RELATIVE GROMOV-WITTEN INVARIANTS AND THE MIRROR FORMULA

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Abstract. Let X be a smooth complex projective variety, and let $Y \subset X$ be a smooth very ample hypersurface such that $-K_Y$ is nef. Using the technique of relative Gromov-Witten invariants, we give a new short and geometric proof of (a version of) the "mirror formula", i.e. we show that the generating function of the genus zero 1-point Gromov-Witten invariants of Y can be obtained from that of X by a certain change of variables (the so-called "mirror transformation"). Moreover, we use the same techniques to give a similar expression for the (virtual) numbers of degree-d plane rational curves meeting a smooth cubic at one point with multiplicity $3d$, which play a role in local mirror symmetry.

For a smooth very ample hypersurface Y of a smooth complex projective variety X, the theory of relative Gromov-Witten invariants gives rise to an algorithm that allows one to compute the genus zero Gromov-Witten invariants of Y from those of X [Ga]. The goal of this paper is to show that in the case when $-K_Y$ is nef, this algorithm can be "solved" explicitly to obtain a formula that expresses the generating function of the 1-point Gromov-Witten invariants of Y in terms of that of X. This so-called "mirror formula" (also denoted "quantum Lefschetz hyperplane theorem" by some authors) has already been known for some time ([Gi], [LLY], [K], [B], [L]). Our approach however is entirely different and essentially "elementary" in the sense that it does not use any of the special techniques that have been used in the previous proofs, like e.g. torus actions, equivariant cohomology, or moduli spaces other than the usual spaces of stable maps to X and their subspaces. This does not only make our proof much simpler than the previous ones, but also hopefully easier to generalize, e.g. to more general hypersurfaces, or to higher genus of the curves.

Let us briefly recall the ideas and results from [Ga]. For $n \geq 0$ and a homology class $\beta \in H_2(X)/\text{torsion}$ we denote by $\bar{M}_n(X,\beta)$ the moduli space of n-pointed genus zero stable maps to X of class β. For any $m \geq 0$ there are closed subspaces $\bar{M}_{(m)}(X,\beta)$ of $\bar{M}_1(X,\beta)$ that can be thought of as parametrizing 1-pointed rational curves in X having multiplicity (at least) m to Y at the marked point. (For simplicity, we suppress in the notation the dependence of these spaces on Y.) These moduli spaces have expected codimension m in $\bar{M}_1(X,\beta)$. In fact, they come equipped with natural virtual fundamental classes $[\bar{M}_{(m)}(X,\beta)]^{virt}$ of this expected dimension. If X is a projective space and Y a hyperplane, then these moduli spaces do have the expected dimension, and their virtual fundamental classes are equal to the usual ones.

The idea is now to raise the multiplicity m of the curves from 0 up to $Y \cdot \beta + 1$ by one at a time. Curves with multiplicity (at least) 0 are just unrestricted curves

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in X, whereas a multiplicity of $Y \cdot \beta + 1$ forces at least the irreducible curves to lie inside Y. In other words, we consider the chain of inclusions

$$\bar{M}_1(Y,\beta) \subset \bar{M}_{(Y\cdot\beta)}(X,\beta) \subset \bar{M}_{(Y\cdot\beta-1)}(X,\beta) \subset \cdots \subset \bar{M}_{(0)}(X,\beta) = \bar{M}_1(X,\beta)$$

of "virtual codimension one". The main theorem of [Ga] describes each of these inclusions explicitly in terms of intersection theory. This gives us a way to describe $\bar{M}_1(Y,\beta)$ inside $\bar{M}_1(X,\beta)$, and hence to compute Gromov-Witten invariants of Y in terms of those of X.

It is easy to write down a naïve guess what these inclusions should look like. A stable map in X has multiplicity at least m to Y if and only if the $(m-1)$-jet of ev^*Y vanishes, where $ev : \bar{M}_1(X,\beta) \to X$ denotes the evaluation map. Hence the cycle $\bar{M}_{(m+1)}(X,\beta)$ inside $\bar{M}_{(m)}(X,\beta)$ should just be the first Chern class of the line bundle of m-jets modulo $(m-1)$-jets of $ev^*\mathcal{O}(Y)$. This Chern class is easily computed to be $ev^*Y + m\psi$, where ψ is the "cotangent line class", i.e. the first Chern class of the line bundle whose fiber at a stable map (C,x,f) is the cotangent space of C at the point x.

However, our above informal description of $\bar{M}_{(m)}(X,\beta)$ as the space of curves with multiplicity at least m to Y at the marked point breaks down at the "boundary", i.e. at those curves where the marked point lies on a component of the curve that lies completely inside Y, so that the multiplicity becomes "infinite". Hence the above calculation receives correction terms from these curves. Their explicit form is given by the following theorem (see [Ga] theorem 2.6).

Theorem 0.1. *For all $m \geq 0$ we have*

$$(ev^*Y + m\psi) \cdot [\bar{M}_{(m)}(X,\beta)]^{virt} = [\bar{M}_{(m+1)}(X,\beta)]^{virt} + [D_{(m)}(X,\beta)]^{virt}.$$

Here, the correction term $D_{(m)}(X,\beta) = \coprod_r \coprod_{B,M} D(X,B,M)$ is a disjoint union of individual terms

$$D(X,B,M) := \bar{M}_{1+r}(Y,\beta^{(0)}) \times_{Y^r} \prod_{i=1}^{r} \bar{M}_{(m^{(i)})}(X,\beta^{(i)})$$

where $r \geq 0$, $B = (\beta^{(0)},\dots,\beta^{(r)})$ with $\beta^{(i)} \in H_2(X)/torsion$ and $\beta^{(i)} \neq 0$ for $i > 0$, and $M = (m^{(1)},\dots,m^{(r)})$ with $m^{(i)} > 0$. The maps to Y^r are the evaluation maps for the last r marked points of $\bar{M}_{1+r}(Y,\beta^{(0)})$ and each of the marked points of $\bar{M}_{(m^{(i)})}(X,\beta^{(i)})$, respectively. The union in $D_{(m)}(X,\beta)$ is taken over all r, B, and M subject to the following three conditions:

$$\sum_{i=0}^{r} \beta^{(i)} = \beta \qquad \text{(degree condition)},$$

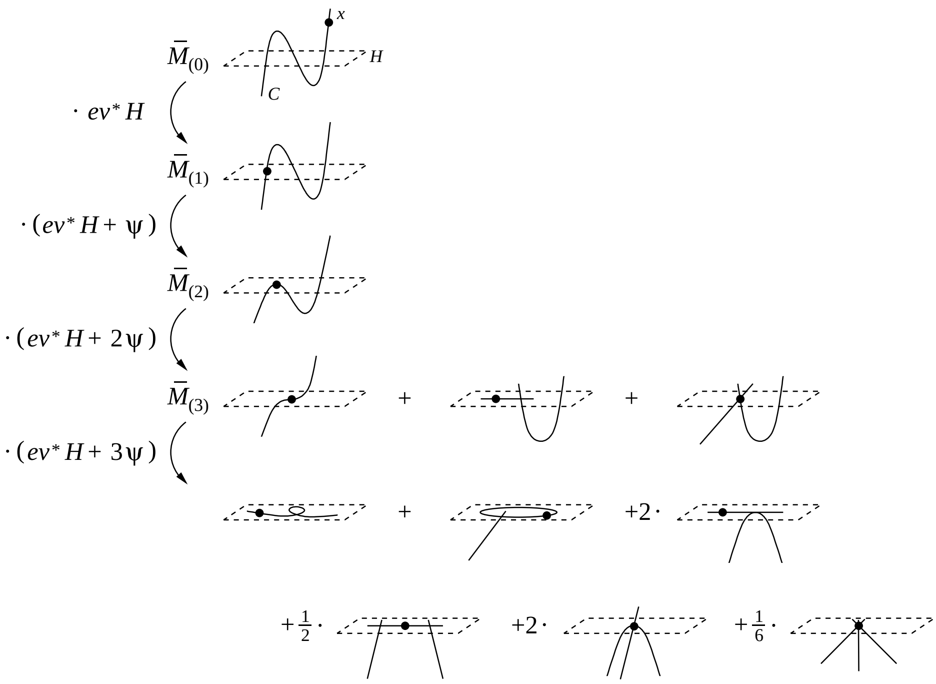
$$Y \cdot \beta^{(0)} + \sum_{i=1}^{r} m^{(i)} = m \qquad \text{(multiplicity condition)},$$

$$\text{if } \beta^{(0)} = 0 \text{ then } r \geq 2 \qquad \text{(stability condition)}.$$

In the equation of the theorem, the virtual fundamental class of the summands $D(X,B,M)$ is defined to be $\frac{m^{(1)}\cdots m^{(r)}}{r!}$ times the class induced by the virtual fundamental classes of the factors $\bar{M}_{1+r}(Y,\beta^{(0)})$ and $\bar{M}_{(m^{(i)})}(X,\beta^{(i)})$. The spaces $D(X,B,M)$ can be considered to be subspaces of $\bar{M}_1(X,\beta)$ (see below), so the equation of the theorem makes sense in the Chow group of $\bar{M}_1(X,\beta)$.

Geometrically speaking, the moduli spaces $D(X,B,M)$ in the correction terms describe curves with $r+1$ irreducible components $C^{(0)},\dots,C^{(r)}$ with homology classes $\beta^{(0)},\dots,\beta^{(r)}$, such that $C^{(0)}$ lies inside Y, and the $C^{(i)}$ for $i>0$ intersect $C^{(0)}$ in a point where they have multiplicity $m^{(i)}$ to Y. The marked point is always on the component $C^{(0)}$. Using this description, the spaces $D(X,B,M)$ can be considered as subspaces of $\bar{M}_1(X,\beta)$. The multiplicity condition ensures that they are actually subspaces of $\bar{M}_{(m)}(X,\beta)$ and have the correct expected dimension. The factor $\frac{1}{r!}$ in the definition of the virtual fundamental class of the correction terms is just combinatorial and corresponds to the choice of order of the components $C^{(1)},\dots,C^{(r)}$. In contrast, the factor $m^{(1)}\cdots m^{(r)}$ is of geometric nature and somewhat tricky to derive.

As an example of the theorem, consider the case where $X=\mathbb{P}^3$, $Y=H$ is a hyperplane, and β is the class of cubic curves in X. Then the equations of the theorem for $m=0,\dots,3$ can be pictured as follows (where we set $\bar{M}_{(m)} := \bar{M}_{(m)}(\mathbb{P}^3,3)$):



(Of course, in the pictures where we have drawn the marked point on a node of the curve, the corresponding stable maps have a contracted component, i.e. we have $\beta^{(0)}=0$.)

So we see that $\bar{M}_1(H,3)$ is equal to $\prod_{i=0}^{3}(ev^*H+i\psi)\cdot\bar{M}_1(\mathbb{P}^3,3)$ plus a bunch of correction terms coming from reducible curves as shown in the picture. This is an equation of 9-dimensional cycles in $\bar{M}_1(\mathbb{P}^3,3)$. To make this into equations for the Gromov-Witten invariants of H, we have to intersect it with some cohomology class γ of codimension 9 that is a polynomial in ev^*H and ψ. Note that in the correction terms this will impose 9 conditions on the component $C^{(0)}$ contained in H. However, in all the terms where the degree of $C^{(0)}$ is at most 2, the moduli space for this component has dimension smaller than 9. Hence all these terms vanish, and it follows that the 1-point Gromov-Witten invariants of H (of degree 3 in this example) are expressible in terms of those of $\mathbb{P}^3$ as

$$I_3^H(\gamma) = I_3^{\mathbb{P}^3}\left(\gamma \cdot \prod_{i=0}^{3}(H + i\psi)\right).$$

The same argument works for higher degree of the curves.

Now let us come back to the case of general X and Y. Can we still hope that the correction terms vanish when we compute the Gromov-Witten invariants? Recall that the reason for the vanishing above was that the dimension of the moduli space of curves in Y quickly gets bigger when the degree of the curves goes up (in the example, the 9 conditions that were needed for Gromov-Witten invariants for cubics in Y were "too many" for lines and conics in Y). Hence, as the (virtual) dimension of the moduli space of stable maps to Y is vdim $\bar{M}_1(Y, \beta) = -K_Y \cdot \beta + \dim Y - 2$, we see that we need that $-K_Y$ is sufficiently positive.

If $-K_Y$ is negative, basically all correction terms that could appear in the computation of the Gromov-Witten invariants will do so. The main nuisance about this is that the correction terms contain the full n-point Gromov-Witten invariants of Y (namely, $n = 1 + r$ in each of the correction terms), and not just the 1-point invariants that we originally wanted to compute. There would be two ways to proceed:

- Use the version of theorem 0.1 for n-point invariants as proven in [Ga].
- Use the WDVV equations to compute the n-point invariants of Y in terms of 1-point invariants whenever they occur.

Both methods can be used without problems to write down an algorithm to compute the Gromov-Witten invariants of Y in terms of those of X. However, we do not know at the moment how to express the result in a nice closed form.

Most interesting are the cases where $-K_Y$ is nef, but yet not "positive enough" to ensure the vanishing of all correction terms. We will show that, whenever $-K_Y$ is nef, the only n-point invariants of Y that might occur in the algorithm are those with fundamental or divisor classes at all but the first marked point. These invariants can of course be reduced immediately to 1-point invariants using the fundamental class and divisor axioms for Gromov-Witten invariants. Thus we arrive at recursion formulas that involve only 1-point invariants. Solving them directly, we obtain a nice expression for the invariants of Y: the "mirror formula".

The necessary computations to achieve this are done in section 1. In section 2 we apply the results to two examples. First of all we rederive the expression for the genus zero Gromov-Witten invariants of the quintic threefold. Secondly, we prove a similar expression for the (virtual) numbers of plane rational curves of degree d having contact of order $3d$ to a smooth cubic. These numbers play a role in local mirror symmetry (see [CKYZ] and [T]). They are a by-product of our work, as they are just simple examples of relative Gromov-Witten invariants. The two main computational lemmas (that have nothing to do with algebraic geometry, but rather are formal statements about certain power series occurring in the calculation) are proved in the appendix.

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1. The mirror transformation

As in the introduction let X be a smooth complex projective variety, and let Y be a smooth very ample hypersurface such that $-K_Y$ is nef. By abuse of notation, we denote by $H^*(X)$ and $H_*(X)$ the groups of algebraic (co-)homology classes modulo torsion. For a class $\beta \in H_2(X)$ we write $\beta \geq 0$ if β is effective, and $\beta > 0$ if $\beta \geq 0$ and $\beta \neq 0$. To keep the notation as simple as possible, we will assume in the following computations that the class of Y generates $H^2(X)$ over $\mathbb{Q}$ (see remark 1.14 for the changes needed in the general case).

For any $\beta > 0$ we denote by $\bar{M}_n(X,\beta)$ the space of n-pointed rational stable maps of class β to X. Let $ev_i : \bar{M}_n(X,\beta) \to X$ be the evaluation maps, and let ψ_i be the cotangent line classes. For cohomology classes $\gamma_i \in H^*(X)$ the corresponding Gromov-Witten invariant is defined to be

$$I_\beta^X(\gamma_1\psi^{k_1} \otimes \cdots \otimes \gamma_n\psi^{k_n}) := ev_1^*\gamma_1 \cdot \psi_1^{k_1} \cdots ev_n^*\gamma_n \cdot \psi_n^{k_n} \cdot [\bar{M}_n(X,\beta)]^{virt} \in \mathbb{Q}$$

if the dimension condition $\sum_i(\text{codim } \gamma_i + k_i) = \text{vdim } \bar{M}_n(X,\beta)$ is satisfied, and zero otherwise. It is usual and convenient to encode all the 1-point invariants of class β in a single cohomology class

$$\begin{aligned} I_\beta^X &:= ev_* \left(\frac{1}{1-\psi} \cdot [\bar{M}_1(X,\beta)]^{virt} \right) \\ &= \sum_{i,j} I_\beta^X(T^i\psi^j) \cdot T_i \qquad \in H^*(X), \end{aligned}$$

where $ev = ev_1$, $\{T^i\}$ is a basis of $H^*(X) \otimes \mathbb{Q}$, and $\{T_i\}$ is the dual basis. Note that the dimension condition ensures that for each i at most one j contributes a non-zero term to the sum above, so all 1-point invariants of X of class β can be reconstructed from the cohomology class I_β^X.

We define the Gromov-Witten invariants I_β^Y of Y in the same way, replacing $\bar{M}_n(X,\beta)$ by $\bar{M}_n(Y,\beta)$, *but keeping the ev_i to denote the evaluation maps to X.* Note that β is still a homology class in X; so strictly speaking $\bar{M}_n(Y,\beta)$ is the space of stable maps to Y of all homology classes whose push-forward to X is β.

For $\beta = 0$, we set $I_0^X := 1$ and $I_0^Y := Y$.

Now consider the moduli spaces $\bar{M}_{(m)}(X,\beta)$ of 1-pointed relative stable maps to X with multiplicity m to Y at the marked point ([Ga] definition 1.1). In the same manner as above, these spaces together with their virtual fundamental classes ([Ga] definition 1.18) give rise to invariants $I_{\beta,(m)}(\gamma\psi^k)$ that can be assembled into a cohomology class

$$I_{\beta,(m)} = ev_* \left(\frac{1}{1-\psi} \cdot [\bar{M}_{(m)}(X,\beta)]^{virt} \right) \quad \in H^*(X).$$

Remark 1.1. For future reference, let us note that (as expected from geometry) $I_{\beta,(0)} = I_\beta^X$ and $I_{\beta,(m)} = 0$ for $m > Y \cdot \beta$ (see [Ga] remark 1.3).

Finally, let $D_{(m)}(X,\beta)$ be the correction terms defined in theorem 0.1, and set

$$J_{\beta,(m)} = ev_* \left(\frac{1}{1-\psi} \cdot [D_{(m)}(X,\beta)]^{virt} \right) + m \cdot ev_*[\bar{M}_{(m)}(X,\beta)]^{virt} \quad \in H^*(X). \tag{1}$$

The surprising additional term will appear in the proof of the following lemma. Geometrically, it corresponds to unstable maps that have two irreducible components

$C^{(0)}$ and $C^{(1)}$, where $C^{(0)}$ is contracted to a point in Y and contains the marked point, and $C^{(1)}$ is a curve with multiplicity m to Y at this point (see the end of the proof of lemma 1.8).

The first thing to do is to rewrite theorem 0.1 in the new simplified notation.

Lemma 1.2. *For all $\beta > 0$ and $m \geq 0$ we have*

$$(Y+m) \cdot I_{\beta,(m)} = I_{\beta,(m+1)} + J_{\beta,(m)} \quad \in H^*(X).$$

Proof. Intersect the equation of theorem 0.1 with $\frac{1}{1-\psi}$ and push it forward by the evaluation map to get

$$\begin{aligned} &ev_* \left((ev^* Y + m\psi) \cdot \frac{1}{1-\psi} \cdot [\bar{M}_{(m)}(X,\beta)]^{virt} \right) \\ &\quad = ev_* \left(\frac{1}{1-\psi} \cdot [\bar{M}_{(m+1)}(X,\beta)]^{virt} \right) + ev_* \left(\frac{1}{1-\psi} \cdot [D_{(m)}(X,\beta)]^{virt} \right). \end{aligned}$$

As $\frac{\psi}{1-\psi} = \frac{1}{1-\psi} - 1$, the left hand side of this equation can be rewritten as

$$(Y+m) \cdot ev_* \left(\frac{1}{1-\psi} \cdot [\bar{M}_{(m)}(X,\beta)]^{virt} \right) - m \cdot ev_* [\bar{M}_{(m)}(X,\beta)]^{virt}.$$

Taking into account the definitions of $I_{\beta,(m)}$ and $J_{\beta,(m)}$, we arrive at the equation stated in the lemma. □

Remark 1.3. In particular,

$$\prod_{i=0}^{Y\cdot\beta} (Y+i) \cdot I_\beta^X = \sum_{m=0}^{Y\cdot\beta} \prod_{i=m+1}^{Y\cdot\beta} (Y+i) \cdot J_{\beta,(m)}.$$

This follows from a recursive application of lemma 1.2, with the start and the end of the recursion given by remark 1.1.

The next thing to do is to evaluate the $J_{\beta,(m)}$ explicitly.

Remark 1.4. Let us first consider the first summand $ev_* \left(\frac{1}{1-\psi} \cdot [D_{(m)}(X,\beta)]^{virt} \right)$ in the definition (1) of $J_{\beta,(m)}$. Using the definition of $D_{(m)}(X,\beta)$ and its virtual fundamental class given in theorem 0.1, we see that this first summand is a sum of individual terms, each of which has the form

$$I^Y_{\beta^{(0)}}(T^i \psi^j \otimes \gamma_1 \otimes \cdots \otimes \gamma_r) \cdot \frac{1}{r!} \prod_{k=1}^r \left(m^{(k)} \cdot I_{\beta^{(k)},(m^{(k)})}(\gamma_k^\vee) \right) \cdot T_i, \tag{2}$$

where $\gamma^\vee$ denotes the dual of a class γ *in* Y. These terms are summed over all i, $j \geq 0$, $r \geq 0$, $\beta^{(k)}$ (with $\beta^{(0)} \geq 0$ and $\beta^{(k)} > 0$ if $k > 0$), and $m^{(k)} > 0$, subject to the conditions

(i) $\beta^{(0)} + \cdots + \beta^{(r)} = \beta$ (degree condition),
(ii) $Y \cdot \beta^{(0)} + m^{(1)} + \cdots + m^{(r)} = m$ (multiplicity condition),
(iii) if $\beta^{(0)} = 0$ then $r \geq 2$ (stability condition).

Moreover, the γ_k have to run over a basis of $H^*(Y) \otimes \mathbb{Q}$ (actually it is sufficient to let them run over a basis of the part of $H^*(Y) \otimes \mathbb{Q}$ induced by X, see [Ga] remark 5.4).

The main simplification of this huge sum is due to the following lemma, which follows from a simple dimension count. It is the only point in our computations where we need that $-K_Y$ is nef.

Lemma 1.5. *The above expression (2) can only be non-zero if all γ_k are fundamental or divisor classes. Moreover, for all k we must have*

$$\begin{aligned} m^{(k)} &= Y \cdot \beta^{(k)} - K_Y \cdot \beta^{(k)} - 1 && \text{if } \gamma_k \text{ is the fundamental class,} \\ m^{(k)} &= Y \cdot \beta^{(k)} - K_Y \cdot \beta^{(k)} && \text{if } \gamma_k \text{ is a divisor class.} \end{aligned}$$

Proof. As the invariants $I_{\beta^{(k)},(m^{(k)})}(\gamma_k^\vee)$ must have dimension zero for all k, it follows that

$$\begin{aligned} \text{codim } \gamma_k &= \dim Y - \text{codim } \gamma_k^\vee \\ &= \dim Y - \dim \bar{M}_{(m^{(k)})}(X, \beta^{(k)}) \\ &= \dim Y - (-K_X \cdot \beta^{(k)} + \dim X - 2 - m^{(k)}) \\ &= -Y \cdot \beta^{(k)} + K_Y \cdot \beta^{(k)} + 1 + m^{(k)} \qquad \text{(by adjunction).} \end{aligned}$$

This shows the equation for the $m^{(k)}$. Moreover, as $-K_Y$ is nef and we must have $m^{(k)} \leq Y \cdot \beta^{(k)}$ for the relative invariant to be non-zero (see remark 1.1), it follows that codim $\gamma_k \leq 1$, as desired. □

Remark 1.6. Obviously, in the same way one can show that:

- If $-K_Y \cdot \beta \geq 1$ for all $\beta > 0$ then all the γ_k have to be fundamental classes. (In the following computations this would mean that all $r_\beta = 0$, which greatly simplifies the calculation.) This is e.g. the case if Y is a hypersurface in $X = \mathbb{P}^n$ of degree at most n.
- If $-K_Y \cdot \beta \geq 2$ for all $\beta > 0$ then no γ_k can exist, i.e. we must always have $r = 0$. Hence in this case we conclude that there are no correction terms in the computation of the Gromov-Witten invariants. The only term on the right hand side of remark 1.3 is I_β^Y (for $r = 0$ and $m = Y \cdot \beta$), so it follows that the "naïve" formula
$$I_\beta^Y = \prod_{i=0}^{Y \cdot \beta} (Y + i) \cdot I_\beta^X$$
is true (as in the case considered in the introduction where $Y \subset X$ is a plane in $\mathbb{P}^3$). This is e.g. the case if Y is a hypersurface in $X = \mathbb{P}^n$ of degree at most $n - 1$.

Remark 1.7. As we have assumed that the class of Y generates $H^2(X)$ over $\mathbb{Q}$, lemma 1.5 states that the only factors that can occur in the k-product in (2) are the numbers

$$\begin{aligned} s_\beta &:= (Y \cdot \beta - K_Y \cdot \beta - 1) \cdot I_{\beta,(Y \cdot \beta - K_Y \cdot \beta - 1)}(1^\vee) \\ \text{and} \quad r_\beta &:= (Y \cdot \beta - K_Y \cdot \beta) \cdot I_{\beta,(Y \cdot \beta - K_Y \cdot \beta)}(Y^\vee) \end{aligned}$$

for some $\beta > 0$. Thus we can then rewrite (2) using multi-index notation as follows. For a multi-index $\mu = (\mu_\beta)$ of non-negative integers indexed by the positive homology classes β of $H^2(X)$, we apply the usual notations

$$\begin{aligned} \textstyle\sum \mu &:= \textstyle\sum_\beta \mu_\beta, & s^\mu &:= \textstyle\prod_\beta s_\beta^{\mu_\beta}, \\ \mu! &:= \textstyle\prod_\beta \mu_\beta!, & |\mu| &:= \textstyle\sum_\beta \mu_\beta \cdot \beta. \end{aligned}$$

Then we can rewrite (2) as

$$I^Y_{\beta^{(0)}}(T^i\psi^j \otimes 1^{\otimes \sum \mu} \otimes Y^{\otimes \sum \nu}) \cdot \frac{1}{r!} \cdot s^\mu r^\nu \cdot T_i, \tag{3}$$

where μ and ν are the multi-indices such that the factors s_β and r_β appear in (2) μ_β and ν_β times, respectively. In particular, $r = \sum \mu + \sum \nu$ is the number of nodes of the curves under consideration.

We are now ready to evaluate the $J_{\beta,(m)}$ explicitly in terms of the 1-point Gromov-Witten invariants I^Y_β of Y and the relative 1-point invariants s_β and r_β.

Lemma 1.8. *With the notation of remark 1.7,*

$$J_{\beta,(m)} = \sum_{\mu,\nu} \left(Y + Y \cdot \beta^{(0)}\right)^{\sum \nu} \cdot \frac{s^\mu}{\mu!} \frac{r^\nu}{\nu!} \cdot I^Y_{\beta^{(0)}}$$

for all $\beta > 0$ and $m \geq 0$, where the sum is taken over all multi-indices μ and ν such that $\beta^{(0)} := \beta - |\mu| - |\nu| \geq 0$ (degree condition) and $m = Y \cdot \beta - K_Y \cdot (|\mu| + |\nu|) - \sum \mu$ (multiplicity condition).

Proof. Inserting expression (3) for (2) in remark 1.4, we see that the first summand in the definition (1) of $J_{\beta,(m)}$ is

$$\begin{aligned} &ev_* \left(\frac{1}{1-\psi} \cdot [D_{(m)}(X,\beta)]^{virt}\right) \\ &\qquad = \sum_{i,j} \sum_{\mu,\nu} I^Y_{\beta^{(0)}}(T^i\psi^j \otimes 1^{\otimes \sum \mu} \otimes Y^{\otimes \sum \nu}) \cdot \frac{s^\mu}{\mu!} \frac{r^\nu}{\nu!} \cdot T_i, \end{aligned}$$

where the sum is taken over all i, j, μ, ν such that

(i) $\beta^{(0)} := \beta - |\mu| - |\nu| \geq 0$ (degree condition),
(ii) $Y \cdot \beta - K_Y \cdot (|\mu| + |\nu|) - \sum \mu = m$ (multiplicity condition — here we inserted the expression of lemma 1.5 for the $m^{(i)}$),
(iii) if $\beta^{(0)} = 0$ then $\sum \mu + \sum \nu \geq 2$ (stability condition).

Now we compute the Gromov-Witten invariant $I^Y_{\beta^{(0)}}(\cdots)$ in terms of 1-point invariants of Y. We claim that for $\beta^{(0)} > 0$

$$\sum_{i,j} I^Y_{\beta^{(0)}}(T^i\psi^j \otimes 1^{\otimes \sum \mu} \otimes Y^{\otimes \sum \nu}) \cdot T_i = (Y + Y \cdot \beta^{(0)})^{\sum \nu} \cdot I^Y_{\beta^{(0)}}. \tag{4}$$

In fact, this follows from the fundamental class axiom

$$\begin{aligned} \sum_{i,j} I^Y_{\beta^{(0)}}(T^i\psi^j \otimes 1 \otimes \cdots) \cdot T_i &= \sum_{i,j\neq 0} I^Y_{\beta^{(0)}}(T^i\psi^{j-1} \otimes \cdots) \cdot T_i \\ &= \sum_{i,j} I^Y_{\beta^{(0)}}(T^i\psi^j \otimes \cdots) \cdot T_i \end{aligned}$$

and the divisor axiom

$$\begin{aligned}\sum_{i,j} I^Y_{\beta^{(0)}}(T^i\psi^j \otimes Y \otimes \cdots)\cdot T_i &= \sum_{i,j}(Y\cdot\beta^{(0)})\cdot I^Y_{\beta^{(0)}}(T^i\psi^j\otimes\cdots)\cdot T_i \\ &\quad + \sum_{i,j\neq 0} I^Y_{\beta^{(0)}}(T^i\cdot Y\;\psi^{j-1}\otimes\cdots)\cdot T_i \\ &= \sum_{i,j}(Y\cdot\beta^{(0)})\cdot I^Y_{\beta^{(0)}}(T^i\psi^j\otimes\cdots)\cdot T_i \\ &\quad + \sum_{i,j\neq 0} I^Y_{\beta^{(0)}}(T^i\psi^{j-1}\otimes\cdots)\cdot(T_i\cdot Y) \\ &= (Y\cdot\beta^{(0)}+Y)\cdot\sum_{i,j} I^Y_{\beta^{(0)}}(T^i\psi^j\otimes\cdots)\cdot T_i\end{aligned}$$

(see e.g. [Ge] proposition 12), where "$\cdots$" denotes any tensor product of cohomology classes (i.e. not including cotangent line classes). In fact, the same formula (4) is also true for $\beta^{(0)}=0$, as in this case

$$\begin{aligned}\sum_{i,j} I^Y_0(T^i\psi^j\otimes 1^{\otimes\sum\mu}\otimes Y^{\otimes\sum\nu})\cdot T_i &= (Y^{\sum\nu})\cdot Y \\ &= Y^{\sum\nu}\cdot I^Y_0\end{aligned}$$

by the "mapping to point axiom". Hence the first summand in the definition (1) of $J_{\beta,(m)}$ is

$$ev_*\left(\frac{1}{1-\psi}\cdot[D_{(m)}(X,\beta)]^{virt}\right)=\sum_{\mu,\nu}\left(Y+Y\cdot\beta^{(0)}\right)^{\sum\nu}\cdot\frac{s^\mu}{\mu!}\frac{r^\nu}{\nu!}\cdot I^Y_{\beta^{(0)}} \tag{5}$$

with the sum taken over all μ,ν satisfying the degree, multiplicity and stability conditions. The second summand is

$$\begin{aligned}m\cdot ev_*[\bar M_{(m)}(X,\beta)]^{virt} &= m\cdot\sum_i I_{\beta,(m)}(T^i)\cdot T_i \\ &= s_\beta\cdot Y\cdot\delta_{m,Y\cdot\beta-K_Y\cdot\beta-1} \\ &\quad + r_\beta\cdot Y^2\cdot\delta_{m,Y\cdot\beta-K_Y\cdot\beta}\end{aligned}$$

by lemma 1.5. As we have defined $I^Y_0=Y$, this adds exactly the terms with $\beta^{(0)}=0$ and $\sum\mu+\sum\nu=1$ to the sum in (5) that were excluded because of the stability condition. It follows that

$$J_{\beta,(m)}=\sum_{\mu,\nu}\left(Y+Y\cdot\beta^{(0)}\right)^{\sum\nu}\cdot\frac{s^\mu}{\mu!}\frac{r^\nu}{\nu!}\cdot I^Y_{\beta^{(0)}},$$

with the sum taken over all μ,ν satisfying the degree and multiplicity conditions. □

Remark 1.9. The multiplicity condition in lemma 1.8 can be replaced by

$$m=Y\cdot\beta-\epsilon\sum\mu,$$

where $\epsilon\in\{0,1\}$ depends only on Y. To see this, recall that the multiplicity condition was obtained from the original one

$$m=Y\cdot\beta^{(0)}+\sum m^{(k)} \tag{6}$$

by inserting the expressions $m^{(k)} = Y \cdot \beta^{(k)} - K_Y \cdot \beta^{(k)}$ (for every $r_{\beta^{(k)}}$) or $m^{(k)} = Y \cdot \beta^{(k)} - K_Y \cdot \beta^{(k)} - 1$ (for every $s_{\beta^{(k)}}$), respectively. But by remark 1.1 we have $r_{\beta^{(k)}} = 0$ if $m^{(k)} = Y \cdot \beta^{(k)} - K_Y \cdot \beta^{(k)} > Y \cdot \beta^{(k)}$. So (as K_Y is nef) $r_{\beta^{(k)}}$ can only be non-zero if $m^{(k)} = Y \cdot \beta^{(k)}$. Hence we can insert this simplified expression for $m^{(k)}$ in (6).

In the same way, $s_{\beta^{(k)}}$ can only be non-zero if $m^{(k)} = Y \cdot \beta^{(k)} - 1$ (in the case $K_Y = 0$) or $m^{(k)} = Y \cdot \beta^{(k)}$ (in the case $K_Y > 0$). In other words, $m^{(k)} = Y \cdot \beta^{(k)} - \epsilon$ with $\epsilon \in \{0, 1\}$ depending only on Y.

If we now take the original multiplicity condition (6) and insert the new simplified expressions $m^{(k)} = Y \cdot \beta^{(k)}$ (for every $r_{\beta^{(k)}}$) and $m^{(k)} = Y \cdot \beta^{(k)} - \epsilon$ (for every $s_{\beta^{(k)}}$), respectively, we arrive at the desired multiplicity condition $m = Y \cdot \beta - \epsilon \sum \mu$.

Remark 1.10. Now we can insert the expression of lemma 1.8 (with the multiplicity condition from remark 1.9) into the formula of remark 1.3. Thus we obtain

$$\prod_{i=0}^{Y\cdot\beta} (Y+i) \cdot I_\beta^X = \sum_{\mu,\nu} \prod_{i=Y\cdot\beta-\epsilon\sum\mu+1}^{Y\cdot\beta} (Y+i) \cdot \Big(Y + Y \cdot \beta^{(0)}\Big)^{\sum\nu} \cdot \frac{s^\mu}{\mu!} \frac{r^\nu}{\nu!} \cdot I_{\beta^{(0)}}^Y$$
$$= \sum_{\mu,\nu} \prod_{i=0}^{\epsilon\sum\mu-1} (Y + Y \cdot \beta - i) \cdot \Big(Y + Y \cdot \beta^{(0)}\Big)^{\sum\nu} \cdot \frac{s^\mu}{\mu!} \frac{r^\nu}{\nu!} \cdot I_{\beta^{(0)}}^Y,$$

where the sum is now taken over all μ, ν satisfying the degree condition $\beta^{(0)} := \beta - |\mu| - |\nu| \geq 0$. Note that this equation is trivially true in the case $\beta = 0$ as well (both sides are equal to Y in this case).

To get rid of the degree condition, we multiply these equations with $q^{Y\cdot\beta}$ (where q is a formal variable) and add them up; so we get

$$\sum_\beta \prod_{i=0}^{Y\cdot\beta} (Y+i) \cdot I_\beta^X \cdot q^{Y\cdot\beta}$$
$$= \sum_{\beta^{(0)}} \sum_{\mu,\nu} \prod_{i=0}^{\epsilon\sum\mu-1} (Y + Y \cdot \beta - i) \cdot \Big(Y + Y \cdot \beta^{(0)}\Big)^{\sum\nu} \cdot \frac{s^\mu}{\mu!} \frac{r^\nu}{\nu!} \cdot I_{\beta^{(0)}}^Y \cdot q^{Y\cdot\beta}, \quad (7)$$

where the sum now runs over all multi-indices μ, ν (and $\beta = \beta^{(0)} + |\mu| + |\nu|$).

Although this equation looks quite complicated, note that all geometric ideas in its derivation are still visible: the left hand side is the "naïve" expression for the Gromov-Witten invariants of Y that we already encountered in the introduction and remark 1.6. The product $\prod_{i=0}^{Y\cdot\beta}(Y+i)$ here corresponds to the process of raising the multiplicity of the curves from 0 to $Y \cdot \beta + 1$. The right hand side of the equation describes the correction terms. They correspond to reducible curves with one component in the hypersurface ($I_{\beta^{(0)}}^Y$) and various others in the ambient space with specified multiplicities to the hypersurface ($s^\mu r^\nu$). The factor $(Y + Y \cdot \beta^{(0)})^{\sum\nu}$ comes from the ($\sum\nu$)-fold application of the divisor axiom that we used to describe the component in the hypersurface by a 1-point invariant instead of by a $(1+r)$-point invariant.

All that remains to be done to arrive at the "mirror formula" is to simplify the right hand side of equation (7). To do so, define $P(t)$ to be "the right hand side with $Y \cdot \beta^{(0)}$ replaced by a formal variable t":

Definition 1.11. Let

$$P(t) := \sum_{\mu,\nu} \prod_{i=0}^{\epsilon \sum \mu - 1} (Y + Y \cdot (|\mu| + |\nu|) + t - i) \cdot (Y+t)^{\sum \nu} \cdot \frac{s^\mu}{\mu!} \frac{r^\nu}{\nu!} \cdot q^{Y \cdot (|\mu|+|\nu|)},$$

so that (7) can be written as

$$\sum_\beta \prod_{i=0}^{Y \cdot \beta} (Y+i) \cdot I_\beta^X \cdot q^{Y \cdot \beta} = \sum_\beta P(Y \cdot \beta) \cdot I_\beta^Y \cdot q^{Y \cdot \beta}. \tag{8}$$

Lemma 1.12. *The power series $P(t)$ of definition 1.11 satisfies the differential equation $\frac{d^2}{dt^2} \ln P = 0$. In particular, if $P(t) = P_0 + P_1 \cdot t + \cdots$ is the Taylor expansion of P then $P(t) = P_0 \exp(\frac{P_1}{P_0} t)$.*

Proof. This can be checked directly from the definition of $P(t)$. The statement does not depend on the special values of r_β and s_β; it is equally true if the r_β and s_β are considered to be formal variables. We give a proof of the statement in appendix A (apply lemma A.1 with the collection of variables x_i being the union of the r_β and s_β, $z = 0$, and t replaced by $t + Y$). □

Corollary 1.13 (Mirror formula)**.** *If we formally set $\tilde{q} = q \cdot \exp \frac{P_1}{P_0}$ with P_0 and P_1 as in lemma 1.12, then*

$$\sum_\beta \prod_{i=0}^{Y \cdot \beta} (Y+i) \cdot I_\beta^X \cdot q^{Y \cdot \beta} = P_0 \cdot \sum_\beta I_\beta^Y \cdot \tilde{q}^{Y \cdot \beta},$$

i.e. the generating function $\sum_\beta I_\beta^Y \cdot q^{Y \cdot \beta}$ of the 1-point Gromov-Witten invariants of Y can be obtained from the "naïve" expression $\sum_\beta \prod_{i=0}^{Y \cdot \beta} (Y+i) \cdot I_\beta^X \cdot q^{Y \cdot \beta}$ by a formal change of variables ($q \to \tilde{q}$) and a scaling factor ($\cdot P_0$).

Proof. Immediately from (8) and lemma 1.12. □

Remark 1.14. In the above computations we assumed that the class of Y generates $H^2(X)$ over $\mathbb{Q}$. In fact, this is not essential. All that happens for higher dimension of $H^2(X)$ is that the notation becomes more complicated at some steps of the calculation. Most importantly, in remark 1.7 there are now more factors that can occur in the k-product of (2). Namely, instead of the r_β we now have

$$r_{i,\beta} = (Y \cdot \beta - K_Y \cdot \beta) \cdot I_{\beta,(Y \cdot \beta - K_Y \cdot \beta)}(\gamma_i^\vee),$$

for $i = 1, \dots, \dim H^2(X) \otimes \mathbb{Q}$, where the γ_i form a basis of $H^2(X) \otimes \mathbb{Q}$, chosen such that $\gamma_1 = Y$. Correspondingly, lemma 1.8 becomes

$$J_{\beta,(m)} = \sum_{\mu,\nu_i} \prod_i \left(\gamma_i + \gamma_i \cdot \beta^{(0)}\right)^{\sum \nu_i} \cdot \frac{s^\mu}{\mu!} \cdot \prod_i \frac{{r_i}^{\nu_i}}{\nu_i!} \cdot I^Y_{\beta^{(0)}}$$

where the ν_i are multi-indices. In the alternative multiplicity condition of remark 1.9, the number ϵ will now depend on β (it is 1 if $K_Y \cdot \beta = 0$ and 0 if $K_Y \cdot \beta > 0$). Hence the multiplicity condition is now $m = Y \cdot \beta - \epsilon\mu$, where ϵ is a multi-index with entries 0 and 1. Finally, we need a formal variable q_i for each γ_i to replace the expression $q^{Y\cdot\beta}$ by $q^\beta := \prod_i q_i^{\gamma_i\cdot\beta}$. Definition 1.11 then becomes

$$P(\{t_i\}) := \sum_{\mu,\nu_i} \prod_{j=0}^{\epsilon\mu-1} (Y + Y\cdot(|\mu| + \sum_i |\nu_i|) + t_1 - j) \cdot \prod_i (\gamma_i + t_i)^{\sum \nu_i}$$
$$\cdot \frac{s^\mu}{\mu!} \cdot \prod_i \frac{{r_i}^{\nu_i}}{\nu_i!} \cdot q^{|\mu|+\sum_i |\nu_i|},$$

with which we obtain the equation (compare to (8))

$$\sum_\beta \prod_{i=0}^{Y\cdot\beta} (Y+i) \cdot I_\beta^X \cdot q^\beta = \sum_\beta P(\{\gamma_i \cdot \beta\}) \cdot I_\beta^Y \cdot q^\beta. \tag{9}$$

The same proof as for lemma 1.12 works to show that $\partial_{t_i}\partial_{t_j} \ln P = 0$ for all i, j, so it follows that $P(t) = P_0 \exp(\frac{\sum P_i t_i}{P_0})$, where $P(\{t_i\}) = P_0 + \sum_i P_i \cdot t_i + \cdots$ is the linear expansion of P. Hence the mirror formula of corollary 1.13 holds in the same way

$$\sum_\beta \prod_{i=0}^{Y\cdot\beta} (Y+i) \cdot I_\beta^X \cdot q^\beta = P_0 \cdot \sum_\beta I_\beta^Y \cdot \tilde{q}^\beta,$$

where $\tilde{q}_i = q_i \cdot \exp \frac{P_i}{P_0}$.

2. Examples

Example 2.1 (Application to the quintic threefold)**.** Let $X = \mathbb{P}^4$, and let $Y \subset X$ be a smooth quintic hypersurface, so that $Y = 5H \in H^*(X)$, where H is the class of a hyperplane. We are interested in the genus zero Gromov-Witten invariants of Y, i.e. in the numbers $n_d = \frac{1}{d} I_d^Y(H)$ (note that H has d points of intersection with a degree-d curve). As this is the H^3-coefficient of I_d^Y (up to a scaling factor), we consider the equation (8) modulo H^4. (This discards the invariants $I_d^Y(\psi)$.)

Since the only Gromov-Witten invariants of Y are $I_d^Y(H)$ (and $I_d^Y(\psi)$), the polynomials I_d^Y have no H^0, H^1, and H^2 terms for $d > 0$. Hence as it is well-known that

$$I_d^X = \prod_{i=1}^d \frac{1}{(H+i)^5},$$

(see e.g. [P] section 1.4) it follows from (8) that

$$\sum_{d\geq 0} 5H \cdot \frac{\prod_{i=1}^{5d}(5H+i)}{\prod_{i=1}^d (H+i)^5} \, q^{5d} = 5H \; P_0 \pmod{H^3}.$$

This is sufficient to reconstruct P: if we expand

$$\sum_{d\geq 0} \frac{\prod_{i=1}^{5d}(5H+i)}{\prod_{i=1}^d (H+i)^5} \, q^{5d} =: F_0 + F_1 H + F_2 H^2 + \cdots \tag{10}$$

then $P|_{t=H=0} = F_0$ and $\partial_H P|_{t=H=0} = F_1$. So as P is a function of $t + 5H$ and satisfies $\partial_t^2 \ln P = 0$, it follows that $\partial_t P|_{t=H=0} = \frac{1}{5} F_1$, and hence

$$P = F_0 \cdot \exp\left(\left(\frac{t}{5} + H\right) \cdot \frac{F_1}{F_0}\right).$$

In particular,

$$\begin{aligned} P_0 &= F_0 \cdot \exp\left(H \frac{F_1}{F_0}\right) \\ &= F_0 + H\, F_1 + \frac{H^2}{2} \frac{F_1^2}{F_0} + \cdots . \end{aligned}$$

So by comparing the H^3-coefficient of (8) we get

$$F_2 = \frac{1}{2} \frac{F_1^2}{F_0} + \frac{1}{5} \sum_{d>0} d n_d q^{5d} F_0 \exp\left(d \frac{F_1}{F_0}\right).$$

Together with (10), this equation determines the n_d recursively and gives the well-known numbers $n_1 = 2875$, $n_2 = 609250 + \frac{2875}{8}$, $\ldots$.

Example 2.2 (Application to plane elliptic curves)**.** We want to compute the (virtual) numbers of rational plane curves of degree d having multiplicity $3d$ to a smooth elliptic plane cubic, i.e. the relative Gromov-Witten invariants $I_{d,(3d)}(1) = \frac{3}{d} r_d$ in the case where $X = \mathbb{P}^2$ and Y is a smooth elliptic cubic. According to [T] remark 1.11 these numbers are related to the local mirror symmetry of [CKYZ].

The computation of the numbers r_d is very similar (yet not identical) to that of the Gromov-Witten invariants of Y in section 1. This time we apply lemma 1.2 recursively only up to multiplicity $3d$ instead of $3d + 1$, so we get

$$\prod_{i=0}^{3d-1} (3H + i)\; I_d^X = I_{d,(3d)} + \sum_{m=0}^{3d-1} \prod_{i=m+1}^{3d-1} (3H + i)\; J_{d,(m)}.$$

Note that $I_d^Y = 0$ for $d > 0$, as there are no rational curves in Y. So if we insert the expression for $J_{d,(m)}$ of lemma 1.8, we get in the same way as in remark 1.10

$$\begin{aligned} \sum_{d>0} \prod_{i=0}^{3d-1} (3H + i)\; I_d^X\, q^{3d} &= \sum_{d>0} \frac{3H^2}{d}\, r_d q^{3d} \\ &\quad + \sum_{\mu,\nu} \prod_{i=3d-\sum \mu+1}^{3d-1} (3H + i)\; (3H)^{\sum \nu} \cdot \frac{s^\mu}{\mu!} \frac{r^\nu}{\nu!} \cdot 3H\, q^{3d} \end{aligned} \tag{11}$$

where we already inserted the expression $m = 3d - \sum \mu$ for Calabi-Yau hypersurfaces (see remark 1.9). Here, in the second line we set $d = |\mu| + |\nu|$, and we obviously only sum over those μ with $\sum \mu \geq 1$.

Similar to definition 1.11 let us set

$$Q(t) := \sum_{\mu} \prod_{i=1}^{\sum \mu - 1} (3H \cdot |\mu| + t - i)\; \frac{s^\mu}{\mu!}\, q^{3H\cdot|\mu|}\, t,$$

where the sum is now taken over all μ — not only those with $\sum \mu \geq 1$. The $\mu = 0$ term contributes a 1 (together with the factor t). The definition of $Q(t)$ is so that $Q(3H) - 1$ yields exactly the $\nu = 0$ terms in the second line of (11).

Similarly to lemma 1.12 the power series $Q(t)$ satisfies a differential equation: by lemma A.2 $\ln Q(t)$ is linear in t, i.e. $Q(t) = \exp(c \cdot t)$. To compute c, we expand as in example 2.1 the left hand side of (11)

$$\sum_{d>0} 3H \cdot \frac{\prod_{i=1}^{3d-1}(3H+i)}{\prod_{i=1}^{d}(H+i)^3}\, q^{3d} =: F_1 H + F_2 H^2 + \cdots$$

(in [T] $F_1(q^3)$ is called $I_2^{(0)}(z)$, and $F_2(q^3)$ is called $I_3^{(0)}(z)$). As the t-expansion of $Q(t)$ is

$$Q(t) = 1 + ct + \frac{1}{2}c^2t^2 + \cdots,$$

comparison of the H^1 terms in (11) gives $F_1 =$ (the H^1 term of $Q(3H)$) $= 3c$; so $Q(t) = \exp(\frac{F_1 \cdot t}{3})$.

Now compare the H^2 term in (11). Note that we must have $\sum \nu \leq 1$ because of the factor $(3H)^{\sum \nu + 1}$. The $\nu = 0$ term is exactly the second coefficient of $Q(3H)$ as remarked above, i.e. $\frac{1}{2}F_1^2$. The terms with $\sum \nu = 1$ can be written as a sum over d, where d is the index of the one non-zero entry of ν. The contribution for a given d is exactly $9 r_d q^{3d} \frac{Q(3d)}{3d} = \frac{3}{d} r_d q^{3d} \exp(d F_1)$, with the $\mu = 0$ term in $Q(3d)$ coming from the right hand side of the first line of (11). Thus we get the equation

$$F_2 = \frac{1}{2}F_1^2 + \sum_{d>0} \frac{3}{d} r_d\, q^{3d} \exp(dF_1),$$

which determines the numbers $\frac{3}{d} r_d = I_{d,(3d)}(1)$. The first few numbers are given in the following table.

d	1	2	3	4	5	6	7	8
$I_{d,(3d)}(1)$	9	$\frac{135}{4}$	244	$\frac{36999}{16}$	$\frac{635634}{25}$	307095	$\frac{193919175}{49}$	$\frac{3422490759}{64}$

This equation is equivalent to the conjecture of remark 1.11 in [T]. Together with [T] theorem 2.1 it proves that $I_{d,(3d)} = (-1)^d 3dK_d$, where K_d is the top Chern class of the rank-$(3d-1)$ bundle on $\bar{M}_0(\mathbb{P}^2, d)$ with fiber $H^1(C, f^*K_{\mathbb{P}^2})$ at the point $(C, f) \in \bar{M}_0(\mathbb{P}^2, d)$. At the moment we do not know of a geometric proof of this statement.

Appendix A. Proof of the main technical lemmas

In this appendix we show that the power series $P(t)$ and $Q(t)$ of definition 1.11 and example 2.2 satisfy certain differential equations.

Lemma A.1. *Let x_i be a collection of variables (possibly infinite), and let $a_i, b_i \in \mathbb{N}$, $c_i \in \mathbb{C}$. Define*

$$P(t,z) = \sum_k \frac{x^k}{k!}\, t^{ak} \prod_{i=0}^{bk-1} (ck + z + t - i),$$

where k is a multi-index, and where we used the usual multi-index notations $ak = \sum_i a_i k_i$, $x^k = \prod_i x_i^{k_i}$, $k! = \prod_i k_i!$. Assume that, for every i, the pair (a_i, b_i) is $(0,0)$, $(1,0)$, or $(0,1)$. Then

$$\partial_t^2 \ln P = \partial_z^2 \ln P = \partial_t \partial_z \ln P = 0.$$

Proof. Step 1. We consider the c_i to be formal variables and show by induction on n that for every i and every $n \geq 0$

$$\begin{array}{lccc} \text{if} & \partial_t^2 \ln P|_{c_i=0} = & \partial_z^2 \ln P|_{c_i=0} = & \partial_t\partial_z \ln P|_{c_i=0} = 0 \\ \text{then} & \partial_{c_i}^n \partial_t^2 \ln P|_{c_i=0} = & \partial_{c_i}^n \partial_z^2 \ln P|_{c_i=0} = & \partial_{c_i}^n \partial_t\partial_z \ln P|_{c_i=0} = 0. \end{array}$$

So assume that

$$\partial_{c_i}^j \partial_t^2 \ln P|_{c_i=0} = \partial_{c_i}^j \partial_z^2 \ln P|_{c_i=0} = \partial_{c_i}^j \partial_t \partial_z \ln P|_{c_i=0} = 0$$

for $j \leq n$. Note that by definition of P we have $\partial_{c_i} P = x_i \partial_{x_i} \partial_z P$. Let ∂_1 and ∂_2 denote either ∂_t or ∂_z. Then it follows that (everything in the following calculation is evaluated at $c_i = 0$):

$$\begin{aligned}
\partial_{c_i}^{n+1} \partial_1 \partial_2 \ln P &= \partial_{c_i}^n \partial_1 \partial_2 \frac{\partial_{c_i} P}{P} \\
&= x_i \partial_{c_i}^n \partial_1 \partial_2 \frac{\partial_{x_i} \partial_z P}{P} \\
&= x_i \partial_{c_i}^n \partial_1 \partial_2 \left(\partial_{x_i} \frac{\partial_z P}{P} - \partial_z P \cdot \partial_{x_i} \frac{1}{P} \right) \\
&= x_i \partial_{c_i}^n \partial_1 \partial_2 \left(\partial_{x_i} \frac{\partial_z P}{P} + \frac{\partial_z P}{P} \cdot \frac{\partial_{x_i} P}{P} \right) \\
&= x_i \partial_{x_i} \partial_z \underbrace{\partial_{c_i}^n \partial_1 \partial_2 \ln P}_{=0} + x_i \partial_{c_i}^n \partial_1 \partial_2 (\partial_z \ln P \cdot \partial_{x_i} \ln P) \\
&= x_i \partial_{c_i}^n (\partial_1 \partial_2 \partial_z \ln P \cdot \partial_{x_i} \ln P + \partial_1 \partial_z \ln P \cdot \partial_2 \partial_{x_i} \ln P \\
&\qquad + \partial_2 \partial_z \ln P \cdot \partial_1 \partial_{x_i} \ln P + \partial_z \ln P \cdot \partial_1 \partial_2 \partial_{x_i} \ln P) \\
&= 0
\end{aligned}$$

(for the last step note that every summand has a factor that contains a $\partial_t^2 \ln P$, $\partial_z^2 \ln P$, or $\partial_t \partial_z \ln P$ that gets at most n ∂_{c_i}'s, so it vanishes by the induction assumption).

Step 2. By step 1 it suffices to prove the lemma in the case $c = 0$. Note that then P becomes a product of two terms of the form

$$R = \sum_k \frac{x^k}{k!} t^{ak} \qquad \text{and} \qquad S = \sum_k \frac{x^k}{k!} \prod_{i=0}^{bk-1} (z + t - i)$$

where the first term contains all the x_i with $(a_i, b_i) = (0,0)$ or $(a_i, b_i) = (1,0)$, and the second term all the x_i with $(a_i, b_i) = (0,1)$. Obviously, it suffices to prove the lemma for R and S separately. But

$$R = \sum_k \prod_i \frac{(x_i t_i^a)^{k_i}}{k_i!} = \exp \Big(\sum_i x_i t^{a_i} \Big)$$

and

$$S = \sum_k \frac{x^k}{k!} \binom{z+t}{\sum k} (\sum k)! = \Big(1 + \sum_i x_i \Big)^{z+t},$$

and in both cases it is obvious that the lemma holds. □

Lemma A.2. *Let x_i be a collection of variables (possibly infinite), and let $c_i \in \mathbb{C}$. Define*

$$Q(t) = \sum_k \frac{x^k}{k!} \, t \prod_{i=1}^{\sum k - 1} (ck + t - i)$$

in multi-index notation, where k is a multi-index. Then $\ln Q(t)$ is linear in t, i.e.

$$(t\partial_t - 1) \ln Q = 0.$$

Proof. The proof is very similar to that of lemma A.1.

Step 1. We consider the c_i to be formal variables and show by induction on n that for every i and every $n \geq 0$

$$\text{if} \quad (t\partial_t - 1) \ln Q|_{c_i=0} = 0 \quad \text{then} \quad \partial_{c_i}^n (t\partial_t - 1) \ln Q|_{c_i=0} = 0.$$

So assume that $\partial_{c_i}^j (t\partial_t - 1) \ln Q|_{c_i=0} = 0$ for $j \leq n$. By definition of Q we have $\partial_{c_i} Q = x_i \partial_{x_i} (\partial_t - \frac{1}{t}) Q$. Hence it follows that (everything in the following calculation is evaluated at $c_i = 0$):

$$\begin{aligned}
\partial_{c_i}^{n+1} (t\partial_t - 1) \ln Q &= \partial_{c_i}^n (t\partial_t - 1) \frac{x_i \partial_{x_i} (\partial_t - \frac{1}{t}) Q}{Q} \\
&= x_i (t\partial_t - 1) \left(\underbrace{\partial_{c_i}^n \left(\partial_t - \frac{1}{t} \right) \partial_{x_i} \ln Q}_{=0} + \partial_{c_i}^n \partial_t \ln Q \cdot \partial_{x_i} \ln Q \right) \\
&= x_i \partial_{c_i}^n \left(\partial_t \ln Q \cdot \partial_{x_i} (t\partial_t - 1) \ln Q + \partial_t (t\partial_t - 1) \ln Q \cdot \partial_{x_i} \ln Q \right) \\
&= 0
\end{aligned}$$

(for the last step note that every summand has a factor that contains a $(t\partial_t - 1) \ln Q$ that gets at most n ∂_{c_i}'s, so it vanishes by the induction assumption).

Step 2. By step 1 it suffices to prove the lemma in the case $c = 0$. But then

$$Q(t) = \sum_k \frac{x^k}{k!} \prod_{i=0}^{\sum k - 1} (t - i) = \Big(1 + \sum_i x_i \Big)^t,$$

which obviously satisfies the statement of the lemma. □

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