, $\operatorname{vdim}(V_1) = \operatorname{rdim}(V_1) = \operatorname{vdim}(V_2) = \operatorname{rdim}(V_2) = 0$, $\operatorname{vdim}(V_0) = 3$, and $\operatorname{rdim}(V_0) = 0$. Moreover, $c_V = 8$, $c_{V_1} = c_{V_2} = 6$, and $c_{V_0} = 3$.

The virtual dimension of a vertex V can be thought of as the expected dimension of the moduli space of curves in $X_V \cong L_{r_V}^{k_V} \times \mathbb{R}^{m_V}$ of degree Σ_V. It includes the dimension of a lineality space coming from translations in $\mathbb{R}^{m_V}$, which is subtracted from the virtual dimension to obtain the resolution dimension (as can be seen in the example above for V_0, which can locally be moved in its 3-dimensional cell spanned by e_0, e_1, e_2). The classification number of Definition 3.4 (c) is a useful number for inductive proofs because it is always non-negative and becomes smaller in resolutions, as the following lemma shows.

Lemma 3.6. *Let α be a resolution of a vertex V of a curve in X. Then the classification number of every vertex W of α is smaller than c_V.*

Proof. Note first that $c_W \leq c_V$ since both summands in the definition of the classification number cannot get bigger when passing from V to W:

(a) As α is a tree, shrinking an edge to zero length will merge two vertices into one, whose valence is the sum of the original valences minus 2. In particular, since the trivial combinatorial type of V is obtained from α by a sequence of such processes, we conclude that $\operatorname{val}(V) \geq \operatorname{val}(W)$, with equality if and only if all vertices of α except W have valence 2.

(b) If W is mapped to the relative interior of a cone σ_W, we have $r_W = \dim X - \dim \sigma_W$. The analogous statement holds for V, and hence

$$r_V = \dim X - \dim \sigma_V \geq \dim X - \dim \sigma_W = r_W$$

since σ_V is a face of σ_W.

If we had equality for both numbers, all vertices W' of α except W must have valence 2 by (a). Moreover, W and V have to lie in the same cell by (b). Hence α is the same combinatorial type as the trivial type V after removing all 2-valent vertices W'. But this means that all two-valent vertices lie in the interior of an edge that is completely mapped to one cell of X. As this is excluded by definition, there can actually be no 2-valent vertices. Hence α is the trivial type V, in contradiction to α being a resolution of V. $\square$

The idea to construct the desired moduli spaces of curves in X is now as follows.

(a) Vertices of negative resolution dimension should not be admitted, since they correspond (locally, and modulo their lineality space) to an algebraic moduli space of curves of negative virtual dimension. We will exclude them in Definition 3.8. For the case of curves, this corresponds to the Riemann-Hurwitz condition as e.g. in [BBM11, Cap14, CMR16, BM16].

(b) Vertices of resolution dimension 0 correspond (again locally and modulo their lineality space) to a 0-dimensional algebraic moduli space. Hence the curves in the corresponding tropical moduli spaces should not allow any resolutions, i.e. these

moduli spaces will consist of only one cell, whose weight is the degree of the algebraic moduli space.

In this paper, we will only consider the tropical situation. The weights of the cells for vertices of resolution dimension 0 are therefore considered as initial input data for our constructions, as in Definition 3.9 below.

(c) Vertices of positive resolution dimension will lead to curves that allow resolutions, and whose moduli spaces therefore consider of several cells. These spaces can be obtained recursively by gluing from the initial ones of (b) using Construction 3.12, so that no additional input data is required for these cases.

However, as for this paper the initial weights of (b) are arbitrary numbers a priori, they need to satisfy some compatibility conditions in order for the gluing process to lead to a well-defined space. These conditions are encoded in the notion of a good vertex in Definition 3.11. They are originally formulated in every resolution dimension; we will see in Corollary 3.18 however that only the conditions in resolution dimension 1 are relevant since the others follow from them.

In this recursive construction of the moduli spaces, the following example shows that the resolution dimension is not necessarily strictly increasing. We will therefore use the classification number for these purposes.

Example 3.7. Consider again two combinatorial types $\alpha > \beta$ as in Example 3.5, however in $X = L_2^3$ and with directions as indicated in the following picture.
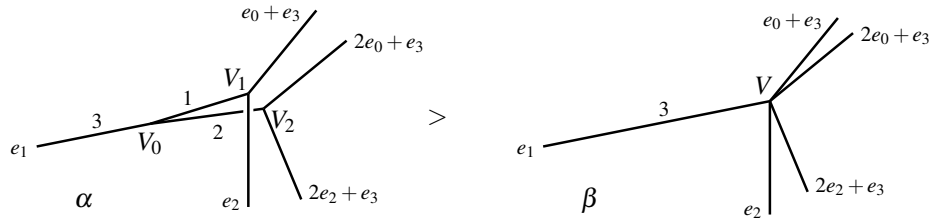
Then $\mathrm{rdim}(V) = \mathrm{rdim}(V_0) = \mathrm{rdim}(V_1) = 1$ and $\mathrm{rdim}(V_2) = 0$, i. e. all resolution dimensions are non-negative and hence will be admitted. The weight for the type α will be defined by a gluing procedure over its three vertices in Construction 3.12. However, V_1 has the same resolution dimension as V, so that a recursive construction over the resolution dimension would not work.

Definition 3.8 (The moduli space $\mathscr{M}_0(X, \Sigma)$ as a polyhedral complex).

(a) An *admissible* combinatorial type of a curve in X is a combinatorial type α such that for all vertices V in α we have $\mathrm{rdim}(V) \geq 0$.

(b) We denote by $\mathscr{M}_0(X, \Sigma)$ the polyhedral subcomplex of $\tilde{\mathscr{M}}_0(X, \Sigma)$ consisting of all closed cells $\overline{\mathscr{M}(\alpha)}$ such that α and all its faces are admissible, and

$$\dim \mathscr{M}(\alpha) = \mathrm{vdim}\,\mathscr{M}_0(X, \Sigma) := |\Sigma| - K_X \cdot \Sigma + \dim X - 3.$$

Note that this dimension is just $\mathrm{vdim}(V)$ if α is a combinatorial type with just one vertex V.

(c) The *neighborhood* of a combinatorial type α in $\mathscr{M}_0(X, \Sigma)$ is defined as

$$\mathscr{N}(\alpha) := \bigcup_{\beta \geq \alpha} \mathscr{M}(\beta),$$

where the union is taken over all combinatorial types $\beta \geq \alpha$ in $\mathscr{M}_0(X, \Sigma)$.

In the following, we will apply this definition also to the case of local curves as in Notation 3.3, in order to obtain moduli spaces $\mathscr{M}_0(X_V, \Sigma_V)$.

Example 3.5 shows that faces of admissible types need not be admissible again, so that the condition of all faces of α being admissible in Definition 3.8 (b) is not vacuous.

Definition 3.9 (Moduli data)**.** *Moduli data* for a smooth variety X are a collection $(\omega_V)_V$ of weights in $\mathbb{Q}$ for every vertex V of a curve in X with $\mathrm{rdim}(V) = 0$. All subsequent constructions and results in this section will depend on the choice of such moduli data.

Construction 3.10 (Vertices of resolution dimension 0)**.** Let V be a vertex of a local curve in $\mathscr{M}_0(X_V, \Sigma_V)$. As $X_V \cong L_{r_V}^{k_V} \times \mathbb{R}^{m_V}$, the trivial combinatorial type of V in $\mathscr{M}_0(X_V, \Sigma_V)$ has dimension m_V, corresponding to a translation along $\mathbb{R}^{m_V}$. But $\mathrm{vdim}(V) = \mathrm{rdim}(V) + m_V$, and thus the trivial combinatorial type is maximal in $\mathscr{M}_0(X_V, \Sigma_V)$ if and only if $\mathrm{rdim}(V) = 0$. In this case, the moduli space $\mathscr{M}_0(X_V, \Sigma_V)$ consists of only one cell, and we equip it with the weight ω_V from our moduli data.

To define the weights on $\mathscr{M}_0(X, \Sigma)$ in general we need Definition 3.11 and Construction 3.12, which depend on each other and work in a combined recursion on the classification number of vertices. The following definition of a good vertex of a certain classification number thus assumes that good vertices of smaller classification number have already been defined. Moreover, for every combinatorial type α in a local moduli space $\mathscr{M}_0(X_V, \Sigma_V)$ all of whose vertices have smaller classification number and are good it assumes from Construction 3.12 below that there is a *gluing cycle* $\mathscr{Z}(\alpha)$ on the neighborhood $\mathscr{N}(\alpha)$.

Definition 3.11 (Good vertices)**.** Let V be a vertex of a (local) curve in $\mathscr{M}_0(X_V, \Sigma_V)$, of classification number c_V. By recursion on c_V, we define V to be a *good* vertex if the following three conditions hold.

(a) Every vertex of every resolution α of V in $\mathscr{M}_0(X_V, \Sigma_V)$ (which has classification number smaller than c_V by Lemma 3.6) is good. (There is then a gluing cycle $\mathscr{Z}(\alpha)$ on $\mathscr{N}(\alpha)$ by Construction 3.12).

(b) $\mathscr{M}_0(X_V, \Sigma_V)$ is a tropical cycle with the following weights:
   - If $\mathrm{rdim}(V) = 0$ we equip the unique cell of $\mathscr{M}_0(X_V, \Sigma_V)$ with the weight from the moduli data as in Construction 3.10.
   - If $\mathrm{rdim}(V) > 0$ the maximal types α in $\mathscr{M}_0(X_V, \Sigma_V)$ are not the trivial one, i. e. they are resolutions of V. The weights on the corresponding cells $\mathscr{M}(\alpha) = \mathscr{N}(\alpha)$ are then the ones of the gluing cycles $\mathscr{Z}(\alpha)$ as in (a).

(c) For every resolution α of V in $\mathcal{M}_0(X_V, \Sigma_V)$ and every maximal type $\beta \geq \alpha$ in $\mathcal{M}_0(X_V, \Sigma_V)$ (which is then also a resolution of V), the weight of the cell $\mathcal{M}(\beta)$ is the same in the gluing cycles $\mathcal{Z}(\alpha)$ and $\mathcal{Z}(\beta)$.

Construction 3.12 (The gluing cycle $\mathcal{Z}(\alpha)$). Fix a (not necessarily maximal) combinatorial type α in a moduli space $\mathcal{M}_0(X, \Sigma)$ as in the picture below on the left, and assume that all its vertices are good. We will now construct a cycle $\mathcal{Z}(\alpha)$ of dimension $\mathrm{vdim}\,\mathcal{M}_0(X, \Sigma)$ on the neighborhood $\mathcal{N}(\alpha)$. An important example of this is when α is a resolution of a vertex in a local moduli space. More technical details on this construction can be found in [Och13, Construction 1.5.13].

For each vertex V of α let σ_V° be the open cell of X in which V lies, and let $X(V) = \bigcup_{\sigma \supset \sigma_V} \sigma^\circ \subset X$; it is an open neighborhood of σ_V. Similarly, for each edge e of α let σ_e° be the open cell of X in which e lies, and set $X(e) = \bigcup_{\sigma \supset \sigma_e} \sigma^\circ$.
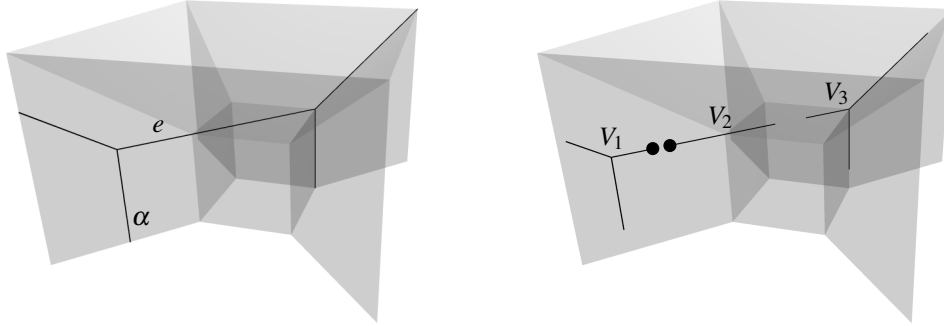
First we will construct local moduli spaces $\mathcal{M}_V$ for each vertex V of α. As V is good the local moduli space $\mathcal{M}_0(X_V, \Sigma_V)$ is a tropical cycle by Definition 3.11 (b). Let $\mathcal{M}_0'(X_V, \Sigma_V)$ be the corresponding moduli cycle of curves with bounded ends, i.e. the pull-back of $\mathcal{M}_0(X_V, \Sigma_V)$ under the quotient map that forgets the lengths of the ends as in Construction 2.20. We denote by

$$\mathcal{M}_V = \{(\Gamma, x_1, \ldots, x_n, h) \in \mathcal{M}_0'(X_V, \Sigma_V) : h(\Gamma) \subset X(V)\}$$

its partially open polyhedral subcomplex of all curves with bounded ends that lie entirely in $X(V)$. A typical element of $\mathcal{M}_V$ is a curve piece as shown in the picture above on the right. (Note that these pieces might also be resolutions of the corresponding vertices.)

Now we glue these pieces $\mathcal{M}_V$ together. For each bounded edge e of α, joining two vertices V_1 and V_2 (as in the picture), there are corresponding bounded ends in $\mathcal{M}_{V_1}$ and $\mathcal{M}_{V_2}$. Let

$$\mathrm{ev}_e : \prod_V \mathcal{M}_V \to X(e) \times X(e)$$

be the evaluation map at these two ends, where the product is taken over all vertices V of α. The product

$$\prod_e \mathrm{ev}_e^* \Delta_{X(e)}$$

over all pull-backs of the diagonals $\Delta_{X(e)}$ along these evaluation maps as in Appendix A can also be written as the product $\prod_e \mathrm{ev}_e^* \Delta_X$ along the extended evaluation maps to $X \times X$

by Remark A.5. We will therefore abbreviate it by $\mathrm{ev}^* \Delta_X$; it is a cocycle on $\prod_V \mathcal{M}_V$. The support of the cycle $\mathrm{ev}^* \Delta_X \cdot \prod_V \mathcal{M}_V$ then consists of points corresponding to curve pieces that can be glued together to a curve in X, i.e. so that e.g. the positions of the dots in the picture above coincide.

However, these curve pieces still carry the information about the position of the gluing points. In order to forget these positions, we take a quotient as follows. For each bounded edge e of α between two vertices V_1 and V_2 as above, there are two vectors u_{V_1} and u_{V_2} in the lineality spaces of $\mathcal{M}_{V_1}$ and $\mathcal{M}_{V_2}$, respectively, that parametrize the lengths of these ends as in Constructions 2.16 and 2.17. The vector $u_e := u_{V_1} - u_{V_2}$ is then in the lineality space of $\prod_V \mathcal{M}_V$, and we denote by L_α the vector space spanned by the vectors u_e for all bounded edges e of α. Let q be the quotient map by L_α; the quotient $q\big(\mathrm{ev}^* \Delta_X \cdot \prod_V \mathcal{M}_V\big)$ then does not contain the information about the gluing points any more.

From this cycle we now obtain an injective morphism that considers the curve pieces as a glued curve in X. It can be defined as

$$f : q\big(\mathrm{ev}^* \Delta_X \cdot \prod_V \mathcal{M}_V\big) \to \mathcal{M}'_{0,n}(\mathbb{R}^N, \Sigma)$$

$$\big((d_{i,j})_{\{i,j\} \in R_V}, a^V\big)_V \mapsto \left(\left(\sum_{\{k,l\} \in R_{i,j}} d_{k,l}\right)_{\{i,j\} \in R}, a^{V_0}\right),$$

where

- a^V denotes the coordinates of the root vertex in the local moduli space $\mathcal{M}_V$ for V, of which we choose V_0 as the root vertex in $\mathcal{M}'_{0,n}(\mathbb{R}^N, \Sigma)$;

- $d_{i,j}$ denotes the distance coordinates on the moduli spaces, with R_V and R the index sets of all pairs of ends in the local moduli spaces $\mathcal{M}_V$ and the full moduli space $\mathcal{M}'_{0,n}(\mathbb{R}^N, \Sigma)$, respectively;

- $R_{i,j} \subset \bigcup_V R_V$ for $\{i,j\} \in R$ is the set of all pairs of ends of the local curves in the moduli spaces $\mathcal{M}_V$ that lie on the unique path between the ends x_i and x_j.

The push-forward cycle

$$\mathcal{Z}'(\alpha) := f_* q\left(\mathrm{ev}^* \Delta_X \cdot \prod_V \mathcal{M}_V\right) := f_*\left(q\left(\mathrm{ev}^* \Delta_X \cdot \prod_V \mathcal{M}_V\right)\right)$$

in $\mathcal{M}'_{0,n}(\mathbb{R}^N, \Sigma)$ will be called the *gluing cycle* (with bounded ends) for α. Finally, taking the quotient by the lineality space corresponding to the ends of α, we obtain a gluing cycle $\mathcal{Z}(\alpha)$ (with unbounded ends) in $\mathcal{M}_{0,n}(\mathbb{R}^N, \Sigma)$. The cells of maximal dimension come with a natural weight in this construction, which we will call the *gluing weight*. It is not clear a priori that this weight is independent of the choice of α, but it will turn out to be so in Theorem 3.16.

Lemma 3.13. *Assume that all vertices occurring in a combinatorial type α in $\mathcal{M}_0(X, \Sigma)$ are good. Then*

$$\dim \mathcal{Z}(\alpha) = \dim X + |\Sigma| - 3 - K_X \cdot \Sigma.$$

Proof. By definition we have

$$\mathscr{Z}'(\alpha) = f_* q \left(\text{ev}^* \Delta_X \cdot \prod_V \mathscr{M}_V \right).$$

Let s denote the number of vertices V in α, so that $s-1$ is the number of bounded edges. As the push-forward preserves dimensions, we only have to compute the dimension of $q\left(\text{ev}^* \Delta_X \cdot \prod_V \mathscr{M}_V\right)$. We have that

$$\dim \prod_V \mathscr{M}_V = \sum_V \left(2\,\text{val}(V) - K_{X_V} \cdot \Sigma_V + \dim X - 3\right)$$

$$= s \dim X - K_X \cdot \Sigma + \sum_V (\text{val}(V) - 3) + \sum_V \text{val}(V)$$

$$= s \dim X - K_X \cdot \Sigma + 2|\Sigma| - 4 + s,$$

where we used that for a tree the number of vertices equals $|\Sigma| - 2 - \sum_V(\text{val}(V) - 3)$. The cycle $\text{ev}^*(\Delta_X) \cdot \prod_V \mathscr{M}_V$ has codimension $(s-1)\dim X$, and taking the quotient q eliminates another $s-1$ dimensions. Passing from $\mathscr{Z}'(\alpha)$ to $\mathscr{Z}(\alpha)$ reduces the dimension by $|\Sigma|$, so in total we get

$$\dim \mathscr{Z}(\alpha) = s \dim X - K_X \cdot \Sigma + 2|\Sigma| - 4 + s - (s-1)\dim X - (s-1) - |\Sigma|$$

$$= \dim X + |\Sigma| - 3 - K_X \cdot \Sigma.$$

$\square$

Lemma 3.14. *Assume that all vertices occurring in a combinatorial type α in $\mathscr{M}_0(X,\Sigma)$ are good. If the gluing cycle $\mathscr{Z}(\alpha)$ is not zero, the support of $\mathscr{Z}(\alpha)$ contains $\mathscr{M}(\alpha)$ and is contained in $\mathscr{N}(\alpha)$ (so in particular also in $\mathscr{M}_0(X,\Sigma)$).*

Proof. In the notation of Construction 3.12, consider the set $q^{-1}(f^{-1}(\mathscr{Z}'(\alpha)))$ of all curve pieces that glue to the combinatorial type α. Under each evaluation map ev_e, this set maps to the linerality space of $\Delta_{X(e)}$ in $X(e) \times X(e)$. The functions used for cutting out the diagonal in Construction A.2 all contain this linerality space in their own linerality space, so their pull-backs are linear on the above set and hence do not subdivide it. These properties are preserved under push-forward with f and taking the quotient q, and thus $\mathscr{Z}(\alpha)$ contains all of $\mathscr{M}(\alpha)$ if it is non-zero.

By construction, the other points in the gluing cycle $\mathscr{Z}(\alpha)$ correspond to curves obtained by resolving each vertex of α, and thus to resolutions of α. Hence, $\mathscr{Z}(\alpha)$ is contained in the union of all neighboring cells of $\mathscr{M}(\alpha)$, and thus by Lemma 3.13 in $\mathscr{N}(\alpha)$. $\square$

Definition 3.15 (The moduli space $\mathscr{M}_0(X,\Sigma)$ as a weighted polyhedral complex)**.** Assume that all vertices occurring in curves in $\mathscr{M}_0(X,\Sigma)$ are good. By Lemma 3.13 and 3.14, each maximal cell α of $\mathscr{M}_0(X,\Sigma)$ is also a maximal cell in the gluing cycle $\mathscr{Z}(\alpha)$. We define the weight of this cell of $\mathscr{M}_0(X,\Sigma)$ to be the corresponding gluing weight of Construction 3.12.

The aim of this section is to show that these weights make $\mathscr{M}_0(X,\Sigma)$ into a tropical variety, i.e. balanced, if all vertices of resolution dimension 1 are good. Examples can be found in Section 4 and [GMO17].

Theorem 3.16. *Assume that all vertices in a combinatorial type α in $\mathcal{M}_0(X,\Sigma)$ are good. If β is a resolution of α in $\mathcal{M}_0(X,\Sigma)$ (so that in particular $\mathcal{N}(\beta) \subset \mathcal{N}(\alpha)$) then the weight of every maximal cell in the gluing cycle $\mathcal{Z}(\beta)$ in $\mathcal{N}(\beta)$ agrees with the weight of the same cell in $\mathcal{Z}(\alpha)$ in $\mathcal{N}(\alpha)$.*

In particular, if α is of virtual codimension 1 then the cycle $\mathcal{M}_0(X,\Sigma)$ with the weights of Definition 3.15 is balanced at α.

Proof. This is basically a straight-forward recuction proof; however, we have to pay attention to several intersection-theoretical details. We start by describing the gluing cycle $\mathcal{Z}'(\beta)$ as in Construction 3.12. For every vertex V of α, let J_V be the set of vertices of β that degenerate to V in α, so that $\cup_V J_V$ is the set of all vertices of β. We denote by $\mathrm{EV}^* \Delta_X$ the product over all pull-backs of the diagonals along evaluation maps belonging to the edges of β, by Q the quotient map on $\mathrm{EV}^* \Delta_X \cdot \prod_V \prod_{W \in J_V} \mathcal{M}_W$ forgetting the gluing points along the bounded edges, and by F the morphism embedding the resulting cycle to $\mathcal{M}_0'(X,\Sigma)$. Then by Construction 3.12 the gluing cycle $\mathcal{Z}'(\beta)$ is given by

$$\mathcal{Z}'(\beta) = F_* Q\left(\mathrm{EV}^* \Delta_X \cdot \prod_V \prod_{W \in J_V} \mathcal{M}_W \right),$$

where the first product is taken over all vertices V of α. We will now decompose the maps Q, F, and EV into contributions coming from the vertices of α as follows. For each vertex V of α, let I_V be the set of bounded edges of β contracting to V in α. We denote by q_V the quotient map that forgets the gluing points on these edges, and by f_V the morphism that embeds the cycle $\prod_{g \in I_V} \mathrm{ev}_g^* \Delta_X \cdot \prod_{W \in J_V} \mathcal{M}_W$ in the local moduli space $\mathcal{M}_0'(X_V, \Sigma_V)$. Furthermore, denote by q and f the quotient and embedding maps for the gluing cycle for α, respectively. With $\tilde{q} = \prod_V q_V$ and $\tilde{f} = \prod_V f_V$ we can then write

$$\mathcal{Z}'(\beta) = f_* q\left(\tilde{f}_* \tilde{q}\left(\prod_e \mathrm{ev}_e^* \Delta_X \cdot \prod_V \prod_{g \in I_V} \mathrm{ev}_g^* \Delta_X \cdot \prod_V \prod_{W \in J_V} \mathcal{M}_W \right) \right),$$

by Lemma 2.15, with the product over e running over all edges of α. By the compatibility of push-forwards and quotient maps with diagonal pull-backs (see Lemma A.3), we may rewrite this as

$$\mathcal{Z}'(\beta) = f_* q\left(\prod_e \mathrm{ev}_e^* \Delta_X \cdot \tilde{f}_* \tilde{q}\left(\prod_V \prod_{f \in I_V} \mathrm{ev}_f^* \Delta_X \cdot \prod_V \prod_{W \in J_V} \mathcal{M}_W \right) \right)$$

$$= f_* q\left(\mathrm{ev}^* \Delta_X \cdot \prod_V \underbrace{f_{V*} q_V\left(\prod_{f \in I_V} \mathrm{ev}_f^* \Delta_X \cdot \prod_{W \in J_V} \mathcal{M}_W \right)}_{=: \tilde{\mathcal{M}}_V} \right), \qquad (*)$$

where $\mathrm{ev}^* \Delta_X$ is the product over all pull-backs of the diagonals along evaluation maps belonging to edges of α. But now all vertices V of α are good by assumption, and therefore by Definition 3.11 the weights of all maximal cells in the gluing cycle $\tilde{\mathcal{M}}_V$ agree with those of $\mathcal{M}_V$. Hence $\tilde{\mathcal{M}}_V$ is an open subcycle of $\mathcal{M}_V$, and as $(*)$ is just the gluing construction for α we conclude that every maximal cell of $\mathcal{Z}'(\beta)$ has the same weight in $\mathcal{Z}'(\alpha)$. Of course, this property is preserved when passing to unbounded ends in $\mathcal{Z}(\beta)$ and $\mathcal{Z}(\alpha)$.

In particular, if α is of virtual codimension 1 and thus β of virtual codimension 0 in $\mathcal{M}_0(X,\Sigma)$, we see that the cycle $\mathcal{M}_0(X,\Sigma)$ is locally around α given by the cycle $\mathcal{Z}(\alpha)$, and thus balanced. $\qquad\square$

Corollary 3.17. *If all vertices that can appear in combinatorial types in $\mathcal{M}_0(X,\Sigma)$ are good, then $\mathcal{M}_0(X,\Sigma)$ is a tropical variety of dimension*

$$\dim \mathcal{M}_0(X,\Sigma) = \dim X + |\Sigma| - 3 - K_X \cdot \Sigma$$

with the weights of Definition 3.15.

Proof. Apply Theorem 3.16 to all combinatorial types of codimension 1. $\qquad\square$

So in order to obtain a tropical variety $\mathcal{M}_0(X,\Sigma)$ from gluing we will have to show that all vertices are good with respect to our given moduli data. The following result tells us that we only have to do this in resolution dimension 1.

Corollary 3.18. *If all vertices V with $\mathrm{rdim}(V) = 1$ in a given moduli space are good, then all vertices are good.*

Proof. Let V be a vertex in a given moduli space, with $X_V \cong L_{r_V}^{k_V} \times \mathbb{R}^{m_V}$. We will prove the statement of the lemma by induction on the classification number c_V.

If $\mathrm{rdim}(V) = 0$ then V does not admit any resolution in $\mathcal{M}_0(X,\Sigma)$ by Construction 3.10, and hence V is good. If $\mathrm{rdim}(V) = 1$ then V is good by assumption. We can therefore assume that $\mathrm{rdim}(V) > 1$. By Definition 3.4 this means that $\mathrm{vdim}(V)$ is at least by 2 bigger than the dimension m_V of the lineality space in $\mathcal{M}_0(X_V,\Sigma_V)$ coming from translations, and hence all combinatorial types α of virtual codimension 1 in $\mathcal{M}_0(X_V,\Sigma_V)$ correspond to (non-trivial) resolutions of V.

Condition (a) of Definition 3.11 of a good vertex now follows by induction on c_V, condition (b) by Theorem 3.16 applied to all these cells of virtual codimension 1, and condition (c) by Theorem 3.16 applied to all resolutions α and maximal cells $\beta > \alpha$. $\qquad\square$

Taking Corollaries 3.17 and 3.18 together, we thus see that in order to obtain a well-defined moduli space $\mathcal{M}_0(X,\Sigma)$ we only have to check that all vertices of resolution dimension 1 are good.

4. MODULI SPACES OF LINES IN SURFACES

In this section we want to construct the moduli spaces $\mathcal{M}_0(X,1)$ of lines in a surface $X \subset \mathbb{R}^3$. By Corollary 3.17, the dimension of these spaces is the same as in the classical case, namely $3 - \deg X$. So it is empty for $\deg X > 3$, and we obtain a finite number of lines counted with multiplicities for $\deg X = 3$.

Let us consider all possible local situations in such a surface. We want to use decorations on the graph of the line to describe the vertices, as introduced in [Vig07]: A bold dot indicates that the line passes through a vertex of X and a bold line indicates that the line passes through an edge of X. This leads to the following combinatorial possibilities.
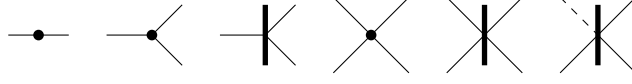
Here the difference between the last two decorations is that either one edge of the line lies on the edge of X (which is indicated by the dashed line in the last picture), or all edges of the line point into maximal cells of X (which is indicated by the second picture from the right). Note that the pictures above do not specify the combinatorial type completely, as there are in general several possibilities for the directions of the ends.

Remark 4.1 (Local degrees). If $d = \deg X$ then $K_X \cdot \Sigma = d$ for the degree Σ of a line. This means that every local degree $K_{X_V} \cdot \Sigma_V$ at a vertex V can be at most $d \leq 3$. Moreover, as V has to be admissible, i. e. must satisfy $\mathrm{rdim}(V) \geq 0$, Definition 3.4 (b) implies that $\mathrm{val}(V) \geq K_{X_V} \cdot \Sigma_V + 1$ for the bold dot decorations and $\mathrm{val}(V) \geq K_{X_V} \cdot \Sigma_V + 2$ for the bold line decorations. This leaves us with the following table, which lists the resolution dimensions of the possible types, and a name of the type in brackets. Impossible types are marked with "X". (The case above type (H) is excluded since there would have to be a maximal cell of X containing two of the four edges, and hence $K_{X_V} \cdot \Sigma_V$ cannot be 1.)

$K_{X_V} \cdot \Sigma_V$						
1	0 (A)	1 (B)	0 (D)	2 (E)	X	1 (I)
2	X	0 (C)	X	1 (F)	0 (H)	0 (J)
3	X	X	X	0 (G)	X	X

Construction 4.2 (Moduli data for resolution dimension 0). For the vertices V of resolution dimension 0 in Remark 4.1, we have to define moduli data as in Definition 3.9. We will fix this according to the situation in algebraic geometry as follows. Assume first that V lies on a vertex of X, so that $X_V \cong L_2^3$ after an integer linear isomorphism. Let $\Sigma = (\sum_{i=0}^3 \alpha_i^j e_i)_{j=1,\dots,n}$ with $\alpha_i^j \geq 0$ be the degree of Σ, where $n = \mathrm{val}(V)$.

We then consider four planes H_0, H_1, H_2, H_3 in $\mathbb{P}^3$ in general position and count rational algebraic stable maps $(C, x_1, \dots, x_n, f)$ relative to these planes with intersection profiles $(\alpha_i^j)_{j=1,\dots,n}$ at H_i for all i. Their (finite) number will be the weight that we assign to the vertex V. In more complicated cases when this number is infinite, we expect that the corresponding (virtual) relative Gromov-Witten invariant would be the correct choice here, but for our situation at hand this problem does not occur.

If V lies on an edge of X, we assign a weight to V analogously after projecting $X_V \cong L_1^2 \times \mathbb{R}$ to L_1^2.

Here are two examples:

  (a) For type (A) in the table above the only possible degree is $\Sigma = (e_0 + e_1, e_2 + e_3)$ up to permutations, corresponding to lines in $\mathbb{P}^3$ passing to the two points $H_0 \cap H_1$ and $H_2 \cap H_3$. As there is exactly one such line, we assign to (A) the weight 1.

(b) For type (G) there are several possible degrees; as an example we will consider $\Sigma = (3e_0 + 2e_1, e_1 + e_2, e_2 + e_3, e_2 + 2e_3)$, and thus count maps with $f^*H_0 = 3x_1$, $f^*H_1 = 2x_1 + x_2$, $f^*H_2 = x_2 + x_3 + x_4$, $f^*H_3 = x_3 + 2x_4$ (in the Chow groups of the corresponding $f^{-1}(H_i)$). Such a map would have to send x_1 to $H_0 \cap H_1$, and x_3 and x_4 to $H_2 \cap H_3$. As f^*H_0 only contains x_1 and f^*H_3 only contains x_3 and x_4, the curve would have to lie completely over the line through those two points. But then x_2 would have to map to both of these points simultaneously, which is impossible. Hence we assign the weight 0 to this type.

Remark 4.3 (Conditions in resolution dimension 1). As the next step, we have to verify that, with the given moduli data, all vertices of resolution dimension 1 are good. To show the general procedure will sketch this here for type (F), more details and the other cases can be found in [Och13, Section 3.3].

Note that the rays in this type must satisfy exactly the same linear relation as the rays of a tropical line. Also, none of the three possible resolutions is allowed to have a bounded edge of higher weight, as this does not occur for lines. It is checked immediately that this leaves only the degree $\Sigma = (2e_0 + e_1, e_1 + e_3, e_2, e_2 + e_3)$, up to isomorphism. It is shown in the picture below, together with its three resolutions in X.

In order to embed the local moduli space into $\mathcal{M}_{0,4} \times \mathbb{R}^3$, we evaluate the position of x_3 in $\mathbb{R}^2 \cong \mathbb{R}^3 / \langle e_2 \rangle$ with coordinate directions e_1 and e_3, and the position of x_2 in $\mathbb{R} \cong \mathbb{R}^3 / \langle e_1, e_3 \rangle$ with coordinate direction e_2 (see Remark 2.21). Then the rays of the local moduli space are spanned by the vectors $v_{\{1,3\}} + e_1 - e_3$, $v_{\{1,4\}} + e_3$, and $v_{\{1,2\}} - e_1$ (in the notation of Construction 2.16) for the three resolutions below, respectively. This is balanced with weights one, which are actually the gluing weights.
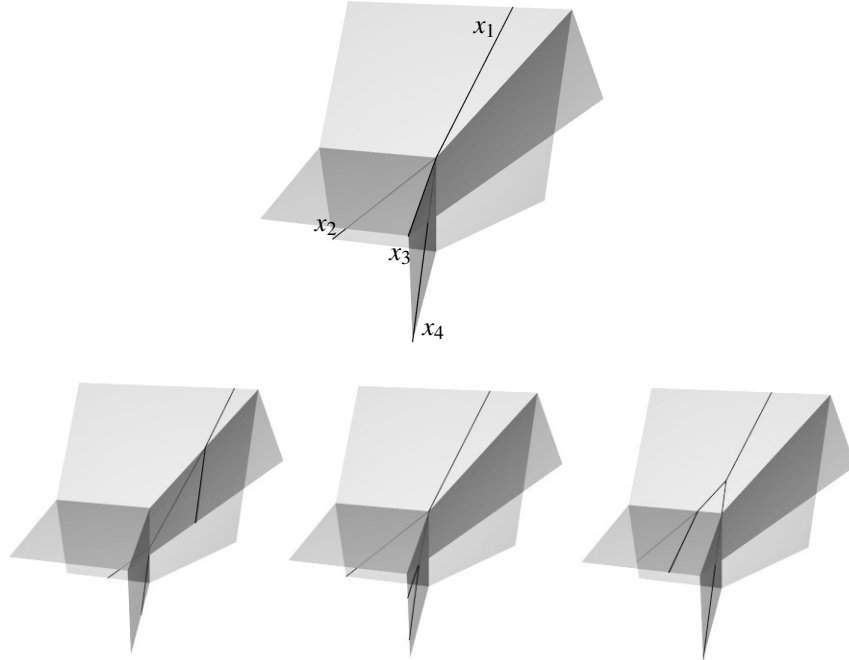
From Corollaries 3.17 and 3.18 we therefore conclude:

Corollary 4.4. *With the moduli data of Construction 4.2, the moduli space $\mathscr{M}_0(X,1)$ of lines in a tropical surface $X \subset \mathbb{R}^3$ is a tropical variety of dimension*

$$\dim \mathscr{M}_0(X,1) = 3 - \deg X$$

(and empty if $\deg X > 3$). In particular, this moduli space consists of finitely many (weighted) points if $\deg X = 3$.

Example 4.5 (Infinitely many lines in a tropical cubic surface [Vig10]). Consider a floor-decomposed generic cubic surface where the three walls (represented by a line, a conic and a cubic) have the following relative position to each other, projected in the e_3-direction:
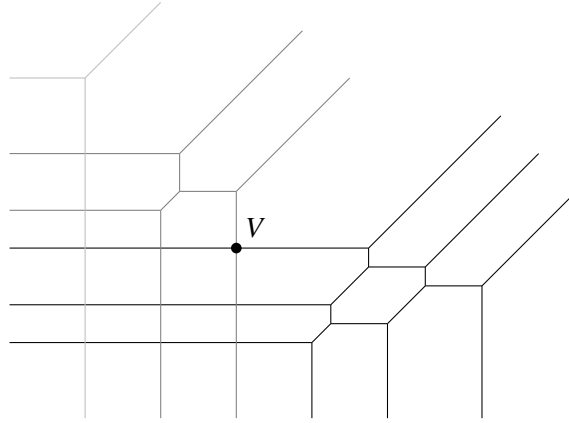
Such a cubic surface contains exactly 27 isolated lines that count with multiplicity 1. In addition, it has a 1-dimensional family of lines not containing any of the others. All lines in this family have a vertex mapping to the point V shown above, while the rest of them is mapped to maximal cells of X. General lines in this family are therefore decorated as in the following picture.
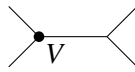
If the vertex mapped to V is 3-valent, it has to be of resolution dimension -1. Therefore the only admissible line in this family, i. e. the only line in the family actually in $\mathscr{M}_0(X,1)$, is the one with no bounded edge, i. e. with a 4-valent vertex mapping to V. This vertex is then of resolution dimension 0. To determine its type, we map the four rays of the cubic at V, which are $-e_1 + e_3$, $e_1 - 2e_3$, $-e_2 - 2e_3$, and $e_2 + 3e_3$, by an integer linear isomorphism to the four unit vectors. This maps the rays of the line to the degree $\Sigma = (3e_0 + 2e_1, e_1 + e_2, e_2 + e_3, e_2 + 2e_3)$. As this line has weight 0 by Construction 4.2 (b), we conclude that the whole family does not contribute to the virtual number, and the degree of the 0-cycle $\mathscr{M}_0(X,1)$ is again 27.

We therefore conjecture:

Conjecture 4.6. *For every smooth cubic surface $X \subset \mathbb{R}^3$ the 0-cycle $\mathscr{M}_0(X, 1)$ has degree 27.*

APPENDIX A. PULLING BACK THE DIAGONAL OF A SMOOTH VARIETY

Let X be a partially open tropical cycle, and let Y be a smooth tropical variety. In order to glue moduli spaces in Section 3 we need the pull-back of the diagonal Δ_Y of Y by some morphism $f : X \to Y \times Y$. But although the diagonal is locally a product of Cartier divisors in this case, tropical intersection theory unfortunately does not yet provide a well defined pull-back for it. This appendix therefore contains the technical details necessary to construct a well-defined pull-back cycle $f^* \Delta_Y$.

First we briefly review some facts about matroids and matroid fans from [FR13]. Let M be a matroid of rank r on a finite ground set E. To every flat F of M we associate a vector $e_F := \sum_{i \in F} e_i \in \mathbb{R}^E$, where the e_i are the negative standard basis vectors. Correspondingly, to every chain of flats $\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_s = E$ we assign a cone, spanned by $e_{F_1}, \ldots, e_{F_s}$, and $-e_{F_s}$. Let $B(M)$ denote the collection of all these cones, where the maximal ones are equipped with weight 1. This simplicial fan defines a tropical variety $B(M)$ whose dimension is the rank of M. We call it the *matroid fan* associated to M.

Of special interest to us is the *uniform matroid* $U_{r,k}$ on a ground set E of cardinality k, with rank function $r(A) = \min(|A|, r)$. Its associated matroid fan is $B(U_{r,k}) \cong L_{r-1}^{k-1} \times \mathbb{R}$.

Construction A.1. By [FR13, Section 4] the diagonal $\Delta_{B(M)}$ in $B(M) \times B(M)$ can be cut out by a product of rational functions: we have $\Delta_{B(M)} = \varphi_1 \cdot \cdots \cdot \varphi_r \cdot (B(M) \times B(M))$ for the rational functions φ_i linear on the cones of $B(M)$ determined by

$$\varphi_i(e_A, e_B) = \begin{cases} -1 & \text{if } r_M(A) + r_M(B) - r_M(A \cup B) \geq i, \\ 0 & \text{else} \end{cases}$$

for flats A, B of M, where r_M is the rank function of M. Moreover, recursively intersecting with the φ_i yields a matroid fan in each intermediate step, hence a locally irreducible tropical variety; this will be important in the construction. If we want to specify the matroid M in the notation, we will write φ_i also as φ_i^M.

We can now give the construction to pull back the diagonal from a smooth tropical fan.

Construction A.2. Consider a morphism $f : X \to Y \times Y$ where $Y \cong L_{r-1}^{k-1} \times \mathbb{R}^m$. Then there is a (non-canonical) isomorphism $\theta : Y \times \mathbb{R} \to B(M) \times \mathbb{R}^m$, where $M = U_{r,k}$, and θ maps the additional factor $\mathbb{R}$ onto the lineality space of the matroid fan. Associated to f we denote by $\tilde{f}$ the composition map

$$X \times \mathbb{R}^2 \xrightarrow{f \times \mathrm{id}} Y \times Y \times \mathbb{R}^2 \cong (Y \times \mathbb{R}) \times (Y \times \mathbb{R}) \xrightarrow{\theta \times \theta} B(M)^2 \times (\mathbb{R}^m)^2.$$

Let $\psi_1, \ldots, \psi_m$ denote functions which cut out the diagonal $\Delta_{\mathbb{R}^m}$, and consider the cocycle

$$\Phi_Y := \varphi_1 \cdot \cdots \cdot \varphi_r \cdot \psi_1 \cdot \cdots \cdot \psi_m \qquad \text{on } B(M)^2 \times (\mathbb{R}^m)^2,$$

where φ_i are the functions on $B(M)^2$ from Construction A.1 above. One verifies immediately that the pull-back $\tilde{f}^*\Phi_Y \cdot (X \times \mathbb{R}^2)$ has the lineality space $L_X := 0 \times \Delta_{\mathbb{R}}$ in $X \times \mathbb{R}^2$. So we can take the quotient by L_X and use the projection $p_X : (X \times \mathbb{R}^2)/L \to X$ to define

$$f^*\Delta_Y := f^*\Delta_Y \cdot X := p_{X*}\left[(\tilde{f}^*\Phi_Y \cdot (X \times \mathbb{R}^2))/L_X\right].$$

As the intermediate steps in Construction A.1 are locally irreducible, it follows from [Fra12, Lemma 3.8.13] that the support of $\tilde{f}^*\Phi_Y \cdot (X \times \mathbb{R}^2)$ lies over the diagonal of $\mathbb{R}$ in $\mathbb{R}^2$ (so that p_X is injective on $(\tilde{f}^*\Phi_Y \cdot (X \times \mathbb{R}^2))/L_X$, in accordance with our convention in Remark 2.12), and that the support of $f^*\Delta_Y$ lies in $f^{-1}(\Delta_Y)$.

Moreover, it can be shown that this definition depends neither on the choice of $\psi_1, \ldots, \psi_m$ [Fra13, Theorem 2.25] nor on the choice of isomorphism θ [Och13, Lemma 1.4.3]. However, it is not known whether it depends on the choice of the rational functions $\varphi_1, \ldots, \varphi_r$ cutting out the diagonal of $B(M)$.

Let us briefly state the main properties of this definition that follow from the compatibilities between the various intersection-theoretic constructions.

Lemma A.3.

(a) *(Projection formula) For two morphisms $Z \xrightarrow{g} X \xrightarrow{f} Y \times Y$, where g is injective and Y a smooth fan, we have*

$$g_*\left[(f \circ g)^*\Delta_Y \cdot Z\right] = f^*\Delta_Y \cdot g_*Z.$$

(b) *(Quotients) Let X be a partially open tropical variety with lineality space L and quotient map $q : X \to X/L$, and let $f : X/L \to Y \times Y$ be a morphism for a smooth fan Y. Then*

$$q\left((f \circ q)^*\Delta_Y \cdot X\right) = f^*\Delta_Y \cdot (X/L).$$

(c) *(Commutativity) For two morphisms $f : X \to Y \times Y$ and $g : X \to Z \times Z$ to smooth fans Y and Z we have*

$$f^*\Delta_Y \cdot (g^*\Delta_Z \cdot X) = g^*\Delta_Z \cdot (f^*\Delta_Y \cdot X).$$

(d) *(Projections) Let X and Z be partially open tropical varieties, and denote by $p : X \times Z \to X$ the projection. For any morphism $f : X \to Y \times Y$ for a smooth fan Y we have*

$$(f \circ p)^*\Delta_Y \cdot (X \times Z) = (f^*\Delta_Y \cdot X) \times Z.$$

(e) *(Products) Let $f : X \to Y \times Y$ and $f' : X' \to \mathbb{R}^k \times \mathbb{R}^k$ be two morphisms, for a smooth fan Y. Then*

$$(f \times f')^*\Delta_{Y \times \mathbb{R}^k} \cdot (X \times X') = (f^*\Delta_Y \cdot X) \times (f'^*\Delta_{\mathbb{R}^k} \cdot X').$$

Proof. All these statements can be checked immediately, see [Och13, Section 1.4] for details. As an example, we show part (a): We have

$$
\begin{aligned}
g_* \left[(f \circ g)^* \Delta_Y \cdot Z \right] = g_* p_{Z*} &\left[\left(\widetilde{f \circ g}^* \Phi_Y \cdot (Z \times \mathbb{R}^2) \right) / L_Z \right] \\
= p_{X*} (g \times \mathrm{id})_* &\left[\left((g \times \mathrm{id})^* \tilde{f}^* \Phi_Y \cdot (Z \times \mathbb{R}^2) \right) / L_Z \right] \quad \text{(functoriality)} \\
= p_{X*} &\left[\left((g \times \mathrm{id})_* (g \times \mathrm{id})^* \tilde{f}^* \Phi_Y \cdot (Z \times \mathbb{R}^2) \right) / L_X \right] \quad \text{(Lemma 2.15)} \\
= p_{X*} &\left[\left(\tilde{f}^* \Phi_Y \cdot (g_*(Z) \times \mathbb{R}^2) \right) / L_X \right] \quad\quad \text{(projection formula)} \\
= f^* \Delta_Y \cdot g_*(Z),
\end{aligned}
$$

where we have used the projection formula for cocycles as in [Fra13, Proposition 2.24 (3)]. □

So far $Y \cong L_{r-1}^{k-1} \times \mathbb{R}^m$ was assumed to be a smooth fan. In order to generalize this to smooth varieties we need the following compatibility statement. Let σ be a cell of Y in the coarsest subdivision, with relative interior σ°. We consider both σ and σ° as subsets of the diagonal $\Delta_Y \subset Y \times Y$. Let $Y(\sigma)$ be the union of all open cells in the matroid subdivision of $Y \times Y$ whose closure intersects σ°, which is then an open neighborhood of σ° in $Y \times Y$. It is also contained in $Y_\sigma \times Y_\sigma$, where Y_σ is the star of Y at σ. As Y_σ is again a smooth fan, we can regard the restriction of f to $X_\sigma := f^{-1}(Y(\sigma))$ also as a morphism $f_\sigma : X_\sigma \to Y_\sigma \times Y_\sigma$, and apply Construction A.2 to this map. As expected, we will now show that over the cell σ this gives the same result as for $f : X \to Y \times Y$.

Lemma A.4 (Compatibility). *Let $f : X \to Y \times Y$ be a morphism, with Y a smooth fan. Moreover, let σ be a cell in the coarsest subdivision of Y. With notations as above, the weights of $f^* \Delta_Y \cdot X$ and $f_\sigma^* \Delta_{Y_\sigma} \cdot X_\sigma$ then agree on all cells of X whose interior is mapped by f to the cell σ° in the diagonal Δ_Y.*

Proof. Let $Y \cong L_{r-1}^{k-1} \times \mathbb{R}^m$, and let $M = U_{r,k}$ be the corresponding uniform matroid on $E = \{1, \dots, k\}$, so that $Y \times \mathbb{R} \cong B(M) \times \mathbb{R}^m$. The cell σ then corresponds to a subset $S \subset E$, i.e. it consists of all cells in the matroid subdivision for chains of flats in S. If $\dim \sigma = s$ then $Y_\sigma \cong L_{r-s-1}^{k-s-1} \times \mathbb{R}^{m+s}$, or more precisely $Y_\sigma \times \mathbb{R} \cong B(M_S) \times \mathbb{R}^s \times \mathbb{R}^m$, where M_S is the uniform matroid of rank $r-s$ on $E \setminus S$.

As Construction A.2 is local, it suffices to show that the rational functions cutting out the diagonal in this construction are the same for the spaces $Y \times Y \times \mathbb{R}^2 \cong B(M)^2 \times (\mathbb{R}^m)^2$ and $Y_\sigma \times Y_\sigma \times \mathbb{R}^2 \cong B(M_S)^2 \times (\mathbb{R}^s)^2 \times (\mathbb{R}^m)^2$ when restricted to the common open neighborhood $Y(\sigma) \times \mathbb{R}^2$ of the cell $\sigma^\circ \times \mathbb{R}$ in the diagonal. For this we need to prove that these rational functions agree on all rays of $\overline{Y(\sigma)}$. As (e_S, e_S) is the only interior ray of $\sigma^\circ \times \mathbb{R}$, the maximal cells of $\overline{Y(\sigma)}$ correspond to maximal chains of flats in $M \oplus M$ containing (S, S), and hence we have to compare the rational functions of Constructions A.1 and A.2 on all rays (e_A, e_B) for flats A, B of M with $A, B \subset S$ or $A, B \supset S$.

The functions cutting out the diagonal of $\mathbb{R}^m$ can obviously be chosen to be the same in both cases. By Construction A.1, the others are $\varphi_1^M, \dots, \varphi_k^M$ for $B(M)^2$, and $\varphi_1^{M_S}, \dots, \varphi_{k-s}^{M_S}$ and $\varphi_1^S, \dots, \varphi_s^S$ for $B(M_S)^2 \times (\mathbb{R}^s)^2$ (where S stands for the uniform matroid of full rank on

S, so that $\varphi_1^S, \ldots, \varphi_s^S$ can be used to cut out the diagonal of $\mathbb{R}^s$). Now, on the rays (e_A, e_B) mentioned above ...

- φ_i^M agrees with $\varphi_{i-s}^{M_S}$ for $i = s+1, \ldots, k$:

  If $A, B \subset S$ then both $r_M(A) + r_M(B) - r_M(A \cup B) \geq i$ and $r_{M_S}(A \backslash S) + r_{M_S}(B \backslash S) - r_{M_S}((A \cup B) \backslash S) \geq i - s$ are never satisfied.

  If $A, B \supset S$ then $r_M(A) = r_{M_S}(A \backslash S) + s$, and similarly for B and $A \cup B$. Hence $r_M(A) + r_M(B) - r_M(A \cup B) \geq i$ is equivalent to $r_{M_S}(A \backslash S) + r_{M_S}(B \backslash S) - r_{M_S}((A \cup B) \backslash S) \geq i - s$.

- φ_i^M agrees with φ_i^S for $i = 1, \ldots, s$:

  If $A, B \subset S$ then $r_M(A) + r_M(B) - r_M(A \cup B) \geq i$ is equivalent to $r_S(A \cap S) + r_S(B \cap S) - r_S((A \cup B) \cap S) \geq i$.

  If $A, B \supset S$ then both $r_M(A) + r_M(B) - r_M(A \cup B) \geq i$ and $r_S(A \cap S) + r_S(B \cap S) - r_S((A \cup B) \cap S) \geq i$ are always satisfied.

$\square$

Remark A.5 (Pullbacks of diagonals of smooth varieties). Lemma A.4 implies that we cannot only pull back diagonals of smooth fans, but also of smooth varieties: Let $f : X \to Y \times Y$ be a morphism from a partially open tropical cycle to a smooth tropical variety.

To assign a weight to a cell τ of dimension $\dim X - \dim Y$ over the diagonal Δ_Y, let σ be the cell in $Y \cong \Delta_Y$ so that the relative interior of τ maps to the relative interior of σ. Choose any face σ' of σ (which might be σ itself), replace Y by the star $Y_{\sigma'}$ at this face and X by the open subcycle of X consisting of all points mapping to σ' or any of its adjacent open cells in both components of $Y \times Y$, and assign to τ its weight in the cycle $f_{\sigma'}^* \Delta_{Y_{\sigma'}} \cdot X_{\sigma'}$. By Lemma A.4 the result does not depend on the choice of σ'.

Using the same local procedure for a cell τ of dimension $\dim X - \dim Y - 1$, we obtain a balanced cycle $f_{\sigma'}^* \Delta_{Y_{\sigma'}} \cdot X_{\sigma'}$ including all adjacent cells of dimension $\dim X - \dim Y$. Hence our local construction glues to give a well-defined cycle $f^* \Delta_Y \cdot X$.

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