The enumerative geometry of rational and elliptic tropical curves and a Riemann-Roch theorem in tropical geometry

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Abstract:

The work is devoted to the study of tropical curves with emphasis on their enumerative geometry. Major results include a conceptual proof of the fact that the number of rational tropical plane curves interpolating an appropriate number of general points is independent of the choice of points, the computation of intersection products of Psi-classes on the moduli space of rational tropical curves, a computation of the number of tropical elliptic plane curves of given genus and fixed tropical j-invariant as well as a tropical analogue of the Riemann-Roch theorem for algebraic curves.

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14N35	Gromov-Witten invariants, quantum cohomology
51M20	Polyhedra and polytopes; regular figures, division of spaces
14N10	Enumerative problems (combinatorial problems)

Keywords:

Tropical geometry, tropical curves, enumerative geometry, metric graphs.

. . . dedicated to my parents — in love and gratitude

Contents

Preface

Tropical geometry

Tropical geometry is a rather new field of mathematics. Although its roots can be traced back to work of G. Bergman [Ber71] and R. Bieri, J. Groves [BG84], it emerged as a subject on its own (and received its name) as recently as 2002. Generally speaking, tropical geometry transforms (algebraic or symplectic) geometric objects to polyhedral ones and allows the study of properties of the former objects by purely combinatorial means.

Despite its rather short history, methods of tropical geometry have proven to be a useful and promising tool in different areas of mathematics, such as

- **Number theory:** W. Gubler's proof of the Bogomolov conjecture for totally degenerate abelian varieties [Gub07a],[Gub07b].
- **Real enumerative geometry:** Work of I. Itenberg, V. Kharlamov and E. Shustin on the positivity of Welschinger invariants and asymptotic enumeration of real rational curves [IKS03], [IKS04], [IKS].
- **Symplectic geometry:** M. Abouzaid's proof of a part of Kontsevich's homological mirror symmetry conjecture [Abo].

One should note that there are a number of different ways to define tropical curves and higher-dimensional generalizations thereof, e.g. as limits of amoeba, as images of varieties over non-archimedean fields under the valuation map, as varieties over the so-called max-plus algebra or as polyhedral complex with some extra structure (see e.g. A. Gathmann's overview article [Gat06]).

In this thesis, we prefer a combinatorial point of view and we define our basic objects of study — tropical curves and tropical fans — to be either abstract polyhedral complexes or polyhedral complexes in some real vector space which satisfy the so-called balancing condition.

Complex enumerative geometry and tropical curves

One of the main motivations to study tropical geometry is its relation to complex enumerative geometry. Complex enumerative geometry deals with the problem to count geometric objects defined over the field of complex numbers, which satisfy given conditions. An important class among these problems is the count of rational plane curves of fixed degree interpolating an appropriate number of points in the projective plane over the complex numbers (where appropriate means that we expect a finite number of such curves). Although some instances of these problems where solved decades or even centuries ago (e.g. there is a unique line through two general points, a unique conic through five general points and the number of rational cubics through eight points equals twelve), the general case of counting curves of arbitrary degree remained open until Maxim Kontsevich gave a recursive formula to enumerate these curves in 1994, using intersection theory of moduli spaces of curves and maps [KM94]. A recursive formula for counting plane curves of arbitrary degree and genus satisfying general point condition has been given by Lucia Caporaso and Joe Harris in 1998 [CH98].

In 2000, M. Kontsevich conjectured that there exists a close relationship between the enumeration of complex plane algebraic curves and the enumeration of plane tropical curves — that is, certain graphs in the real plane. This relationship was made precise by Grigory Mikhalkin, who introduced the concept of a multiplicity of a plane tropical curve and proved what is now known as Mikhalkin's correspondence theorem: the number of plane complex algebraic curves (of fixed genus and degree) interpolating an appropriate number of general points equals the number of plane tropical curves (of the same genus and degree and counted with multiplicities) interpolating the same number of general points. Furthermore, G. Mikhalkin presented an algorithm based on lattice paths in polygons to compute these numbers for arbitrary degree and genus [Mik05].

One should note that in contrast to complex algebraic geometry (where enumerative numbers are intersection numbers of cycles on suitable moduli spaces), it is not obvious that the number of tropical curves counted with multiplicity does not depend on the choice of general points. In earlier work, the invariance of this numbers has been shown by G. Mikhalkin (via relating the tropical numbers to its counterparts in enumerative geometry [Mik05]) and by A. Gathmann and H. Markwig (via ad-hoc computations in tropical geometry [GM07b]).

Results

In this thesis, we extend and generalize the work of G. Mikhalkin, A. Gathmann and H. Markwig mentioned above in a number of ways:

- We introduce the concept of tropical fans and their morphisms to prove basic results in tropical intersection theory.
- We show that the moduli spaces of rational curves and maps are tropical fans.
- We use the tropical fan structure of the moduli space of rational maps to give a conceptual proof (using tropical intersection theory) of the fact that the number of rational curves interpolating an approriate number of points counted with multiplicity does not depend on the choice of general points.
- We extend Mikhalkin's correspondence theorem to count elliptic curves with fixed tropical j-invariant.
- We modify Mikhalkin's lattice path algorithm for rational curves such that less paths have to be taken into account.

In addition, we use the structure of the moduli space of rational tropical curves to compute intersection products of Psi-classes, we present a (non-sharp) upper bound for the number of rational tropical curves (counted without multiplicities) and we present a tropical analogue of the Riemann-Roch formula for algebraic curves.

Chapter Synopsis

The results of this thesis are presented in three chapters, which are in principle self-contained and may be read separately:

- **Chapter 1: Moduli spaces of rational tropical curves and maps**

  We start in section 1.1 by introducing the concept of tropical fans (the "local building blocks of tropical varieties") and their morphisms. For such a morphism of tropical fans of the same dimension we show that the number of inverse images (counted with suitable tropical multiplicities) does not depend on the chosen point. As an application of this theory of tropical fans, we show in section 1.2 resp. section 1.4 that the moduli spaces of n-marked rational tropical curves resp. maps have the structure of tropical fans. We use this structure in section 1.3 to compute intersection products of Psi-classes defined by G. Mikhalkin [Mik] using tropical intersection theory developed by Lars Allermann and Johannes Rau [AR] and in section 1.5 to present two examples of new, easy and unified proofs of two independence statements in tropical enumerative geometry, e.g. that the number of rational tropical curves through given general points is independent of the choice of points.

- **Chapter 2: Counting tropical elliptic plane curves with fixed j-invariant**

  In complex algebraic geometry, the problem of enumerating plane elliptic curves of given degree and with fixed complex structure has been solved by Rahul Pandharipande [Pan97]. In this chapter, we treat the tropical analogue of this problem, namely the determination of the number of tropical elliptic plane curves of fixed degree and given tropical j-invariant interpolating an appropriate number of general points. In order to solve this problem, we start in section 2.1 by introducing the concept of weighted polyhedral complexes and their morphisms. We show in section 2.2 that the moduli space of n-marked elliptic curves has the structure of a weighted polyhedral complex. We use this structure in section 2.3 to compute the number of plane elliptic tropical curves with fixed tropical j-invariant and interpolating an appropriate number of general points. We show that this number is independent of the choice of the value of j and is equal to its analogue in algebraic geometry. In section 2.4, we use this result to simplify G. Mikhalkin's algorithm to count curves via lattice paths in the case of rational tropical curves and to get an upper bound on the number of rational tropical curves of fixed degree (counted without multiplicity) interpolating an appropriate number of general points.

- **Chapter 3: A Riemann-Roch theorem in tropical geometry**

  In this chapter, we extend a graph-theoretic Riemann-Roch theorem due to Matt Baker and Sergey Norine [BN07] to metric graphs and establish a Riemann-Roch theorem for divisors on (abstract) tropical curves.

Publication of the results

The results of this thesis are the outcome of joint work with Andreas Gathmann and Hannah Markwig and have appeared or are going to appear in several journal articles [KMa], [KMb], [GK08], [GKM]. In this joint work, it is not easy to separate the contributions each of us made. As far as it can be told, main contributions of Andreas Gathmann include the main ideas of the sections 1.1 and 3.3 and main contributions of Hannah Markwig include sections 1.4 and 1.5 as well as parts of chapter 2, which is heavily based on our joint work [KMa]. My main contributions include sections 1.2, 1.3, 3.2 as well as parts of chapter 2.

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1 Moduli spaces of rational tropical curves and maps

Introduction

In this chapter, we study the moduli spaces of rational curves and maps. In order to do this, we start in section 1.1 by introducing the concept of tropical fans and morphisms thereof. The main result of this section states that for a morphism of tropical fans of the same dimension, with the target being irreducible, the sum of the multiplicities of the inverse image points of a general point P in the target is independent of the choice of P.

In section 1.2, we study the moduli space of abstract n-marked rational tropical curves. Such curves are metric trees with n labelled leaves and without 2-valent vertices and they can be parametrized by the combinatorial structure of the underlying (non-metric) tree and the length of each interior edge. The tropical moduli space $\mathcal{M}_{0,n}$ (the space which parametrizes these curves) has the structure of a polyhedral complex, obtained by gluing several copies of the positive orthant $\mathbb{R}_{\geq 0}^{n-3}$ — one copy for each 3-valent combinatorial tree with n leaves. The main result of this section states that the space $\mathcal{M}_{0,n}$ can be embedded as a tropical fan into a real vector space Q_n.

Recently, G. Mikhalkin introduced tropical Psi-classes on the moduli space of rational tropical curves as follows: for $k \in [n]$, the tropical Psi-class Ψ_k is the subcomplex of cones of $\mathcal{M}_{0,n}$ corresponding to tropical curves which have the property that the leaf labelled with the number k is adjacent to a vertex of valence at least four. In section 1.3, we apply the tropical intersection theory developed by L. Allermann and J. Rau to compute intersection products of an arbitrary number of tropical Psi-classes on the moduli space of rational tropical curves. As a special case, we show that in the case of zero-dimensional (stable) intersections, the resulting numbers agree with the intersection numbers of Psi-classes on the moduli space of n-marked rational curves computed in algebraic geometry.

Extending the results of section 1.2, we construct the tropical analogues of the moduli spaces of stable maps to a toric variety in section 1.4. Furthermore, we equip these spaces with the structure of a tropical fan. This structure is used in section 1.5 to give simple and generalized proofs of two statements in the enumerative geometry of rational curves that have occurred earlier in the literature: the statement that the number of rational tropical curves of given degree and through an appropriate number of general points does not depend on the choice of these points and the statement that the degree of combined evaluation and forgetful map occurring in the proof of the tropical Kontsevich formula is independent of the choice of the points [GM08, proposition 4.4].

1.1 Tropical fans

In this section, let Λ denote a finitely generated free abelian group and $V := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ the corresponding real vector space containing Λ as a lattice. The dual lattice in the dual vector space $V^\vee$ will be denoted $\Lambda^\vee$.

Definition 1.1.1 ((Weighted) fans)
A *fan* X is a collection of *cones*

$$\sigma = \{x \in V;\ f_i(x) = 0,\ f_j(x) \geq 0 \text{ for a set of functions } f_i, f_j \in \Lambda^\vee\} \subset V$$

such that all faces of the cones in X are also contained in X and the intersection of any two cones in X is a face of both. We denote by $V_\sigma \subset V$ the smallest vector subspace of V that contains σ, and by $\Lambda_\sigma := V_\sigma \cap \Lambda \subset \Lambda$ the smallest sublattice of Λ that contains $\sigma \cap \Lambda$. The *dimension* of a cone σ is defined to be $\dim \sigma := \dim V_\sigma$, and the set of all k-dimensional cones will be denoted $X^{(k)}$. The *dimension* $\dim(X)$ of the fan X is the maximal dimension of a cone in X. The fan X is *pure* if each inclusion-maximal cone in X has the same dimension. The union of all cones in X will be denoted $|X| \subset V$. Furthermore, let $\leq$ denote the order relation on the set of cones of X given by inclusion. A *weighted fan* in V is a pair (X, ω_X) where X is a fan of some pure dimension n in V, and $\omega_X : X^{(n)} \to \mathbb{Z}_{>0}$ is a map. We call $\omega_X(\sigma)$ the *weight* of the cone $\sigma \in X^{(n)}$ and write it simply as $\omega(\sigma)$ if no confusion can result. Also, by abuse of notation we will write a weighted fan (X, ω_X) simply as X if the weight function ω_X is clear from the context.

Example 1.1.2 (Half-space fan)
For $f \in \Lambda^\vee \backslash \{0\}$ the three cones

$$\{x \in V;\ f(x) = 0\}, \quad \{x \in V;\ f(x) \geq 0\}, \quad \{x \in V;\ f(x) \leq 0\}$$

form a fan. We call it the *half-space fan* H_f.

Definition 1.1.3 (Subfans)
Let X be a fan in V. A fan Y in V is called a *subfan* of X (denoted $Y \subset X$) if each cone of Y is contained in a cone of X. In this case we denote by $C_{Y,X} : Y \to X$ the map that sends a cone $\sigma \in Y$ to the (unique) inclusion-minimal cone of X that contains σ. Note that for a subfan $Y \subset X$ we obviously have $|Y| \subset |X|$ and $\dim C_{Y,X}(\sigma) \geq \dim \sigma$ for all $\sigma \in Y$.

Definition 1.1.4 (Refinements)
Let (X, ω_X) and (Y, ω_Y) be weighted fans in V. The fan (Y, ω_Y) is a *refinement* of (X, ω_X) if

- $Y \subset X$;
- $|Y| = |X|$ (so in particular $\dim Y = \dim X$); and
- $\omega_Y(\sigma) = \omega_X(C_{Y,X}(\sigma))$ for all maximal cones $\sigma \in Y$.

Two weighted fans (X, ω_X) and (Y, ω_Y) are *equivalent* (written $(X, \omega_X) \cong (Y, \omega_Y)$) if there exists a common refinement.

Definition 1.1.5 (Integer linear maps)
Let $V = \Lambda \otimes \mathbb{R}$ and $V' = \Lambda' \otimes \mathbb{R}$. An *integer linear map* $f : V \to V'$ is a linear map such that $f(\Lambda) \subset \Lambda'$ (i.e. the matrix representation of f with respect to lattice bases contains only integer entries). Note that the absolute value of the determinant of f equals the lattice index $|\Lambda'/f(\Lambda)|$; this number will be denoted by $|\det(f)|$.

Definition 1.1.6 (Morphisms of (weighted) fans)
Let X be a fan in $V = \Lambda \otimes \mathbb{R}$, and let Y be a fan in $V' = \Lambda' \otimes \mathbb{R}$. A *morphism* $f : X \to Y$ is a map from $|X| \subset V$ to $|Y| \subset V'$ induced by an integer linear map. By abuse of notation we will often denote the corresponding linear maps from Λ to Λ' and from V to V' by the same letter f (note however that these latter maps are not determined uniquely by $f : X \to Y$ if $|X|$ does not span V). A morphism of weighted fans is a morphism of fans (i.e. there are no conditions on the weights). As the notion of morphism is invariant under refinements of weighted fans, morphisms from (X, ω_X) to (Y, ω_Y) are the same as morphisms from any refinement of (X, ω_X) to any refinement of (Y, ω_Y).

Construction 1.1.7 (Image fan)
Let X be a purely n-dimensional fan in $V = \Lambda \otimes \mathbb{R}$, and let Y be any fan in $\Lambda' \otimes \mathbb{R}$. For any morphism $f : X \to Y$ we will construct an *image fan* $f(X)$ in V' of pure dimension n as follows. Consider the collection of cones

$$Z := \{f(\sigma);\ \sigma \in X \text{ contained in a maximal cone of } X \text{ on which } f \text{ is injective}\}$$

in V'. Note that Z is in general not a fan in V', since e.g. the images of some maximal cones might overlap in a region of dimension n. To make it into one choose linear forms $g_1, \ldots, g_N \in \Lambda'$ such that each cone $f(\sigma) \in Z$ can be written as

$$f(\sigma) = \{x \in V';\ g_i(x) = 0,\ g_j(x) \geq 0,\ g_k(x) \leq 0 \text{ for all } i \in I_\sigma, j \in J_\sigma, k \in K_\sigma\} \qquad (*)$$

for suitable index sets $I_\sigma, J_\sigma, K_\sigma \subset [N]$. Now we replace X by the fan $\tilde{X} = X \cap H_{g_1 \circ f} \cap \cdots \cap H_{g_N \circ f}$ (see example 1.1.2). This fan satisfies $|\tilde{X}| = |X|$, and by definition each cone of $\tilde{X}$ is of the form

$$\{x \in V;\ x \in \sigma,\ g_i(f(x)) = 0,\ g_j(f(x)) \geq 0,\ g_k(f(x)) \leq 0 \text{ for all } i \in I, j \in J, k \in K\}$$

for some $\sigma \in X$ and a partition $I \cup J \cup K = [N]$. By $(*)$ the image of such a cone under f is of the form

$$\begin{aligned}&\{x \in V';\ x \in f(\sigma),\ g_i(x) = 0,\ g_j(x) \geq 0,\ g_k(x) \leq 0 \text{ for all } i \in I, j \in J, k \in K\}\\ &= \{x \in V';\ g_i(x) = 0,\ g_j(x) \geq 0,\ g_k(x) \leq 0 \text{ for all } i \in I', j \in J', k \in K'\}\end{aligned}$$

for some partition $I' \cup J' \cup K' = [N]$, i.e. it is a cone of the fan $H_{g_1} \cap \cdots \cap H_{g_N}$. Hence the set of cones

$$\tilde{Z} := \{f(\sigma);\ \sigma \in \tilde{X} \text{ contained in a maximal cone of } \tilde{X} \text{ on which } f \text{ is injective}\}$$

consists of cones in the fan $H_{g_1} \cap \cdots \cap H_{g_N}$ and $\tilde{Z}$ is a pure fan in V' of dimension n.

If moreover X is a weighted fan then $\tilde{X}$ will be a weighted fan as well. In this case we make $\tilde{Z}$ into a weighted fan by setting

$$\omega_{\tilde{Z}}(\sigma') := \sum_{\sigma \in \tilde{X}^{(n)}: f(\sigma) = \sigma'} \omega_{\tilde{X}}(\sigma) \cdot |\Lambda'_{\sigma'}/f(\Lambda_\sigma)|$$

for all $\sigma' \in \tilde{Z}^{(n)}$— note that $f(\Lambda_\sigma)$ is a sublattice of $\Lambda'_{\sigma'}$ of full rank.

It is clear from the construction that the equivalence class of $(\tilde{Z}, \omega_{\tilde{Z}})$ remains unchanged if we replace the $g_1, \ldots, g_N$ by any larger set of linear forms (this would just lead to a refinement), and hence also if we replace them by any other set of linear forms satisfying $(*)$. Hence the equivalence class of $(\tilde{Z}, \omega_{\tilde{Z}})$ does not depend on any choices we made. We denote it by $f(X)$ and call it the *image fan* of f. It is obvious that (the equivalence class) of $f(X)$ depends only on the equivalence class of X.

By abuse of notation we will usually drop the tilde from the above notation and summarize the construction above as follows: given a weighted fan X of dimension n, an arbitrary fan Y, and a morphism $f : X \to Y$, we may assume after passing to an equivalent fan for X that

$$f(X) = \{f(\sigma);\ \sigma \in X \text{ contained in a maximal cone of } X \text{ on which } f \text{ is injective}\}$$

is a fan. We consider this to be a weighted fan of dimension n by setting

$$\omega_{f(X)}(\sigma') := \sum_{\sigma \in X^{(n)}: f(\sigma) = \sigma'} \omega_X(\sigma) \cdot |\Lambda'_{\sigma'}/f(\Lambda_\sigma)|$$

for $\sigma' \in f(X)^{(n)}$. This weighted fan is well-defined up to equivalence.

Construction 1.1.8 (Normal vectors)
Let $\tau < \sigma$ be cones in V with $\dim(\tau) = \dim(\sigma) - 1$. By definition there is a linear form $g \in \Lambda^\vee$ that is zero on τ, non-negative on σ, and not identically zero on σ. Then g induces an isomorphism $V_\sigma/V_\tau \to \mathbb{R}$ that is non-negative and not identically zero on σ/V_τ, showing that the cone σ/V_τ lies in a unique half-space of $V_\sigma/V_\tau \cong \mathbb{R}$. As moreover $\Lambda_\sigma/\Lambda_\tau \subset V_\sigma/V_\tau$ is isomorphic to $\mathbb{Z}$ there is a unique generator of $\Lambda_\sigma/\Lambda_\tau$ lying in this half-space. We denote it by $u_{\sigma/\tau} \in \Lambda_\sigma/\Lambda_\tau$ and call it the *(primitive) normal vector* of σ relative to τ.

Definition 1.1.9 (Tropical fan)
A *tropical fan* in V is a weighted fan (X, ω_X) in V such that for all $\tau \in X^{(\dim X-1)}$ the *balancing condition* holds

$$\sum_{\sigma > \tau} \omega_X(\sigma) \cdot u_{\sigma/\tau} = 0 \quad \in V/V_\tau.$$

Definition 1.1.10 (Irreducible fans)
A tropical fan X in V is *irreducible* if there is no tropical fan Y in V of dimension $\dim(X)$ such that $|Y| \subsetneq |X|$. As the condition of irreducibility remains unchanged under refinements, it is well-defined on equivalence classes of tropical fans.

Definition 1.1.11 (Product of fans)
Let X and X' be fans in $V = \Lambda \otimes \mathbb{R}$ and $V' = \Lambda' \otimes \mathbb{R}$, respectively. Then their product $\{\sigma \times \sigma' \,:\, \sigma \in X, \sigma' \in X'\}$ is a fan in $V \times V'$ which is called the *product* of the two fans X and X' and denoted by $X \times X'$. Obviously, we have $|X \times X'| = |X| \times |X'|$.

Proposition 1.1.12
The product of two irreducible tropical fans is irreducible.

Proof. Let X and X' be irreducible tropical fans of dimensions n and n' in $V = \Lambda \otimes \mathbb{R}$ and $V' = \Lambda' \otimes \mathbb{R}$, respectively, and let Y be a tropical fan of dimension $n + n'$ in $V \times V'$ with $|Y| \subsetneq |X \times X'|$. By passing from Y to the refinement $Y \cap (X \times X')$ we may assume that $Y \subset X \times X'$. For any $\sigma \times \sigma' \in (X \times X')^{(n+n')}$ we claim that $A(\sigma \times \sigma') := C^{-1}_{Y,X\times X'}(\sigma \times \sigma') \cap Y^{(n+n')}$ is either empty or consists of cones that cover $\sigma \times \sigma'$ and all have the same weight. In fact, if this was not the case then there would have to be a cone $\tau \in Y^{(n+n'-1)}$ with $C_{Y,X\times X'}(\tau) = \sigma \times \sigma'$ so that to the two sides of τ in $\sigma \times \sigma'$ there is either only one cone of $A(\sigma \times \sigma')$ or two cones with different weights — and in both cases the balancing condition for Y would be violated at τ.

Let us now define a purely $(n + n')$-dimensional fan Z in $V \times V'$ by taking all cones $\sigma \times \sigma' \in (X \times X')^{(n+n')}$ for which $A(\sigma \times \sigma') \neq \emptyset$, together with all faces of these cones. Associating to such a cone $\sigma \times \sigma' \in Z^{(n+n')}$ the weight of the cones in $A(\sigma \times \sigma')$ we make Z into a tropical fan of dimension $n + n'$ — in fact it is just a tropical fan of which Y is a refinement.

As $\dim Z = n + n'$ and $|Z| = |Y| \subsetneq |X \times X'|$ there must be cones $\sigma_1 \times \sigma_1' \in Z$ and $\sigma_0 \times \sigma_0' \in (X \times X')\backslash Z$. We distinguish two cases:

- $\sigma_0 \times \sigma_1' \in (X \times X')\backslash Z$: Then we construct a purely n-dimensional fan X_0 in V by taking all cones $\sigma \in X^{(n)}$ with $\sigma \times \sigma_1' \in Z$, together with all faces of these cones. Setting $\omega_{X_0}(\sigma) := \omega_Z(\sigma \times \sigma_1')$ the fan X_0 becomes an n-dimensional tropical fan, with the balancing condition inherited from Z (in fact, the balancing condition around a cone $\tau \in X_0^{(n-1)}$ follows from the one around $\tau \times \sigma_1'$ in Z). As $|X_0|$ is neither empty (since $\sigma_1 \in X_0$) nor all of $|X|$ (since $\sigma_0 \notin X_0$) this is a contradiction to X being irreducible.

- $\sigma_0 \times \sigma_1' \in Z$: This case follows in the same way by considering the purely n'-dimensional fan Y_0 in V' given by all cones $\sigma' \in Y^{(n')}$ with $\sigma_0 \times \sigma' \in Z$, together with all faces of these cones — leading to a contradiction to X' being irreducible.

□

Lemma 1.1.13
Let X and Y be tropical fans of dimension n in V. Assume that $|Y| \subset |X|$, and that X is irreducible. Then $Y \cong \lambda \cdot X$ for some $\lambda \in \mathbb{Q}_{>0}$.

Proof. As X is irreducible we have $|Y| = |X|$. Replacing X and Y by the refinements $X \cap Y$ and $Y \cap X$ respectively we may assume that X and Y consist of the same cones (with possibly different weights). Let $\lambda := \min_{\sigma \in X^{(n)}} \omega_Y(\sigma)/\omega_X(\sigma) > 0$, and let $\alpha \in \mathbb{Z}_{>0}$ with $\alpha\lambda \in \mathbb{Z}$. Consider the new weight function $\omega(\sigma) = \alpha(\omega_Y(\sigma) - \lambda\,\omega_X(\sigma))$ for $\sigma \in X^{(n)}$. By construction it takes values in $\mathbb{Z}_{\geq 0}$, with value 0 occurring at least once. Construct a new weighted fan Z from this weight function by taking all cones $\sigma \in X^{(n)}$ with $\omega(\sigma) > 0$, together with all faces of these cones. Then (Z, ω) is a tropical fan as the balancing condition is linear in the weights. It does not cover $|X|$ since at least one maximal cone has been deleted from X by construction. Hence it must be empty as X was assumed to be irreducible. This means that $\omega_Y(\sigma) - \lambda\,\omega_X(\sigma) = 0$ for all $\sigma \in X^{(n)}$. □

Proposition 1.1.14
Let X be an n-dimensional tropical fan in $V = \Lambda \otimes \mathbb{R}$, Y an arbitrary fan in $V' = \Lambda' \otimes \mathbb{R}$, and $f : X \to Y$ a morphism. Then $f(X)$ is an n-dimensional tropical fan as well (if it is not empty).

Proof. Since we have already seen in construction 1.1.7 that $f(X)$ is a weighted fan of dimension n we just have to check the balancing condition of definition 1.1.9. As in this construction we may assume that $f(X)$ just consists of the cones $f(\sigma)$ for all $\sigma \in X$ contained in a maximal cone on which f is injective. Let $\tau' \in f(X)^{(n-1)}$, and let $\tau \in X^{(n-1)}$ with $f(\tau) = \tau'$. Applying f to the balancing condition for X at τ we get

$$\sum_{\sigma > \tau} \omega_X(\sigma) \cdot f(u_{\sigma/\tau}) = 0 \quad \in V'/V'_{\tau'}$$

for all $\tau \in X^{(n-1)}$. Now let $\sigma' \in f(X)$ be a cone with $\sigma' > \tau'$, and $\sigma \in X$ with $\sigma > \tau$ such that $f(\sigma) = \sigma'$. Note that the primitive normal vector $u_{\sigma'/\tau'}$ is related to the (possibly non-primitive) vector $f(u_{\sigma/\tau})$ by

$$f(u_{\sigma/\tau}) = |\Lambda'_{\sigma'}/(\Lambda'_{\tau'} + \mathbb{Z}\, f(u_{\sigma/\tau}))| \cdot u_{\sigma'/\tau'}$$

if f is injective on σ, and $f(u_{\sigma/\tau}) = 0$ otherwise. Inserting this into the above balancing condition, and using the exact sequence

$$0 \longrightarrow \Lambda'_{\tau'}/f(\Lambda_\tau) \longrightarrow \Lambda'_{\sigma'}/f(\Lambda_\sigma) \longrightarrow \Lambda'_{\sigma'}/(\Lambda'_{\tau'} + \mathbb{Z}\, f(u_{\sigma/\tau})) \longrightarrow 0$$

we conclude that

$$\sum_{\sigma>\tau} \omega_X(\sigma) \cdot |\Lambda'_{\sigma'}/f(\Lambda_\sigma)| \cdot u_{\sigma'/\tau'} = 0 \quad \in V'/V'_{\tau'},$$

where the sum is understood to be taken over only those $\sigma > \tau$ on which f is injective. Let us now sum these equations up for all τ with $f(\tau) = \tau'$. The above sum then simply becomes a sum over all σ with $f(\sigma) > \tau'$ (note that each such σ occurs in the sum exactly once since f is injective on σ so that σ cannot have two distinct codimension-1 faces that both map to τ'). Splitting this sum up according to the cone $f(\sigma)$ we get

$$\sum_{\sigma'>\tau'} \sum_{\sigma: f(\sigma)=\sigma'} \omega_X(\sigma) \cdot |\Lambda'_{\sigma'}/f(\Lambda_\sigma)| \cdot u_{\sigma'/\tau'} = 0 \quad \in V'/V'_{\tau'}.$$

But using the definition of the weights of $f(X)$ of construction 1.1.7 this is now simply the balancing condition

$$\sum_{\sigma'>\tau'} \omega_{f(X)}(\sigma') \cdot u_{\sigma'/\tau'} = 0 \quad \in V'/V'_{\tau'}$$

for $f(X)$. □

Corollary 1.1.15
Let X and Y be tropical fans of the same dimension n in $V = \Lambda \otimes \mathbb{R}$ and $V' = \Lambda' \otimes \mathbb{R}$, respectively, and let $f : X \to Y$ be a morphism. Assume that Y is irreducible. Then there is a fan Y_0 in V' of smaller dimension with $|Y_0| \subset |Y|$ such that

(a) each point $Q \in |Y| \backslash |Y_0|$ lies in the interior of a cone $\sigma'_Q \in Y$ of dimension n;

(b) each point $P \in f^{-1}(|Y| \backslash |Y_0|)$ lies in the interior of a cone $\sigma_P \in X$ of dimension n;

(c) for $Q \in |Y| \backslash |Y_0|$ the sum

$$\sum_{P \in |X| : f(P)=Q} \operatorname{mult}_P f$$

does not depend on Q, where the multiplicity $\operatorname{mult}_P f$ *of f at P is defined to be*

$$\operatorname{mult}_P f := \frac{\omega_X(\sigma_P)}{\omega_Y(\sigma'_Q)} \cdot |\Lambda'_{\sigma'_Q}/f(\Lambda_{\sigma_P})| = \frac{\omega_X(\sigma_P)}{\omega_Y(\sigma'_Q)} \cdot |\det(f)|$$

Proof. Consider the tropical fan $f(X)$ in V' (see construction 1.1.7 and proposition 1.1.14). If $f(X) = \emptyset$ (i.e. if there is no maximal cone of X on which f is injective) the statement of the corollary is trivial. Otherwise $f(X)$ has dimension n and satisfies $|f(X)| \subset |Y|$, so as Y is irreducible it follows by lemma 1.1.13 that $f(X) \cong \lambda \cdot Y$ for some $\lambda \in \mathbb{Q}_{>0}$. After passing to equivalent fans we may assume that $f(X)$ and Y consist of the same cones, and that these are exactly the cones of the form $f(\sigma)$ for $\sigma \in X$ contained in a maximal cone on which f is injective. Now let Y_0 be the fan consisting of all cones of Y dimension less than n. Then (a) and (b) hold by construction. Moreover, each $Q \in |Y| \backslash |Y_0|$ lies in the interior of a unique n-dimensional cone σ', and there is a 1:1 correspondence between points $P \in f^{-1}(Q)$ and n-dimensional cones σ in X with $f(\sigma) = \sigma'$. So we conclude that

$$\sum_{P: f(P)=Q} \operatorname{mult}_P f = \sum_{\sigma: f(\sigma)=\sigma'} \frac{\omega_X(\sigma)}{\omega_Y(\sigma')} \cdot |\Lambda'_{\sigma'}/f(\Lambda_\sigma)| = \frac{\omega_{f(X)}(\sigma')}{\omega_Y(\sigma')} = \lambda$$

does not depend on Q. □

Definition 1.1.16 (Marked fans)
A *marked fan* in V is a pure-dimensional simplicial fan X in V together with the data of vectors $v_\sigma \in (\sigma \backslash \{0\}) \cap \Lambda$ for all $\sigma \in X^{(1)}$ (i.e. v_σ is an integral vector generating the edge σ).

Construction 1.1.17
Let X be a marked fan of dimension n.

(a) Let $\sigma \in X^{(k)}$ be a k-dimensional cone in X. As σ is simplicial by assumption, there are exactly k edges $\sigma_1, \dots, \sigma_k \in X^{(1)}$ that are faces of σ, and that the vectors $v_{\sigma_1}, \dots, v_{\sigma_k}$ generating these edges are linearly independent. Hence

$$\tilde{\Lambda}_\sigma := \mathbb{Z}\, v_{\sigma_1} + \cdots + \mathbb{Z}\, v_{\sigma_k}$$

is a sublattice of Λ_σ of full rank, and consequently $\Lambda_\sigma/\tilde{\Lambda}_\sigma$ is a finite abelian group. We set $\omega(\sigma) := |\Lambda_\sigma/\tilde{\Lambda}_\sigma| \in \mathbb{Z}_{>0}$. In particular, this makes the marked fan X into a weighted fan. In this paper marked fans will always be considered to be weighted fans in this way.

(b) Let $\sigma \in X^{(k)}$ and $\tau \in X^{(k-1)}$ with $\sigma > \tau$. As in (a) there are then exactly $k-1$ edges in $X^{(1)}$ that are faces of τ, and k edges that are faces of σ. There is therefore exactly one edge $\sigma' \in X^{(1)}$ that is a face of σ but not of τ. The corresponding vector $v_{\sigma'}$ will be denoted $v_{\sigma/\tau} \in \Lambda$; it can obviously also be thought of as a "normal vector" of σ relative to τ. Note however that in contrast to the normal vector $u_{\sigma/\tau}$ of construction 1.1.8 it is defined in Λ and not just in Λ/Λ_τ, and that it need not be primitive.

Lemma 1.1.18
Let X be a marked fan of dimension n (and hence a weighted fan by construction 1.1.17 (a)). Then X is a tropical fan if and only if for all $\tau \in X^{(n-1)}$ the balancing condition

$$\sum_{\sigma > \tau} v_{\sigma/\tau} = 0 \quad \in V/V_\tau$$

holds (where the vectors $v_{\sigma/\tau}$ are as in construction 1.1.17 (b)).

Proof. We have to show that the given equations coincide with the balancing condition of definition 1.1.9. For this it obviously suffices to prove that

$$\omega(\sigma) \cdot u_{\sigma/\tau} = \omega(\tau) \cdot v_{\sigma/\tau} \quad \in \Lambda_\sigma/\Lambda_\tau \qquad \text{(and hence in } V_\sigma/V_\tau\text{)}$$

for all $\sigma \in X^{(n)}$ and $\tau \in X^{(n-1)}$ with $\sigma > \tau$. Using the isomorphism $\Lambda_\sigma/\Lambda_\tau \cong \mathbb{Z}$ of construction 1.1.8 this is equivalent to the equation

$$\omega(\sigma) = \omega(\tau) \cdot |\Lambda_\sigma/(\Lambda_\tau + \mathbb{Z}\, v_{\sigma/\tau})|$$

in $\mathbb{Z}$, i.e. using construction 1.1.17 (a) and the relation $\tilde{\Lambda}_\sigma = \tilde{\Lambda}_\tau + \mathbb{Z}\, v_{\sigma/\tau}$ to the equation

$$|\Lambda_\sigma/(\tilde{\Lambda}_\tau + \mathbb{Z}\, v_{\sigma/\tau})| = |\Lambda_\tau/\tilde{\Lambda}_\tau| \cdot |\Lambda_\sigma/(\Lambda_\tau + \mathbb{Z}\, v_{\sigma/\tau})|.$$

But this follows immediately from the exact sequence

$$0 \longrightarrow \Lambda_\tau/\tilde{\Lambda}_\tau \longrightarrow \Lambda_\sigma/(\tilde{\Lambda}_\tau + \mathbb{Z}\, v_{\sigma/\tau}) \longrightarrow \Lambda_\sigma/(\Lambda_\tau + \mathbb{Z}\, v_{\sigma/\tau}) \longrightarrow 0.$$

□

Corollary 1.1.19
In the situation and with the notation of corollary 1.1.15 assume moreover that X and Y are marked fans as in definition 1.1.16, and that their structure of tropical fans has been induced by these data as in construction 1.1.17 (a). Let $\sigma_1, \dots, \sigma_n \in X^{(1)}$ be the 1-dimensional faces of σ_P, and let $\sigma'_1, \dots, \sigma'_n \in Y^{(1)}$ be the 1-dimensional faces of $\sigma'_Q = \sigma'_{f(P)}$. Then $\operatorname{mult}_P f$ is equal to the absolute value of the determinant of the matrix for the linear map $f|_{V_{\sigma_P}} : V_{\sigma_P} \to V'_{\sigma'_Q}$ in the bases $\{v_{\sigma_1}, \dots, v_{\sigma_n}\}$ and $\{v_{\sigma'_1}, \dots, v_{\sigma'_n}\}$.

Proof. It is well-known that $|\Lambda'_{\sigma'_Q}/f(\Lambda_{\sigma_P})|$ is the determinant of the linear map $f|_{V_{\sigma_P}} : V_{\sigma_P} \to V'_{\sigma'_Q}$ with respect to lattice bases of Λ_{σ_P} and $\Lambda'_{\sigma'_Q}$. The statement of the corollary now follows since the base change to $\{v_{\sigma_1}, \dots, v_{\sigma_n}\}$ and $\{v_{\sigma'_1}, \dots, v_{\sigma'_n}\}$ clearly leads to factors in the determinant of $\omega_X(\sigma_P)$ and $1/\omega_Y(\sigma'_Q)$, respectively. □

1.2 The space of rational curves

In the following, for a graph Γ, we will denote by $E(\Gamma)$ resp. $V(\Gamma)$ its set of edges resp. vertices. An edge of Γ which is not a leaf of Γ will be called a non-leaf; the set of non-leaves of a graph Γ will be denoted $\operatorname{NL}(\Gamma)$.

Definition 1.2.1 (n-marked rational tropical curve)
Let $n \geq 3$ be a natural number. An *n-marked rational tropical curve* is a metric tree Γ (that is, a tree together with a length function $l : \operatorname{NL}(\Gamma) \to \mathbb{R}_{>0}$ assigning to each non-leaf a positive real number) with exactly n leaves, labelled by variables $\{x_1, \dots, x_n\}$. The space of all n-marked rational tropical curves is denoted $\mathcal{M}_{0,n}$.

Remark 1.2.2
The space $\mathcal{M}_{0,n}$ of all n-marked tropical curves (also known as the space of phylogenetic trees, see [BHV01]) has the structure of a polyhedral complex of dimension $n-3$ obtained by gluing copies of the space $\mathbb{R}^k_{>0}$ for $0 \le k \le n-3$ — one copy for each combinatorial type of a tree with n leaves and exactly k non-leaves. Its face lattice is given by $\tau \le \sigma$ if and only if the tree corresponding to τ is obtained from the tree corresponding to σ by contracting non-leaves.

Construction 1.2.3
The space $\mathcal{M}_{0,n}$ can be embedded into a real vector space by the following construction. Consider the space $\mathbb{R}^{\binom{n}{2}}$ indexed by the set $\mathcal{T}$ of all subsets $T \subset [n]$ with $|T| = 2$ and define a map

$$\begin{array}{rcl} \mathrm{dist}_n : \mathcal{M}_{0,n} & \longrightarrow & \mathbb{R}^{\binom{n}{2}} \\ (\Gamma, x_1, \ldots, x_n) & \longmapsto & (\mathrm{dist}_\Gamma(x_i, x_j))_{\{i,j\} \in \mathcal{T}} \end{array}$$

where $\mathrm{dist}_\Gamma(x_i, x_j)$ denotes the distance between the leaf with label x_i and the leaf with label x_j, that is, the sum of the lengths of all non-leaves on the (unique) path between x_i and x_j. Furthermore, define a linear map Φ_n by

$$\begin{array}{rcl} \Phi_n : \mathbb{R}^n & \longrightarrow & \mathbb{R}^{\binom{n}{2}} \\ (a_1, \ldots, a_n) & \longmapsto & (a_i + a_j)_{\{i,j\} \in \mathcal{T}}. \end{array}$$

Denote by Q_n the $\left(\binom{n}{2} - n\right)$-dimensional quotient vector space $\mathbb{R}^{\binom{n}{2}} / \mathrm{Im}(\Phi_n)$, and by $q_n : \mathbb{R}^{\binom{n}{2}} \to Q_n$ the canonical projection.

Theorem 1.2.4 ([SS04], theorem 4.2)
The map $\varphi_n := q_n \circ \mathrm{dist}_n : \mathcal{M}_{0,n} \to Q_n$ is an embedding, and the image $\varphi_n(\mathcal{M}_{0,n}) \subset Q_n$ is a simplicial fan of pure dimension $n-3$. The interior of its m-dimensional cells corresponds to combinatorial types of graphs with n marked leaves and exactly m non-leaves.

Construction 1.2.5
For each subset $I \subset [n]$ of cardinality $1 < |I| < n-1$, let T_I denote the tree with exactly one edge E of length one, such that the labels in one of the connected components of $T_I \setminus \{E\}$ are exactly the labels x_i with $i \in I$. Let $v_I := \varphi(T_I) \in Q_n$, i.e. v_I is the residue class in Q_n of the vector

$$(v_I)_{\{i,j\}} := \begin{cases} 1, & \text{if } |I \cap \{i,j\}| = 1 \\ 0, & \text{otherwise.} \end{cases}$$

By theorem 1.2.4, the v_I generate the edges of the simplicial fan $\varphi_n(\mathcal{M}_{0,n})$. Denote by $\Lambda_n := \langle v_I \rangle_{\mathbb{Z}} \subset Q_n$ the lattice in Q_n generated by all vectors v_I.

Theorem 1.2.6
The marked fan $(\varphi_n(\mathcal{M}_{0,n}), \{v_I\})$ is a tropical fan in Q_n with lattice Λ_n. In other words, using the embedding φ_n the space $\mathcal{M}_{0,n}$ can be thought of as a tropical fan of dimension $n-3$.

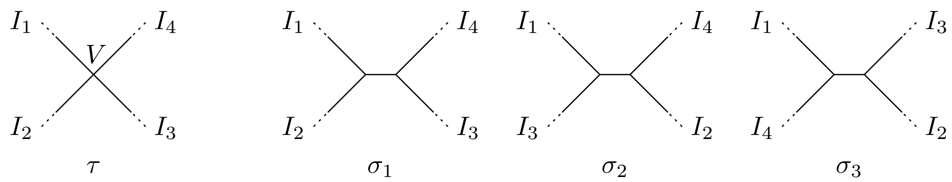
Proof. By lemma 1.1.18 we have to prove that

$$\sum_{\sigma > \tau} v_{\sigma/\tau} = 0 \in Q_n/V_\tau$$

for all $\tau \in \varphi_n(\mathcal{M}_{0,n})^{(n-4)}$ and $v_{\sigma/\tau}$ as in construction 1.1.17 (b).

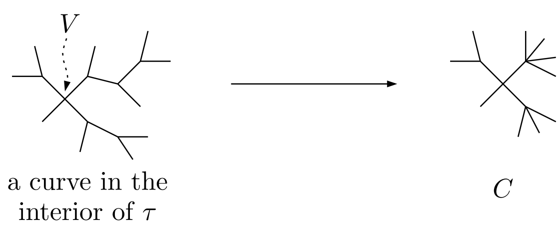
In order to prove the equation above, fix a ridge $\tau \in \varphi_n(\mathcal{M}_{0,n})$. As τ is a cell of codimension one, the curves C parametrized by points in the interior of τ have exactly one vertex V of valence $\text{val}(V) = 4$ obtained by collapsing an edge E, while all other vertices are trivalent. Let $E_1, \ldots, E_4$ denote the edges adjacent to V and for $k \in [4]$, let I_k denote the set of all $i \in [n]$ such that x_i lies in the connected component of $C \setminus \{V\}$ containing E_k.

By construction, there are exactly three facets $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{M}_{0,n}$ such that $\tau < \sigma_i$, corresponding to the three possibilities of partitioning the set $\{I_1, \ldots, I_4\}$ into two subsets of cardinality two.



Note that $v_{I_1 \cup I_2}$ is an edge of σ_1 which is not an edge of τ; hence we have $v_{\sigma_1/\tau} = v_{I_1 \cup I_2}$, and similarly $v_{\sigma_2/\tau} = v_{I_1 \cup I_3}$ and $v_{\sigma_3/\tau} = v_{I_1 \cup I_4}$.

Now let $C \in \tau$ be the point corresponding to the curve obtained from the combinatorial type τ by setting the length of each non-leaf E to 1 if E contains the vertex V, and 0 otherwise:



Furthermore, let $a \in \mathbb{R}^n$ be the vector with $a_i = 1$ if the marked leaf x_i is adjacent to V, and $a_i = 0$ otherwise. We claim that

$$v_{\sigma_1/\tau} + v_{\sigma_2/\tau} + v_{\sigma_3/\tau} = \Phi_n(a) + \text{dist}_n(C) \in \mathbb{R}^{\binom{n}{2}},$$

from which the required balancing condition then follows after passing to the quotient by $\operatorname{Im}(\Phi_n) + V_\tau$. We check this equality coordinate-wise for all $T = \{i, j\} \in \mathcal{T}$. We may assume that i and j do not lie in the same I_k since otherwise the T-coordinate of every term in the equation is zero. Then the T-coordinate on the left hand side is 2 since x_i and x_j are on different sides of the newly inserted edge for exactly two of the types $\sigma_1, \sigma_2, \sigma_3$. On the other hand, if we denote by $0 \le c \le 2$ the number how many of the two leaves x_i and x_j are adjacent to V, then the T-coordinates of $\Phi_n(a)$ and $\operatorname{dist}_n(C)$ are c and $2 - c$ respectively; so the proposition follows. □

Remark 1.2.7
A slightly different proof of this balancing condition has been given independently by G. Mikhalkin [Mik, section 2].

Example 1.2.8
For $n = 4$ the space $\mathcal{M}_{0,4}$ is embedded by φ_4 in $Q_4 = \mathbb{R}^{\binom{4}{2}} / \operatorname{Im} \Phi_4 \cong \mathbb{R}^6/\mathbb{R}^4 \cong \mathbb{R}^2$, and the lattice $\Lambda_4 \subset Q_4$ is spanned by the three vectors $v_{\{1,2\}} = v_{\{3,4\}}$, $v_{\{1,3\}} = v_{\{2,4\}}$, and $v_{\{1,4\}} = v_{\{2,3\}}$. The space $\mathcal{M}_{0,4}$ is one-dimensional and has three facets, each spanned by one of the three vectors above.

Using the fan structure of the space $\mathcal{M}_{0,n}$, we show that the forgetful maps introduced by A. Gathmann and H. Markwig [GM08, definition 4.1] are morphism of tropical fans.

Definition 1.2.9 (Forgetful maps)
Let $n \ge 4$ be an integer. We have a *forgetful map* ft from $\mathcal{M}_{0,n}$ to $\mathcal{M}_{0,n-1}$ which assigns to an n-marked curve $(C, x_1, \dots, x_n)$ the (stabilization of the) $(n-1)$-marked curve $(C, x_1, \dots, x_{n-1})$.

Proposition 1.2.10
With the tropical fan structure of theorem 1.2.6 the forgetful map $\operatorname{ft} : \mathcal{M}_{0,n} \to \mathcal{M}_{0,n-1}$ *is a morphism of fans in the sense of definition 1.1.6.*

Proof. Let $\operatorname{pr} : \mathbb{R}^{\binom{n}{2}} \to \mathbb{R}^{\binom{n-1}{2}}$ denote the projection to those coordinates $T \in \mathcal{T}$ with $n \notin T$. As $\operatorname{pr}(\operatorname{Im}(\Phi_n)) = \operatorname{Im}(\Phi_{n-1})$, the map pr induces a linear map $\widetilde{\operatorname{pr}} : Q_n \to Q_{n-1}$.

We claim that $\widetilde{\operatorname{pr}}_{|\varphi_n(M_{0,n})}$ is the map induced by ft, hence we have to show the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{M}_{0,n} & \xrightarrow{\operatorname{ft}} & \mathcal{M}_{0,n-1} \\ \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n-1}} \\ Q_n & \xrightarrow{\widetilde{\operatorname{pr}}} & Q_{n-1}. \end{array}$$

So let $C = (\Gamma, x_1, \dots, x_n) \in \mathcal{M}_{0,n}$ be an abstract n-marked rational tropical curve. If x_n is not adjacent to exactly one non-leaf and one leaf, then forgetting x_n does not change any of the distances $\operatorname{dist}_C(x_i, x_j)$ for $i, j \neq n$; so in this case we are done.

Now assume that x_n is adjacent to one leaf x_k and one non-leaf E of length $l(E)$. Then the distances between marked leaves are given by

$$\mathrm{dist}_{\mathrm{ft}(C)}(x_i, x_j) = \begin{cases} \mathrm{dist}_C(x_i, x_j) - l(E) & \text{if } j = k \\ \mathrm{dist}_C(x_i, x_j) & \text{if } j \neq k, \end{cases}$$

hence $\widetilde{\mathrm{pr}}(\varphi_n(C)) = \varphi_{n-1}(\mathrm{ft}(C)) + \Phi_{n-1}(l(E) \cdot e_k)$, where e_k denotes the k-th standard unit vector in $\mathbb{R}^{n-1}$. We conclude that $\widetilde{\mathrm{pr}} \circ \varphi_n = \varphi_{n-1} \circ \mathrm{ft}$. Hence the diagram is commutative.

It remains to check that $\widetilde{\mathrm{pr}}(\Lambda_n) \subset \Lambda_{n-1}$. It suffices to check this on the generators v_I; and we can assume $n \in I$ since $v_I = v_{[n]\setminus I}$. We get

$$\widetilde{\mathrm{pr}}(v_I) = \begin{cases} v_{I \setminus \{n\}} & \text{, if } |I| \geq 3 \\ 0 & \text{, if } |I| = 2. \end{cases}$$

□

Remark 1.2.11
In example 1.2.8, we observed the following three properties of the set

$$V_1 = \{v_I \in Q_4 \ : \ 1 \notin I \text{ and } |I| = 2\} = \{v_{\{3,4\}}, v_{\{2,4\}}, v_{\{2,3\}}\} \subset Q_4$$

(a) The linear span of V_1 equals the space Q_4.

(b) The sum of the elements of V_1 equals 0.

(c) Every vector $w \in Q_4$ has a unique representation of the form

$$w = \sum_{v_I \in V_1} \lambda_I v_I$$

with $\lambda_I \geq 0$ for every $v_I \in V_1$ and $\lambda_I = 0$ for at least one vector $v_I \in V_1$.

An important step in the computation of intersection products of Psi-classes in the next section is the generalization of these observations about the set $V_1 \subset Q_4$ to the set

$$V_k = \{v_I \in Q_n \ : \ k \notin I \text{ and } |I| = 2\} \subset Q_n,$$

which is the content of the following lemmas.

Lemma 1.2.12
The linear span of the set $V_k \subset Q_n$ equals $Q_n = \mathbb{R}^{\binom{n}{2}} / \mathrm{Im}(\Phi_n)$.

Proof. We prove that for $S = \{s_1, s_2\} \in \mathcal{T}$, the S-th standard unit vector $e_S \in \mathbb{R}^{\binom{n}{2}}$ is the sum of a linear combination of elements in V_k and an element of $\mathrm{Im}(\Phi_n)$.

First assume that $k \notin S = \{s_1, s_2\}$. Then it follows immediately from the definitions of v_S and Φ_n that $e_S = (-v_S + \Phi_n(e_{s_1} + e_{s_2}))/2$, where e_{s_i} denotes the s_i-th unit vector in $\mathbb{R}^n$.

Now assume that $S = \{s_1, k\}$. We claim that

$$e_S = \frac{1}{2}\left(\left(\sum_{I \in \mathcal{T}\,:\, I \cap S = \{s_1\}} v_I\right) - \Phi_n(a)\right), \tag{1.1}$$

where $a \in \mathbb{R}^n$ is the vector with entries

$$a_i := \begin{cases} n-4, & \text{if } i = s_1 \\ 0, & \text{if } i = k \\ 1 & \text{otherwise} \end{cases}$$

We check this equality in each component $T = \{t_1, t_2\}$ where the entry is equal to one if and only if $|I \cap T| = 1$ — note that $s_1 \in I$.

If $S \cap T = \emptyset$, then $\Phi_n(a)_T = 2$ and we have $|I \cap T| = 1$ iff I contains one element of T.
If $S \cap T = \{k\}$, then $\Phi_n(a)_T = 1$ and we have $|I \cap T| = 1$ iff $I = \{s_1\} \cup T \setminus \{k\}$.
If $S \cap T = \{s_1\}$, then $\Phi_n(a)_T = n - 3$ and we have $|I \cap T| = 1$ iff $I \neq T$.
If $S \cap T = \{s_1, k\}$, then $\Phi_n(a)_T = n - 4$ and we have $|I \cap T| = 1$ for all choices of I.
It follows that for $T \neq S$, we have

$$\left(\sum_{I \in \mathcal{T}\,:\, I \cap S = \{s_1\}} v_I - \Phi_n(a)\right)_T = 0$$

and for $T = S$, we have

$$\left(\sum_{I \in \mathcal{T}\,:\, I \cap S = \{s_1\}} v_I - \Phi_n(a)\right)_T = n - 2 - (n-4) = 2,$$

hence equation (1.1) holds. □

Lemma 1.2.13
The sum over all elements $v_S \in V_k$ is an element in $\mathrm{Im}(\Phi_n)$, hence

$$\sum_{v_S \in V_k} v_S = 0 \in Q_n.$$

Proof. If $k \notin T$, then there are $\binom{n-3}{1}\binom{2}{1}$ two-element subsets $S \in [n] \setminus \{k\}$ with the property that $|S \cap T| = 1$, hence $(\sum v_s)_T = 2(n-3)$ in this case. If $k \in T$, then the number of subsets $S \in [n] \setminus \{k\}$ satisfying $|S \cap T| = 1$ equals $n-2$, hence $(\sum v_s)_T = n-2$ in this case. It follows that

$$\sum_{v_S \in V_k} v_S = \Phi_n(n-3, \ldots, n-3, 1, n-3, \ldots, n-3)$$

with entry 1 at position k. □

Remark 1.2.14
As $|V_k| = \binom{n-1}{2} = \dim(Q_n) + 1$, every element $w \in Q_n$ has a unique representation

$$w = \sum_{v_S \in V_k} \lambda_S v_S$$

with $\lambda_S \geq 0$ for all S and $\lambda_S = 0$ for at least one $S \in \mathcal{T}$ which will be called a *positive representation of w with respect to V_k*. Given a representation

$$w = \sum_{v_S \in V_k} \lambda_S v_S$$

with $\lambda_S > 0$ for all S, the unique positive representation with respect to V_k is obtained by subtracting $\sum_{v_S \in V_k} (\min \lambda_S) v_S$.

Lemma 1.2.15
Let $I \subset [n]$ with $1 < |I| < n-1$ and assume without restriction that $k \notin I$. Then a positive representation of $v_I \in Q_n$ with respect to V_k is given by

$$v_I = \sum_{S \subset I, v_S \in V_k} v_S.$$

Proof. We claim that

$$v_I = \left(\sum_{S \subset I, v_S \in V_k} v_S \right) - (|I| - 2) \cdot \Phi_n \left(\sum_{i \in I} e_i \right)$$

and we check this equality in each component $T = \{t_1, t_2\}$. If $T \subset I$, then $(v_I)_T = 0$. There are $|I| - 2$ choices for v_S such that S contains t_1 and not t_2, and the same number of choices such that S contains t_2 and not t_1. Hence the first sum of the right hand side contributes $2(|I| - 2)$. As

$$\Phi_n \left(\sum_{i \in I} e_i \right)_T = 2$$

we get 0 altogether. If $|T \cap I| = 1$, then $(v_I)_T = 1$. On the right hand side, there are $|I| - 1$ choices of S such that S contains $T \cap I$, and

$$\Phi_n \left(\sum_{i \in I} e_i \right)_T = 1.$$

If $T \cap I = \emptyset$, both sides are equal to 0. □

1.3 Intersection products of tropical Psi-classes

Following G.Mikhalkin ([Mik], definition 3.1), for $k \in [n]$ the tropical Psi-class $\Psi_k \subset \mathcal{M}_{0,n}$ is the polyhedral complex whose cells are the closures of the $(n-4)$-dimensional cones in $\mathcal{M}_{0,n}$ corresponding to curves with the property that the leaf with label x_k is adjacent to a vertex of valence four and where the weight of each cell is equal to one. In order to apply the tropical intersection theory developed by L. Allermann and J. Rau to compute intersection products of tropical Psi-classes, we represent (a multiple of) Ψ_k as defined above as the Weil-divisor associated to a convex piecewise-linear function f_k.

Definition 1.3.1
Let $k \in [n]$ and let $w \in Q_n$. The map f_k is the piecewise-linear convex function given as the extension of the map $V_k \ni v_S \mapsto 1$ to the space Q_n, i.e. given by

$$f_k(w) := \sum \lambda_S f_k(v_S),$$

where $w = \sum \lambda_S v_S$ is a positive representation of w with respect to V_k.

Remark 1.3.2
The definition above can be rephrased as

$$f(w) = \min \sum \lambda_S, \ \text{ such that } \lambda_S \geq 0 \ \text{ and } w = \sum \lambda_S v_S,$$

observing that this minimum is attained if and only if there is a $v_S \in V_k$ such that $\lambda_S = 0$. It follows that for $w = \sum \lambda_S v_S \in Q_n$ not necessarily given in standard representation, we can compute $f(w)$ to be

$$f(w) = \sum \lambda_S - (\min \lambda_S \ \cdot \ |V_k|) = \sum \lambda_S - \left(\min \lambda_S \ \cdot \ \binom{n-1}{2} \right).$$

Lemma 1.3.3
The map f_k is linear on each cone of $\mathcal{M}_{0,n}$.

Proof. Let τ be a cone and C the tropical curve of the combinatorial type corresponding to τ with all lengths equal to one. The cone τ is generated by vectors v_I corresponding to trees T_I (see construction 1.2.5) where all but one bounded edge of C are contracted. Assume τ is generated by $v_{I_1}, \ldots, v_{I_r}$, and let $k \notin I_i$ for all i. Then each point p in τ is given by a linear combination $p = \sum_{i=1}^{r} \mu_i v_{I_i}$, with $\mu_i \in \mathbb{R}_{\geq 0}$. By lemma 1.2.15, we have a representation

$$v_{I_i} = \sum_{v_S \in V_k} \lambda_{i,S} v_S,$$

where each $\lambda_{i,S}$ is either 1 or 0, depending on whether $S \subset I_i$ or not.

We claim that

$$p = \sum_{v_S \in V_k} \big(\sum_{i=1}^{r} \mu_i \lambda_{i,S} \big) v_S \tag{1.2}$$

is a positive representation of p with respect to V_k. As all the numbers $\sum_{i=1}^{r} \mu_i \lambda_{i,S}$ are non-negative, it remains to show that there is at least one set S such that $\sum_{i=1}^{r} \mu_i \lambda_{i,S} = 0$.

Let $C \setminus \{\overline{x_k}\}$ be the graph obtained from C by removing the closure of the leaf x_k, and let x_a, x_b be leaves in different connected components of the graph $C \setminus \{\overline{x_k}\}$. Then the set $\{a, b\}$ is not contained in any of the sets I_i by construction. Hence $\lambda_{i,\{a,b\}} = 0$ for all i and it follows that $\sum_{i=1}^{r} \mu_i \lambda_{i,\{a,b\}} = 0$. We conclude that equation (1.2) is a positive representation and it follows that

$$f_k(p) = f_k\left(\sum_{v_S \in V_k} \Big(\sum_{i=1}^{r} \mu_i \lambda_{i,S}\Big) v_S\right) = \sum_{v_S \in V_k} \Big(\sum_{i=1}^{r} \mu_i \lambda_{i,S}\Big) f_k(v_S),$$

hence f_k is linear on τ. □

Remark 1.3.4
Let Z be a cycle in $\mathcal{M}_{0,n}$ and let τ be a ridge of Z. Let σ_i be the facets of Z containing τ, let $\omega(\sigma_i)$ be the corresponding weights and let $v_{\sigma_i/\tau} = v_{I_i}$ denote the corresponding normal vectors, where we assume without loss of generality that $k \notin I_i$.

Using lemma 1.2.15 and the construction of tropical intersection products due to L. Allermann and J. Rau, the weight of τ in $\operatorname{div}(f_k) \cdot Z$ equals

$$\begin{aligned}
\omega(\tau) &= \left(\sum_i f_k(\omega(\sigma_i) u_{\sigma_i/\tau})\right) - f_k\left(\sum_i \omega(\sigma_i) u_{\sigma_i/\tau}\right) \\
&= \min_{v_T \in V_k} \left(\sum_i \omega(\sigma_i) \lambda_{i,T}\right) \cdot \binom{n-1}{2} \\
&= \min_{v_T \in V_k} \left(\sum_{i:\, T \subset I_i} \omega(\sigma_i)\right) \cdot \binom{n-1}{2}
\end{aligned}$$

where $\lambda_{i,T}$ equals 1 or 0 depending on whether $T \subset I_i$.

Proposition 1.3.5
The divisor of f_k in $\mathcal{M}_{0,n}$ is

$$\operatorname{div}(f_k) = \binom{n-1}{2} \Psi_k.$$

Proof. We compute the numbers $\omega(\tau)$ for all ridges τ. Using the notation of the proof of theorem 1.2.6, let V be the (unique) vertex of valence val(V)=4, let $E_1, \ldots, E_4$ denote the edges adjacent to V, let $I_k \subset [n]$ denote the set of all $i \in [n]$ such that x_i lies in the connected component of $C \setminus \{V\}$ containing E_k and let $\sigma_1, \sigma_2, \sigma_3$ denote the facets containing τ. Introducing $A_1 = I_3 \cup I_4, A_2 = I_2 \cup I_4, A_3 = I_2 \cup I_3$, we have $v_{\sigma_i/\tau} = v_{A_i}$.

Using remark 1.3.4 and the fact that $\omega(\sigma_i) = 1$ by definition of $\mathcal{M}_{0,n}$, we have to compute

$$\begin{aligned}
\omega(\tau) &= \sum_{i=1}^{3} f_k\left(v_{\sigma_i/\tau}\right) - f_k\left(\sum_{i=1}^{3} v_{\sigma_i/\tau}\right) \\
&= \min_{v_T \in V_k} \left(\sum_{i:\, T \subset A_i} \omega(\sigma_i)\right) \cdot \binom{n-1}{2} \\
&= \min_{v_T \in V_k} |\{i : T \subset A_i\}| \cdot \binom{n-1}{2}.
\end{aligned}$$

Assume without loss of generality that $k = 1$ and $1 \in I_1$. If x_1 is not adjacent to V, then the set $I_1 \setminus \{1\}$ is non-empty, hence contains an element s. As s is not contained in any set A_i, we conclude that $\omega(\tau) = 0$. If x_1 is adjacent to V, then $I_1 = \{1\}$ and every subset $T \in \mathcal{T}$ with $1 \notin T$ is contained in at least one of the sets A_i. On the other hand, choosing $i_2 \in I_2$ and $i_3 \in I_3$, the set $\{i_2, i_3\}$ is contained in exactly one set A_i, namely A_3. It follows that $\omega(\tau) = \binom{n-1}{2}$ in this case. □

In the rest of this section, given integers $k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}$ and a subset $I \subset [n]$, we denote $K(I) = \sum_{i \in I} k_i$. Using this notation, the main theorem of this section reads as follows:

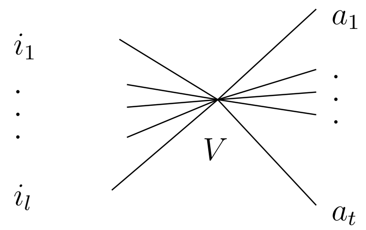
Theorem 1.3.6
The intersection $\Psi_1^{k_1} \cdot \ldots \cdot \Psi_n^{k_n}$ is the subfan of $\mathcal{M}_{0,n}$ consisting of the closure of the cones of dimension $n - 3 - K([n])$ corresponding to abstract tropical curves C such that for all vertices V of C the equality $\operatorname{val}(V) = K(I_V) + 3$ holds, where I_V denotes the set

$$I_V = \{i \in [n] \ : \ \textit{leaf } x_i \textit{ is adjacent to } V \textit{ and } k_i \geq 1\} \subset [n].$$

The weight of the facet $\sigma(C)$ containing the point $\varphi_n(C)$ equals

$$\omega(\sigma(C)) = \frac{\prod_{V \in V(C)} K(I_V)!}{\prod_{i=1}^{n} k_i!}$$

Proof. We prove the theorem by induction on $K([n])$, that is, we compute the weight $\omega(\tau)$ of a codimension one cell $\tau \subset \prod_{i=1}^n \psi_i^{k_i}$ in the intersection product $\psi_1 \prod_{i=1}^n \psi_i^{k_i}$. Let τ be a ridge of $\prod_{i=1}^n \psi_i^{k_i}$ and let C be a curve parametrized by τ. As τ is of codimension one, there is by construction exactly one vertex V of C of valence one higher than expected, i.e. $\operatorname{val}(V) = K(I_V) + 4$ — apart from the leaves x_i with $i \in I_V = \{i_1, \ldots, i_l\}$ there are exactly $K(I_V) + 4 - |I_V|$ edges $a_1, \ldots a_t$ adjacent to V as indicated in the picture below.



Using remark 1.3.4, we compute

$$\omega(\tau) = \min_{v_T \in V_k} \left(\sum_{i:\, T \subset A_i} \omega(\sigma_i) \right)$$

where σ_i denote the facets containing τ and $v_{\sigma_i/\tau} = v_{A_i}$ denote their normal vectors.

Facets σ containing the ridge τ parametrize combinatorial curves C' such that C is obtained from C' by collapsing an edge E with vertices V_1 and V_2 to the vertex V. By assumption, we know that the weight of such a facet equals

$$\omega(\sigma_i) = \frac{\prod K(I_{V'})!}{\prod k_i!} \cdot K(I_{V_1})!K(I_{V_2})! =: W \cdot K(I_{V_1})!K(I_{V_2})!,$$

where the first product goes over all vertices V' of C' different from V_1 and V_2. The normal vector of such a cone σ_i is by definition given by $v_{\sigma_i/\tau} = v_{A_i}$, where A_i is the set of labels of one connected component of $V \setminus \{E\}$. Assume without loss of generality that x_1 is adjacent to V_1 and let $A_i \subset [n]$ denote the subset of labels of the connected component of $V \setminus \{E\}$ not containing x_1.

Assume first that x_1 is not adjacent to V. Then there exists a leaf $x_s \neq x_1$ in the connected component of $C \setminus \{V\}$ containing x_1, hence the label s is not contained in any set A_i. Consequently, $\omega(\tau) = 0$.

Assume that x_1 is adjacent to V, let $s_1, s_2 \in [n] \setminus \{1\}$ be labels in different connected components of $C \setminus \{V\}$ and let $T = \{s_1, s_2\}$ (choosing s_1, s_2 to lie in the same connected component would imply that T is contained in (m)any subsets A_i). It suffices to prove that

$$\sum_{T \subset A_i} \omega(\sigma_i) = \frac{\prod K(I_{V'})!}{\prod k_i!} \cdot \frac{(K(I_V)+1)!}{k_1+1} = W \cdot \frac{(K(I_V)+1)!}{k_1+1}$$

where the first product goes over all vertices V' of C different from V.

Note that a cell σ satisfying the conditions above corresponds to a partition $J_{V_1} \cup J_{V_2} = I_V \setminus \{1, s_1, s_2\} =: M$ and a distribution of the labels a_i among the leaves not labelled by some x_i, $i \in M \cup \{1, s_1, s_2\}$. The total number of labels a_i equals $K(I_V) + 4 - |M|$ and there are exactly $K(J_{V_1}) + k_1 + 1 - |J_{V_1}|$ non-labelled edges at vertex V_1. Hence we get

$$\begin{aligned} \omega(\tau) &= W \cdot \sum_{J_{V_1} \cup J_{V_2} = M} (K(J_{V_1}) + k_1)!(K(J_{V_2}) + k_{s_1} + k_{s_2})! \binom{K(I_V) + 1 - |M|}{K(J_{V_1}) + k_1 + 1 - |J_{V_1}|} \\ &= W \cdot \frac{(K(I_V)+1)!}{k_1+1} \end{aligned}$$

where the last equality follows from the combinatorial identity

$$\sum_{I\subset[m]} (K(I)+a)!(K-(K(I)+a))!\binom{K+1-m}{K(I)+a+1-|I|} = \frac{(K+1)!}{a+1}$$

setting $M=[m], I=J_{V_1}, K=K(I_V)$ and $a=k_1$. □

Corollary 1.3.7
If $K([n])=n-3$, then $\Psi_1^{k_1}\cdot\ldots\cdot\Psi_n^{k_n}=\{0\}$ with weight

$$\omega(\{0\})=\frac{(n-3)!}{\prod_{i=1}^n k_i!}=\binom{n-3}{k_1,\ldots,k_n}.$$

Remark 1.3.8
For $n\geq 3$, let $\overline{M}_{0,n}$ denote the space of n-pointed stable rational curves, that is, tuples $(C,p_1,\ldots,p_n)$ consisting of a connected algebraic curve C of arithmetic genus 0 with simple nodes as only singularities and a collection $p_1,\ldots,p_n$ of distinct smooth points on C such that the number of automorphisms of C with the property that the points p_i are fixed is finite.

Define the line bundle $\mathcal{L}_i$ on $\overline{M}_{0,n}$ to be the unique line bundle whose fiber over each pointed stable curve $(C,p_1,\ldots,p_n)$ is the cotangent space of C at p_i and let $\Psi_i\in A^1(\overline{M}_{0,n})$ denote its first Chern class. For $1\leq i\leq n$, let $k_i\in\mathbb{Z}_{\geq 0}$ such that $\sum_{i=1}^n k_i=\dim(\overline{M}_{0,n})=n-3$. Then the following equation holds (see e.g. [HM98], section 2.D)

$$\int_{\overline{M}_{0,n}}\Psi_1^{k_1}\Psi_2^{k_2}\ldots\Psi_n^{k_n}=\frac{(n-3)!}{\prod_{i=1}^n k_i!}.$$

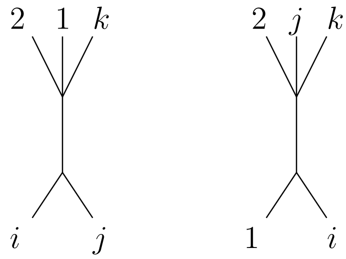
It follows that the 0-dimensional intersection products of Psi-classes on the moduli space of n-marked rational algebraic curves coincides with its tropical counterpart.

Example 1.3.9 (Psi-classes on $\mathcal{M}_{0,5}$)
In order to compute the intersection products Ψ_1^2 and $\Psi_1\cdot\Psi_2$, we compute the value of the function f_1 on all facets of Ψ_1 containing the cell $\{0\}$. These facets correspond by definition to tropical curves with five leaves and the property that the leaf with label x_1 is adjacent to a vertex V of valence $\mathrm{val}(V)=4$.
There are $\binom{4}{2}=6$ of these cells and each cell is generated by a normal vector $v_{\{i,j\}}$, where $i,j\in[n]\setminus\{1\}$ and the sum over all those normal vectors equals 0, the weight of $\{0\}$ is given by

$$\sum_{i,j\in[n]\setminus\{1\},i\neq j} f_1(v_{\{i,j\}})-f_1(0)=6$$

Using proposition 1.3.5, we conclude that the weight of $\{0\}$ in $\Psi_1\cdot\Psi_1$ equals 1.

Now we compute the weight of $\{0\}$ in $f_1\cdot\Psi_2$. Three of the cones containing $\{0\}$ correspond to a curve as on the right, the other three to a curve as on the left:

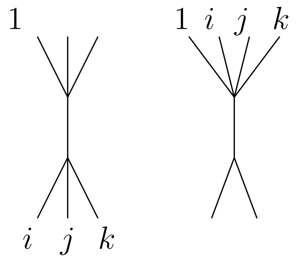


The normal vectors of the first type are $v_{\{i,j\}}$ and they are given as a positive combination with respect to V_1. A positive representation for the normal vectors of the second type is $v_{\{2,j\}} + v_{\{2,k\}} + v_{\{j,k\}}$. Hence the weight of $\{0\}$ is given by

$$\begin{aligned}&f_1(v_{\{3,4\}}) + f_1(v_{\{3,5\}}) + f_1(v_{\{4,5\}})\\ +&f_1(v_{\{2,3\}} + v_{\{2,4\}} + v_{\{3,4\}}) + f_1(v_{\{2,3\}} + v_{\{2,5\}} + v_{\{3,5\}}) + f_1(v_{\{2,4\}} + v_{\{2,5\}} + v_{\{4,5\}}) - f_1(0)\\ =&1 + 1 + 1 + 3 + 3 + 3 = 12.\end{aligned}$$

Thus the weight of $\{0\}$ in $\Psi_1 \cdot \Psi_2$ is 2.

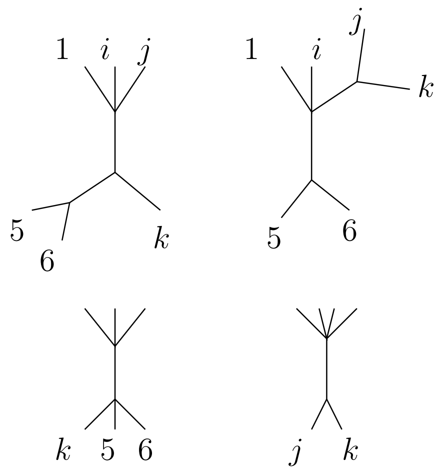
Example 1.3.10 (Psi-classes on $\mathcal{M}_{0,6}$)
In order to compute Ψ_1^2, respectively $f_1 \cdot \Psi_1$, on $\mathcal{M}_{0,6}$ we have to compute the weight of a cone of codimension 1 in Ψ_1. Such a cone corresponds to a tropical curve with either another 4-valent vertex (as on the left) or with a 5-valent vertex, to which 1 is adjacent (as on the right):



A cone corresponding to the curve on the left is contained in three facets of Ψ_1, corresponding to the three possible resolutions of the lower vertex. These three cones are generated by the normal vectors $v_{\{i,j\}}$, $v_{\{i,k\}}$ and $v_{\{j,k\}}$. Thus the weight of such a cone is

$$f_1(v_{\{i,j\}}) + f_1(v_{\{i,k\}}) + f_1(v_{\{j,k\}}) - f_1(v_{\{i,j\}} + v_{\{i,k\}} + v_{\{j,k\}}) = 0,$$

and it does not belong to Ψ_1^2. A cone corresponding to the curve on the right (where we assume now $\{i, j, k\} = \{2, 3, 4\}$ for simplicity) has 6 neighbors in Ψ_1, 3 as on the right and 3 as on the left (below the curve, the corresponding normal vector for the cones are shown):



Hence, the weight of this cone is

$$\begin{aligned}
&f_1(v_{\{2,5\}} + v_{\{2,6\}} + v_{\{5,6\}}) + f_1(v_{\{3,5\}} + v_{\{3,6\}} + v_{\{5,6\}}) + f_1(v_{\{4,5\}} + v_{\{4,6\}} + v_{\{5,6\}}) \\
&+ f_1(v_{\{2,3\}}) + f_1(v_{\{2,4\}}) + f_1(v_{\{3,4\}}) \\
&- f_1(v_{\{2,5\}} + v_{\{2,6\}} + v_{\{5,6\}} + v_{\{3,5\}} + v_{\{3,6\}} + v_{\{5,6\}} \\
&\quad + v_{\{4,5\}} + v_{\{4,6\}} + v_{\{5,6\}} + v_{\{2,3\}} + v_{\{2,4\}} + v_{\{3,4\}}) \\
&= 3 + 3 + 3 + 1 + 1 + 1 - f_1(2v_{\{5,6\}}) = 12 - 2 = 10,
\end{aligned}$$

because the sum over all normal vectors is not automatically written as a positive combination with respect to V_1 and we have to subtract $\sum_{i,j\in[6]\setminus\{1\},i\neq j} v_{\{i,j\}}$ to make a positive combination. By proposition 1.3.5, the weight of each such cone in Ψ_1^2 is one.

1.4 Moduli spaces of rational tropical maps

In this section we consider tropical analogues of the algebro-geometric moduli spaces of (rational) stable maps introduced by A. Gathmann and H. Markwig [GM08]. The points of these spaces correspond to marked abstract rational tropical curves together with a suitable map to $\mathbb{R}^r$. We equip these spaces with the structure of tropical fans using the fan structure of $\mathcal{M}_{0,n}$ developed in section 1.2 to derive enumerative results in section 1.5.

Definition 1.4.1 (Tropical $\mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^r, \Delta)$)
Let $n \geq 3$ be a natural number. A *(parametrized) labelled n-marked rational tropical curve in* $\mathbb{R}^r$ is a tuple $(\Gamma, x_1, \ldots, x_N, h)$ for some $N \geq n$, where $(\Gamma, x_1, \ldots, x_N)$ is an abstract N-marked rational tropical curve and $h : \Gamma \to \mathbb{R}^r$ is a continuous map satisfying:

(a) For each edge $E \in E(\Gamma)$, we have $h_{|E} = a + t \cdot v$ for some $a \in \mathbb{R}^r$ and $v \in \mathbb{Z}^r$. The vector v will be denoted $v(E, V)$ and called the *direction* of E (at the vertex V). If E is a leaf, we denote its direction by $v(E)$.

(b) For every vertex V of Γ the *balancing condition* holds:

$$\sum_{E|V\in\partial E} v(E,V)=0.$$

(c) $v(x_i)=0$ for $i\in[n]$ and $v(x_i)\neq 0$ for $i\notin[n]$.

Two labelled n-marked tropical curves $(\Gamma,x_1,\dots,x_N,h)$ and $(\tilde{\Gamma},\tilde{x}_1,\dots,\tilde{x}_N,\tilde{h})$ in $\mathbb{R}^r$ are called isomorphic (and will from now on be identified) if there is an isomorphism $\varphi:(\Gamma,x_1,\dots,x_N)\to(\tilde{\Gamma},\tilde{x}_1,\dots,\tilde{x}_N)$ of the underlying abstract curves such that $\tilde{h}\circ\varphi=h$.

The *degree* of a labelled n-marked tropical curve is defined to be the $(N-n)$-tuple $\Delta=(v(x_{n+1}),\dots,v(x_N))\in(\mathbb{Z}^r\backslash\{0\})^{N-n}$. Its *combinatorial type* is defined to be the combinatorial type of the underlying abstract marked tropical curve $(\Gamma,x_1,\dots,x_N)$ together with the directions of all edges. For the rest of this section, we let $N=n+|\Delta|$ denote the total number of leaves of an n-marked curve in $\mathbb{R}^r$ of degree Δ.

The space of all labelled n-marked tropical curves of a given degree Δ in $\mathbb{R}^r$ will be denoted $\mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^r,\Delta)$. For the special choice

$$\Delta=(-e_0,\dots,-e_0,\dots,-e_r,\dots,-e_r)$$

with $e_0:=-e_1-\cdots-e_r$ and where each e_i occurs exactly d times this space is denoted by $\mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^r,d)$ and we will refer to its elements as *curves of degree d*.

Construction 1.4.2 (Tropical $\mathcal{M}_{0,n}(\mathbb{R}^r,\Delta)$)
Let $N\geq n\geq 0$, and let $\Delta=(v_{n+1},\dots,v_N)\in(\mathbb{Z}^r\backslash\{0\})^{N-n}$. It follows directly from the definitions that the subgroup G of the symmetric group $\mathbb{S}_{N-n}$ consisting of all permutations σ of the set $[N]\setminus[n]$ such that $v_{\sigma(i)}=v_i$ for all $i\in[m]$ acts on the space $\mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^r,\Delta)$ by relabelling the leaves labelled by x_i with $i\in[N]\setminus[n]$. We denote the quotient space $\mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^r,\Delta)/G$ by $\mathcal{M}_{0,n}(\mathbb{R}^r,\Delta)$ and refer to this space as the space of *(unlabeled) n-marked tropical curves* in $\mathbb{R}^r$ of degree Δ. Its elements can obviously be thought of as n-marked tropical curves in $\mathbb{R}^r$ for which we have only specified how many of its leaves have a given direction, but where no labelling of these leaves is given. Consequently, when considering unlabeled curves we can (and usually will) think of Δ as a multiset $\{v_{n+1},\dots,v_N\}$ instead of as a vector $(v_{n+1},\dots,v_N)$.

Note that this definition of $\mathcal{M}_{0,n}(\mathbb{R}^r,\Delta)$ agrees with the one given by A. Gathmann and H. Markwig in [GM08], and that for curves of degree d in $\mathbb{R}^r$, the group G constructed above equals $(\mathbb{S}_d)^{r+1}$.

Remark 1.4.3
Note that it follows from the remarks above and a result of B. Siebert and T. Nishinou [NS06, proposition 2.11] that the number of combinatorial types in any given moduli space $\mathcal{M}_{0,n}(\mathbb{R}^r,\Delta)$ (and thus also in $\mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^r,\Delta)$) is finite. Furthermore, by results of A. Gathmann and H. Markwig, the subset of curves of a fixed combinatorial type is a cone in a real vector space [GM08, proposition 2.11] and these cones can be glued locally

[GM08, example 2.13]. Hence it follows that the moduli spaces of labelled or unlabeled n-marked tropical curves in $\mathbb{R}^r$ are polyhedral complexes.

Remark 1.4.4
In the following, we will use the tropical fan structure of $\mathcal{M}_{0,n}$ to make the moduli spaces $\mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^r, d)$ of labelled curves in $\mathbb{R}^r$ into tropical fans as well. The idea is the same as in the classical algebro-geometric case: the non-contracted leaves of the tropical curves in $\mathbb{R}^r$ can be thought of as their intersection points with the $r+1$ coordinate hyperplanes of $\mathbb{P}^r$; so passing from unlabeled to labelled curves corresponds to labelling these intersection points. The moduli spaces of stable maps is obtained by taking the quotient by the group of possible permutations of the labels. Note that this makes the spaces of stable maps into stacks instead of varieties (as the group action is in general not free). Similarly, our moduli spaces of unlabeled tropical curves in $\mathbb{R}^r$ can only be thought of as "tropical stacks" instead of tropical varieties. To avoid this notion we will work with labelled tropical curves in this paper. This is no problem for enumerative purposes since we can count labelled curves and divide the result by $|G|$.

Let ψ denote the forgetful map

$$\begin{aligned} \psi: \quad \mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^r, \Delta) &\to \mathcal{M}_{0,n+|\Delta|} \\ (\Gamma, x_1, \ldots, x_N, h) &\mapsto (\Gamma, x_1, \ldots, x_N) \end{aligned}$$

that forgets the map to $\mathbb{R}^r$.

Lemma 1.4.5
The forgetful map ψ induces a bijection of combinatorial types of the two moduli spaces $\mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^r, \Delta)$ and $\mathcal{M}_{0,N}$ with $N = n + |\Delta|$.

Proof. It is clear that ψ induces a well-defined map between combinatorial types of labelled and abstract tropical curves. By definition the only additional data in the combinatorial type of a labelled curve compared to the underlying abstract curve is the directions of the edges. As the directions of the leaves are fixed by Δ it therefore suffices to show that for a given (combinatorial type of a) graph Γ and fixed directions of the leaf there is a unique choice of directions for the bounded edges compatible with the balancing condition of definition 1.4.1. But this is clear, as the direction of a bounded edge E from vertex V_1 to vertex V_2 is given by sum over all directions of leaves in the connected component of $\Gamma \setminus \{E\}$ containing the vertex V_2. □

For the rest of this paper we will assume for simplicity that $n > 0$, i.e. that there is at least one contracted leaf.

Proposition 1.4.6
With notations as above, the map

$$\begin{aligned} \mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^r, \Delta) &\to Q_N \times \mathbb{R}^r \\ C &\mapsto (\varphi_N(\psi(C)), \text{ev}_1(C)) \end{aligned}$$

is an embedding whose image is the tropical fan $\varphi_N(\mathcal{M}_{0,N}) \times \mathbb{R}^r$*. So* $\mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^r, \Delta)$ *can be thought of as a tropical fan of dimension* $r + N - 3 = r + n + |\Delta| - 3$*, namely as the fan* $\mathcal{M}_{0,n+|\Delta|} \times \mathbb{R}^r$*.*

Proof. It is clear that the given map is a continuous map of polyhedral complexes. By lemma 1.4.5 it suffices to check injectivity and compute its image for a fixed combinatorial type α. By an argument of A. Gathmann and H. Markwig [GM08, proposition 2.11] the cell of $\mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^r, \Delta)$ corresponding to curves of type α is given by $\mathbb{R}^k_{>0} \times \mathbb{R}^r$, where the first k positive coordinates are the lengths of the bounded edges, and the last r coordinates are the position of a root vertex (that we choose to be x_1 here). The map of the proposition obviously sends this cell bijectively to the product of the corresponding cell of $\varphi_N(\mathcal{M}_{0,N})$ and $\mathbb{R}^r$. □

Definition 1.4.7 (Evaluation map)
For $i \in [n]$, we define the the *i-th evaluation map* to be the map

$$\begin{array}{rcl} \text{ev}_i : \mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^r, \Delta) & \longrightarrow & \mathbb{R}^r \\ (\Gamma, x_1, \ldots x_N, h) & \longmapsto & h(x_i). \end{array}$$

Proposition 1.4.8
With the tropical fan structure of proposition 1.4.6 the evaluation maps ev_i *are morphisms of fans (in the sense of definition 1.1.6).*

Proof. As usual let $N = n + |\Delta|$, and identify $\mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^r, \Delta)$ with the space $\mathcal{M}_{0,N} \times \mathbb{R}^r$ as in proposition 1.4.6. For all $1 \le i \le n$ consider the linear map

$$\begin{array}{rcl} \text{ev}'_i : \mathbb{R}^{\binom{N}{2}} \times \mathbb{R}^r & \longrightarrow & \mathbb{R}^r \\ (a_{\{1,2\}}, \ldots, a_{\{N-1,N\}}, b) & \longmapsto & b + \frac{1}{2} \sum_{\substack{k=2 \\ k \neq i}}^{N} \left(a_{\{1,k\}} - a_{\{i,k\}}\right) v_k, \end{array}$$

where v_k is the direction of leaf x_k i.e. $v_1 = \cdots = v_n = 0$ and $(v_{n+1}, \ldots, v_N) = \Delta$. As $\text{ev}'_i(\text{Im}(\Phi_N) \times \{0\}) = \{0\}$, the map ev'_i induces a linear map $\widetilde{\text{ev}'_i} : Q_N \times \mathbb{R}^r \to \mathbb{R}^r$.

We claim that $\widetilde{\text{ev}'_i}|_{\mathcal{M}_{0,N} \times \mathbb{R}^r}$ is the map induced by ev_i, hence we have to show the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{M}_{0,N} \times \mathbb{R}^r & \xrightarrow{\text{ev}_i} & \mathbb{R}^r \\ \downarrow{\scriptstyle \varphi_N \times \text{id}} & \nearrow{\scriptstyle \widetilde{\text{ev}'_i}} & \\ Q_N \times \mathbb{R}^r & & \end{array}$$

As the case $i = 1$ is trivial, and using the additivity of the function $\sum_{k=1}^{N} v_k \,(a_{1k} - a_{ik})$, we may assume that there exists only one edge E, and that this edge separates x_1 and x_i. Let V be the vertex adjacent to x_1, and let $I_1 \subset [N]$ be the set of labels of leaves in the connected component of $\Gamma \setminus \{E\}$ containing x_1 (so that $1 \in I_1$ and $i \notin I_1$). Since by the balancing condition of definition 1.4.1 the equation $v(E,V) = \sum_{k \notin I_1} v_k = -\sum_{k \in I_1} v_k$ holds, we get

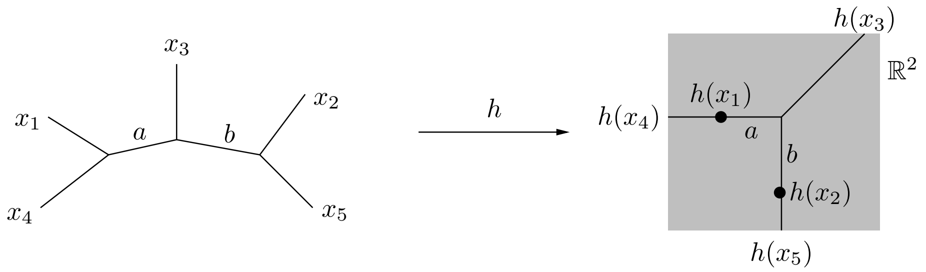
$$\begin{aligned}\left(\widetilde{\mathrm{ev}}'_i \circ (\varphi_N \times id)\right)(\Gamma, x_1, \ldots, x_N, h(x_1)) &= h(x_1) + \frac{1}{2}\left[-\sum_{k \in I_1} l(E) v_k + \sum_{k \notin I_1} l(E) v_k\right] \\ &= h(x_1) + l(E)\, v(E,V) \\ &= h(x_i)\end{aligned}$$

as required (where $l(E)$ denotes the length of E). It remains to check that $\widetilde{\mathrm{ev}}'_i(\Lambda_N \times \mathbb{Z}^r) \subset \mathbb{Z}^r$. It suffices to check that $\widetilde{\mathrm{ev}}'_i(v_I, b) \in \mathbb{Z}^r$ for the generators v_I of Λ_N (see construction 1.2.5) and all $b \in \mathbb{Z}^r$. Assuming $1 \in I$, we get

$$\widetilde{\mathrm{ev}}'_i(v_I, b) = \begin{cases} b - \sum_{k \in I} v_k & \text{, if } i \notin I \\ b & \text{, if } i \in I, \end{cases}$$

which finishes the proof as $v_k \in \mathbb{Z}^r$. □

Example 1.4.9
As an example of the calculation in the above proof we consider the space $\mathcal{M}^{\mathrm{lab}}_{0,2}(\mathbb{R}^2, 1) = \mathcal{M}_{0,5} \times \mathbb{R}^2$, so that the curves have $N = 5$ leaves with directions $v_1 = v_2 = 0$, $v_3 = (1,1)$, $v_4 = (-1,0)$, $v_5 = (0,-1)$. We consider curves of the combinatorial type drawn in the following picture:



We can read off immediately from the picture that $\mathrm{ev}_2(C) = h(x_1) + a(1,0) + b(0,-1)$.

On the other hand, we compute using the formula of the proof of proposition 1.4.8:

$$\begin{aligned}&\left(\widetilde{\mathrm{ev}}'_2 \circ (\varphi_N \times id)\right)(C, x_1, \ldots, x_5, h(x_1)) \\ &\quad = \widetilde{\mathrm{ev}}'_2(\mathrm{dist}_C(x_1,x_2), \ldots, \mathrm{dist}_C(x_4,x_5), h(x_1)) \\ &\quad = h(x_1) + \frac{1}{2}\sum_{k=3}^{5} (\mathrm{dist}_C(x_1,x_k) - \mathrm{dist}_C(x_i,x_k))\, v_k \\ &\quad = h(x_1) + \frac{1}{2}\left[(a-b)v_3 + (0-(a+b))v_4 + ((a+b)-0)v_5\right] \\ &\quad = h(x_1) + \frac{1}{2}\left[a(v_3 - v_4 + v_5) + b(-v_3 - v_4 + v_5)\right] \\ &\quad = h(x_1) + a(1,0) + b(0,-1).\end{aligned}$$

Remark 1.4.10
As in section 1.2, one can define forgetful maps on the moduli space of labelled tropical curves in $\mathbb{R}^r$ as well. We can forget certain contracted leaves, but also we can forget all non-contracted leaves and the $\mathbb{R}^r$ factor in $\mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^r, \Delta) = \mathcal{M}_{0,N} \times \mathbb{R}^r$, which corresponds to forgetting the map h. The same argument as in proposition 1.2.10 shows that these forgetful maps are morphisms as well.

Remark 1.4.11
In proposition 1.4.6 we have constructed the tropical fan structure on the moduli space $\mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^r, \Delta)$ using the evaluation at the first (contracted) marked leaf. If we use a different contracted leaf x_i with $i \in [n] \setminus \{1\}$ instead, the two embeddings in $Q_N \times \mathbb{R}^r$ only differ by addition of $\text{ev}_i(C) - \text{ev}_1(C)$ in the $\mathbb{R}^r$ factor. As this is an isomorphism by proposition 1.4.8 it follows that the tropical fan structure defined in proposition 1.4.6 is natural in the sense that it does not depend on the choice of contracted leaf.

1.5 Applications to the enumerative geometry of rational curves

In this final section, we apply the results obtained in this chapter to reprove and generalize two statements in tropical enumerative geometry. The first application concerns the number of rational tropical curves in $\mathbb{R}^r$ through given general points.

Theorem 1.5.1 ([GM07b, theorem 4.8])
Let $r \geq 2$, let Δ be a degree of tropical curves in $\mathbb{R}^r$, and let $n \in \mathbb{Z}_{>0}$ be such that $r + n + |\Delta| - 3 = nr$. Then the number of rational tropical curves of degree Δ in $\mathbb{R}^r$ through n points in general position (counted with multiplicities) does not depend on the position of the points.

Proof. Let $\text{ev} := \text{ev}_1 \times \cdots \times \text{ev}_n : \mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^r, \Delta) \to \mathbb{R}^{nr}$. By proposition 1.4.6 the space $\mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^r, \Delta)$ is a tropical fan, and by proposition 1.4.8 the map ev is a morphism of fans (of the same dimension). As $\mathbb{R}^{nr}$ is irreducible, corollary 1.1.15 implies that for general $Q \in \mathbb{R}^{nr}$ the number

$$\sum_{C \in |\mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^r, \Delta)| : \text{ev}(C) = Q} \text{mult}_C \, \text{ev}$$

does not depend on Q. So if we define the tropical multiplicity of a curve C occurring in this sum to be $\text{mult}_C \, \text{ev}$ we conclude that the number of labelled curves of degree Δ in $\mathbb{R}^r$ through the n points in $\mathbb{R}^r$ specified by Q does not depend on the choice of (general) Q. The statement for unlabeled curves now follows by dividing the resulting number by the order $|G|$ of the group G defined in construction 1.4.2. $\square$

Remark 1.5.2
In the proof of theorem 1.5.1 we have defined the tropical multiplicity of a curve to be the multiplicity of the evaluation map as in corollary 1.1.15. By corollary 1.1.19 this multiplicity can be computed on a fixed cell of $\mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^r, \Delta)$ as the absolute value of the determinant of the matrix obtained by expressing the evaluation map in terms of the basis on the source given by the vectors v_I (see construction 1.2.5) that span the given cell. But note that the coordinates on the given cell with respect to this basis are simply the lengths of the bounded edges. It follows that the determinant that we have to consider is the same as the one used by A. Gathmann and H. Markwig [GM08, section 3]. In particular, in case $r = 2$ this notion of multiplicity of curves agrees with the notion of multiplicity introduced by G. Mikhalkin [Mik05, definition 4.15, definition 4.16].

Remark 1.5.3
For simplicity we have formulated theorem 1.5.1 only for the case of counting tropical curves through given points. Of course the very same proof can be used for counting curves through given affine linear subspaces (with rational slopes) if one replaces the evaluation maps ev_i by their compositions with the quotient maps $\mathbb{R}^r \to \mathbb{R}^r/L_i$, where L_i is the linear subspace chosen at the i-th contracted leaf. This setup has been considered e.g. by T. Nishinou and B. Siebert [NS06].

As a second application we consider the map introduced by A. Gathmann and H. Markwig in the proof of the tropical analogue of Kontsevich's recursion formula to compute the number of rational plane curves of fixed degree through given general point conditions [GM08, proposition 4.4].

Theorem 1.5.4
Let $d \geq 1$, and let $n = 3d$. Define

$$\pi := \text{ev}_1^1 \times \text{ev}_2^2 \times \text{ev}_3 \times \cdots \times \text{ev}_n \times \text{ft}_4 : \mathcal{M}_{0,n}(\mathbb{R}^2, d) \to \mathbb{R}^{2n-2} \times \mathcal{M}_{0,4},$$

i.e. π describes the first coordinate of the first marked point, the second coordinate of the second marked point, both coordinates of the other marked points, and the point in $\mathcal{M}_{0,4}$ defined by the first four marked points. Then (with $\text{mult}_\pi(P)$ *is defined in [GM08, definition 3.1.]), the number*

$$\deg_\pi(Q) := \sum_{P \in \pi^{-1}(Q)} \text{mult}_\pi(P)$$

does not depend on a general point Q.

Proof. We define the map

$$\pi' := \text{ev}_1^1 \times \text{ev}_2^2 \times \text{ev}_3 \times \cdots \times \text{ev}_n \times \text{ft}_4 : \mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^2, d) \to \mathbb{R}^{2n-2} \times \mathcal{M}_{0,4}$$

obtained from π by replacing $\mathcal{M}_{0,n}(\mathbb{R}^2, d)$ by $\mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^2, d)$. Then for each inverse image $P \in \mathcal{M}_{0,n}(\mathbb{R}^2, d)$ with $\pi(P) = Q$ there exist $|\mathbb{S}_d^3|$ different $P' \in \mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^2, d)$ with $\pi'(P') = Q$ of the same multiplicity $\text{mult}_{\pi'}(P') = \text{mult}_\pi(P)$ (for the different labellings of the non-marked leaves), hence

$$|\mathbb{S}_d^3| \cdot \deg_\pi(Q) = \deg_{\pi'}(Q).$$

and it suffices to show that $\deg_{\pi'}(Q)$ does not depend on Q. By proposition 1.4.6 the space $\mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^2, d)$ is a tropical fan. As $\mathcal{M}_{0,4}$ is a irreducible tropical fan by example 1.2.8, and $\mathbb{R}^{2n-2}$ is an irreducible tropical fan (consisting of just one cone), too, we can conclude using proposition 1.1.12 that $Y := \mathbb{R}^{2n-2} \times \mathcal{M}_{0,4}$ is an irreducible tropical fan as well. Obviously, the source and target of π' are of the same dimension. As π' is a morphism of fans by propositions 1.2.10 and 1.4.8 we can apply corollary 1.1.15 and conclude that

$$\sum_{P \in |\mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^2, d)| : \pi'(P) = Q} \text{mult}_P \, \pi'$$

does not depend on Q. So it remains to show that $\text{mult}_P \, \pi' = \text{mult}_{\pi'}(P)$. But this follows from corollary 1.1.19 in the same way as in remark 1.5.2. □

Remark 1.5.5
In the same way as theorem 1.5.1, theorem 1.5.4 can be generalized immediately to the case of rational curves in $\mathbb{R}^r$ of arbitrary degree Δ and with various linear subspaces as conditions.

2 Counting tropical elliptic plane curves with fixed j-invariant

Introduction

In classical algebraic geometry, the problem of enumerating complex plane elliptic curves of fixed degree d and with fixed j-invariant interpolating $3d-1$ general points has been solved by R. Pandharipande [Pan97], who computed this number, using Gromov-Witten theory, to be

$$E(d,j) = \binom{d-1}{2} N_d$$

for $j \notin \{0, 1728\}$, where N_d denotes the number of irreducible rational curves of degree d interpolating $3d-1$ general points. In this chapter, we consider the tropical analogue of this problem, namely we compute the number $E_{\text{trop}}(d)$ of tropical elliptic plane curves of degree d with fixed tropical j-invariant interpolating $3d-1$ general points in the plane.

In order to compute these numbers $E_{\text{trop}}(d)$, we start in section 2.1 by introducing the concept of weighted polyhedral complexes (i.e. a polyhedral complex together with (rational) weights) and morphisms thereof. In section 2.2, we define the appropriate moduli space of (relevant) elliptic curves and equip this space with suitable weights.

In section 2.3, we define the tropical j-invariant and derive theorem 2.3.9, which states that the number $E_{\text{trop}}(d)$ of tropical elliptic plane curves of degree d with fixed tropical j-invariant interpolating $3d-1$ general points (counted with multiplicities) does depend neither on the choice of j nor on the choice of the general points. Furthermore, we compute this number to be

$$E_{\text{trop}}(d) = \binom{d-1}{2} N_d = E(d,j).$$

In section 2.4, we apply the results obtained in the previous sections to derive new results concerning the enumerative geometry of tropical rational curves. Namely, we get a new formula to enumerate these curves (see lemma 2.4.1). Combined with Mikhalkin's lattice path algorithm (see [Mik05]), this formula leads to a modified lattice path count for tropical rational curves which has the advantage that fewer paths have to be taken into account. Furthermore, this formula is used to derive an algorithm to compute upper bounds for the maximal absolute number $T_d(\mathcal{P})$ of tropical rational curves of degree d interpolating a configuration $\mathcal{P}$ of $3d-1$ points in general position (where absolute means counted without multiplicities).

2.1 Weighted polyhedral complexes

Definition 2.1.1 ((weighted) polyhedral complexes)
Let $N \in \mathbb{Z}_{>0}$ be a natural number. For $k \in [N]$, let Λ_k be a finitely generated free abelian group and let $\sigma_k \subset V_k := \Lambda_k \otimes \mathbb{R}$ be an open rational polyhedron (that is, the relative interior of a rational polyhedron). We denote by $V_{\sigma_k} \subset V_k$ the smallest affine vector subspace of V_k that contains σ_k. The *dimension* of the open polyhedron σ_k is defined to be $\dim \sigma_k := \dim V_{\sigma_k}$.

A *polyhedral complex* with cells $\sigma_1, \ldots, \sigma_N$ is a topological space X together with continuous inclusion maps $i_k : \overline{\sigma_k} \to X$ such that X is the disjoint union of the sets $i_k(\sigma_k)$ and the "coordinate changing maps" $i_k^{-1} \circ i_l$ are integer affine linear maps (where defined), i.e. a map of the form $f(v) + b$ where f is an integer linear map and b is an element in V_k. We will usually drop the inclusion maps i_k in the notation and say that the cells σ_k are contained in X.

The *dimension* $\dim X$ of a polyhedral complex X is the maximum of the dimensions of its cells. We say that X is *pure* of dimension $\dim X$ if every cell is contained in the closure of a cell of dimension $\dim X$. A point of X is said to be *in general position* if it is contained in a cell of dimension $\dim X$. For a point P in general position, we denote the cell of dimension $\dim X$ in which it is contained by σ_P. The set of all m-dimensional cells of X will be denoted $X^{(m)}$.

A *weighted polyhedral complex* is a pair (X, ω_X) where X is a pure polyhedral complex of dimension n and $\omega_X : X^{(n)} \to \mathbb{Q}_{\geq 0}$ is a map. We call $\omega_X(\sigma)$ the *weight* of the cell $\sigma \in X^{(n)}$ and write it simply as $\omega(\sigma)$ if no confusion can result. Also, by abuse of notation we will write a weighted polyhedral complex (X, ω_X) simply as X if the weight function ω_X is clear from the context.

Definition 2.1.2 (morphisms of (weighted) polyhedral complexes)
A *morphism* of two polyhedral complexes X and Y is a continuous map $f : X \to Y$ such that for each cell $\sigma_k \subset X$ the image $f(\sigma_k)$ is contained in only one cell of Y, and $f|_{\sigma_k}$ is an integer affine linear map. A morphism of weighted polyhedral complexes is a morphism of polyhedral complexes (i.e. there are no conditions on the weights).

Definition 2.1.3 (multiplicity (cf. corollary 1.1.15))
Let $f : X \to Y$ be a morphism of weighted polyhedral complexes of dimension n, and let $P \in X$ be a point such that both P and $f(P)$ are in general position. Then the restriction $f|_{\sigma_P}$ induces an integer affine linear map f_P between the n-dimensional affine

vector spaces V_{σ_P} and $V_{\sigma_{f(P)}}$. The *multiplicity of f at P* is defined to be

$$\text{mult}_f(P) = \frac{\omega_X(\sigma_P)}{\omega_Y(\sigma_{f(P)})} \cdot |\det(f_P)|.$$

As the multiplicity depends only on the cell σ_P, we will refer to this number also as the *multiplicity of f in the cell σ_P*, denoted $\text{mult}_f(\sigma_P)$.

Definition 2.1.4 (f-general position, degree)
A point $Q \in Y$ is said to be *in f-general position* if Q is in general position in Y and all points of $f^{-1}(Q)$ are in general position in X. If $Q \in Y$ is a point in f-general position we define the *degree* of f at Q to be

$$\deg_f(Q) := \sum_{P \in f^{-1}(Q)} \text{mult}_f(P).$$

Note that $\deg_f(Q)$ is a non-negative rational number.

2.2 The space of elliptic curves

In the following, for a graph Γ, we will denote by $E(\Gamma)$ resp. $V(\Gamma)$ its set of edges resp. vertices. An edge of Γ which is not a leaf of Γ will be called a non-leaf; the set of non-leaves of a graph Γ will be denoted $\text{NL}(\Gamma)$.

Definition 2.2.1 (n-marked (elliptic) tropical curves (cf. definition 1.2.1))
Let $n \geq 3$ be a natural number. An *n-marked abstract tropical curve* is a connected metric graph Γ (that is, a graph together with a length function $l : \text{NL}(\Gamma) \to \mathbb{R}_{>0}$ assigning each non-leaf E a positive length $l(E)$) with exactly n leaves, labelled by variables $\{x_1, \ldots, x_n\}$. The first Betti number of the graph Γ will be denoted by $g(\Gamma)$ and referred to as the genus of the tropical curve Γ. If $g(\Gamma) = 0$ (resp. $g(\Gamma) = 1$) then Γ is said to be a rational (resp. elliptic) tropical curve. Note that in the latter case, Γ contains exactly one cycle, which will be denoted by $\text{cycle}(\Gamma)$.

Definition 2.2.2 ((combinatorial types of) plane elliptic curves (cf. definition 1.4.1))
Let $n \geq 3$ be a natural number. A *(parametrized) labelled n-marked tropical curve in $\mathbb{R}^r$* is a tuple $(\Gamma, x_1, \ldots, x_N, h)$ for some $N \geq n$, where $(\Gamma, x_1, \ldots, x_N)$ is an N-marked abstract tropical curve and $h : \Gamma \to \mathbb{R}^r$ is a continuous map satisfying:

(a) For each edge $E \in E(\Gamma)$, we have $h|_E = a + t \cdot v$ for some $a \in \mathbb{R}^r$ and $v \in \mathbb{Z}^r$. The vector v will be denoted $v(E, V)$ and called the *direction* of E (at the vertex V). If E is a leaf, we denote its direction by $v(E)$.

(b) For every vertex V of Γ the *balancing condition* holds:

$$\sum_{E | V \in \partial E} v(E, V) = 0.$$

(c) $v(x_i) = 0$ for $i \in [n]$ and $v(x_i) \neq 0$ for $i \notin [n]$.

Two labelled n-marked tropical curves $(\Gamma, x_1, \ldots, x_N, h)$ and $(\tilde{\Gamma}, \tilde{x}_1, \ldots, \tilde{x}_N, \tilde{h})$ in $\mathbb{R}^r$ are called isomorphic (and will from now on be identified) if there is an isomorphism $\varphi : (\Gamma, x_1, \ldots, x_N) \to (\tilde{\Gamma}, \tilde{x}_1, \ldots, \tilde{x}_N)$ of the underlying abstract curves such that $\tilde{h} \circ \varphi = h$.

A *n-marked plane tropical curve of degree d* is a labelled n-marked tropical curve in $\mathbb{R}^2$ such that $N = n + 3d$ and the equality

$$(v(x_{n+1}), \ldots, v(x_N)) = (-e_0, \ldots, -e_0, e_1, \ldots e_1, -e_2, \ldots, -e_2),$$

holds, where e_i denotes the i-th standard unit vector in $\mathbb{R}^2$, $e_0 := -e_1 - e_2$ and each e_i occurs exactly d times. Its *combinatorial type* is defined to be the combinatorial type of the underlying abstract marked tropical curve $(\Gamma, x_1, \ldots, x_N)$ together with the directions of all edges.

For $n \geq 3$ and $d > 0$, we define $\mathcal{M}_{1,n}(d)$ to be the set of isomorphism classes of n-marked plane tropical curves $(\Gamma, x_1, \ldots, x_N, h)$ of degree d such that $g(\Gamma) \leq 1$.

Definition 2.2.3 (map γ, deficiency)
Let α be a combinatorial type in $\mathcal{M}_{1,n}(d)$. We define an *integer linear map* γ_α by

$$\begin{aligned} \gamma_\alpha : \mathbb{R}^{2+|\operatorname{NL}(\alpha)|} &\longrightarrow \mathbb{R}^2 \\ (a_1, a_2, l(E_1), \ldots, l(E_{|\operatorname{NL}(\alpha)|})) &\longmapsto \begin{cases} \sum_{E \in cycle(\alpha)} l(E) v_E & \text{if } g(\alpha) = 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

We define the *deficiency of the combinatorial type* α to be $\operatorname{def}(\alpha) = 2 - \operatorname{rk}(\gamma_\alpha)$.

Lemma 2.2.4
The space $\mathcal{M}^\alpha_{1,n}(d) \subset \mathcal{M}_{1,n}(d)$ parametrizing curves Γ of type α can be identified with an open polyhedron of dimension

$$\dim(\mathcal{M}^\alpha_{1,n}(d)) = 3d + n + g(\alpha) - 1 - \sum_{V \in V(\alpha)} (\operatorname{val}(V) - 3) + \operatorname{def}(\alpha)$$

in the space $\mathbb{R}^{2+|\operatorname{NL}(\alpha)|} = \mathbb{Z}^{2+|\operatorname{NL}(\alpha)|} \otimes \mathbb{R}$.

Proof. Let Γ be a tropical curve of type α. After choosing a root vertex V, a curve $(\Gamma, x_1, \ldots x_N, h)$ is fixed by specifying the two coordinates of $h(V) \in \mathbb{R}^2$ and the length $l(E) > 0$ of each non-leaf E in α. The number of non-leaves of Γ equals either $3d + n - \sum(\operatorname{val}(V) - 3)$ or $3d + n - 3 - \sum(\operatorname{val}(V) - 3)$, depending whether $g(\alpha) = 1$ or $g(\alpha) = 0$. If Γ is an elliptic curve, the fact that the cycle in Γ has to map to a cycle in $\mathbb{R}^2$ imposes $2 - \operatorname{def}(\alpha)$ additional conditions. □

Proposition 2.2.5
Let α be a combinatorial type in $\mathcal{M}_{1,n}(d)$. Then every point in $\overline{\mathcal{M}^\alpha_{1,n}(d)}$ can naturally be identified with an element in $\mathcal{M}_{1,n}(d)$. The corresponding map

$$i_\alpha : \overline{\mathcal{M}^\alpha_{1,n}(d)} \longrightarrow \mathcal{M}_{1,n}(d)$$

maps the boundary $\partial\mathcal{M}^{\alpha}_{1,n}(d)$ to a union of strata $\mathcal{M}^{\alpha'}_{1,n}(d)$ where α' is a combinatorial type with fewer non-leaves than α. Moreover, the restriction of i_α to the inverse image of a cell $\mathcal{M}^{\alpha'}_{1,n}(d)$ is an integer affine linear map. Hence $\mathcal{M}_{1,n}(d)$ is a polyhedral complex.

Proof. By the proof of lemma 2.2.4 a point in the boundary of the open polyhedron $\mathcal{M}^{\alpha}_{1,n}(d) \subset \mathbb{R}^{|\operatorname{NL}(\alpha)|+2}$ corresponds to a type α' obtained from α by collapsing (a non-empty set of) non-leaves. The tuple $(\Gamma', x_1, \ldots, x_N, h|_{\Gamma'})$ is an n-marked (rational or elliptic) curve. It is clear that the restriction of the map i_α to the inverse image of each cell $\mathcal{M}^{\alpha'}_{1,n}(d)$ is an integer affine linear map since the affine structure on each cell is given by the position of the curve in the plane and the lengths of the non-leaves. □

Definition 2.2.6 (moduli space of elliptic curves)
The *moduli space of (relevant) elliptic curves* $\mathcal{M}'_{1,n}(d)$ is the $(3d+n)$-skeleton of the polyhedral complex $\mathcal{M}_{1,n}(d)$, where the weights of the facets are defined to be zero except for the following cases:

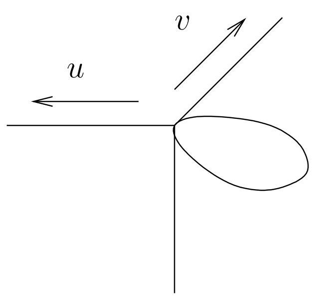
- If $g(\alpha) = 1$ and $\operatorname{def}(\alpha) = 0$, then the weight of the corresponding facet equals

$$\omega(\mathcal{M}^{\alpha}_{1,n}(d)) := \operatorname{index}\left(\gamma_\alpha : \mathbb{Z}^{2+\operatorname{NL}(\alpha)} \to \mathbb{Z}^2\right).$$

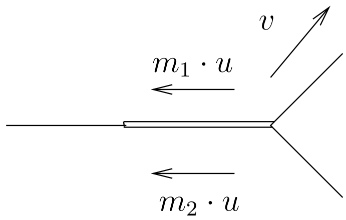
- If α contains a loop at a 5-valent vertex V, then the weight of the corresponding facet equals

$$\omega(\mathcal{M}^{\alpha}_{1,n}(d)) := \frac{1}{2}(|\det(u,v)| - 1),$$

  with u and v direction vectors of two non-loop edges emanating from V.



- If α locally looks as follows where in the notations below, $m_1 \cdot u$, $m_2 \cdot u$ and v denote the direction vectors of the corresponding edges (with $\gcd(m_1, m_2) = 1$)



then the weight of the corresponding facet equals

$$\omega(\mathcal{M}^{\alpha}_{1,n}(d)) := \begin{cases} \frac{1}{2}|\det(u,v)| & \text{if } m_1 = m_2 = 1 \text{ and } |E(\text{cycle}(C)) = 2|, \\ |\det(u,v)| & \text{otherwise.} \end{cases}$$

Remark 2.2.7
An example of a type which is not contained in the space $\mathcal{M}'_{1,n}$ would be e.g. a type α which contains a loop at a 4-valent vertex V.

2.3 The number of tropical elliptic curves with fixed j-invariant

In the next two sections, in order to count curves with fixed j-invariant, we fix $n = 3d-1$.

Definition 2.3.1 (j-invariant, evaluation map ev, ev $\times j$)
Let Γ be a curve in $\mathcal{M}'_{1,n}(d)$ of combinatorial type α. We define its *j-invariant* to be

$$j(\Gamma) = \begin{cases} \sum_{E \in \text{cycle}(\alpha)} l(E), & \text{if } g(\Gamma) = 1 \\ 0, & \text{if } g(\Gamma) = 0. \end{cases}$$

Defining the *evaluation map* ev to be

$$\begin{array}{rcl} \text{ev} : \mathcal{M}'_{1,n}(d) & \longrightarrow & \mathbb{R}^{2n} \\ (\Gamma, x_1, \ldots, x_N, h) & \longmapsto & (h(x_1), \ldots, h(x_n)), \end{array}$$

we obtain a map

$$\text{ev} \times j : \mathcal{M}'_{1,n}(d) \longrightarrow \mathbb{R}^{2n} \times \mathbb{R}_{\geq 0}.$$

Lemma 2.3.2
The restriction of the map ev $\times j$ to any cell $\mathcal{M}^{\alpha}_{1,n}(d)$ is a integer affine linear map. It follows that the map ev $\times j$ *is a morphism of weighted polyhedral complex of the same dimension $2n+1$ (by interpreting the space $\mathbb{R}^{2n} \times \mathbb{R}_{\geq 0}$ as a polyhedral complex with one cell of weight 1).*

Proof. The coordinates on $\mathcal{M}^{\alpha}_{1,n}(d)$ are by lemma 2.2.4 given by a root vertex V and an order on the non-leaves. These coordinates do not need to be independent, but if they are not, they satisfy a linear condition themselves. As Γ is connected, there exist a path from vertex V to each leaf $x_i, i \in [n]$, and the position of $h(x_i)$ is given by

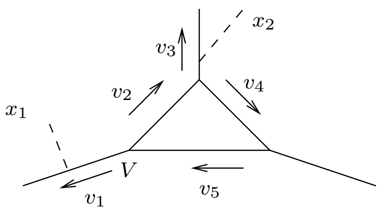
$$h(x_i) = h(V) + \sum_{E} v(E) \cdot l(E),$$

where the sum goes over all edges E in the path from V to x_i, hence the position $h(x_i)$ is given by two affine linear expressions in the coordinates of $\mathcal{M}_{1,n}^{\alpha}(d)$. By definition, the j-invariant $j(C)$ is given as the sum of the lengths of all bounded edges of Γ contained in cycle(α). $\square$

Remark 2.3.3
Let α be a combinatorial type such that $g(\alpha) = 1$, $\operatorname{def}(\alpha) = 0$ and the curves parametrized by α are 3-valent. We can extend the map $\operatorname{ev} \times j$ to a map $\widetilde{\operatorname{ev} \times j}$ on the whole space $\mathbb{R}^{2+NL(\alpha)}$ containing the polyhedron $\mathcal{M}_{1,n}^{\alpha}(d)$. A result of J. Rau [Rau06, Example 1.7] states the following equality:

$$\operatorname{mult}_{\operatorname{ev} \times j}(\mathcal{M}_{1,n}^{\alpha}(d)) = |\det(\operatorname{ev} \times j)| \cdot \operatorname{index}(\gamma_\alpha) = |\det(\widetilde{\operatorname{ev} \times j} \times \gamma_\alpha)|.$$

Example 2.3.4
We compute $|\det(\widetilde{\operatorname{ev} \times j} \times \gamma_\alpha)|$ for the following (local) picture of an elliptic curve.



Let V be the root vertex. We choose the following paths: for x_1, we pass v_1. For x_2, we pass v_2 and v_3. For γ_α, we pass v_2, v_4 and v_5. Then the matrix reads:

$$\begin{pmatrix} \mathrm{Id}_2 & v_1 & 0 & 0 & 0 & 0 \\ \mathrm{Id}_2 & 0 & v_2 & v_3 & 0 & 0 \\ 0 & 0 & v_2 & 0 & v_4 & v_5 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Each row except the last represent two rows, the first column represents two columns, Id_2 stands for the 2×2 unit matrix.

The following lemmas help to determine the $(\operatorname{ev} \times j)$-multiplicity of some curves C. Following G. Mikhalkin, we define the multiplicity $\operatorname{mult}(V)$ of a 3-valent vertex V with adjacent edges E_1, E_2, E_3 of directions v_1, v_2, v_3 to be $\operatorname{mult}(V) := |\det(v_i, v_j)|$ with $i \neq j$ — by the balancing condition, this number does not depend on the choice of i and j. Furthermore, for a rational curve C', we define $\operatorname{mult}(C') := \prod_{V \in V(C')} \operatorname{mult}(V)$. By a result of A. Gathmann and H. Markwig, the number $\operatorname{mult}(C')$ equals the absolute value of the determinant of the evaluation in the marked leaves $x_i, i \in [n]$ of the rational curve C' [GM08, proposition 3.8].

Lemma 2.3.5
Let C be a curve in general position that contains a loop L adjacent to some vertex V

and let C' denote the rational curve obtained from C by removing the edge L. Then the $(\mathrm{ev} \times j)$*-multiplicity of C equals*

$$\mathrm{mult}_{\mathrm{ev} \times j}(C) = \begin{cases} \frac{1}{2}\big(\mathrm{mult}(V) - 1\big) \cdot \mathrm{mult}(C'), & \text{if } \mathrm{val}(V) = 5, \\ 0, & \text{otherwise.} \end{cases}$$

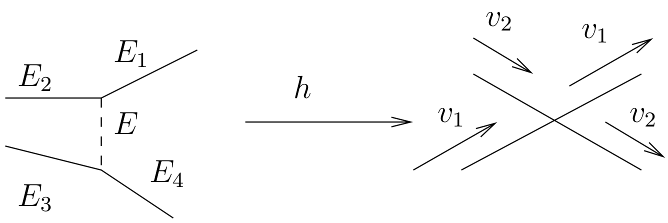
Proof. A matrix representation of $\mathrm{ev} \times j$ can be written down after choosing a root vertex and an order of the non-leaves. Note that in the j-row there is a single entry 1 at the coordinate of the contracted edge, as no other edge is contained in the cycle. To compute the determinant, we can therefore remove this last line and the column of the coordinate of the loop L. The matrix obtained in this way does not take L into account and describes the evaluation in the marked leaves $x_i, i \in [n]$ of the rational curve C' obtained by removing the edge L. Using the result of A. Gathmann and H. Markwig mentioned above and multiplying with the weight of the corresponding cell (see definition 2.2.6) yields the desired result.

□

Lemma 2.3.6
Let C be a curve in general position with a contracted non-leaf $E \in \mathrm{cycle}(C)$ adjacent to vertices V_1 and V_2, $V_1 \neq V_2$. Let C' denote the rational curve obtained from C by removing E and straightening the 2-valent vertices V_1 and V_2. Let v_i denote the direction of a remaining edge adjacent to vertex V_i. Then the $(\mathrm{ev} \times j)$*-multiplicity of C equals*

$$\mathrm{mult}_{\mathrm{ev} \times j}(C) = |\det(v_1, v_2)| \cdot \mathrm{mult}(C').$$

Proof. By the balancing condition, at each of these two vertices V_1 and V_2 the two other adjacent edges are mapped to opposite directions:



In order to compute the $(\mathrm{ev} \times j)$-multiplicity using remark 2.3.3, we determine a matrix representation of $\mathrm{ev} \times j \times \gamma_\alpha$. The edge E is only needed for the map j — as it is contracted by the map h it is not needed to describe a position of the image of a marked leaf. That is, in the column for E, we have a 1 at the row of the map j and 0 in every other row. Note that $\mathrm{cycle}(C)$ contains exactly one element of the two sets $\{E_1, E_2\}$ and $\{E_3, E_4\}$, assume without loss of generality $E_1, E_3 \in \mathrm{cycle}(C)$. Note that if E_1 is used in a path to a marked leaf, then also E_2, and if E_3, then also E_4. Let l_i denote the length of the edge E_i, and l denote the length of the edge E. Assume that the directions are labelled as in the picture. Then the matrix representation of $\mathrm{ev} \times j \times \gamma_\alpha$ is

	$h(V)$	l_1	l_2	l_3	l_4	l	other edges
marked leaves $x_i, i \in [n]$ using neither of the E_i	Id_2	0	0	0	0	0	$*$
marked leaves $x_i, i \in [n]$ using E_1	Id_2	v_1	v_1	0	0	0	$*$
marked leaves $x_i, i \in [n]$ using E_3	Id_2	0	0	v_2	v_2	0	$*$
γ_α	0	v_1	0	v_2	0	0	$*$
j	0	1	0	1	0	1	$*$

We perform the following operations which do not change the absolute value of the determinant: we delete the last row and the l-column. We subtract the l_2-column from the l_1-column and we change the place of the l_1-column: it shall appear as first column. Then, we subtract the l_4-column from the l_3-column and move the l_3-column to the second place. At last, we put the row γ_α on top. After these operations, we obtain

	l_1	l_3	$h(V)$	l_2	l_4	other edges
γ_α	v_1	v_2	0	0	0	$*$
marked leaves $x_i, i \in [n]$ using neither of the E_i	0	0	Id_2	0	0	$*$
marked leaves $x_i, i \in [n]$ using E_1	0	0	Id_2	v_1	0	$*$
marked leaves $x_i, i \in [n]$ using E_3	0	0	Id_2	0	v_2	$*$

Note that this is a block matrix with the matrix of the evaluation map in the $3d-1$ marked leaves $x_i, i \in [n]$ of the rational curve C' on the lower right. It follows that

$$\mathrm{mult}_{\mathrm{ev}\times j}(C) = |\det(v_1, v_2)| \cdot \mathrm{mult}(C').$$

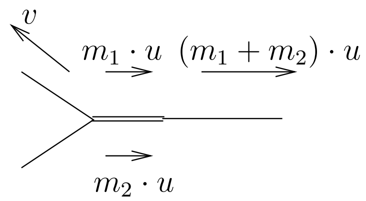
Our argument here assumes that the edges $E_1, \ldots, E_4$ are non-leaves. However, one can prove the same if some of these edges are leaves. Their lengths do not appear as coordinates then, but also they cannot be contained in a path connecting V with any leaf x_i. $\square$

Lemma 2.3.7
Let C be a curve in general position with $\mathrm{def}(C) = 1$ such that $|E(\mathrm{cycle}(C)) = 2|$. Then the $(\mathrm{ev} \times j)$-multiplicity of C equals (with notations as indicated in the picture below)

$$\mathrm{mult}_{\mathrm{ev}\times j}(C) = \begin{cases} 0, & \textit{if no 4-valent vertex is contained in } \mathrm{cycle}(C) \\ |\det(u,v)| \cdot \mathrm{mult}(C') & \textit{if } m_1 = m_2 = 1 \\ (m_1+m_2) \cdot |\det(u,v)| \cdot \mathrm{mult}(C') & \textit{otherwise,} \end{cases}$$

where C' denotes the rational curve obtained by gluing the two edges that form the cycle to one edge of direction $(m_1 + m_2) \cdot u$.

Proof. The following picture shows the cycle of the curve C. We choose m_1 and m_2 such that $\gcd(m_1, m_2) = 1$.



To determine the matrix of $\operatorname{ev} \times j$, we need a lattice basis of $\mathcal{M}^{\alpha}_{1,n}(d)$. As the equations of the cycle are given by $l_1 \cdot m_1 \cdot u - l_2 \cdot m_2 \cdot u$, we can choose unit vectors for all coordinates except l_1 and l_2, plus the vector with m_2 at the l_1-coordinate and m_1 at the l_2-coordinate. As $\gcd(m_1, m_2) = 1$, this is a lattice basis. The j-invariant of C is given by $j(C) = l_1 + l_2$. That is, in the j-row of the matrix, we have only zeros except for the column which belongs to the vector with m_2 at l_1 and m_1 at l_2, there we have the entry $m_1 + m_2$. But then we can delete the j-row and this column. The determinant we want to compute is equal to $(m_1 + m_2)$ times the determinant of the matrix obtained by deleting. This matrix is the matrix of evaluating the points of the rational curve C' which is obtained by identifying the two edges which form the cycle, hence its determinant is equal to $\operatorname{mult}(C')$ and multiplying by the weight of the corresponding cell yields the desired result. □

Lemma 2.3.8
Let C be a curve with $\operatorname{def}(C) = 0$ and $|\operatorname{cycle}(C)| = 3$. Furthermore, let C' be the rational curve obtained from C by shrinking the cycle to a vertex V. Then the $(\operatorname{ev} \times j)$-multiplicity of C equals

$$\operatorname{mult}_{\operatorname{ev} \times j}(C) = \operatorname{mult}(V) \cdot \operatorname{mult}(C').$$

Proof. Let $E(\operatorname{cycle}(C)) = \{E_1, E_2, E_3\}$ and let v_{E_i} be the direction vector of E_i. Using remark 2.3.3, we compute the absolute value of the determinant of the matrix $\operatorname{ev} \times j \times \gamma_\alpha$. This matrix has a block form with a 0 block on the bottom left (as the equations of the cycle and the j-invariant only need the three length coordinates of E_1, E_2 and E_3). The block on the top left is the evaluation of the rational curve C' at the marked leaves $x_i, i \in [n]$ while the block on the bottom right is the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ v_{E_1} & v_{E_2} & v_{E_3} \end{pmatrix}.$$

As the absolute value of the determinant of this submatrix equals

$$|\det(v_{E_1}, v_{E_2})| + |\det(v_{E_1}, v_{E_3})| + |\det(v_{E_2}, v_{E_3})| = \operatorname{mult}(V),$$

the desired result follows. □

Theorem 2.3.9
The numbers $\deg_{\operatorname{ev} \times j}(\mathcal{P})$ do not depend on the choice of a general point $\mathcal{P} \in \mathbb{R}^{2n} \times \mathbb{R}_{\geq 0}$. This number will be referred to as the number of tropical elliptic curves of degree d with fixed j-invariant (through $3d-1$ general points), denoted by $E_{\text{trop}}(d)$.

Definition 2.3.10 (string in a tropical curve)
A *string* in a tropical curve C is a subgraph of Γ homeomorphic either to $\mathbb{R}$ or to S^1 (that is, a "path" starting and ending with a leaf, or a path around a loop) that does not intersect the closures $\overline{x_i}$ of the marked leaves with $i \in [n]$ (cf. [Mar06, definition 4.47]). If the number of marked leaves $x_i, i \in [n]$ on C is less than $3d + g - 1$, then C has a string ([Mar06, lemma 4.50]).

Proof. Analogously [GM08, proof of theorem 4.4], the degree of ev $\times j$ is *locally* constant on the subset of $\mathbb{R}^{2n} \times \mathbb{R}_{\geq 0}$ of points in (ev $\times j$)-general position since at any curve that counts for $\deg_{\text{ev} \times j}(\mathcal{P})$ with a non-zero multiplicity the map ev $\times j$ is a local isomorphism. The points in ev $\times j$-general position are the complement of a polyhedral complex of codimension 1, that is they form a finite number of top-dimensional regions separated by "walls" that are polyhedra of codimension 1. Hence it remains to show that $\deg_{\text{ev} \times j}$ is locally constant at general points on these walls. Such a point is the image under ev $\times j$ of a general tropical curve C of a type α such that $\mathcal{M}^{\alpha}_{1,n}(d)$ is of codimension 1. More precisely, if $\mathcal{P}$ is a point on a wall, and C is a curve through $\mathcal{P}$, we have to show that the sum of the (ev $\times j$)-multiplicities of the curves through $\mathcal{P}'$ near $\mathcal{P}$ and close to C does not depend on the choice of $\mathcal{P}'$.
Using lemma 2.2.4, we determine types α of codimension 1:

(a) $\text{def}(\alpha) = 0$, $g(\alpha) = 1$ and there is one vertex of valence 4.
(b) $\text{def}(\alpha) = 1$ and α has two vertices of valence 4.
(c) $\text{def}(\alpha) = 1$ and α has one vertex of valence 5;
(d) $\text{def}(\alpha) = 2$ and α has three vertices of valence 4;
(e) $\text{def}(\alpha) = 2$ and α has one vertex of valence 5 and one vertex of valence 4;
(f) $\text{def}(\alpha) = 2$ and α has one vertex of valence 6.

Note that the codimension 1 case that α is the type of a rational curve is missing here: the reason is that we do not "cross" such a wall consisting of rational curves, we can only enlarge the j-invariant if $j = 0$, not make it smaller. More precisely, the curves which pass through a configuration $\mathcal{P}'$ in the neighborhood of a point configuration through which a rational curve passes, are always of the same types; the types (and with them, the multiplicities with which we count) do not depend on $\mathcal{P}'$.

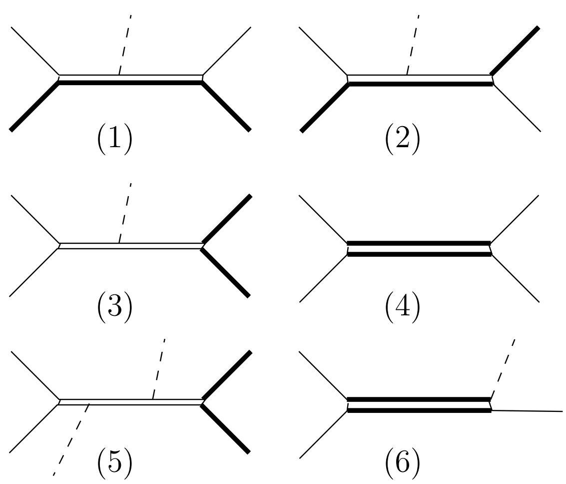
For each of the cases in the list, we have to prove separately that $\deg_{\text{ev} \times j}$ is locally constant around a curve C of type α. The proof of case (a) is similar to the proof [GM08, theorem 4.4], the only difference being the fact that the matrices differ as they contain three lines corresponding to the maps γ_α and j.

In case (b) we can assume that at least one of the 4-valent vertices is adjacent to the cycle as otherwise all types with α in the boundary would have weight 0. If exactly one vertex V is adjacent to the cycle, then the only curves with α in the boundary which do count with a non-zero weight are those obtained by resolving the vertex V and the proof is analogous to case (a), with the only difference that matrices of ev $\times j$ instead of the matrices of ev $\times j \times \gamma_\alpha$ are used. If both 4-valent vertices V_1 and V_2 are adjacent to the cycle and neither V_1 nor V_2 is adjacent to a marked leaf $x_i, i \in [n]$, then C has a string: Consider the connected components of $\Gamma \setminus \bigcup_i \overline{x_i}$. Remove the closures of the marked leaves $\overline{x_1}, \ldots, \overline{x_n}$ from Γ one after the other. We only remove edges at 3-valent vertices. Therefore each removal can either separate one more component, or break a cycle. Assume that all connected components are rational (else C contains a string). Then one of our removals must have broken the cycle. As C is marked by $3d-1$ points, we end up with $3d-1$ connected components. But then there has to be one connected component which contains two leaves, hence C contains a string.

If C has at least two strings then C moves in an at least 2-dimensional family with the images of the marked leaves $x_i, i \in [n]$ fixed. As $\mathbb{R}_{\geq 0}$ is one-dimensional this means that C moves in an at least 1-dimensional family with the image point under j fixed. But then also the curves close to C are not fixed, hence they count 0. So we do not have to consider this case. Also, if for all curves C' which contain C in their boundary the string does not involve an edge of the cycle, then C (and all curves C') move in a family of dimension at least 1 with the image point under j fixed. So we do not have to consider this case either.

So we assume now that C lies in the boundary of a type which has exactly one string that involves (at least) one of the edges of the flat cycle.

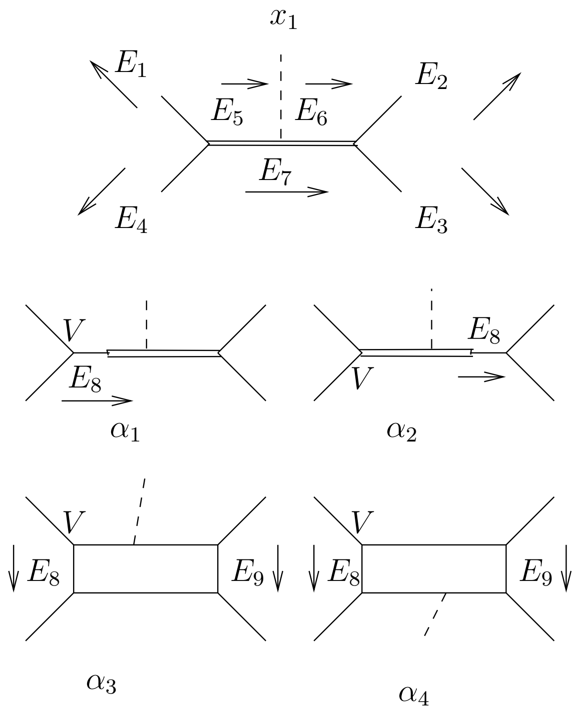
There are (up to symmetry) six possibilities for the string as shown in the following local picture.



Assume now that there is a marked leaf $x_i, i \in [n]$ adjacent to a 4-valent vertex of the flat cycle. Then the removal of this marked leaf $x_i, i \in [n]$ both breaks a cycle and separates two components. So we cannot conclude that C has a string. However, we can conclude that there is no other marked leaf adjacent to the cycle, as otherwise two marked leaves would map to the same line. Hence in this case the curve looks locally like our sixth picture above.

As the cycle is not a string in the cases (1)-(3) and (5), there must be a marked leaf adjacent to it. Two marked leaves adjacent to the flat cycle are only possible if the string does not involve any edge of the flat cycle (as in (3) and (5)).

In each of the six cases, there are four types which contain α in their boundary (see remark 2.2.5). The following picture shows the four types $\alpha_1, \ldots, \alpha_4$ for case (1). We will give our argument only for case (1), it is analogous in all other five cases.



Our first aim is to show that we can choose bases of the corresponding strata $\mathcal{M}^{\alpha}_{1,n}(d)$ such that the matrices $\mathrm{ev} \times j$ (and the matrices $\mathrm{ev} \times j \times \gamma_\alpha$ for the types α_3 and α_4) contain a block which involves only the edges locally around the flat cycle. We can then make statements about the $(\mathrm{ev} \times j)$-multiplicity using remark 2.3.3 and these smaller blocks.

Choose the root vertex for the four types to be V as indicated in the picture above. Also choose the labeling for the edges around the cycle as above. Let v_i be the directions of the E_i (as indicated in the picture). We then have $v_5 = v_6$. As E_5, E_6 and E_7 are mapped to the same line in $\mathbb{R}^2$, we can choose m_1 and m_2 with $v_5 = v_6 = m_2 \cdot u$ and $v_7 = m_1 \cdot u$ such that $\gcd(m_1, m_2) = 1$. We will consider a matrix representation A_i $(i = 1, 2)$ of $\mathrm{ev} \times j$ for the types α_1 and α_2 and a matrix representation B_i $(i = 3, 4)$ of $\mathrm{ev} \times j \times \gamma_\alpha$ for α_3 and α_4. The $(\mathrm{ev} \times j)$-multiplicity for α_1 is then given by $|\det(u, v_2) \cdot \det A_1|$, the $(\mathrm{ev} \times j)$-multiplicity for α_2 is $|\det(u, v_1) \cdot \det A_2|$ by definition 2.2.6. For α_3, it is due to remark 2.3.3 given by $|\det B_3|$ and for α_4 by $|\det B_4|$. Later on, we will also need to consider matrix representations A_3 and A_4 of $\mathrm{ev} \times j$ for the types α_3 and α_4. We will however not choose a lattice basis for those, so they are not useful for the computation of the $(\mathrm{ev} \times j)$-multiplicity. We will specify later on what bases we choose for A_3 and A_4.

We choose a basis of the subspace $\mathcal{M}_{1,n}^{\alpha_i}(d) \subset \mathbb{R}^{2+\mathrm{NL}(\alpha_i)}$ for $i = 1, 2$ consisting of two unit vectors for the root vectors and unit vectors for all bounded edges except E_5, E_6 and E_8. In addition, we take two vectors with E_5, E_6 and E_7-coordinates as follows: $(1, -1, 0)$ and $(0, m_1, m_2)$. In fact, this is a lattice basis: As $\gcd(m_1, m_2) = 1$, we can find integer numbers such that $am_1 + bm_2 = 1$. Then we can complete our basis with the vector $(0, b, a)$ (at E_5, E_6, E_7) and get a lattice basis of $\mathbb{Z}^{2+\mathrm{NL}(\alpha_i)}$. For $i = 3, 4$, we choose a basis of $\mathcal{M}_{1,n}^{\alpha_i}(d) \subset \mathbb{R}^{2+\mathrm{NL}(\alpha_i)}$ consisting of only unit vectors except three vectors involving the coordinates of $E_5, \ldots, E_9$.

Because the bases we choose for the A_i and for the B_i differ only by a few vectors, there will be a block in which the matrices A_i and B_i ($i = 3, 4$) do not differ. The following argumentation works therefore analogously for all six matrices $A_1, \ldots, A_4$, B_3 and B_4.

Assume that d_1 leaves $x_i, i \in [N] \setminus [n]$ can be reached from V via E_1, d_2 via E_2 and so on. As the only string passes via E_4 and E_3, there must be d_1 marked leaves $x_i, i \notin [n]$ which can be reached from V via E_1, d_2 marked leaves $x_i, i \notin [n]$ via E_2, $d_3 - 1$ marked leaves $x_i, i \notin [n]$ via E_3 and $d_4 - 1$ via E_4. Note that the marked points which can be reached via E_1 and E_2 do not need any of the length coordinates of edges via E_3 or E_4. As there are $2 \cdot (d_3 - 1 + d_4 - 1)$ rows for the marked points via E_3 and E_4 and $2d_3 - 1 + 1 - 3 + 1 + 2d_4 - 2$ bounded edges via E_3 and E_4, all six matrices have a 0 block on the top right. For B_3 and B_4, we also put the equation γ_α of the cycle in the first block of rows.

	$h(V)$	other edges	edges via E_3 and E_4
x_1, points behind E_1 and E_2 and j-coord	Id_2	$*$	0
points behind E_3 and E_4	Id_2	$*$	$*$

The block on the bottom right is the same for all six matrices. So we can disregard it and only consider the top left block given by the marked points which can be reached via E_1 and E_2, the j-coordinate, γ_α for B_3 and B_4 and the length coordinates of $E_1, E_2, E_5, \ldots E_8/E_9$ plus the length coordinates of non-leaves via E_1 respectively E_2. Choose a leaf x_2 which can be reached via E_1 and a leaf x_3 which can be reached via E_2. Choose the following order for the rows: begin with the marked points $x_1, \ldots, x_3$, then take the j-coordinate (and for the matrices B_3 and B_4 the equation γ_α of the cycle). Then take the remaining marked leaves. Choose the following order for the columns: begin with $h(V)$, E_1 and E_2. For the types α_1 and α_2, take the two basis vectors involving $E_5, \ldots, E_7$ and then E_8. For the types α_3 and α_4, take the three basis vectors involving $E_5, \ldots, E_9$. For B_3 and B_4, take the length coordinates of $E_5, \ldots, E_9$. Then take the remaining length coordinates. Note that each marked leaf which can be reached via E_1 has the same entries in the first 7 (respectively, 9 for B_3 and B_4) columns as x_2. Each marked leaf which can be reached via E_2 has the same entries in the first 7 (respectively, 9) columns as x_3. That is, we can subtract the x_2-rows from all rows of marked points via E_1 and the x_3-rows from all rows of marked points via E_2. Then we have a 0 block on the bottom left. Note that the bottom right block is equal for all six matrices. That is, we can now go on with the four 7×7-matrices and the two 9×9 matrices. The determinants of the original six matrices only differ by the factor which is equal to the determinants of the corresponding 7×7-matrices (respectively, 9×9) matrices. Let us call these blocks A'_i, respectively B'_i. Here are the four blocks A'_1, A'_2, B'_3 and B'_4 and their determinants:

$$A'_1 = \begin{pmatrix} \mathrm{Id}_2 & 0 & 0 & m_2 \cdot u & 0 & (m_1+m_2)\cdot u \\ \mathrm{Id}_2 & v_1 & 0 & 0 & 0 & 0 \\ \mathrm{Id}_2 & 0 & v_2 & 0 & m_1 \cdot m_2 \cdot u & (m_1+m_2)\cdot u \\ 0 & 0 & 0 & 0 & m_1+m_2 & 0 \end{pmatrix}$$

$$\det(A'_1) = -m_2 \cdot (m_1+m_2)^2 \cdot \det(u,v_1) \cdot \det(u,v_2)$$

$$A'_2 = \begin{pmatrix} \mathrm{Id}_2 & 0 & 0 & m_2 \cdot u & 0 & 0 \\ \mathrm{Id}_2 & v_1 & 0 & 0 & 0 & 0 \\ \mathrm{Id}_2 & 0 & v_2 & 0 & m_1 \cdot m_2 \cdot u & (m_1+m_2)\cdot u \\ 0 & 0 & 0 & 0 & m_1+m_2 & 0 \end{pmatrix}$$

$$\det(A'_2) = -m_2 \cdot (m_1+m_2)^2 \cdot \det(u,v_1) \cdot \det(u,v_2)$$

$$B'_3 = \begin{pmatrix} \mathrm{Id}_2 & 0 & 0 & m_2 \cdot u & 0 & 0 & 0 & 0 \\ \mathrm{Id}_2 & v_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathrm{Id}_2 & 0 & v_2 & m_2 \cdot u & m_2 \cdot u & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 \cdot u & m_2 \cdot u & -m_2 \cdot u & v_1 + m_2 \cdot u & -v_2 + m_2 \cdot u \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\det(B'_3) = -\det(u,v_1) \cdot \det(u,v_2) \cdot m_2 \cdot \big((m_2^2 + m_1 m_2)(\det(u,v_1) + \det(u,v_2)) + m_2 \det(v_1,v_2)\big)$$

$$B'_4 = \begin{pmatrix} \mathrm{Id}_2 & 0 & 0 & m_2 \cdot u & 0 & 0 & -v_1 - m_1 \cdot u & 0 \\ \mathrm{Id}_2 & v_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathrm{Id}_2 & 0 & v_2 & m_2 \cdot u & m_2 \cdot u & 0 & -v_1 - m_1 \cdot u & v_2 - m_1 \cdot u \\ 0 & 0 & 0 & m_2 \cdot u & m_2 \cdot u & -m_1 \cdot u & -v_1 - m_1 \cdot u & v_2 - m_1 \cdot u \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\det(B'_4) = \det(u,v_1) \cdot \det(u,v_2) \cdot m_2 \cdot \big((m_1^2 + m_1 m_2)(\det(u,v_1) + \det(u,v_2)) - m_2 \det(v_1,v_2)\big)$$

A computation shows that

$$\det(u,v_2) \cdot \det A'_1 + \det(u,v_1) \cdot \det A'_2 - \det B_3 + \det B_4 = 0. \tag{2.1}$$

Note that in the cases (4) and (6) *without marked leaves* $x_i, i \notin [n]$ *adjacent to the flat cycle* we have to make a difference if $m_1 = m_2 = 1$. In this case, definition 2.2.6 states that we have to multiply the types analogous to α_1 and α_2 (which still contain a flat cycle) with the factor $\frac{1}{2} \cdot \det(u, v_1)$ respectively $\frac{1}{2} \cdot \det(u, v_2)$, instead of $\det(u, v_1)$ and $\det(u, v_2)$. However, the types α_3 and α_4 are not different in these cases, so we count them only once. Altogether, the weighted sum of determinants as above is still 0.

We still need to check which types occur for a given point configuration $\mathcal{P}''$ near $\mathcal{P}'$. Let $\mathcal{P}'' \subset \mathbb{R}^{2n} \times \mathbb{R}_{\geq 0}$ be a configuration. If there exists a curve C of type α_i through $\mathcal{P}''$, then $A_i^{-1} \cdot \mathcal{P}''$ gives us the coordinates of C in $\mathcal{M}_{1,n}^{\alpha_i}(d)$ in the basis $\{v_{i,1}, \ldots, v_{i,2n-1}\}$. That is, the first two coordinates of the vector

$$\sum_j (A_i^{-1} \cdot \mathcal{P}'')_j \cdot v_{i,j} \subset \mathcal{M}_{1,n}^{\alpha_i}(d)$$

denote the position of the root vertex, and all other coordinates the lengths of the non-leaves of C. A curve of type α_i exists if and only if all coordinates of the vector $\sum_j (A_i^{-1} \cdot \mathcal{P}'')_j \cdot v_{i,j} \subset \mathcal{M}_{1,n}^{\alpha_i}(d)$ which correspond to lengths are positive. Choose $\mathcal{P}''$ close to the configuration $\mathcal{P}'$, through which a curve of type α exists. By continuity of A_i^{-1}, all coordinates of $\sum_j (A_i^{-1} \cdot \mathcal{P}'')_j \cdot v_{i,j} \subset \mathcal{M}_{1,n}^{\alpha_i}(d)$ except the length of E_8 $(i = 1, 2)$, respectively of E_8 and E_9 $(i = 3, 4)$, are positive.

Note that there is a curve of type α_i $(i = 1, 2)$ through $\mathcal{P}''$ if and only if the E_8-coordinate of $A_i^{-1} \cdot \mathcal{P}''$ is positive.

Now we specify which bases we choose for the types α_3 and α_4. For α_3, begin again with the two unit vectors for the position of the root vertex. Take unit vectors for all non-leaves which are not contained in the cycle. Let

$$\begin{aligned} M_1 &:= -\det(v_1, v_2) + m_2 \cdot \det(v_1, u) - m_2 \cdot \det(u, v_2), \\ M_2 &:= -m_2 \cdot \det(u, v_2) \text{ and} \\ M_3 &:= m_2 \cdot \det(u, v_1). \end{aligned}$$

Take the three vectors with entries

$$\begin{aligned} &(-1, 1, 0, 0, 0), \\ &(0, m_1, m_2, 0, 0) \text{ and} \\ &(0, M_1, 0, -M_2, M_3) \end{aligned}$$

at the coordinates of $E_5, \ldots, E_9$. (Let the vector with the entries $(0, M_1, 0, -M_2, M_3)$ be the last basis vector.) These three vectors are linearly independent and satisfy the conditions given by the cycle. However, we cannot say whether this basis is a lattice basis of $\mathcal{M}_{1,n}^{\alpha_3}(d)$. So we do not know whether the determinant of the matrix A_3 is equal to the $(\text{ev} \times j)$-multiplicity of α_3. But we are not interested in the determinant of A_3 here, we just want to use A_3 to check whether there is a curve of type α_3 through $\mathcal{P}''$ or not. Note that the last basis vector is the only one which involves the lengths of E_8 and

E_9. As due to the balancing condition we have $\det(u, v_1) > 0$ and $\det(u, v_2) > 0$, the two entries of this vector corresponding to these two lengths are positive. That is, there is a curve of type α_3 through $\mathcal{P}''$ if and only if the last coordinate of $A_3^{-1} \cdot \mathcal{P}''$ (that is, the coordinate with which we have to multiply our last basis vector to get the lengths) is positive.

For α_4, choose besides the unit vectors the three vectors with entries

$$(-1, 1, 0, 0, 0),$$
$$(0, m_1, m_2, 0, 0) \text{ and}$$
$$(0, M_1', 0, M_2, -M_3)$$

at the coordinates of $E_5, \ldots, E_9$, where

$$M_1' := -\det(v_1, v_2) - m_1 \cdot \det(u, v_1) - m_1 \cdot \det(u, v_2).$$

The two entries of $(0, M_1', 0, M_2, -M_3)$ corresponding to the lengths of E_8 and E_9 are negative. Hence there is a curve of type α_4 through $\mathcal{P}''$ if and only if the last coordinate of $A_4^{-1} \cdot \mathcal{P}''$ (that is, the coordinate with which we have to multiply our last basis vector to get the lengths) is negative.

For all four types, we are interested in the last coordinate of $A_i^{-1} \cdot \mathcal{P}''$. By Cramer's rule, this last coordinate is equal to $\det \tilde{A}_i / \det A_i$, where $\tilde{A}_i$ denotes the matrix where the last column of A_i is cancelled and replaced by $\mathcal{P}''$. Note that the four matrices $A_1, \ldots, A_4$ only differ in the last column. Hence the matrices $\tilde{A}_i$ do not depend on i, and we can decide whether there is a curve of type α_i through $\mathcal{P}''$ by determining the sign of $\det A_i$. (This argument is analogous to the proof of proposition 4.4 in [GM08].)

Recall that $|\det A_i|$ is a product of a factor which does not differ for all four types and a factor which is equal to the determinant of a 7×7-matrix A_i' which describes a curve of type α_i "locally around the cycle".

Here are the two matrices A_3' and A_4' and their determinants:

$$A_3' = \begin{pmatrix} \mathrm{Id}_2 & 0 & 0 & m_2 \cdot u & 0 & 0 \\ \mathrm{Id}_2 & v_1 & 0 & 0 & 0 & 0 \\ \mathrm{Id}_2 & 0 & v_2 & 0 & m_1 \cdot m_2 \cdot u & -M_2 \cdot (-v_1 - m_2 \cdot u) + M_3 \cdot (v_2 - m_2 \cdot u) \\ 0 & 0 & 0 & 0 & m_1 + m_2 & M_1 - M_2 + M_3 \end{pmatrix}$$

$$\det(A_3') = \det(u, v_1) \cdot \det(u, v_2) \cdot m_2^2 \cdot \big((m_2^2 + m_1 m_2)(\det(u, v_1) + \det(u, v_2)) + m_2 \det(v_1, v_2)\big)$$

$$A_4' = \begin{pmatrix} \mathrm{Id}_2 & 0 & 0 & m_2 \cdot u & 0 & M_2 \cdot (-v_1 - m_1 \cdot u) \\ \mathrm{Id}_2 & v_1 & 0 & 0 & 0 & 0 \\ \mathrm{Id}_2 & 0 & v_2 & 0 & m_1 \cdot m_2 \cdot u & 0 \\ 0 & 0 & 0 & 0 & m_1 + m_2 & M_1' + M_2 - M_3 \end{pmatrix}$$

$$\det(A_4') = -\det(u,v_1)\cdot\det(u,v_2)\cdot m_2^2\cdot\big((m_1^2+m_1m_2)(\det(u,v_1)+\det(u,v_2))-m_2\det(v_1,v_2)\big)$$

We know that $\det(u,v_1)\geq 0$, $\det(u,v_2)\geq 0$ and $\det(v_1,v_2)\geq 0$. So there are now two cases to distinguish:

- $\big((m_1^2+m_1m_2)(\det(u,v_1)+\det(u,v_2))-m_2\det(v_1,v_2)\big)\geq 0$ — then $\det A_4'$ is negative. As we have seen, a curve of type α_4 exists if and only if the last coordinate of $A_4'^{-1}\cdot\mathcal{P}''$ is negative, hence if and only if $\det\tilde{A}_i$ (a matrix which depends only on $\mathcal{P}''$, not on i) is positive. $\det A_1'$, $\det A_2'$ are both negative, a curve of one of these types exists if the last coordinate of $A_i'^{-1}\cdot\mathcal{P}''$ $(i=1,2)$ is positive, hence if $\det\tilde{A}_i$ is negative. $\det A_3'$ is positive, and a curve of this type exists if the last coordinate of $A_3'^{-1}\cdot\mathcal{P}''$ is positive, hence if $\det\tilde{A}_i$ is positive. Hence α_1 and α_2 are on one side of the "wall", α_3 and α_4 on the other. But as in this case equation 2.1 from above reads

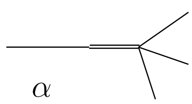
  $$-|\det(u,v_2)\cdot\det A_1'|-|\det(u,v_1)\cdot\det A_2'|+|\det B_3|+|\det B_4|=0$$

  we have that the sum of the $(\mathrm{ev}\times j)$-multiplicities of the curves through a configuration near the wall stays constant.
- $\big((m_1^2+m_1m_2)(\det(u,v_1)+\det(u,v_2))-m_2\det(v_1,v_2)\big)\leq 0$ — then $\det A_4'$ is positive. A curve of type α_4 exists if and only if $\det\tilde{A}_i$ is negative. So in this case α_1, α_2 and α_4 are on one side of the "wall" and α_3 on the other. But equation 2.1 from above reads

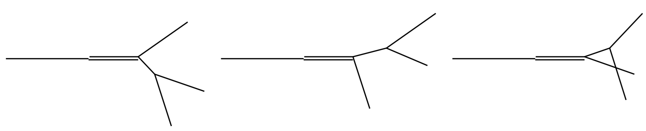
  $$-|\det(u,v_2)\cdot\det A_1'|-|\det(u,v_1)\cdot\det A_2'|+|\det B_3|-|\det B_4|=0$$

  and we have again that the sum of the $(\mathrm{ev}\times j)$-multiplicities of the curves through a configuration near the wall stays constant.

Let us now come to case (c). As before we can argue that only those curves count, where the 5-valent vertex is adjacent to the flat cycle. Then the following curves contain α in the boundary and do not contribute 0:



α

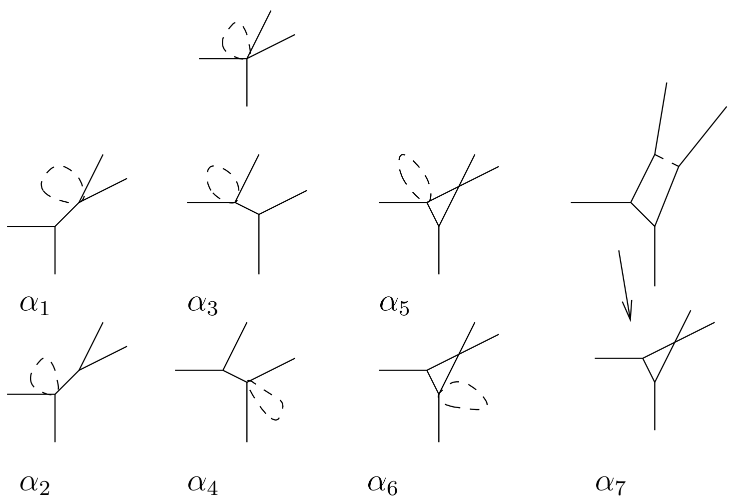


The proof is here again analogous to case (a), only using the "small" matrices of $\mathrm{ev}\times j$.

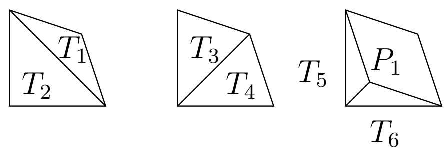
In case (d), all curves which have α in their boundary contribute 0. In case (e), there is only one possibility with curves that do not contribute 0: those where the cycle is adjacent to the 5-valent vertex. Then the curves which have α in their boundary are the curves where the 4-valent vertex is resolved, as in (a). The proof is analogous to case (a), except that we use the "small" matrices for $\mathrm{ev} \times j$.

In case (f), the 6-valent vertex has to be adjacent to the cycle, because otherwise every curve which has α in its boundary would count 0. So we can now assume that there is a 6-valent vertex, where two of the adjacent edges are of direction 0 and form a loop. To resolve this 6-valent vertex, we can either form a 5-valent vertex with a loop and a 3-valent vertex (these curves are contained in strata of top dimension then), or we can resolve it to four 3-valent vertices. (We cannot form a flat cycle from the given 6 edges: the contracted edge must be part of the cycle, and it can either be the whole cycle itself, or it forces the cycle to span $\mathbb{R}^2$.) In the second case, two of the four 3-valent vertices are connected by the contracted edge and therefore mapped to the same image point in $\mathbb{R}^2$. Now we want to use the statement that the number of rational curves through given points does not depend on the position of the points for our case here (see [GM07b], respectively use the analogous proof as for proposition 4.4 of [GM08]). More precisely, if there is a point configuration through which a curve with a 4-valent vertex passes, and we disturb the point configuration slightly, then we always get the same number of tropical curves (counted with multiplicity) passing through the new point configuration.

The image of the 6-valent vertex (and its adjacent edges) in $\mathbb{R}^2$ looks like a 4-valent vertex. The types with one 5-valent and one 3-valent vertex are mapped to two 3-valent vertices, and the type with four 3-valent vertices is mapped to two 3-valent vertices and a crossing of two line segments. That is, the images of the 6-valent vertex as well as of all types which contain it in their boundary look like the possible resolutions of a 4-valent vertex. We know that there are three types which contain a 4-valent vertex in their boundary, and we only have to check how we can add contracted non-leaves to these 3 types, and with which multiplicity they are counted. The following picture shows the seven possible ways to add contracted non-leaves to the three types:



Note that we can vary the length of the contracted non-leaf in each type, so the curves of these types can have any possible j-invariant. The question whether there is a curve of type α_i through a configuration $\mathcal{P} = (p_1, \ldots, p_n, l) \in \mathbb{R}^{2n} \times \mathbb{R}_{\geq 0}$ depends therefore only on the question whether the image of the curve passes through $(p_1, \ldots, p_n)$. If a curve of type α_1 passes through $\mathcal{P}$, then also a curve of type α_2 and vice versa. The same holds for α_3 and α_4, and for α_5, α_6 and α_7. So we only have to see that the sum of the $(\text{ev} \times j)$-multiplicities of the types whose images are equal can be written as a factor times the multiplicity of the rational curve which arises after removing the contracted non-leaf (and straightening the other edges). Then the statement follows from the statement that the number of rational curves through given points does not depend on the position of the points (see [GM07b], respectively proposition 4.4 of [GM08]). To see this, we pass to the dual pictures (see [Mar06, section 5.1]). The 4-valent vertex is dual to a quadrangle. The images of curves of type α_1 and α_2 are dual to a subdivision of this quadrangle in two triangles. The same holds for the images of curves of type α_3 and α_4, however the two triangles arise here by adding the other diagonal. Images of curves of type α_5, α_6 and α_7 are dual to a subdivision consisting of one parallelogram and two triangles.



Lemma 2.3.5 states that the $(\text{ev} \times j)$-multiplicity of a curve of type α_1 is equal to $(\text{Area}(T_1) - \frac{1}{2})$ times the multiplicity of the rational curve which is obtained by removing the contracted non-leaf. (Recall that the multiplicity of a vertex is by definition equal to $2 \cdot \text{Area}(T)$, where T denotes the dual triangle.) Analogously, the $(\text{ev} \times j)$-multiplicity of a curve of type α_2 is equal to $(\text{Area}(T_2) - \frac{1}{2})$ times the multiplicity of the same rational curve. The sum is equal to $(\text{Area}(Q) - 1)$ times the multiplicity of the rational curve, where Q denotes the quadrangle. We get the same for curves of type α_3 and α_4. The sum of the $(\text{ev} \times j)$-multiplicities of α_5, α_6 and α_7 is again by lemma 2.3.5 and lemma 2.3.6 equal to $(\text{Area}(T_5) - \frac{1}{2}) + (\text{Area}(T_6) - \frac{1}{2}) + \text{Area}(P_1) = (\text{Area}(Q) - 1)$ times the multiplicity of the corresponding rational curve. Hence the statement follows. □

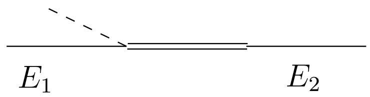
Proposition 2.3.11
Let $\mathcal{P} = (p_1, \ldots, p_n, l) \in \mathbb{R}^{2n} \times \mathbb{R}_{\geq 0}$ *with* $l \gg 0$. *Then every curve* $C \in (\text{ev} \times j)^{-1}(\mathcal{P})$ *with* $\text{mult}_{\text{ev} \times j}(C) \neq 0$ *has a contracted non-leaf.*

Proof. We have to show that the set of all points $j(C) \in \mathbb{R}_{\geq 0}$ is bounded, where C runs over all curves in $\mathcal{M}'_{1,n}(d)$ with non-zero $\text{ev} \times j$-multiplicity that have no contracted non-leaf and satisfy the given incidence conditions at the marked points. As there are only finitely many combinatorial types we can restrict the considerations to curves of a fixed (but arbitrary) combinatorial type α. Since $\mathcal{P}$ is in $(\text{ev} \times j)$-general position we can assume that the curves are 3-valent, respectively contain a flat cycle adjacent to a 4-valent vertex.

Assume first that C is 3-valent. As C is marked by $3d-1$ points we can conclude with [Mar06, lemma 4.5] that C has a string. Analogously to the proof of theorem 2.3.9 above, we get that there is precisely one string Γ' in C.

If $\Gamma' \approx \mathbb{R}$, then analogously to the proof of [GM08, proposition 5.1], the movement of the string is bounded by the adjacent non-leaves, except if the string consists of only two neighboring leaves. But in this case the only length which is not bounded cannot contribute to the j-invariant. It follows that $j(C)$ is bounded.
If $\Gamma' \approx S^1$, then there have to be non-leaves adjacent to the cycle who restrict the movement of the cycle. Again $j(C)$ is bounded.

Now assume C has a flat cycle and a 4-valent vertex adjacent to it, and assume no marked leaf is adjacent to that 4-valent vertex. Then all marked leaves are adjacent to a 3-valent vertex, and hence we can analogously to [Mar06, lemma 4.50] see that the curve contains a string. As above, there is exactly one string and its movement is bounded. Now assume that there is a marked leaf adjacent to the flat cycle. Also in this case, the image of the cycle can still not grow arbitrary large:



The edge E_2 has to be bounded: its direction is not a primitive integer vector, as it is equal to the sum of the directions of the two edges of the flat cycle, and therefore it cannot be a leaf. It follows that the cycle cannot grow arbitrary large. □

As we know that the number of curves $C \in (\mathrm{ev} \times j)^{-1}(\mathcal{P})$ (counted with multiplicity) does not depend on $\mathcal{P}$ by theorem 2.3.9, we can now choose a special configuration $\mathcal{P} = (p_1, \ldots, p_n, l)$ where the j-invariant l is very large. Then by proposition 2.3.11 we can conclude that all curves $C \in (\mathrm{ev} \times j)^{-1}(\mathcal{P})$ contain a contracted non-leaf, which is contained in the cycle. As in lemma 2.3.5 and lemma 2.3.6, this contracted non-leaf can either be a loop itself (which is then adjacent to a 5-valent vertex), or the contracted non-leaf is adjacent to two 3-valent vertices. In both cases, we know that we can form a rational curve C' out of C by removing the contracted edge and straightening other edges, and we can compute the $(\mathrm{ev} \times j)$-multiplicity of C in terms of the multiplicity of this rational curve C'. Note that a rational curve which is obtained in this way is 3-valent, as we take an elliptic curve of codimension 0. The following lemma shows that we can also “go back”: we can form elliptic curves out of a given rational curve interpolating a configuration $(p_1, \ldots, p_n)$.

Proposition 2.3.12 ([Mik05])
There exists a configuration $(p_1, \ldots p_n)$ such that the number of rational tropical curves C' of degree d interpolating these points is finite and the associated Newton subdivisions $\mathrm{SubDiv}(C')$ *consist only of triangles and parallelograms.*

Lemma 2.3.13
Let $\mathcal{P} = (p_1, \dots, p_n, l) \in \mathbb{R}^{2n} \times \mathbb{R}_{\geq 0}$ be general with $l \gg 0$ and $(p_1, \dots, p_n)$ as above (proposition 2.3.12). Let C' be a rational curve interpolating the points $p_1, \dots, p_n$. Then there are several ways to construct an elliptic curve C with j-invariant l out of C', and the sum of the ev $\times j$*-multiplicities of these elliptic curves is equal to $\binom{d-1}{2} \cdot \operatorname{mult}(C')$.*

Proof. Let $(C', x_1, \dots, x_N, h)$ be a rational curve interpolating the points p_i and let V be a vertex of C'. Let $(C, x_1, \dots, x_N, \tilde{h})$ be the elliptic curve obtained from C' by adding a loop L of length $l(L) = l$ at vertex V and extend the map h to a map $\tilde{h}$ defined on C by defining $\tilde{h}(L) = h(V)$. By lemma 2.3.5, we get

$$\operatorname{mult}_{\operatorname{ev} \times j}(C) = \frac{1}{2}(\operatorname{mult}(V) - 1) \cdot \operatorname{mult}(C') = (\operatorname{Area}(T) - \frac{1}{2}) \cdot \operatorname{mult}(C'),$$

where T denotes the triangle dual to V.

Assume that there exist edges E_1 and E_2 such that $h(E_1)$ and $h(E_2)$ intersect in one point, i.e. there exist points $p_1 \in E_1$ and $p_2 \in E_2$ such that $h(p_1) = h(p_2)$. Then we can construct an elliptic curve C with prescribed j-invariant l by subdividing each edge E_i at p_i and inserting an edge E connecting p_1 and p_2. By lemma 2.3.6, we get

$$\operatorname{mult}_{\operatorname{ev} \times j}(C) = |\det(v_1, v_2)| \cdot \operatorname{mult}(C') = \operatorname{Area}(P) \cdot \operatorname{mult}(C'),$$

where v_i is the direction of E_i and P denotes the parallelogram dual to the crossing of E_1 and E_2.

For a lattice polytope $P \subset \mathbb{R}^2$, let $i(P)$ denote the number of integer point in the interior of P and let $b(P)$ denote the number of integer points on the boundary of P which are not vertices of P. Using Pick's theorem (see e.g. [Ful93, section 5.3]), we compute

$$\begin{aligned}
&\sum_T \big(\operatorname{Area}(T) - \frac{1}{2}\big) + \sum_P \operatorname{Area}(P) = \\
&\sum_T \big(i(T) + \frac{b(T)}{2} + \frac{1}{2} - \frac{1}{2}\big) + \sum_P \big(i(P) + \frac{b(P)}{2} + 1\big) = \\
&\sum_T \big(i(T) + \frac{b(T)}{2}\big) + \sum_P \big(i(P) + \frac{b(P)}{2}\big) + \#\{P | P \text{ parallelogram in the subdiv}\} = \\
&\sum_T \big(i(T) + \frac{b(T)}{2}\big) + \sum_P \big(i(P) + \frac{b(P)}{2}\big) + \#\{\text{lattice points of the subdiv}\}
\end{aligned}$$

where the last equality holds, because C' is rational and the genus of a tropical curve satisfying the conditions of proposition 2.3.12 is equal to the number of points of the subdivision minus the number of parallelograms. Now we know that the interior lattice points of the big triangle Δ_d (which is the Newton polygon of curves of degree d) that are not contained in the subdivision must either be interior points of a triangle or a parallelogram or on the boundary of a triangle or parallelogram. In the first case, they are counted in $i(T)$ respectively $i(P)$ of a polygon. In the latter case, as they are interior points of Δ_d, they are part of the boundary of exactly two polygons. That is, in our above sum, they are counted as $b(T)/2$ respectively $b(P)/2$ for two polygons. Hence the first part of the sum counts all interior points which are not part of the subdivision. So we have

$$\sum_T \big(i(T) + \frac{b(T)}{2}\big) + \sum_P \big(i(P) + \frac{b(P)}{2}\big) + \#\{\text{lattice points of the subdiv}\} =$$
$$\#\{\text{lattice points not contained in the subdiv}\}+$$
$$\#\{\text{lattice points of the subdiv}\}$$
$$= \#\{\text{interior points of } \Delta_d\} = \binom{d-1}{2}.$$

□

Theorem 2.3.14
The number of tropical elliptic curves of degree d with fixed j-invariant equals

$$E_{\text{trop}}(d) = \binom{d-1}{2} \cdot N_{\text{trop}}(d)$$

where $N_{\text{trop}}(d)$ denotes the number of rational curves interpolating $3d-1$ general points (counted with multiplicity).

Proof. By theorem 2.3.9, the number $E_{\text{trop}}(d)$ is independent of the choice of a general configuration $\mathcal{P} = (p_1, \ldots, p_n, l)$. Choosing a configuration $\mathcal{P}$ with $l \gg 0$ as in proposition 2.3.11, every elliptic curve passing through this configuration has a contracted non-leaf. From each such elliptic curve with a contracted non-leaf we can form a rational curve by removing the contracted edge and straightening divalent vertices, if necessary. Lemma 2.3.13 tells us that we can go "backwards" and form an elliptic curve with prescribed j-invariant l from each rational curve through the configuration $(p_1, \ldots, p_n)$, and that each rational curve contributes with the factor $\binom{d-1}{2}$ to the sum of elliptic curves. It follows that there are $\binom{d-1}{2} \cdot N_{\text{trop}}(d)$ elliptic curves with j-invariant l interpolating the points $p_1, \ldots, p_n$. □

Corollary 2.3.15
The numbers $E_{\text{trop}}(d)$ and $E(d, j)$ (as defined in Introduction) coincide, if $j \notin \{0, 1728\}$.

Proof. Theorem 2.3.14 states that $E_{\text{trop}}(d) = \binom{d-1}{2} N_{\text{trop}}(d)$. This number is equal to $\binom{d-1}{2} N(d)$ by Mikhalkin's Correspondence Theorem [Mik05, theorem 1] and due to Pandharipande's count [Pan97], this is equal to $E(d, j)$. □

2.4 Applications to the enumerative geometry of rational curves

In this section, we apply the results obtained in this chapter so far to the enumerative geometry of rational tropical curves.

Lemma 2.4.1
The number of elliptic curves with a fixed j-invariant and passing through $3d-1$ points in tropical general position is equal to

$$\sum_{C'}\left(\sum_{T}\big(2\,\mathrm{Area}(T)^2-\frac{1}{2}\big)\cdot\mathrm{mult}(C')\right)$$

where C' goes over all rational curves through the $3d-1$ points and T goes over all triangles in the Newton subdivision dual to C'.

Proof. Given a rational curve C', we ask in how many ways such a curve can be interpreted as an elliptic curve with j-invariant 0. In other words, we ask how elliptic curves C with a small j-invariant which have C' in their boundary look like.

We have to distinguish different cases, depending on the deficiency of C.

If $\mathrm{def}(C)=0$, then the cycle of C has to degenerate to a (3-valent) vertex of C', hence it must be formed by three edges. For each 3-valent vertex V of the rational curve C', there are $i(T_V)$ different possibilities to deform C to an elliptic curve C with given (small) j-invariant, where $i(T_V)$ denotes the number of interior points of the triangle T_V dual to the vertex V. Due to lemma 2.3.8 the $(\mathrm{ev}\times j)$-multiplicity of the curve C equals $\mathrm{mult}(V)\cdot\mathrm{mult}(C')$.

Let $\mathrm{def}(C)=1$. If E is an edge of C' with weight higher than 1, then there can be a small flat cycle at both sides of E. The edge E is dual to an edge with interior points in the dual Newton subdivision, and it is in the boundary of two triangles T_1 and T_2, dual to the vertices V_1 and V_2 of E. Assume the flat cycle is adjacent to the vertex V_1, and assume that it is formed by two edges with directions $m_1\cdot u$ and $m_2\cdot u$, with $\gcd(m_1,m_2)=1$ and $(m_1+m_2)\cdot u=v$, where v denotes the direction of E. Then by lemma 2.3.7 the $(\mathrm{ev}\times j)$-multiplicity of this curve is

$$\begin{aligned}&(m_1+m_2)\cdot\det(u,v_1)\cdot\mathrm{mult}(C')=\det((m_1+m_2)u,v_1)\cdot\mathrm{mult}(C')\\&=\det(v,v_1)\cdot\mathrm{mult}(C')=2\,\mathrm{Area}(T_1)\cdot\mathrm{mult}(C'),\end{aligned}$$

where v_1 denotes the direction of another edge adjacent to V_1. Respectively, if $m_1=m_2=1$ the $(\mathrm{ev}\times j)$-multiplicity is

$$\begin{aligned}&\det(u,v_1)\cdot\mathrm{mult}(C')=\det(\frac{1}{2}v,v_1)\cdot\mathrm{mult}(C')\\&=\mathrm{Area}(T_1)\cdot\mathrm{mult}(C').\end{aligned}$$

Assume $\omega(E)$ is even. Then there are $\frac{\omega(E)}{2} - 1$ possibilities to separate E to two edges with different directions (that is, with $m_1 \neq m_2$). Each counts with the factor $2\,\mathrm{Area}(T_1)$. Also, there is one possibility to separate it to two edges with the same direction, which counts for $\mathrm{Area}(T_1)$. Altogether, we have to count the rational curve with the factor

$$\left(\frac{\omega(E)}{2} - 1\right) \cdot 2\,\mathrm{Area}(T_1) + \mathrm{Area}(T_1) = (\omega(E) - 1) \cdot \mathrm{Area}(T_1).$$

Assume $\omega(E)$ is odd. Then there are $\frac{\omega(E)-1}{2}$ possibilities to split E to two edges with different direction, and each counts with the factor $2\,\mathrm{Area}(T_1)$. In any case, we have to count with the factor $(\omega(E) - 1) \cdot \mathrm{Area}(T_1)$. Note that $\omega(E) - 1$ is equal to the number of lattice points on the side of the boundary of T_1 which is dual to E. But as we have to count these possibilities for all edges of higher weight, we have to add it for all three sides of T_1, that is, altogether, we get $b(T_1) \cdot \mathrm{Area}(T_1)$. Hence, to count the elliptic curves with a flat cycle we have to count each rational curve C' with the factor $\sum_T b(T) \cdot \mathrm{Area}(T)$ where T goes over all triangles in the Newton subdivision dual to C'.

Let $\mathrm{def}(C) = 2$. By lemma 2.3.5 we have to count with the factor $\mathrm{Area}(T) - \frac{1}{2}$. Hence, to count the elliptic curves with a contracted cycle we have to count each rational curve C' with the factor $\sum_T \mathrm{Area}(T) - \frac{1}{2}$ where T goes over all triangles in the Newton subdivision dual to C.

Summing up, we get

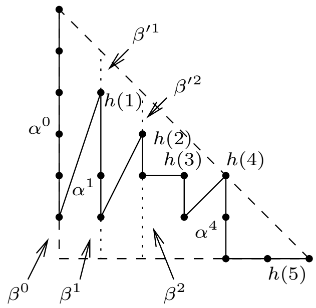
$$\begin{aligned}
&\left(\sum_T i(T) \cdot 2\,\mathrm{Area}(T) + \sum_T b(T) \cdot \mathrm{Area}(T) + \sum_T \big(\mathrm{Area}(T) - \frac{1}{2}\big)\right) \cdot \mathrm{mult}(C') \\
= &\left(\sum_T \big((2i(T) + b(T) + 1) \cdot \mathrm{Area}(T) - \frac{1}{2}\big)\right) \cdot \mathrm{mult}(C') \\
= &\left(\sum_T \big(2\,\mathrm{Area}(T)^2 - \frac{1}{2}\big)\right) \cdot \mathrm{mult}(C')
\end{aligned}$$

where T goes over all triangles in the Newton subdivision dual to C'. □

Lemma 2.4.1 can be applied to speed up Mikhalkins lattice point algorithm to count the number of plane curves in the special case of rational tropical curves. In order to do this, let $\lambda(x, y) = x - \varepsilon y$ with $\varepsilon > 0$ sufficiently small. By a result of G. Mikhalkin [Mik05, theorem 2], the number of λ-increasing paths in the triangle Δ_d is equal to the number of tropical curves through a certain point configuration $\mathcal{P}_\lambda$.

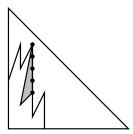
The tropical curves through $\mathcal{P}_\lambda$ are dual to a set of Newton subdivisions. By results of A. Gathmann and H. Markwig [GM07a, proposition 3.8, remark 3.9], the number of these Newton subdivisions equals the column-wise Newton subdivisions of the path. Furthermore, the set of Newton subdivisions which appear as dual subdivisions of a tropical curve through $\mathcal{P}_\lambda$ and the column-wise Newton subdivisions only differ in the location of some parallelograms, while size and locations of the triangles coincide. As

the formula of lemma 2.4.1 only uses triangles, we can use the column-wise Newton subdivisions.
By [GM07a, remark 3.7], a path can only have steps which move one column to the right (with a simultaneous up or down movement), or steps which stay in the same column and move down. The following picture shows such a path and recalls the notations from the article of A. Gathmann and H. Markwig [GM07a].



For the path in the picture, we have $\alpha^0 = 5$, $\alpha^1 = (1,1)$, $\alpha^2 = 1$, $\alpha^3 = 1$, $\alpha^4 = 2$, $\alpha^5 = 0$ and $\alpha^6 = 0$; and $h(1) = 4$, $h(2) = 3$, $h(3) = 2$, $h(4) = 2$, $h(5) = 0$. The only possibilities for the sequences β' are: $\beta'^1 = 1$, $\beta'^2 = 1$, $\beta'^3 = 1$, $\beta'^4 = 0$, $\beta'^5 = 1$. The only possibilities for the sequences β are: $\beta^0 = 1$, $\beta^1 = 1$, $\beta^2 = 2$, $\beta^3 = 1$, $\beta^4 = 0$, $\beta^5 = 0$.

Using [GM07a, proposition 3.8], we get a formula to compute the number of column-wise Newton subdivisions times the multiplicity for a path. To get the desired number, we only have to multiply with the factor $(2\,\text{Area}(T)^2 - \frac{1}{2})$ for each triangle. But note [GM07a, remark 3.9] that the position of the triangles below a path are such that they lie in one column and point to the left. That is, they do not have any interior lattice points, and their area is equal to $\frac{1}{2}$ times the length of their right side:



There is an analogous statement for triangles above the path, of course. So, including this factor, the number $N(d)$ of rational curves of degree d interpolating $3d-1$ points can be expressed as follows:

Corollary 2.4.2
Let $d \geq 3$. Then the following equation holds:

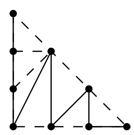
$$N(d) = \frac{1}{\binom{d-1}{2}} \cdot \sum_{\gamma} \sum_{(\beta^0,\dots,\beta^d),(\beta'^0,\dots,\beta'^d)} \binom{\alpha^{i+1}+\beta^{i+1}}{\beta^i} \cdot \binom{\alpha^i+\beta'^i}{\beta'^{i+1}}$$
$$\cdot\, I^{\alpha^{i+1}+\beta^{i+1}-\beta^i} \cdot I^{\alpha^i+\beta'^i-\beta'^{i+1}}$$
$$\cdot \left(\frac{I^2-1}{2} \cdot (\alpha^{i+1}+\beta^{i+1}-\beta^i) + \frac{I^2-1}{2} \cdot (\alpha^i+\beta'^i-\beta'^{i+1}) \right)$$

where the first sum goes over all paths γ and the second sum goes over all sequences $(\beta^0, \dots, \beta^d)$ and $(\beta'^0, \dots, \beta'^d)$ such that $\beta^0 = (d - \alpha^0, 0 \dots, 0)$, $I\alpha^i + I\beta^i = h(i)$, $\beta'^0 = 0$ and $d - i - I\beta'^i = h(i)$, and where for a sequence $\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots)$ $\frac{I^2-1}{2} \cdot \alpha$ denotes the sum $\frac{2^2-1}{2} \cdot \alpha_2 + \frac{3^2-1}{2} \cdot \alpha_3 + \dots$.

Proof. Using Mikhalkin's Correspondence Theorem [Mik05, theorem 2] we conclude that $N(d) = N_{\text{trop}}(d)$. Furthermore, $\binom{d-1}{2} N_{\text{trop}}(d) = E_{\text{trop}}(d) = \deg_{\text{ev} \times j}(\mathcal{P})$, where we can choose any point configuration $\mathcal{P}$ by theorem 2.3.14 and theorem 2.3.9. So $N(d) = \frac{1}{\binom{d-1}{2}} E_{\text{trop}}(d)$ and it remains to argue why the right hand side of the formula above (times $\binom{d-1}{2}$) is equal to $E_{\text{trop}}(d)$. We can choose a point $\mathcal{P} = (p_1, \dots, p_n, l)$ with a small last coordinate l for the cycle length, and such that $(p_1, \dots, p_n)$ are in the position described in [Mik05, theorem 2]. We apply lemma 2.4.1, which states $E_{\text{trop}}(d) = \sum_C \left(\sum_T \left(2 \operatorname{Area}(T)^2 - \frac{1}{2}\right) \cdot \operatorname{mult} C\right)$, where C goes over all rational curves through the $3d - 1$ points and T goes over all triangles in the Newton subdivision dual to C. The Newton subdivision dual to the rational curves through $(p_1, \dots, p_n)$ differ from the column-wise Newton subdivisions (see [GM07a, remark 3.9]) only in the location of some parallelograms. Size and location of the triangles coincide. Therefore the above sum is equal to $\sum_N \left(\sum_T \left(2 \operatorname{Area}(T)^2 - \frac{1}{2}\right) \cdot \operatorname{mult}(N)\right)$, where N goes over all column-wise Newton subdivisions arising from Newton subdivisions dual to rational tropical curves through $(p_1, \dots, p_n)$. A result of A. Gathmann and H. Markwig [GM07a, proposition 3.8] gives us a formula to compute the number of column-wise Newton subdivisions times their multiplicity. We only have to multiply this formula with the factor $(2 \operatorname{Area}(T)^2 - \frac{1}{2})$ for each triangle. As in [GM07a, remark 3.9] the positions of the triangles in a column-wise Newton subdivision are such that they lie in one column and point to the left. That is, they do not have any interior lattice points, and their area is equal to $\frac{1}{2}$ times the length of their right side. The factor $\binom{\alpha^{i+1}+\beta^{i+1}}{\beta^i} \cdot \binom{\alpha^i+\beta'^i}{\beta'^{i+1}}$ counts the possibilities to arrange parallelograms below and above the path (hence the number of Newton subdivisions). The factor $I^{\alpha^{i+1}+\beta^{i+1}-\beta^i} \cdot I^{\alpha^i+\beta'^i-\beta'^{i+1}}$ counts the double areas of the triangles - hence the multiplicity of the curves dual to the path (see also [GM07a, remark 3.9]). The factor $\frac{I^2-1}{2} \cdot (\alpha^{i+1} + \beta^{i+1} - \beta^i) + \frac{I^2-1}{2} \cdot (\alpha^i + \beta'^i - \beta'^{i+1})$ is the factor $(2 \operatorname{Area}(T)^2 - \frac{1}{2})$ for each triangle. □

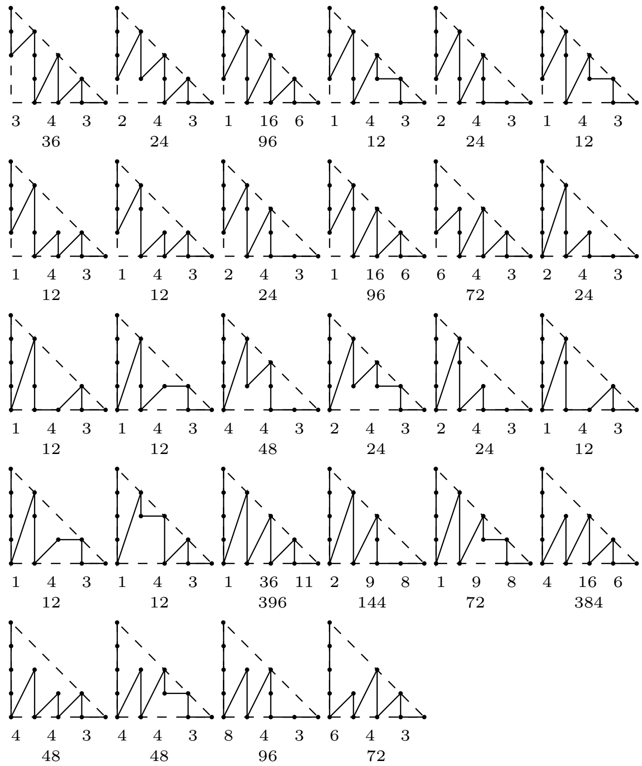
Note that even though this sum looks at the first glance more complicated than the sum from [GM07a, proposition 3.8], it is easier to compute, because a lot of paths count with the factor 0 — all paths with only steps of size 1.

Example 2.4.3
For $d = 3$, there is only one lattice path with a step of size bigger than one.



There is only one possible Newton subdivision for this path, as indicated in the picture. There are two triangles of area 1. Both contribute $\frac{3}{2} \cdot \operatorname{mult} C = \frac{3}{2} \cdot 4 = 6$. Altogether, we get $6 + 6 = 12 = N(3)$, as expected.

Example 2.4.4
For $d = 4$, we only have to consider the paths below, because all other paths have only steps of size 1.



There are three numbers in the first row below each path: the first number is the number of possible Newton subdivisions. The second number is the multiplicity of the tropical curves dual to these Newton subdivisions. (Hence the product of the first two numbers is the multiplicity of the path.) The third number is the factor $\sum_T \left(2 \operatorname{Area}(T)^2 - \frac{1}{2}\right)$ with which we have to count here. The fourth number, in the second row, is the product of the three numbers above, so we have to count each path with that number. The sum of the numbers is the second row is $1860 = 3 \cdot 620 = 3 \cdot N(d)$, as claimed.

In the sequel, we use Welschinger invariants W_d to get obstructions to the possible number of rational tropical curves of fixed degree d interpolating $3d-1$ general points. These invariants have been introduced by J.-Y. Welschinger [Wel05] and can be computed using tropical geometry (see [IKS03]).

Definition 2.4.5
Let $d > 0$ and let $\mathcal{P} = \{p_1, \ldots, p_n\} \subset \mathbb{R}^2$ be a general configuration. For a rational curve

C' interpolating $\mathcal{P}$ we define the Welschinger-multiplicity of C' to be

$$w(C') = \begin{cases} 0, & \text{if } \operatorname{mult}(C') \text{ is even,} \\ \prod_{V \in V(C')} (-1)^{i(T)} & \text{otherwise.} \end{cases}$$

where $i(T)$ denotes the number of interior points in the triangle dual to V.

Theorem 2.4.6
The number

$$W_d(\mathcal{P}) = \sum_{C'} w(C')$$

does not depend on $\mathcal{P}$ *and is called the* d-th Welschinger number*.*

Proposition 2.4.7
The number $T_d(\mathcal{P})$ *of tropical curves of degree* $d \geq 4$ *interpolating a general configuration* $\mathcal{P} = \{p_1, \ldots, p_n\}$ *counted without multiplicities satisfies the strict inequality*

$$T_d(\mathcal{P}) < \frac{N_d + W_d}{2}.$$

Proof. Using only Welschinger invariants and Mikhalkin's multiplicity of tropical curves as constraints, maximality of $T_d(\mathcal{P})$ would be achieved by choosing a configuration $\mathcal{P}$ such that the curves interpolating $\mathcal{P}$ would be the following:

- $x_1 = \frac{1}{4}(N_d + 3W_d)$ curves C'_1 such that the subdivision contains only parallelograms and triangles of area $\frac{1}{2}$.
- $x_2 = \frac{1}{4}(N_d - W_d)$ curves C'_2 such that the subdivision contains one triangle of area $\frac{3}{2}$ and all other cells are either parallelograms or triangles of area $\frac{1}{2}$. Hence the maximal number of curves equals

$$x_1 + x_2 = \frac{1}{4}(N_d + 3W_d) + \frac{1}{4}(N_d - W_d) = \frac{1}{2}(N_d + W_d).$$

But such a configuration $\mathcal{P}$ cannot exist. To see this, we define for a rational curve C'

$$e(C') := \sum_T \left(2\,\text{Area}(T)^2 - \frac{1}{2}\right) \cdot \operatorname{mult}(C'),$$

where the sum goes over all triangles T in the dual subdivision of C', in order to get a contradiction to lemma 2.4.1:

$$\begin{aligned} \binom{d-1}{2} N_d &= \sum_{C'} \left(\sum_T \left(2\,\text{Area}(T)^2 - \frac{1}{2}\right) \cdot \operatorname{mult}(C') \right) \\ &= x_1 e(C'_1) + x_2 e(C'_2) \\ &= 0 + x_2 \cdot (4 \operatorname{mult}(C'_2)) \\ &= \frac{1}{4}(N_d - W_d)(4 \cdot 3) \\ &= 3(N_d - W_d) < \binom{d-1}{2} N_d \quad \text{for all } d \geq 4. \end{aligned}$$

□

Remark 2.4.8
Combining Mikhalkin's correspondence theorem (which states that $\sum_{C'} \mathrm{mult}(C') = N_d$), lemma 2.4.1 (which states that $\sum_{C'} e(C') = \binom{d-1}{2} N_d$) and theorem 2.4.6 (which states that $\sum_{C'} w(C') = W_d$), we get the following algorithmic approach to compute an upper bound for the number $T_d(\mathcal{P})$:

- Identify the possible subdivisions of the triangle $\Delta_d = \mathrm{conv}\{(0,0),(d,0),(0,d)\}$ into triangles and parallelograms such that the dual curve C_i' is rational.
- Compute the numbers $e(C')$, $w(C')$, $\mathrm{mult}(C')$.
- Solve the integer linear problem

$$\begin{aligned} \sum_i x_i \cdot \mathrm{mult}(C_i') &= N_d \\ \sum_i x_i \cdot e(C_i') &= \binom{d-1}{2} \cdot N_d \\ \sum_i x_i \cdot w(C_i') &= W_d \\ 0 \leq x_i &\in \mathbb{Z}. \end{aligned}$$

Then we can conclude that $T_d(\mathcal{P}) \leq \sum_i x_i$.

Example 2.4.9 (Rational cubics ($d = 3$))
To compute an upper bound on $T_3(\mathcal{P})$ using $W_3 = 8$, we need a list of possible subdivisions of curves of degree 3 (we only indicate the special cell in the subdivision, all other cells in the subdivision are triangles of area $\frac{1}{2}$):

Type C	$\mathrm{mult}(C_i)$	$e(C_i)$	$w(C_i)$
One parallelogram (C_1)	1	0	1
Two triangles of area one (C_2)	4	12	0
One triangle with one int. point (C_3)	3	12	-1

We ask how many elements of the set of curves interpolating are of type C_i and call these numbers x_i, in other words, we solve the corresponding integer linear program

$$\begin{aligned} \sum_i x_i \cdot \mathrm{mult}(C_i) &= 12 = N_3 \\ \sum_i x_i \cdot e(C_i) &= 12 = \binom{d-1}{2} \cdot N_3 \\ \sum_i x_i \cdot w(C_i) &= 8 = W_3 \\ 0 \leq x_i &\in \mathbb{Z} \end{aligned}$$

and get an upper bound

$$T_3(\mathcal{P}) \leq 10 = \frac{N_3 + W_3}{2}$$

(with $x_1 = 8$, $x_2 = 1$, $x_3 = 0$ or $x_1 = 9$, $x_2 = 0$, $x_3 = 1$). Note that there exists a configuration $\mathcal{P} = \{p_1, \ldots, p_n\}$ such that $T_3(\mathcal{P}) = 10$ (cf. [Mik06, Example 7.1]).

Example 2.4.10 (Rational quadrics ($d = 4$))
To compute an upper bound on $T_4(\mathcal{P})$ using $W_4 = 240$ (as calculated by I. Itenberg, V. Kharlamov and E. Shustin [IKS03]), we need a list of possible subdivisions of curves of degree 4:

Type C	$\text{mult}(C_i)$	$e(C_i)$	$w(C_i)$
Three parallelograms (C_1)	1	0	1
Two triangles of area one and two paral. (C_2)	4	12	0
Four triangles of area one and one paral. (C_3)	16	96	0
Two triangles with common edge of length three (C_4)	9	144	1
One triangle with one int. point (C_5)	3	12	-1
Two triangles with one int. point (C_6)	9	72	1

Using a computer, we solve the corresponding integer linear program

$$\text{maximize} \quad \sum_{i=1}^{6} x_i \quad \text{subject to}$$

$$\begin{aligned}
\sum_i x_i \cdot \text{mult}(C_i) &= 620 = N_4 \\
\sum_i x_i \cdot e(C_i) &= 1860 = \binom{d-1}{2} \cdot N_4 \\
\sum_i x_i \cdot w(C_i) &= 240 = W_4 \\
0 &\leq x_i \in \mathbb{Z}
\end{aligned}$$

and get an upper bound

$$T_4(\mathcal{P}) \leq 406 < 430 = \frac{N_4 + W_4}{2}$$

(for example with $x_1 = 317$, $x_4 = 6$, $x_5 = 83$, $x_2 = x = 3 = x_6 = 0$).
It is not known if there exists a configuration $\mathcal{P}$ such that $T_4(\mathcal{P}) = 406$.

3 A Riemann-Roch theorem in tropical geometry

Introduction

Matt Baker and Sergey Norine have recently proven a graph-theoretic analogue of the well-known Riemann-Roch formula for Riemann surfaces [BN07]. Given a connected graph $G = (V(G), E(G))$, they define the set $\mathrm{Div}(G)$ of divisors on G to be the free abelian group on $V(G)$. To a given divisor $D \in \mathrm{Div}(G)$, they associate a number $r(D)$ which plays the rôle of the dimension of a linear system in algebraic geometry. Using these notations, their main theorem states that the equality

$$r(D) - r(K_G - D) = \deg(D) + 1 - g(G)$$

holds, where $K_G \in \mathrm{Div}(G)$ denotes the canonical divisor of the graph G (see [Zha93]), $\deg(D)$ denotes the degree of the divisor D and $g(G) := |E(G)| - |V(G)| + 1$ is the genus of the graph G.

In this chapter, we extend this result to metric graphs and tropical curves, hence prove a Riemann-Roch theorem in tropical geometry. We start by introducing the language of divisors and rational functions on tropical curves in section 3.1. In section 3.2, we interpret the results of M. Baker and S. Norine in this language and extend it to metric graphs where the length of each edge is a rational number. Finally, we extend this result in section 3.3 to general metric graphs and tropical curves.

3.1 Tropical rational functions and divisors

In the following, let $\Gamma = (V(\Gamma), E(\Gamma))$ be a finite connected multigraph, not necessarily loop-free. The first Betti number $g(\Gamma) = |E(G)| - |V(G)| + 1$ is called the *genus* of Γ.

Definition 3.1.1 (abstract tropical curves)
An *abstract tropical curve* is a "metric graph with possibly unbounded ends", i.e. a pair (Γ, l) where Γ is a graph and l is a length function $l : E(\Gamma) \to \mathbb{R}_{>0} \cup \{\infty\}$ such that $l(E) = \infty$ implies that E is a leaf of Γ. Each edge of length $l(E)$ is identified with

the real interval $[0, l(E)]$ and each leaf of length ∞ is identified with the real interval $[0, \infty] = \mathbb{R}_{\geq 0} \cup \{\infty\}$ in such a way that the ∞ end of the edge has valence 1. These infinity points of will be called the *(unbounded) ends* of Γ.

Remark 3.1.2
Note that in contrast to abstract rational and elliptic tropical curves introduced in the two chapters before, we allow abstract tropical curves to have vertices of valence 1 and 2, and we add "points at infinity" at each unbounded edge. Note also that every metric graph is a tropical curve.

Definition 3.1.3 (Divisors)
A *divisor* on a tropical curve Γ is an element of the free abelian group generated by the points of (the geometric representation of) Γ. The group of all divisors on Γ is denoted $\mathrm{Div}(\Gamma)$. The *degree* $\deg D$ of a divisor $D = \sum_i a_i P_i$ (with $a_i \in \mathbb{Z}$ and $P_i \in \Gamma$) is defined to be the integer $\sum_i a_i$ and obviously gives rise to a morphism $\deg : \mathrm{Div}(\Gamma) \to \mathbb{Z}$. The *support* $\operatorname{supp} D$ of D is defined to be the set of all points of Γ occurring in D with a non-zero coefficient. A divisor is called *effective* if all its coefficients a_i are non-negative. Following S. Zhang [Zha93] we define the canonical divisor of Γ to be

$$K_\Gamma := \sum_{P \in V(\Gamma)} (\mathrm{val}(P) - 2) \cdot P.$$

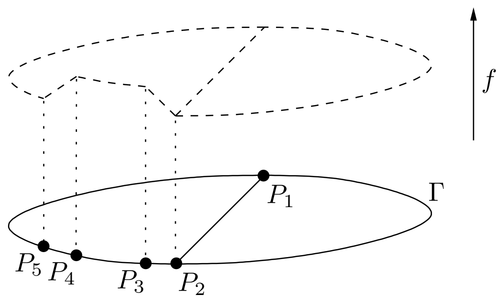
Definition 3.1.4 (Rational functions)
A *rational function* on a tropical curve Γ is a continuous function $f : \Gamma \to \mathbb{R} \cup \{\pm\infty\}$ such that the restriction of f to any edge of Γ is a piecewise linear function with integral slope and only a finite number of pieces. In particular, f can take on the values $\pm\infty$ only at the unbounded ends of Γ. For a rational function f as above and a point $P \in \Gamma$ the *order* $\mathrm{ord}_P f \in \mathbb{Z}$ of f at P will be the sum of the outgoing slopes of f over all segments of Γ emanating from P (of which there are $\mathrm{val}(P)$ if $P \in V(\Gamma)$ and 2 otherwise). In particular, if P is an unbounded end of Γ lying on an unbounded edge E then the order of f at P equals the negative of the slope of f at a point on E sufficiently close to P. Borrowing from algebraic geometry, if $\mathrm{ord}_P f > 0$, we say that the function f has a zero (of order $\mathrm{ord}_f P$) at the point P and if $\mathrm{ord}_P f < 0$, we say that the function f has a pole (of order $-\mathrm{ord}_P f$) at the point P.

Note that $\mathrm{ord}_P f = 0$ for all points $P \in \Gamma \backslash V(\Gamma)$ at which f is locally linear and thus for all but finitely many points. We can therefore define the *divisor associated to* f to be

$$(f) := \sum_{P \in \Gamma} \mathrm{ord}_P f \cdot P \in \mathrm{Div}(\Gamma)$$

as in classical geometry.

Example 3.1.5
Let Γ be the graph indicated in the picture below and assume that f is a function such that the slope of f on the edges connecting P_5, P_4 and P_2, P_3 equals 1 and that f is constant on all other edges of Γ.



Then the divisor of associated to f equals

$$(f) = \sum_{P \in \Gamma} \mathrm{ord}_P\, f \cdot P \in \mathrm{Div}(\Gamma) = P_2 - P_3 - P_4 + P_5.$$

Remark 3.1.6
If f is a rational function on a tropical curve Γ then the degree of its associated divisor (f) is $\deg(f) = \sum_{P \in \Gamma} \mathrm{ord}_P\, f$. By definition of the order this expression can be written as a sum over all segments of Γ on which f is linear, where each such segment counts with the sum of the outgoing slopes of f on it at the two end points of the segment. But as these two slopes are obviously just opposite numbers on each such edge we can conclude that $\deg(f) = 0$ — again analogous to the case of compact curves in classical geometry.

Definition 3.1.7 ($\mathbb{Z}$- and $\mathbb{Q}$-graph and -divisors)
Let (Γ, l) be a tropical curve. If $l(E(\Gamma)) \subset \mathbb{Z}$ (resp. $l(E(\Gamma)) \subset \mathbb{Q}$), we call Γ a $\mathbb{Z}$-*graph* (resp. $\mathbb{Q}$-*graph*). In this case the points of (the geometric representation of) Γ with integer (resp. rational) distance to the vertices are called $\mathbb{Z}$-*points* (resp. $\mathbb{Q}$-*points*) of Γ. We denote the set of these points by $\Gamma_{\mathbb{Z}}$ and $\Gamma_{\mathbb{Q}}$, respectively. A divisor D on a $\mathbb{Z}$-graph (resp. $\mathbb{Q}$-graph) will be called a $\mathbb{Z}$-*divisor* (resp. $\mathbb{Q}$-*divisor*) if $\mathrm{supp}\, D \subset \Gamma_{\mathbb{Z}}$ (resp. $\mathrm{supp}\, D \subset \Gamma_{\mathbb{Q}}$). The canonical divisor on a $\mathbb{Z}$-graph (resp. $\mathbb{Q}$-graph) is obviously a $\mathbb{Z}$-divisor (resp. $\mathbb{Q}$-divisor).

Definition 3.1.8 (Spaces of functions associated to a divisor)
Let D be a divisor of degree n on a tropical curve Γ.

(a) We denote by $R(D)$ the set of all rational functions f on Γ such that the divisor $(f) + D$ is effective. Note that for any such $f \in R(D)$ the divisor $(f) + D$ is a sum of exactly $\deg((f) + D) = \deg D = n$ points by remark 3.1.6. So if we define

$$\begin{aligned} S(D) := \{ & (f, P_1, \dots, P_n);\ f \text{ a rational function on } \Gamma, \\ & P_1, \dots, P_n \in \Gamma \text{ such that } (f) + D = P_1 + \cdots + P_n \} \end{aligned}$$

then we obviously have $R(D) = S(D)/S_n$, where the symmetric group S_n acts on $S(D)$ by permutation of the points P_i.

(b) If Γ is a $\mathbb{Z}$-graph and D a $\mathbb{Z}$-divisor we define a "discrete version" of (a) as follows: let $\tilde{R}(D)$ be the set of all rational functions f on Γ such that $(f) + D$ is an effective

$\mathbb{Z}$-divisor, and set

$$\tilde{S}(D) := \{(f, P_1, \dots, P_n);\ f \text{ a rational function on } \Gamma,\\ P_1, \dots, P_n \in \Gamma_{\mathbb{Z}} \text{ such that } (f) + D = P_1 + \cdots + P_n\},$$

so that again $\tilde{R}(D) = \tilde{S}(D)/S_n$.

If we want to specify the curve Γ in the notation of these spaces we will also write them as $R_\Gamma(D)$, $S_\Gamma(D)$, $\tilde{R}_\Gamma(D)$, and $\tilde{S}_\Gamma(D)$, respectively.

Remark 3.1.9
The spaces $R(D)$, $S(D)$, $\tilde{R}(D)$, $\tilde{S}(D)$ of definition 3.1.8 have the following obvious properties:

(a) all of them are empty if $\deg D < 0$;

(b) $R(D - P) \subset R(D)$ and $\tilde{R}(D - P) \subset \tilde{R}(D)$ for all $P \in \Gamma$;

(c) $\tilde{R}(D) \subset R(D)$ and $\tilde{S}(D) \subset S(D)$ if D is a $\mathbb{Z}$-divisor on a $\mathbb{Z}$-graph Γ.

We want to see now that $R(D)$ and $S(D)$ are *polyhedral complexes* in the sense of definition 2.1.1, i.e. spaces that can be obtained by gluing finitely many polyhedra along their boundaries, where a polyhedron is defined to be a subset of a real vector space given by finitely many linear equalities and strict inequalities. To do this we first need a lemma that limits the combinatorial possibilities for the elements of $R(D)$ and $S(D)$. For simplicity we will only consider the case of metric graphs here (but it is in fact easy to see with the same arguments that lemmas 3.1.10 and 3.1.11 hold as well for tropical curves, i.e. in the presence of unbounded ends).

Lemma 3.1.10
Let $p > 0$ be an integer, and let f be a rational function on a metric graph Γ that has at most p poles (counted with multiplicities). Then the absolute value of the slope of f at any point of Γ (which is not a vertex and where f is differentiable) is bounded by a number that depends only on p and the combinatorial structure of the graph Γ.

Proof. To simplify the notation of this proof we will consider all zeroes and poles of f to be vertices of Γ (by making them into 2-valent vertices in case they happen to lie in the interior of an edge).

Let E be any edge of Γ on which f is not constant. Construct a path γ along Γ starting with E in the direction in which E is increasing, and then successively following the edges of Γ, at each vertex continuing along an edge on which the outgoing slope of f is maximal.

By our convention on 2-valent vertices above the function f is affine linear on each edge of Γ. Let us now study how the slope of f changes along γ when we pass a vertex $P \in \Gamma$. By definition we have $\lambda_1 + \cdots + \lambda_n = \operatorname{ord}_P f$, where $\lambda_1, \dots, \lambda_n$ are the outgoing slopes of f on the edges $E_1, \dots, E_n$ adjacent to P. Now let N be the maximal valence of a vertex occurring in Γ, and assume that our path γ approaches P along the edge E_1 on which f has incoming slope $-\lambda_1$ greater or equal to $(N+p)^\alpha$ for some $\alpha \geq 1$. It then follows that

$$\begin{aligned}\lambda_2 + \cdots + \lambda_n &= -\lambda_1 + \operatorname{ord}_P f \\ &\geq (N+p)^\alpha - p \\ &= N\,(N+p)^{\alpha-1} + p\,((N+p)^{\alpha-1} - 1) \\ &\geq N\,(N+p)^{\alpha-1},\end{aligned}$$

which means that the biggest of the numbers $\lambda_2, \ldots, \lambda_n$, i.e. the outgoing slope of f along γ when leaving P, is at least $(N+p)^{\alpha-1}$ (recall that $n \leq N$ and that λ_1 can never be the biggest of the $\lambda_1, \ldots, \lambda_n$ since it is negative by assumption whereas at least one of the $\lambda_2, \ldots, \lambda_n$ is positive).

So if we assume that the slope of f is at least $(N+p)^\alpha$ on the edge E this means by induction that the slope of f on γ is at least $(N+p)^{\alpha-i}$ after crossing i vertices, i.e. in particular that f is strictly increasing on the first $\alpha+1$ edges of γ. But this is only possible if α is less than the number of edges of Γ: otherwise at least one edge must occur twice among the first $\alpha+1$ edges of γ, in contradiction to f being strictly increasing on γ in this range. As the initial edge E was arbitrary this means that the slope of f on any edge is bounded by $(N+p)^\alpha$, with α being the number of edges of Γ. □

Lemma 3.1.11
For any divisor D on a metric graph Γ the spaces $R(D)$ and $S(D)$ are polyhedral complexes.

Proof. We will start with $S(D)$. For each edge E of Γ we choose an adjacent vertex that we will call the starting point of E. To each element $(f, P_1, \ldots, P_n)$ of $S(D)$ we associate the following discrete data:

(a) the information on which edge or vertex P_i lies for all $i = [n]$;

(b) the (integer) slope of f on each edge at its starting point;

and the following continuous data:

(c) the distance of each P_i that lies on an edge from the starting point of this edge;

(d) the value of f at a chosen vertex.

These data obviously determine f uniquely: on each edge we know the starting slope of f as well as the position and orders of all zeroes and poles, so f can be reconstructed on each edge if its starting value on the edge is given. As Γ is connected by assumption we can thus reconstruct the whole function from the starting value (d).

Since there are only finitely many choices for (a) and (b) (use lemma 3.1.10 for (b)), we get a stratification of $S(D)$ with finitely many strata. The data (c) and (d) are given by finitely many real variables in each stratum, so each stratum is a subset of a real vector space. Finally, the condition on the given data to be compatible is given by several linear equalities and inequalities (the distances (c) must be positive and less than the length of the corresponding edges, and the values of f at the boundary points of the edges must be so that we get a well-defined continuous function on Γ), so that $S(D)$ is indeed a polyhedral complex.

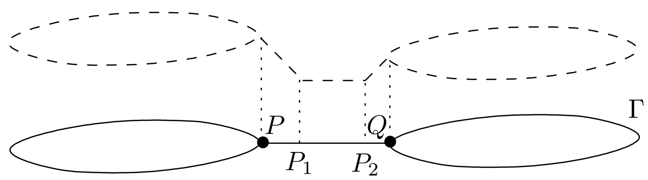
The space $R(D)$ is then simply the quotient of $S(D)$ by the affine linear action of the permutation group of the $P_1, \ldots, P_n$, and hence is a polyhedral complex as well. $\square$

Remark 3.1.12
For the spaces $\tilde{R}(D)$ and $\tilde{S}(D)$ the same argument as in the proof of lemma 3.1.11 holds, with the only exception that the data (c) becomes discrete since the points in (f) are required to be $\mathbb{Z}$-points. Hence the only continuous parameter left is the additive constant (d), i.e. both $\tilde{R}(D)$ and $\tilde{S}(D)$ are finite unions of real lines. We can thus regard $\tilde{R}(D)$ and $\tilde{S}(D)$ as "discrete versions" of the spaces $R(D)$ and $S(D)$.

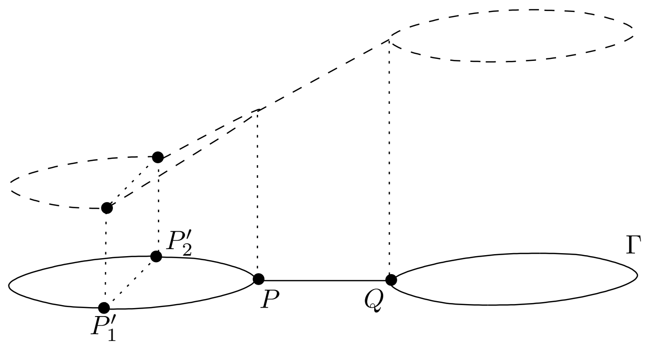
The following example shows that the polyhedral complexes $R(D)$ and $S(D)$ are in general not pure-dimensional, i.e. there may exist inclusion-maximal cells of different dimensions:

Example 3.1.13
Consider the canonical divisor $K_\Gamma = P + Q$ of the metric graph Γ obtained by connecting two cycles C_1 and C_2 of length 1 by an edge E of length $l(E) \in \mathbb{Z}_{>0}$ (see the picture below). Furthermore, let f be a rational function on Γ such that $(f) + K_\Gamma = P_1 + P_2$.

Assume first that both P_1 and P_2 lie in the interior of the edge E. Note that for all such choices of the points P_i there exists (up to an additive constant) exactly one rational function with zeros at P_1 and P_2 and poles at the prescribed points P and Q. It follows that the corresponding cell in $S(K_\Gamma)$ can be identified with $[0, l(E)] \times [0, l(E)] \times \mathbb{R}$, where the first two factors represent the position of the points P_1 and P_2, and the last factor parametrizes the additive constant. Hence the dimension of this cell in $S(K_\Gamma)$ is 3.

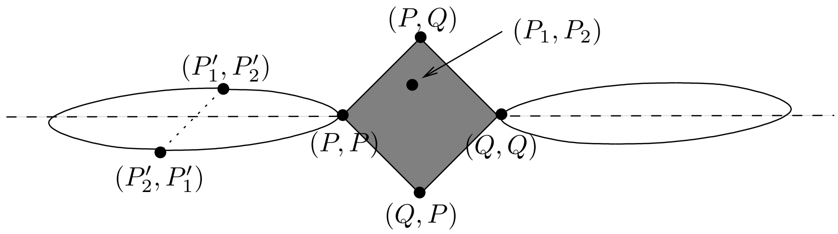


Next, assume that $(f, P_1', P_2') \in S(K_\Gamma)$ such that P_1' is not on the closure of E but rather in the interior of a cycle C_i. We will see in lemma 3.2.2 that P_2' must then lie on the same cycle. Moreover, it is easy to check that this requires P_2' to be the point on C_i "opposite" to P_1' as in the following picture:



Hence for each choice of P_1' on one of the cycles there exists exactly one point P_2' such that $(f, P_1', P_2') \in S(K_\Gamma)$. It follows that this cell of $S(K_\Gamma)$ can be identified with $C_i \times \mathbb{R}$, where the second factor parametrizes the additive constant as above. In particular, the dimension of this cell is 2.

Putting all this we obtain the following schematic picture of the polyhedral complex $S(K_\Gamma)$, where for simplicity we have omitted the factor $\mathbb{R}$ corresponding to the additive constant in all cells:



The space $R(K_\Gamma)$ is then obtained from this by dividing out the action of the symmetric group on two elements, which can be realized geometrically by "folding $S(K_\Gamma)$ along the dashed line above". In particular, both $R(K_\Gamma)$ and $S(K_\Gamma)$ are not pure-dimensional, but rather have components of dimensions 2 and 3.

The above example shows that when formulating a Riemann-Roch type statement about the dimensions of the spaces $R(D)$ we have to be careful since these dimensions are ill-defined in general. The following definition will serve as a replacement:

Definition 3.1.14 ($r(D)$, $\tilde{r}(D)$)
Let D be a divisor of degree n on a tropical curve Γ.

(a) We define $r(D)$ to be the number

$$r(D) = \min_{E \geq 0,\ R(D-E)=\emptyset} \deg(E) - 1.$$

(b) If D is a $\mathbb{Z}$-divisor on a $\mathbb{Z}$-graph Γ there is also a corresponding "discrete version":

$$\tilde{r}(D) = \min_{E \geq 0,\ \tilde{R}(D-E)=\emptyset} \deg(E) - 1,$$

where E is a $\mathbb{Z}$-divisor on Γ.

If we want to specify the curve Γ in the notation of these numbers we will also write them as $r_\Gamma(D)$ and $\tilde{r}_\Gamma(D)$, respectively.

Example 3.1.15 (a) By remark 3.1.9 (a) it is clear that $r(D) = -1$ if $\deg D < 0$, and $r(D) \leq \deg D$ otherwise. The same statement holds for $\tilde{r}(D)$ for $\mathbb{Z}$-divisors on $\mathbb{Z}$-graphs.

(b) For the canonical divisor of the metric graph in example 3.1.13 we have $r(K_\Gamma) = 1$ since we have seen that

- for all points $P_1 \in \Gamma$ there is a rational function f with $(f) + K_\Gamma = P_1 + P_2$ (i.e. $f \in R(K_\Gamma - P_1)$);
- for some choice of $P_1, P_2 \in \Gamma$ (e.g. P_1 and P_2 in the interior of the circles C_1 and C_2, respectively) there is no rational function f with $(f)+K_\Gamma = P_1+P_2$.

(c) Let Γ be a metric graph, and let $\lambda \in \mathbb{R}_{>0}$. By a *rescaling* of Γ by λ we mean the metric graph of the same combinatorics as Γ where we replace each edge E of length $l(E)$ by an edge of length $\lambda \cdot l(E)$. Note that any divisor (resp. rational function) on Γ gives rise to an induced divisor (resp. rational function) on the rescaling by also rescaling the positions of the points (resp. the values of the function). In particular, the numbers $r(D)$ for a divisor D on Γ remain constant under rescalings. Note that rescalings by positive integers take $\mathbb{Z}$-graphs and $\mathbb{Z}$-divisors again to $\mathbb{Z}$-graphs and $\mathbb{Z}$-divisors, but that $\tilde{r}(D)$ may change in this case since the rescaling introduces new $\mathbb{Z}$-points.

Remark 3.1.16
By the proof of lemma 3.1.11 the continuous parameters for the elements $(f, P_1, \dots, P_n)$ of $S(D)$ are the positions of the points P_i and the value of f at a chosen vertex. In particular, when passing from $S(D)$ to $S(D-P)$ for a generic choice of P this fixes one of the P_i and thus makes each cell of $S(D)$ (disappear or) one dimension smaller. It follows that the maximal dimension of the cells of $S(D)$ (and $R(D)$) is always at least $r(D)+1$ (with the $+1$ coming from the additive constant, i.e. the value of the functions at the chosen vertex).

Remark 3.1.17
There is another interpretation of the numbers $r(D)$ that we will need later: let D be a divisor of degree n on a tropical curve Γ, let $i \in \{0, \dots, n\}$, and assume that $S(D) \neq \emptyset$. Consider the forgetful maps

$$\pi_i : S(D) \to \Gamma^i, \quad (f, P_1, \dots, P_n) \mapsto (P_1, \dots, P_i).$$

Note that these maps are morphisms of polyhedral complexes in the sense of definition 2.1.2 (i.e. they map each cell of the source to a single cell in the target by an affine linear map). It is clear by definition that the number $r(D)$ can be interpreted using these maps as the biggest integer i such that the map π_i is surjective.

Example 3.1.18

Consider again the metric graph Γ of example 3.1.13, but now the spaces $R(D)$ and $S(D)$ for the divisor $D = P' + Q'$, where P' are Q' are interior points of the cycles C_1 and C_2, respectively. In this case lemma 3.2.2 will tell us that (f, P_1, P_2) can only be in $S(D)$ if each cycle C_i contains one of the points P_1, P_2, which is then easily seen to require that in fact $\{P, Q\} = \{P_1, P_2\}$, i.e. that f is a constant function. It follows that $R(D)$ is simply the real line, whereas $S(D)$ is two disjoint copies of $\mathbb{R}$ (i.e. both spaces have pure dimension 1). It also follows in the same way that $r(D) = 0$.

In particular, when comparing this to the result of examples 3.1.13 and 3.1.15 (b) (which can be regarded as the limit case when $P' \to P$ and $Q' \to Q$) we see that $r(D)$ can jump, and that the spaces $R(D)$ and $S(D)$ can change quite drastically under "continuous deformations of D". So as in the classical case it is really only the number $r(D) - r(K_\Gamma - D)$, and not $r(D)$ alone, that will turn out to depend on the degree of D and the genus of Γ only.

3.2 Riemann-Roch for $\mathbb{Q}$-divisors

We will now start with the study of Riemann-Roch theorems. Our basic ingredient is the Riemann-Roch theorem for finite (non-metric) graphs of M. Baker and S. Norine that is easily translated into our set-up:

Theorem 3.2.1 ([BN07] theorem 1.11)
Let Γ be a $\mathbb{Z}$-graph of genus g all of whose edge lengths are bigger than 1. Then for every $\mathbb{Z}$-divisor D on Γ we have $\tilde{r}(D) - \tilde{r}(K_\Gamma - D) = \deg D + 1 - g$.

Sketch of proof. We start by replacing each edge e of Γ by a chain of $l(e)$ edges of length 1, arriving at a graph whose geometric representation is the same as before, and where all $\mathbb{Z}$-points that were in the interior of an edge have been turned into 2-valent vertices. Note that by the condition that all edge lengths of the original graph are bigger than 1 this implies that the new graph has no loops, i.e. no edges whose two boundary points coincide (an assumption made throughout in [BN07]). As it is clear by definition that none of the terms in the Riemann-Roch equation changes under this transformation it suffices to prove the theorem for the new graph. By abuse of notation we will also denote it by Γ.

Note that every rational function f on Γ whose divisor is a $\mathbb{Z}$-divisor is uniquely determined by its values on the vertices (since it is just given by linear interpolation on the edges). Moreover, up to a possibly non-integer global additive constant all these values of f on the vertices are integers. Conversely, every integer-valued function on the vertices of Γ gives rise to a rational function on Γ (by linear interpolation) whose divisor is a $\mathbb{Z}$-divisor. As all edge lengths in Γ are 1 the divisor (f) can then be rewritten using this correspondence as

$$(f) = \sum_{\overline{PQ}} (f(Q) - f(P)) \cdot (P - Q) \qquad (*)$$

where the sum is taken over all edges of Γ (and P and Q denote the boundary vertices of these edges in any order). In particular, for a $\mathbb{Z}$-divisor D the number $\tilde{r}(D)$ can also be defined as the maximum number k such that for each choice of vertices $P_1, \dots, P_k$ of Γ there is an integer-valued function f on the vertices of Γ such that $(f) + D$ is effective, where (f) is defined by $(*)$.

This is the approach taken by M. Baker and S. Norine [BN07]. They establish the Riemann-Roch theorem in this set-up, thus proving the theorem as stated above. To prove their theorem their first step is to show its equivalence to the following two statements:

- $\tilde{r}(K_\Gamma) \geq g - 1$; and
- for any $\mathbb{Z}$-divisor $D \in \operatorname{Div}(\Gamma)$ there exists a $\mathbb{Z}$-divisor $E \in \operatorname{Div}(\Gamma)$ with $\deg(E) = g - 1$ and $\tilde{r}(E) = -1$ such that exactly one of the sets $\tilde{R}(D)$ and $\tilde{R}(E - D)$ is empty.

The central idea in the proof of these two statements is then to consider total orderings on the vertices of Γ. For each such ordering there is an associated divisor

$$E = \sum_{e \in E(\Gamma)} m(e) - \sum_{P \in V(\Gamma)} P$$

where $m(e)$ denotes the boundary point of e that is the bigger one in the given ordering — the divisor E in the second statement above can for example be taken to be of this form for a suitable ordering (that depends on D). For details of the proof see [BN07]. □

In order to pass from the "discrete case" (the spaces $\tilde{R}(D)$) to the "continuous case" (the spaces $R(D)$) we need a few lemmas first.

Lemma 3.2.2
Let D be a $\mathbb{Z}$-divisor on a $\mathbb{Z}$-graph Γ, and let $(f, P_1, \dots, P_n) \in S(D)$. Assume moreover that some P_i is not a $\mathbb{Z}$-point. Then on every cycle of Γ containing P_i there is another point P_j (with $i \neq j$) that is also not a $\mathbb{Z}$-point.

Proof. Assume that C is a cycle containing exactly one simple zero $P = P_i \in \Gamma \backslash \Gamma_{\mathbb{Z}}$ (note that if P is a multiple zero then we are done). Consider the cycle C to be the interval $[0, l(C)]$ with the endpoints identified such that the zero point lies on a vertex, and let $x \in \{1, \dots, l(C)\}$ be the integer such that $P \in (x - 1, x)$ with this identification. By adding a suitable constant to f we may assume that $f(x - 1) \in \mathbb{Z}$. Since $P \in (x - 1, x)$ and the slope of f on the interval $[x-1, P]$ differs from that on the interval (P, x) by 1 we conclude that $f(x) \notin \mathbb{Z}$. As all other points of non-differentiability of f on $[0, l(C)]$ are

$\mathbb{Z}$-points by assumption it follows that $f(Q) \in \mathbb{Z}$ for all $Q = 0, \dots, x-1$ and $f(Q) \notin \mathbb{Z}$ for all $Q = x, \dots, l(C)$. In particular, we see that $f(0) \neq f(l(C))$, in contradiction to the continuity of f. □

Lemma 3.2.3
For every $\mathbb{Z}$-divisor D on a $\mathbb{Z}$-graph Γ with $R(D) \neq \emptyset$ we have $\tilde{R}(D) \neq \emptyset$.

Proof. We will prove the statement by induction on $n := \deg D$.

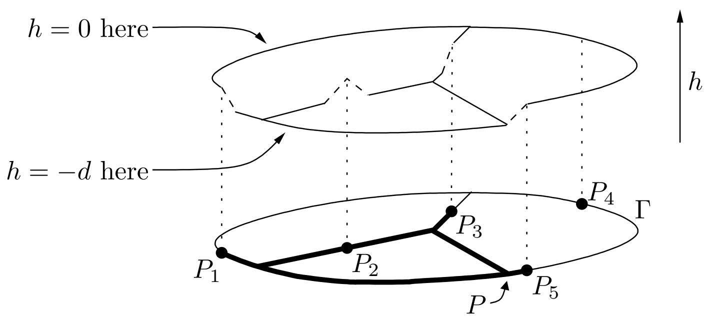
Let $f \in R(D)$, so that $(f) + D = P_1 + \cdots + P_n$ for some (not necessarily distinct) points $P_i \in \Gamma$. In particular, this requires of course that $n \geq 0$. Moreover, if $n = 0$ then $(f) = -D$ is a $\mathbb{Z}$-divisor and hence $f \in \tilde{R}(D)$. As this finishes the proof in the case $n \leq 0$ we can assume from now on that $n > 0$, and that the statement of the lemma is true for all divisors of degree less than n.

If $P_i \in \Gamma_{\mathbb{Z}}$ for some i then $f \in R(D - P_i)$ and hence $\tilde{R}(D - P_i) \neq \emptyset$ by the induction assumption. As this implies $\tilde{R}(D) \neq \emptyset$ we have proven the lemma in this case and may thus assume from now on that none of the P_i is a $\mathbb{Z}$-point of the curve.

After possibly relabeling the points P_i we may assume in addition that $0 < \operatorname{dist}(P_n, \Gamma_{\mathbb{Z}}) \leq \operatorname{dist}(P_i, \Gamma_{\mathbb{Z}})$ for all $i \in [n]$, i.e. that P_n is a point among the P_i that minimizes the distance to the $\mathbb{Z}$-points of the curve. Let $P \in \Gamma_{\mathbb{Z}}$ be a point with $\operatorname{dist}(P_n, P) = \operatorname{dist}(P_n, \Gamma_{\mathbb{Z}}) =: d$, and let $\Gamma' \subset \Gamma$ be the connected component of $\Gamma \backslash \{P_1, \dots, P_n\}$ that contains P. With this notation consider the rational function

$$h : \Gamma \to \mathbb{R}, \quad Q \mapsto \begin{cases} -\min(d, \operatorname{dist}(Q, \{P_1, \dots, P_n\})) & \text{if } Q \in \Gamma', \\ 0 & \text{otherwise.} \end{cases}$$

The following picture shows an example of this construction. In this example we have assumed for simplicity that all edges of the graph have length 1 so that $\Gamma_{\mathbb{Z}}$ is just the set of vertices. The distance from P_5 to P is smallest among all distances from the P_i to a vertex, and the subset $\Gamma' \subset \Gamma$ is drawn in bold.



We claim that $f + h \in R(D - P)$. In fact, this will prove the lemma since $R(D - P) \neq \emptyset$ implies $\tilde{R}(D - P) \neq \emptyset$ and thus also $\tilde{R}(D) \neq \emptyset$ by the induction assumption.

To prove that $f + h \in R(D - P)$ we have to show that $(f + h) + D - P \geq 0$, or in other words that $(h) + P_1 + \cdots + P_n - P \geq 0$. Let us assume that this statement is false, i.e. that there is a point $Q \in \Gamma$ that is contained in the divisor $(h) + P_1 + \cdots + P_n - P$ with a negative coefficient. Note that Q cannot be the point P since $\operatorname{ord}_P h \geq 1$ by construction. So Q must be a pole of h. But again by construction h can only have poles at the points P_i, and the order of the poles can be at most 2 since the slope of h is 0 or ± 1 everywhere. So the only possibility is that Q is a point with $\operatorname{ord}_Q h = -2$ that occurs only once among the P_i (as it is the case for $Q = P_2$ in the example above). But this means that Γ' contains both sides of Q, and thus (since Γ' is connected) that $\Gamma' \cup \{Q\}$ contains a cycle on which Q is the only point in (f) that is not a $\mathbb{Z}$-point. But this is a contradiction to lemma 3.2.2 and hence finishes the proof of the lemma. □

Proposition 3.2.4
Let D be a $\mathbb{Z}$-divisor on a $\mathbb{Z}$-graph Γ. Then there is an integer $N \geq 1$ such that $r(D) = \tilde{r}(D)$ on every rescaling of Γ by an integer multiple of N (see example 3.1.15 (c)).

Proof. Let $n := \deg D$ and $m := r(D) + 1$, and assume first that $m \leq n$. Consider the map $\pi_m : S(D) \to \Gamma^m$ of remark 3.1.17. As π_m is a morphism of polyhedral complexes its image is closed in Γ^m. Since π_m is not surjective by remark 3.1.17 this means that $\Gamma^m \backslash \pi_m(S(D))$ is a non-empty open subset of Γ^m that consequently must contain an element $(P_1, \ldots, P_m)$ with rational coordinates. For this element we have $S(D - P_1 - \cdots - P_m) = \emptyset$ by construction.

Now let N be the least common multiple of the denominators of these coordinates. Then $P_1, \ldots, P_m$ become $\mathbb{Z}$-points on each rescaling of Γ by a multiple of N, and thus we also have $\tilde{S}(D - P_1 - \cdots - P_m) = \emptyset$ on each such rescaling by remark 3.1.9 (c). By definition this then means that $\tilde{r}(D) \leq m - 1 = r(D)$ on these rescalings. This proves the "$\tilde{r}(D) \leq r(D)$" part of the proposition in the case $m \leq n$. But note that this part is trivial if $m > n$, since then $\tilde{r}(D) \leq n \leq m - 1 = r(D)$ by example 3.1.15 (a) (on any rescaling). So we have in fact proven the "$\tilde{r}(D) \leq r(D)$" part of the proposition in any case.

To show the opposite inequality "$\tilde{r}(D) \geq r(D)$" (which in fact holds for any rescaling) we just have to show that $\tilde{R}(D - P_1 - \cdots - P_{r(D)}) \neq \emptyset$ for any choice of $\mathbb{Z}$-points $P_1, \ldots, P_{r(D)}$. But this now follows immediately from lemma 3.2.3 since $R(D - P_1 - \cdots - P_{r(D)}) \neq \emptyset$ by definition. □

We are now ready to prove the Riemann-Roch theorem for $\mathbb{Q}$-divisors on $\mathbb{Q}$-graphs.

Corollary 3.2.5 (Riemann-Roch for $\mathbb{Q}$-graphs)
Let D be a $\mathbb{Q}$-divisor on a $\mathbb{Q}$-graph Γ. Then the following equation holds:

$$r(D) - r(K_\Gamma - D) = \deg D + 1 - g.$$

Proof. Note that it suffices by example 3.1.15 (c) to prove the statement after a rescaling of the curve.

As Γ has only finitely many edges and D contains only finitely many points we can assume after such a rescaling that D is in fact a $\mathbb{Z}$-divisor on a $\mathbb{Z}$-graph Γ, and that all edge lengths of Γ are bigger than 1. By proposition 3.2.4 we can then assume after possibly two more rescalings that both $r(D) = \tilde{r}(D)$ and $r(K_\Gamma - D) = \tilde{r}(K_\Gamma - D)$. The corollary now follows from theorem 3.2.1. □

3.3 Riemann-Roch for tropical curves

We will now extend our Riemann-Roch theorem for $\mathbb{Q}$-graphs (corollary 3.2.5) in two steps, first to metric graphs (i.e. graphs whose edge lengths need not be rational numbers) and then to tropical curves (i.e. graphs with possibly unbounded edges).

Proposition 3.3.1 (Riemann-Roch for metric graphs)
For any divisor D on a metric graph Γ of genus g we have

$$r(D) - r(K_\Gamma - D) = \deg D + 1 - g.$$

Proof. Let $D = a_1 Q_1 + \cdots + a_m Q_m$, and let $n = \deg D$. The idea of the proof is to find a "nearby" $\mathbb{Q}$-graph Γ' with a $\mathbb{Q}$-divisor D' on it such that $r_{\Gamma'}(D') = r_\Gamma(D)$ and $r_{\Gamma'}(K_{\Gamma'} - D') = r(K_\Gamma - D)$, and then to apply the result of corollary 3.2.5 to this case.

To do so we will set up a relative version of the spaces $S(D)$ of definition 3.1.8 and the interpretation of $r(D)$ of remark 3.1.17 in terms of these spaces. We fix $\varepsilon \in \mathbb{Q}_{>0}$ smaller than all edge lengths of Γ and denote by $A(\Gamma)$ the set of all metric graphs that are of the same combinatorial type as Γ and all of whose edge lengths are greater or equal to ε. For such a metric graph $\Gamma' \in A(\Gamma)$ we denote by $B(\Gamma')$ the set of all divisors on Γ' that can be written as $a_1 Q'_1 + \cdots + a_m Q'_m$ for some $Q'_1, \dots, Q'_m \in \Gamma'$ and the same $a_1, \dots, a_m$ as in D. With these notations we set

$$\begin{aligned} S &:= \{(\Gamma', D', f, P_1, \dots, P_n);\ \Gamma' \in A(\Gamma),\ D' \in B(\Gamma'),\ f \text{ a rational function on } \Gamma', \\ &\qquad\qquad P_1, \dots, P_n \in \Gamma' \text{ such that } (f) + D' = P_1 + \cdots + P_n\}, \\ M_i &:= \{(\Gamma', D', P_1, \dots, P_i);\ \Gamma' \in A(\Gamma),\ D' \in B(\Gamma'),\ P_1, \dots, P_i \in \Gamma'\} \qquad \text{for } i = 0, \dots, n, \\ M &:= \{(\Gamma', D');\ \Gamma' \in A(\Gamma),\ D' \in B(\Gamma')\} \end{aligned}$$

In the same way as in lemma 3.1.11 we see that all these spaces are polyhedral complexes — the only difference is that there is some more discrete data (corresponding to fixing the edges or vertices on which the points in D' lie) and some more continuous data (corresponding to the edge lengths of Γ' and the positions of the points in D' on their respective edges). There are obvious forgetful morphisms of polyhedral complexes (i.e. continuous maps that send each cell of the source to a single cell of the target by an affine linear map)

$$\pi_i : S \to M_i, \quad (\Gamma', D', f, P_1, \dots, P_n) \mapsto (\Gamma', D', P_1, \dots, P_i)$$

and

$$p_i : M_i \to M, \quad (\Gamma', D', P_1, \dots, P_i) \mapsto (\Gamma', D').$$

As in remark 3.1.17 we have $r_{\Gamma'}(D') \geq i$ for a divisor $D' \in B(\Gamma')$ on a metric graph $\Gamma' \in A(\Gamma)$ if and only if $\pi_i(S)$ contains $(\Gamma', D', P_1, \dots, P_i)$ for all $P_1, \dots, P_i \in \Gamma'$, or equivalently if and only if $(\Gamma', D') \in M \backslash p_i(M_i \backslash \pi_i(S))$.

Since S is a polyhedral complex and π_i a morphism of polyhedral complexes it follows that the image $\pi_i(S) \subset M_i$ is a union of closed polyhedra. Consequently, $M_i \backslash \pi_i(S)$ is a union of open polyhedra (i.e. an open subset of M_i whose intersection with each polyhedron of M_i can be written as a union of spaces given by finitely many strict linear inequalities).

Next, note that the map p_i is open as it is locally just a linear projection. It follows that $p_i(M_i \backslash \pi_i(S))$, i.e. the locus in M of all (Γ', D') such that $r_{\Gamma'}(D') < i$, is a union of open polyhedra as well. Consequently, its complement $M \backslash p_i(M_i \backslash \pi_i(S))$, i.e. the locus in M of all (Γ', D') such that $r_{\Gamma'}(D') \geq i$, is a union of closed polyhedra. Finally, note that all polyhedral complexes and morphisms involved in our construction are defined over $\mathbb{Q}$, so that the locus of all (Γ', D') with $r_{\Gamma'}(D') < i$ (resp. $r_{\Gamma'}(D') \geq i$) is in fact a union of *rational* open (resp. closed) polyhedra in M. Of course, the same arguments hold for $r_{\Gamma'}(K_{\Gamma'} - D')$ as well.

We are now ready to finish the proof of the proposition. By what we have said above the locus of all (Γ', D') in M such that $r_{\Gamma'}(D') < r_\Gamma(D)+1$ and $r_{\Gamma'}(K_{\Gamma'}-D') < r_\Gamma(K_\Gamma-D)+1$ is an open neighborhood U of (Γ, D). Conversely, the locus of all (Γ', D') in M such that $r_{\Gamma'}(D') \geq r_\Gamma(D)$ and $r_{\Gamma'}(K_{\Gamma'} - D') \geq r_\Gamma(K_\Gamma - D)$ is a union V of rational closed polyhedra. In particular, this means that the rational points of V are dense in V. As $U \cap V$ is non-empty (it contains the point (Γ, D)) it follows that there is a rational point in $U \cap V$, i.e. a $\mathbb{Q}$-graph Γ' with a $\mathbb{Q}$-divisor D' on it such that $r_{\Gamma'}(D') = r_\Gamma(D)$ and $r_{\Gamma'}(K_{\Gamma'} - D') = r_\Gamma(K_\Gamma - D)$. As Γ' and Γ have the same genus, and D' and D the same degree, the proposition now follows from corollary 3.2.5. □

So far we have only considered metric graphs, i.e. tropical curves in which every edge is of finite length. In our final step of the proof of the Riemann-Roch theorem we will now extend this result to arbitrary tropical curves (with possibly infinite edges). In order to do this we will first introduce the notion of equivalence of divisors.

Definition 3.3.2
Two divisors D and D' on a tropical curve Γ are called *equivalent* (written $D \sim D'$) if there exists a rational function f on Γ such that $D' = D + (f)$.

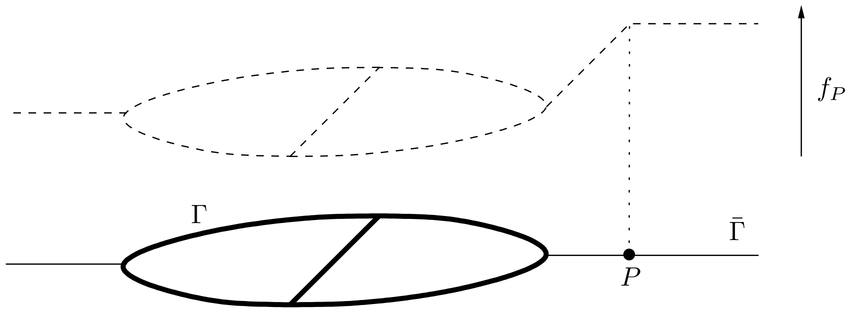
Remark 3.3.3
If $D \sim D'$, i.e. $D' = D + (f)$ for a rational function f, then it is obvious that the map $R(D') \to R(D)$, $g \mapsto g + f$ is a bijection. In particular, this means that $r(D) = r(D')$, i.e. that the function $r : \mathrm{Div}(\Gamma) \to \mathbb{Z}$ depends only on the equivalence class of D.

Lemma 3.3.4
Let $\bar{\Gamma}$ be a tropical curve, and let Γ be the metric graph obtained from $\bar{\Gamma}$ by removing all unbounded edges. Then every divisor $D \in \operatorname{Div}(\bar{\Gamma})$ is equivalent on $\bar{\Gamma}$ to a divisor D' with $\operatorname{supp} D' \subset \Gamma$. *Moreover, if D is effective then D' can be chosen to be effective as well.*

Proof. To any $P \in \bar{\Gamma}$, we associate a rational function f_P as follows. If $P \in \Gamma$, we define f_P to be the zero function. Otherwise, if P lies on some unbounded edge E, we define

$$f_P : \bar{\Gamma} \to \mathbb{R} \cup \{\infty\}, \quad Q \mapsto \begin{cases} \min(\operatorname{dist}(P,\Gamma), \operatorname{dist}(Q,\Gamma)) & \text{if } Q \in E, \\ 0 & \text{if } Q \notin E. \end{cases}$$

If $P \notin \Gamma$, then the function f_P has a simple pole at P and no other zeros or poles away from Γ. The following picture shows an example of such a function, where the metric graph Γ is drawn in bold:



So if $D = a_1P_1 + \cdots + a_nP_n$ and we set $f = \sum_i a_i f_{P_i}$ then $D + (f)$ is a divisor equivalent to D with no zeros or poles away from Γ. Moreover, if D is effective then $D + (f)$ is effective as well since all poles of f are cancelled by D by construction. □

Remark 3.3.5
With notations as above, let $P_1, \cdots, P_n$ denote the end points of the unbounded edges E_i of $\bar{\Gamma}$, and consider the function $f = \sum_i f_{P_i}$. Then f is zero on the graph Γ and has slope one on each unbounded edge. If we denote for all $i \in [n]$ the point $E_i \cap \Gamma$ by Q_i, then $(f) = \sum Q_i - \sum P_i$. Hence $K_\Gamma + (f) = K_{\bar{\Gamma}}$, i.e. $K_\Gamma \sim K_{\bar{\Gamma}}$ on $\bar{\Gamma}$.

Lemma 3.3.6
As in the previous lemma let $\bar{\Gamma}$ be a tropical curve, and let Γ be the metric graph obtained from $\bar{\Gamma}$ by removing all unbounded edges. Moreover, let D be a divisor on Γ (that can then also be thought of as a divisor on $\bar{\Gamma}$ with support on Γ). Then $R_\Gamma(D) \neq \emptyset$ if and only if $R_{\Gamma'}(D) \neq \emptyset$.

Proof. "⇒": Let f be a rational function in $R_\Gamma(D)$. Extend f to a rational function $\bar{f}$ on $\bar{\Gamma}$ so that it is constant on each unbounded edge. Then $\bar{f} \in R_{\bar{\Gamma}}(D)$.

"⇐": Let $\bar{f} \in R_{\bar{\Gamma}}(D)$, and set $f = \bar{f}|_\Gamma$. Let e be an unbounded edge of $\bar{\Gamma}$, and let $P = \Gamma \cap e$ be the vertex where e is attached to Γ. Since $\bar{f}$ has no poles on e it follows that $\bar{f}|_e$ is (not necessarily strictly) decreasing if we identify e with the real interval $[0, \infty]$. Hence the order of f on Γ at P cannot be less than the order of $\bar{f}$ on $\bar{\Gamma}$ at P, and so it follows that $f \in R_\Gamma(D)$. □

Remark 3.3.7
Let $\bar{\Gamma}$, Γ, and D as in lemma 3.3.6. By lemma 3.3.4 any effective divisor $P_1 + \cdots + P_k$ on $\bar{\Gamma}$ is equivalent to an effective divisor $P'_1 + \cdots + P'_k$ with support on Γ. So by remark 3.3.3 the number $r_{\bar{\Gamma}}(D)$ can also be thought of as the biggest integer k such that $R_{\bar{\Gamma}}(D - P_1 - \cdots - P_k) \neq \emptyset$ for all $P_1, \ldots, P_k \in \Gamma$ (instead of for all $P_1, \ldots, P_k \in \bar{\Gamma}$). By lemma 3.3.6 we can therefore conclude that $r_{\bar{\Gamma}}(D) = r_\Gamma(D)$.

With these results we are now able to prove a Riemann-Roch theorem for tropical curves:

Corollary 3.3.8 (Riemann-Roch for tropical curves)
For any divisor D on a tropical curve $\bar{\Gamma}$ of genus g we have $r(D) - r(K_{\bar{\Gamma}} - D) = \deg D + 1 - g$.

Proof. Let Γ be the metric graph obtained from $\bar{\Gamma}$ by removing all unbounded edges. By lemma 3.3.4 and remark 3.3.3 we may assume that $\operatorname{supp} D \subset \Gamma$. Moreover, by remark 3.3.5 we can replace $K_{\bar{\Gamma}}$ by K_Γ (which also has support in Γ) in the Riemann-Roch equation. Finally, using remark 3.3.7 we may replace $r_{\bar{\Gamma}}(D)$ and $r_{\bar{\Gamma}}(K_\Gamma - D)$ by $r_\Gamma(D)$ and $r_\Gamma(K_\Gamma - D)$ respectively, so that the statement follows from proposition 3.3.1. □

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