Gromov-Witten invariants of hypersurfaces

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Preface

Over the last decade Gromov-Witten invariants have become an invaluable tool in modern enumerative geometry. The goal of the theory is to count curves with some given conditions in a complex projective manifold, where the conditions are chosen so that the expected number of curves satisfying them is finite and non-zero.

The most famous example of an enumerative problem has in fact been inspired by physics: let X be a quintic threefold, i.e. a smooth hypersurface of degree 5 in complex projective 4-space $\mathbb{P}^4$. For any $g \geq 0$ and $d > 0$ a naïve dimension count shows that one expects a finite number $N_{g,d}$ of curves of genus g and degree d on X. In 1991 the string theorists Candelas et al. encountered this problem in their study of conformal field theories, and their computations led them to predictions for the numbers $N_{0,d}$ in genus zero [**COGP**]. Their main tool was a certain equivalence of topological field theories that is known to physicists as "mirror symmetry". The discovery of this relation between physics and algebraic geometry was the starting point of modern enumerative geometry.

The analysis of Candelas et al. showed already that in order to count *curves* embedded in X it is actually more natural to study *maps* from a (varying) source curve to the target X. These two notions are of course very much related as every map from a curve to X determines an image curve, and conversely every curve in X arises as the image of some map from a curve to X. There are subtle differences however: the map could e.g. be a k-fold cover onto its image, in which case the degree of the map is k times the degree of the set-theoretic image curve. In fact, the physicists' mirror symmetry computations first led to a different set of numbers $n_{0,d}$ that are supposed to count maps of degree d from a source curve of genus $g = 0$ to X. These numbers are called the Gromov-Witten invariants. They are the coefficients of a generating function that can be obtained from a certain hypergeometric series by an explicit variable transformation. The enumerative numbers $N_{0,d}$ are then related to the Gromov-Witten invariants $n_{0,d}$ by a correction formula that takes care of the contributions from multiple covers. The precise form of this correction formula was simply guessed by the time the paper by Candelas et al. was written.

In the following years physicists have vastly generalized the above results. Bershadsky et al. considered the case of higher genus. They conjectured a generating function for the Gromov-Witten invariants $n_{1,d}$ in genus 1 [**BCOV1**] and an algorithm that can in principle compute the invariants $n_{g,d}$ in any genus [**BCOV2**]. Gopakumar and Vafa have conjectured the general form of the transformation that relates the Gromov-Witten invariants $n_{g,d}$ to the enumerative numbers $N_{g,d}$ for any g [**GoV1, GoV2**]. A non-trivial check for these conjectures is that the resulting numbers $N_{g,d}$ actually turned out to be positive integers as far as they have been computed numerically — a statement that is far from being obvious from the arithmetic of the algorithms.

In the meantime mathematicians were (and to a large extent still are) just puzzled by the physicists' results. To start from the very basics of the problem, it has not even been proven yet that the enumerative number of genus-0 curves in a general quintic threefold X is at all finite. For higher genus the numbers are most definitely not finite since e.g. intersections of X with planes in $\mathbb{P}^4$ lead to infinite families of curves. Neither are the numbers of maps from curves to X of given genus and degree finite, not even in genus 0: multiple covers have ramification points whose position on the curve can vary continuously. So before mathematicians could try to compute the numbers $n_{g,d}$ or $N_{g,d}$ their first problem was simply to define them rigorously.

The problem of defining Gromov-Witten invariants mathematically has been solved around 1995 by the theory of stable maps initiated by Kontsevich [**K2**]. For any projective manifold X, genus $g \geq 0$, and homology class β of a curve in X, Behrend and Manin have constructed a "nice" moduli space (read: separated and proper Deligne-Mumford stack) that parametrizes curves of genus g together with a map of class β to X [**BM**]. The dimension one would expect such a moduli space to have by a naïve dimension count is called its *virtual dimension*; it is zero for the quintic threefold for any g and β. The actual dimension of the moduli space may in general well be bigger. But in any case one can construct a so-called *virtual fundamental class* of the moduli space: a cycle of the virtual dimension that replaces the ordinary fundamental class in homology [**BF, B, LT1**]. The virtual fundamental class thus makes the moduli space appear to have the virtual dimension for intersection-theoretic purposes. The idea is roughly the same as that of an excess intersection product: if Y and Z are subvarieties of X then the dimension of the intersection $Y \cap Z$ may be bigger than the expected (virtual) dimension $\dim Y + \dim Z - \dim X$, but in any case there is an intersection product $Y \cdot Z$: a homology class of the virtual dimension in $Y \cap Z$.

The Gromov-Witten invariants $n_{g,d}$ of a quintic threefold can now simply be defined to be the degrees of the (zero-dimensional) virtual fundamental classes of the moduli spaces of stable maps of genus g and degree d to the quintic. More generally, let X be any projective manifold and pick a genus g and homology class β such that the corresponding moduli space M of stable maps has non-negative virtual dimension. Moreover, choose a cohomology class on M whose codimension is equal to the virtual dimension of M. Usually this cohomology class is chosen so that it describes the condition that the curves in X (or more precisely: stable maps to X) pass through some fixed given subvarieties of X. Evaluating it on the virtual fundamental class gives a zero-dimensional cycle whose degree we call again a Gromov-Witten invariant. It can be interpreted geometrically as the number of curves in X satisfying the given conditions. (Note that this count may be virtual though in the cases when the virtual fundamental class of M is not equal to the ordinary one.)

Having defined the Gromov-Witten invariants of a projective manifold X the next questions are of course what their structure is and how they can be computed. In general this question is very difficult however and far from being solved. The goal of this thesis is to attack this problem in the case of hypersurfaces. More precisely, our main question can be posed as follows:

> *Let Y be a smooth hypersurface of a complex projective manifold X.*
> *Can we compute the Gromov-Witten invariants of Y from those of X?*

This will mean two things: in a first step, we would like to find an algorithm that can be programmed on a computer and that enables us to compute every possible Gromov-Witten invariant of Y numerically if we know the invariants of X. A second question is whether the relation between the invariants of X and Y can then be phrased in some "nice" way, e.g. in the form of an explicit relation between the generating functions of the invariants for the two spaces. Moreover, we will compute all Gromov-Witten invariants of projective spaces directly. This way every relation between the invariants of a hypersurface and the ambient space that we get can be applied immediately to compute Gromov-Witten invariants of hypersurfaces in projective spaces, including the most interesting case of the quintic threefold.

Before we sketch our strategy to compute the Gromov-Witten invariants of hypersurfaces let us briefly comment on the history of this problem. In 1995 Kontsevich had the idea to apply the Atiyah-Bott localization formula to the natural $(\mathbb{C}^*)^{n+1}$ action on the moduli spaces of stable maps to $\mathbb{P}^n$ [**K2**]. He could thus reduce questions about curves in (hypersurfaces in) $\mathbb{P}^n$ to the study of the fixed point loci of this group action. Even if the combinatorics of these fixed point loci are very complicated the resulting integrals could be evaluated on a computer in some easy cases.

Kontsevich was able to compute the first few rational Gromov-Witten invariants of the quintic threefold numerically in this way. His results agreed with the conjecture of the physicists.

The idea of using the Atiyah-Bott localization formula on moduli spaces of stable maps was quickly taken on. It was applied to slightly different group actions, and the structure of the resulting fixed point loci was studied thoroughly. Using these techniques several people were able to give proofs of the physicists' conjectural "mirror formula" for the rational Gromov-Witten invariants of the quintic threefold, and more generally to prove formulas for the rational invariants of hypersurfaces with non-negative anticanonical bundle in projective spaces [**Be, Gi1, LLY1, LLY2**]. As expected, these methods generalized to hypersurfaces with non-negative anticanonical bundle in general projective manifolds [**L, LLY3**]. Only very modest results could be achieved however in the case of hypersurfaces with negative anticanonical bundle [**J**]. Some generalizations to curves of higher genus were constructed, but none of them has been able yet to produce usable results in practice [**GP, LLY4**].

Meanwhile a completely different technique emerged that was originally not even meant to be applicable to Gromov-Witten invariants of hypersurfaces. Caporaso and Harris were able to compute the Gromov-Witten invariants of $\mathbb{P}^2$ in any genus by degenerating plane curves with given incidence conditions so that they split off a given line in $\mathbb{P}^2$ as an irreducible component [**CH**]. The invariants could then be computed recursively by applying the same techniques again to the remaining components. It turned out that in order to obtain a closed recursion one needs to consider additional invariants that are nowadays called relative Gromov-Witten invariants. In the Caporaso-Harris case they can be thought of geometrically as the numbers of plane curves of specified genus and degree that have given local orders of contact to a fixed line.

Shortly afterwards these degeneration techniques have been generalized by Vakil to rational and elliptic curves in projective spaces of higher dimension [**Va1**]. The relative invariants in this case correspond to numbers of curves in $\mathbb{P}^n$ with given local orders of contact to a fixed hyperplane. However, Vakil's constructions did not make use of virtual fundamental classes to arrive at well-defined invariants in the case of moduli spaces of too big dimension. Instead, he analyzed the moduli spaces directly and simply discarded the components that had too big dimension and did not give rise to an enumerative contribution. As a consequence, Vakil's results are not the Gromov-Witten invariants but rather the enumerative numbers.

Very recently Li has constructed relative Gromov-Witten invariants in full generality, i.e. for curves of any genus in a projective manifold X with fixed local orders of contact to a given hypersurface $Y \subset X$ [**Li1, Li2**]. Li's construction fits well in the original Gromov-Witten picture as he first defines moduli spaces of stable relative maps (which are separated and proper Deligne-Mumford stacks), then constructs natural virtual fundamental classes on them, and finally defines relative Gromov-Witten invariants to be intersection products on these spaces evaluated on the virtual fundamental classes. For actual computations however his constructions have some disadvantages. For example, the moduli spaces are not subspaces of the spaces of ordinary stable maps to X, and moduli spaces for curves with several connected components are not simply the products of the moduli spaces for the individual components. These disadvantages can be overcome by studying slightly different moduli spaces that we will call moduli spaces of collapsed stable relative maps. They can be obtained by a certain blow-down of Li's moduli spaces. If the hypersurface Y is very ample and the genus of the curves is zero then these moduli spaces of collapsed stable relative maps can also be constructed directly without Li's machinery. This direct construction actually predates the work of Li. We will present it in this thesis.

We will then apply the theory of stable relative maps to compute Gromov-Witten invariants of hypersurfaces. The idea is still the same as in the starting work of Caporaso and Harris: we degenerate curves in X so that they split off a component contained in Y. This will give us relations between curves in X and curves in Y, i.e. between the Gromov-Witten invariants of X and Y. In genus zero we will show that these relations are always sufficient to compute the invariants of the hypersurface from those of the ambient space. If the anticanonical bundle of Y is non-negative we will use these relations to give an alternative proof of the "mirror theorem", i.e. of a closed formula that relates the generating functions for the invariants of X to that of Y. The advantage of our methods is that they seem to generalize better to curves of higher genus. In fact, we will show that the same ideas work in genus 1 as well, yielding e.g. the elliptic Gromov-Witten invariants of the quintic threefold from those of $\mathbb{P}^4$. A further generalization to curves of arbitrary genus seems possible. In fact, some of our results are already statements about curves of any genus; the relations just do not suffice yet to reconstruct the invariants from the hypersurface from those of the ambient space.

We should mention that many constructions mentioned above have been studied independently in the language of symplectic geometry. For example, moduli spaces of stable maps have been constructed in [**LT2**], and moduli spaces of stable relative maps in [**IP1, IP2, LR**]. In this thesis we will only be concerned with the algebro-geometric picture.

This thesis is organized as follows. In chapter 1 we will review the construction of Gromov-Witten invariants and briefly prove their basic properties. Using the Virasoro conditions and the topological recursion relations we will then state and prove an explicit algorithm that computes all Gromov-Witten invariants of projective spaces. As an example, the resulting numbers are used to check an integrality conjecture of Pandharipande concerning certain linear combinations of invariants of $\mathbb{P}^3$.

Chapter 2 then starts with the study of relative Gromov-Witten invariants. Let $Y \subset X$ be a very ample hypersurface. For curves of genus zero we construct the moduli spaces of collapsed stable relative maps and their virtual fundamental classes. The main theorem of this chapter describes how degenerations of curves in X to Y can be used to relate the Gromov-Witten invariants of the two spaces. We prove that these relations are sufficient to compute the rational invariants of Y from those of X by an explicit algorithm.

In chapter 3 we then study this algorithm in more detail in the case when the anti-canonical bundle of Y is non-negative. Organizing the invariants and equations in a suitable way we are able to prove the "mirror formula" that relates the generating functions for the invariants of X and Y through an explicit variable transformation. In particular, this gives a proof of the formula for the rational Gromov-Witten invariants of the quintic threefold as conjectured by Candelas et al.

We have mentioned already that (relative) Gromov-Witten invariants are defined using virtual fundamental classes. If these virtual fundamental classes are non-trivial then the enumerative interpretation of the invariants is not a priori obvious. In chapter 4 we will study the geometric meaning of relative Gromov-Witten invariants in the special case of plane conics having prescribed local orders of contact to a given curve. This example is particularly interesting because it allows to check a conjecture concerning family Gromov-Witten invariants of K3 surfaces for non-primitive homology classes.

Finally, chapter 5 gives some extensions of our earlier results to curves of higher genus. We review the construction of the moduli spaces of (non-collapsed) stable relative maps and prove some of their basic properties. Using virtual localization techniques for these moduli spaces in the case of $\mathbb{P}^1$-bundles we arrive at equations that can be used to relate Gromov-Witten invariants of hypersurfaces to those of the ambient space in any genus. As an example, we show that these relations are sufficient to compute the elliptic Gromov-Witten invariants of the quintic threefold from those of $\mathbb{P}^4$. The numbers that we obtain agree numerically with the conjecture of Bershadsky et al. in [**BCOV1**].

Some parts of this thesis have already been published elsewhere. The algorithm in chapter 1 to compute the Gromov-Witten invariants of projective spaces has been described in [**Ga4**]. The main parts of chapters 2, 3, and 4 have appeared in [**Ga1**], [**Ga2**], and [**Ga3**], respectively. The material of chapter 5 has not been published elsewhere yet.

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CHAPTER 1

Gromov-Witten invariants of projective spaces

Let X be a smooth projective variety over the field of complex numbers. The first thing we have to do to solve enumerative problems about curves in X is to construct suitable moduli spaces that parametrize curves in X (or rather maps from curves to X). In modern enumerative geometry the standard moduli spaces are the so-called moduli spaces of stable maps. To construct them we need to fix integers $g,n \geq 0$ and a homology class β of an algebraic curve in X. The corresponding moduli space of stable maps $\bar{M}_{g,n}(X,\beta)$ can then be thought of as a compactification of the space of all tuples $(C,x_1,\dots,x_n,f)$, where

- C is a smooth curve of genus g;
- $x_1,\dots,x_n$ are distinct points on C;
- $f : C \to X$ is a morphism of class $f_*[C] = \beta$.

In the beginning of this chapter we will give a short review of the construction of these spaces. Roughly speaking the compactification is obtained by allowing C to become a nodal curve with several irreducible components. The points in the moduli space corresponding to such reducible stable maps are said to lie in the boundary of $\bar{M}_{g,n}(X,\beta)$.

Enumerative problems on X now simply correspond to intersection products on these moduli spaces. For example, let $\mathrm{ev}_i : \bar{M}_{g,n}(X,\beta) \to X$ be the evaluation at the i-th marked point, i.e. the morphism that sends a stable map $(C,x_1,\dots,x_n,f)$ to $f(x_i)$. If $V_1,\dots,V_n$ are subvarieties of X then the product $\mathrm{ev}_1^*[V_1]\cdots\mathrm{ev}_n^*[V_n]$ on $\bar{M}_{g,n}(X,\beta)$ can be thought of as describing the curves in X that intersect all the V_i. If the resulting cycle has dimension 0 then its degree should just give a count of the curves in X passing through the V_i. (In practice things are not so simple because the intersection product could get non-enumerative contributions from the boundary of the moduli space.)

In addition to these evaluation conditions we will have to consider one more type of cohomology classes on the moduli spaces. Namely, for $1 \leq i \leq n$ we let $\psi_i \in A^1(\bar{M}_{g,n}(X,\beta))$ be the first Chern class of the line bundle on $\bar{M}_{g,n}(X,\beta)$ whose fiber at a point $(C,x_1,\dots,x_n,f)$ is simply the cotangent space $T^\vee_{C,x_i}$. The class ψ_i

is called the *i*-th cotangent line class. We can use these classes in addition to the above evaluation classes to form intersection products on the moduli spaces. If such an intersection product has dimension 0 we will call its degree the corresponding Gromov-Witten invariant.

After having defined moduli spaces of stable maps and Gromov-Witten invariants we will briefly state and prove their main properties. Of particular importance to us will be the so-called topological recursion relations: equations in the Chow group of $\bar{M}_{g,n}(X,\beta)$ that express products of cotangent line classes in terms of boundary classes (i.e. classes on moduli spaces of *reducible* stable maps).

It has been proven recently by E. Ionel that any product of at least g cotangent line classes on $\bar{M}_{g,n}(X,\beta)$ is a sum of boundary cycles [**I**]. Unfortunately, the corresponding topological recursion relations are not yet known explicitly for general g. The $g \leq 2$ cases can be found in [**Ge1**]. In theory, it should be possible to derive the equations for other (at least low) values of g from Ionel's work. As g grows however, the terms in the topological recursion relations become very complicated, and their number seems to grow exponentially. Consequently, Ionel's result is barely useful for actual computations, although it is of course very interesting from a theoretical point of view.

In this chapter we will use a much weaker topological recursion relation that expresses only a product of at least $3g-1$ cotangent line classes at the same point in terms of boundary cycles. The idea to obtain this relation is simple: we just pull back the obvious relation $\psi_1^{3g-1} = 0$ on $\bar{M}_{g,1}$ to $\bar{M}_{g,n}(X,\beta)$ using the transformation rule for cotangent line classes under forgetful maps. The result is a topological recursion relation that is extremely easy to state and apply (see proposition 1.3.12). It has appeared first in [**EX, Ge2, Lu**].

The application of this relation that we have in mind in this chapter is the Virasoro conditions for the Gromov-Witten invariants of projective spaces. It has been proven recently by Givental that the Gromov-Witten potential of a projective space $\mathbb{P}^r$ satisfies an infinite series of differential equations called the Virasoro conditions [**Gi2**]. It is easily checked that these equations allow for recursion over the genus and the number of marked points in the following sense: given $g > 0$, $n \geq 1$, cohomology classes $\gamma_2, \dots, \gamma_n \in A^*(X)$, and non-negative integers $m_2, \dots, m_n$, the Virasoro conditions can express linear combinations of genus-g degree-d invariants (in dimension 0)

$$\mathrm{ev}_1^*\gamma \cdot \psi_1^m \cdot \mathrm{ev}_2^*\gamma_2 \cdot \psi_2^{m_2} \cdot \dots \cdot \mathrm{ev}_n^*\gamma_n \cdot \psi_n^{m_n}$$

(where $m \geq 0$, $\gamma \in A^*(X)$, and the degree $d \geq 0$ vary) in terms of other invariants with either smaller genus, or the same genus and smaller number of marked points.

There is one such invariant for every choice of m, i.e. $r+1$ invariants for every choice of d. There is however only one non-trivial Virasoro condition for every d. Consequently, the Virasoro conditions alone are not sufficient to compute the Gromov-Witten invariants.

This is where the topological recursion relations come to our rescue. By inserting them into the Virasoro conditions, we can effectively bound the value of m in the set of unknown invariants above, leaving only the invariants with $0 \le m < 3g-1$. This way we arrive at infinitely many linear Virasoro conditions (one for every choice of d) for only $3g-1$ invariants. It is now of course strongly expected that this system should be solvable, i.e. that the coefficient matrix of this system of linear equations has maximal rank $3g-1$. We will show that this is indeed always the case. In fact, we will show that *any* choice of $3g-1$ distinct non-trivial Virasoro conditions leads to a system of linear equations that determines the invariants uniquely. We do this by computing the determinant of the corresponding coefficient matrix: if we pick the Virasoro conditions associated to the degrees $d_0,\dots,d_{3g-2}$ and reduce the cotangent line powers by our topological recursion relations, we arrive at a system of $3g-1$ linear equations for $3g-1$ invariants whose determinant is

$$\frac{\prod_{i>j}(d_i-d_j)}{\prod_{i=1}^{3g-2} i!} \cdot \prod_{i=1}^{3g-2} \left(i+\frac{1}{2}\right)^{3g-1-i},$$

which is obviously always non-zero. Therefore the Virasoro recursion works, i.e. we have found a constructive way to compute the Gromov-Witten invariants of $\mathbb{P}^r$ in any genus. This is the main result of this chapter (see theorem 1.4.4). The emphasis here lies on the word “constructive” as it has been shown earlier by Dubrovin and Zhang that the Virasoro conditions together with the topological recursion relations determine in principle all Gromov-Witten invariants [**DZ**].

We have written a C++ program that implements the algorithm mentioned above to compute the Gromov-Witten invariants of projective spaces in any genus [**Ga5**]. At the end of this chapter we will give several numerical examples. Of particular interest are certain linear combinations of invariants of $\mathbb{P}^3$ that have been conjectured by Pandharipande to be integers (and maybe to have an enumerative significance) as a generalization of the Gopakumar-Vafa conjecture [**P3**]. Using our algorithm we have verified this conjecture numerically up to genus 4 and degree 6.

This chapter is organized as follows. In section 1.1 we introduce the moduli spaces of stable curves and stable maps, which will be the basic object of study in this work. These moduli spaces are then used to define Gromov-Witten invariants in section 1.2. We will list and prove their basic properties in section 1.3. Section 1.4 discusses the Virasoro conditions on $\mathbb{P}^r$ and describes how to combine them with the topological recursion relations to get systems of linear equations for the Gromov-Witten invariants. The proof that these systems of equations are always solvable (i.e. the computation of the determinant mentioned above) is given in section 1.5. Finally, we will list some numerical results obtained with our method in section 1.6.

1.1. Stable curves and stable maps

DEFINITION 1.1.1. An **(n-pointed) pre-stable curve** is a tuple $(C,x_1,\dots,x_n)$ where

(i) C is a compact connected curve with at most nodes (i.e. ordinary double points) as singularities,
(ii) $x_1,\dots,x_n$ are distinct smooth points of C, called the **marked points** of the curve.

A morphism $\varphi : (C,x_1,\dots,x_n) \to (C',x'_1,\dots,x'_n)$ of n-pointed pre-stable curves is a morphism $\varphi : C \to C'$ with $\varphi(x_i) = x'_i$ for all $i = 1,\dots,n$. An n-pointed pre-stable curve is called **stable** if its group of automorphisms is finite. The genus of a pre-stable curve is defined to be the arithmetic genus $g(C) = h^1(C,\mathcal{O}_C)$ of C. Curves of genus 0 (resp. 1) are called rational (resp. elliptic).

We denote the set of all n-pointed stable curves of genus g by $\bar{M}_{g,n}$. The subset of $\bar{M}_{g,n}$ of all smooth stable curves is denoted $M_{g,n}$. Sometimes we will label the marked points by a finite set I instead of by the numbers $\{1,\dots,n\}$. In this case we denote the corresponding spaces by $\bar{M}_{g,I}$ and $M_{g,I}$, respectively.

REMARK 1.1.2. Every pre-stable curve can be obtained by the following procedure. Let $C_1,\dots,C_r$ be *smooth* compact connected curves, and denote their disjoint union by $\tilde{C}$. Now let $x_1,\dots,x_n,y_1,\dots,y_s,y'_1,\dots,y'_s$ be distinct points on $\tilde{C}$, and let C be the curve obtained from $\tilde{C}$ by identifying y_i with y'_i for all $i = 1,\dots,s$. If C is connected then $(C,x_1,\dots,x_n)$ is an n-pointed pre-stable curve of genus

$$s - r + 1 + \sum_{i=1}^{r} g(C_i)$$

with s nodes $y_i = y'_i$. The curves C_i are called the **components** of the pre-stable curve. The points $x_1,\dots,x_n,y_1,\dots,y_s,y'_1,\dots,y'_s$ (i.e. the marked points and the nodes) are called the **special points** of the pre-stable curve. We say that C is obtained by **gluing** the components C_i in the points $y_i = y'_i$.

REMARK 1.1.3. Note that the automorphism group of a smooth curve of genus g has dimension 3 for $g = 0$, 1 for $g = 1$, and 0 for $g > 1$. Hence a pre-stable curve is stable if and only if every rational (resp. elliptic) component has at least three (resp. one) special point lying on it. The following picture shows some examples. The components of the curves are labeled by their genus.

The curves (A) and (B) are stable, whereas the curve (C) is not (as the middle component has genus zero but only two special points). The genus of the curves is 2, 1, and 2, respectively.

Note that $\bar{M}_{g,n}$ and $M_{g,n}$ are empty if $g = 0$ and $n < 3$, or $g = 1$ and $n < 1$ (i.e. if $n \leq 2-2g$). The space $\bar{M}_{0,3}$ is a single point.

CONSTRUCTION 1.1.4. Assume that $n > 2-2g$, and let $(C, x_1, \dots, x_n)$ be a pre-stable curve of genus g. Then we can construct an associated stable curve, called its **stabilization** and denoted $s(C, x_1, \dots, x_n)$, as follows.

(i) Any rational component of C that has only one special point (which by the assumption $n > 2-2g$ must then be a node) is simply dropped.
(ii) Any rational component of C that has exactly two special points (at least one of which must then be a node) is dropped, and the two special points are identified.

For example, the curve (A) in remark 1.1.3 above is the stabilization of (C). We say that $s(C, x_1, \dots, x_n)$ is obtained from $(C, x_1, \dots, x_n)$ by contracting the unstable components.

REMARK 1.1.5. Similarly to remark 1.1.2 we can also glue a finite number of stable curves in some marked points. The easiest case of this is the following. Let $I_1 \cup I_2 = \{1, \dots, n\}$ be a partition, and let $g_1, g_2 \geq 0$ be integers such that $g_1 + g_2 = g$. Then there is an injective map

$$D(g_1, I_1 \,|\, g_2, I_2) := \bar{M}_{g_1, \{0\} \cup I_1} \times \bar{M}_{g_2, \{0\} \cup I_2} \to \bar{M}_{g,n}$$

given by gluing a stable curve of genus g_1 with marked points x_0 and $\{x_i \,;\, i \in I_1\}$ to a stable curve of genus g_2 with marked points x_0 and $\{x_i \,;\, i \in I_2\}$ in the point

x_0. By the criterion of remark 1.1.3 this new curve will be stable, so the above map is well-defined. We can therefore regard the $D(g_1, I_1 \mid g_2, I_2)$ as subsets of $\bar{M}_{g,n}$. They describe the stable curves that are unions of two curves (that may themselves be reducible) of genus g_1 and g_2 that are glued in one point and that contain the marked points x_i with $i \in I_1$ and $i \in I_2$, respectively.

THEOREM 1.1.6. *For all $g, n \geq 0$ the space $\bar{M}_{g,n}$ of stable curves is an irreducible, smooth, proper, separated Deligne-Mumford stack of dimension $3g - 3 + n$ (unless $n \leq 2 - 2g$, in which case it is empty). The subspace $M_{g,n}$ of smooth stable curves is a dense open substack of $\bar{M}_{g,n}$.*

PROOF. See [**DM**]. □

REMARK 1.1.7. To be more precise, theorem 1.1.6 means the following. We say that a family of n-pointed stable curves over a base scheme S is a flat morphism $\pi : X \to S$ together with n sections $\sigma_1, \dots, \sigma_n : S \to X$ such that the geometric fibers $(\pi^{-1}(P), \sigma_1(P), \dots, \sigma_n(P))$ are stable curves for all $P \in S$. Theorem 1.1.6 now asserts that the functor

$$\text{Schemes} \to \text{Sets}$$
$$S \mapsto \text{families of } n\text{-pointed (pre-)stable curves over } S$$

is representable by a Deligne-Mumford stack with the stated properties.

After having defined the moduli spaces of stable curves let us now move on to stable maps.

DEFINITION 1.1.8. Let X be a smooth projective variety. An **(n-pointed) pre-stable map** to X is a tuple $(C, x_1, \dots, x_n, f)$, where

(i) $(C, x_1, \dots, x_n)$ is a pre-stable curve,
(ii) $f : C \to X$ is a morphism.

A morphism $\varphi : (C, x_1, \dots, x_n, f) \to (C', x'_1, \dots, x'_n, f')$ of n-pointed pre-stable maps is a morphism φ of the underlying pre-stable curves such that $f \circ \varphi = f'$. An n-pointed pre-stable map is called **stable** if its group of automorphisms is finite. The **class** of a pre-stable map $(C, x_1, \dots, x_n, f)$ is defined to be the element $f_*[C] \in H_2^+(X)$, where $H_2^+(X)$ denotes the semigroup of homology classes of algebraic curves modulo torsion.

We denote the set of all n-pointed stable maps to X of class $\beta \in H_2^+(X)$ and genus g by $\bar{M}_{g,n}(X, \beta)$. The subset of $\bar{M}_{g,n}(X, \beta)$ of all smooth stable maps (i.e. stable maps whose underlying pre-stable curve is smooth) is denoted $M_{g,n}(X, \beta)$. In the same way as for stable curves we will write $\bar{M}_{g,I}(X,\beta)$ and $M_{g,I}(X,\beta)$ if the marked points are labeled by a finite index set I instead of by $\{1,\dots,n\}$.

When drawing stable maps we will often only draw the image curve in X, keeping in mind that the abstract curve C and the morphism f are actually part of the data needed to specify the stable map.

Sometimes stable maps will also be called **stable absolute maps** to distinguish them from the stable relative maps that we will introduce in chapters 2 and 5.

REMARK 1.1.9. Note that $\bar{M}_{g,n}(X,\beta) = \bar{M}_{g,n}$ by definition if X is a point (and hence $\beta = 0$). In other words, stable curves are special cases of stable maps.

REMARK 1.1.10. Note that the underlying pre-stable curve of a stable map need not be stable. The following picture shows an example of this (f is constant on the middle elliptic component):

The left and right components of C are rational but have only two special points each. Therefore (C,x_1,x_2) is not a stable curve. But nevertheless (C,x_1,x_2,f) is a stable map: if $\varphi : (C,x_1,x_2,f) \mapsto (C,x_1,x_2,f)$ is an automorphism then by definition we must have $\varphi \circ f = f$. This means that φ must be the identity on the left and right components of C. On the middle component (on which f is constant) φ need not be the identity — but this component is stable as a curve, so there are only finitely many automorphisms φ.

In general, we see that (similarly to remark 1.1.3) a pre-stable map $(C,x_1,\dots,x_n,f)$ is stable if and only if every rational (resp. elliptic) component of C *on which f is constant* has at least three (resp. one) special points.

In particular, $\bar{M}_{g,n}(X,\beta)$ and $M_{g,n}(X,\beta)$ are empty if $\beta = 0$ and $n \le 2-2g$. Note also that $\bar{M}_{g,n}(X,0)$ is just $\bar{M}_{g,n} \times X$. In particular, $\bar{M}_{0,3}(X,0) = X$.

CONSTRUCTION 1.1.11. Assume that $\beta \neq 0$ or $n > 2-2g$, and let $(C,x_1,\dots,x_n,f)$ be a pre-stable map of genus g and class β. Then we can construct an associated stable map $s(C,x_1,\dots,x_n,f) \in \bar{M}_{g,n}(X,\beta)$ by "contracting the unstable components", i.e. by applying constructions 1.1.4 (i) and (ii) to the rational components of C with at most two special points on which f is constant. The resulting stable map is called the stabilization of $(C,x_1,\dots,x_n,f)$.

CONSTRUCTION 1.1.12. There are various maps related to the spaces of stable maps:

(i) For any $1 \leq i \leq n$ the map

$$\begin{array}{rrcl} \mathrm{ev}_i : & \bar{M}_{g,n}(X,\beta) & \to & X \\ & (C,x_1,\dots,x_n,f) & \mapsto & f(x_i) \end{array}$$

is called the **evaluation** map at the i-th marked point.

(ii) Assume that $\beta \neq 0$ or $n > 3-2g$. For any $1 \leq i \leq n$ the map

$$\begin{array}{rrcl} \pi_i : & \bar{M}_{g,n}(X,\beta) & \to & \bar{M}_{g,n-1}(X,\beta) \\ & (C,x_1,\dots,x_n) & \mapsto & s(C,x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) \end{array}$$

is called a **forgetful map** (that forgets the i-th marked point and stabilizes the result). Note that the stabilization is necessary because removing a marked point may make the component on which it lies unstable.

(iii) Assume that $n > 2-2g$. The map

$$\begin{array}{rrcl} \pi : & \bar{M}_{g,n}(X,\beta) & \to & \bar{M}_{g,n} \\ & (C,x_1,\dots,x_n,f) & \mapsto & s(C,x_1,\dots,x_n) \end{array}$$

is also called a forgetful map (that forgets the map and stabilizes the resulting curve).

(iv) Let $F : X \to Y$ be a morphism of smooth projective varieties, and assume that $F_*\beta \neq 0$ or $n > 2-2g$. Then there are functorial maps

$$\begin{array}{rrcl} F_* : & \bar{M}_{g,n}(X,\beta) & \to & \bar{M}_{g,n}(Y,F_*\beta) \\ & (C,x_1,\dots,x_n,f) & \mapsto & s(C,x_1,\dots,x_n,F\circ f). \end{array}$$

The maps of (ii) and (iii) can obviously be composed to obtain forgetful maps that forget several of the marked points, and maybe also the map.

REMARK 1.1.13. Stable maps can be glued in points in the same way as stable curves in remark 1.1.5, provided that the points to be identified map to the same image point in X. This gives rise to injective gluing maps

$$D(g_1,I_1,\beta_1\,|\,g_2,I_2,\beta_2) := \bar{M}_{g_1,\{0\}\cup I_1}(X,\beta_1)\times_X \bar{M}_{g_2,\{0\}\cup I_2}(X,\beta_2) \to \bar{M}_{g,n}(X,\beta)$$

for all $g_1+g_2 = g$, $I_1 \dot\cup I_2 = \{1,\dots,n\}$, $\beta_1+\beta_2 = \beta$, where the two maps to X in the fiber product are given by evaluation at the point x_0.

THEOREM 1.1.14. *Let X be a smooth projective variety, and let $\beta \in H_2^+(X)$. For all $g,n \geq 0$ the space $\bar{M}_{g,n}(X,\beta)$ is a proper and separated Deligne-Mumford stack. (It is in general not smooth, irreducible, connected, reduced, or of constant dimension.)*

PROOF. The fact that $\bar{M}_{g,n}(X,\beta)$ is a Deligne-Mumford stack has been proven in [**BM**] theorem 3.14. (Similarly to remark 1.1.7 one has to define families of stable maps over arbitrary base schemes, and then prove that the associated functor from the category of schemes to the category of sets is representable by a Deligne-Mumford stack.) It is proper and separated by [**FP**] section 4.2. □

PROPOSITION 1.1.15. *The evaluation, forgetful, and functorial maps of construction 1.1.12, as well as the gluing map of remark 1.1.13, are actually morphisms of Deligne-Mumford stacks.*

PROOF. For the evaluation maps see [**BM**] proposition 5.5. The other cases are included in the functoriality statement in the remark after theorem 3.14 in [**BM**]. □

EXAMPLE 1.1.16. Consider the forgetful map $\pi_{n+1} : \bar{M}_{g,n+1}(X,\beta) \to \bar{M}_{g,n}(X,\beta)$ that forgets the last marked point. It has been shown in [**BM**] corollary 4.6 that π_{n+1} is the universal curve over $\bar{M}_{g,n}(X,\beta)$ (and hence is a flat morphism). The section $\sigma_i : \bar{M}_{g,n}(X,\beta) \mapsto \bar{M}_{g,n+1}(X,\beta)$ corresponding to the i-th marked point is given as a special case of the gluing construction of remark 1.1.13: it is the composition of the isomorphisms

$$\begin{array}{lll} & \bar{M}_{g,n}(X,\beta) & \\ \cong & \bar{M}_{g,\{0,1,\dots,i-1,i+1,\dots,n\}}(X,\beta) & \text{(by relabeling } x_i \mapsto x_0) \\ \cong & \bar{M}_{g,\{0,1,\dots,i-1,i+1,\dots,n\}}(X,\beta) \times_X \bar{M}_{0,\{0,i,n+1\}}(X,0) & \text{(as } \bar{M}_{0,3}(X,0) \cong X) \\ \cong & D(g,\{1,\dots,i-1,i+1,\dots,n\},\beta\,|\,0,\{i,n+1\},0) & \subset \bar{M}_{g,n+1}(X,\beta). \end{array}$$

Geometrically, σ_i is given by replacing the i-th marked point by a contracted rational component with the points x_i and x_{n+1} on it:

We will denote the image of σ_i by

$$D_i := D(g,\{1,\dots,i-1,i+1,\dots,n\},\beta\,|\,0,\{i,n+1\},0).$$

Obviously, π_{n+1} maps D_i isomorphically to $\bar{M}_{g,n}(X,\beta)$.

1.2. Gromov-Witten invariants

The idea of Gromov-Witten invariants (and of enumerative geometry in general) is to consider intersection products on the moduli spaces of stable maps that correspond to certain geometric conditions on the curves. Usually one wants these intersection products to have dimension 0, so that their degree is a rational number that has a geometric interpretation as some number of curves in a given variety with certain conditions.

REMARK 1.2.1. In order to make this idea work one first of all needs to know the dimensions of the moduli spaces of stable maps. This is not so easy however, as these moduli spaces usually have several components of different dimensions. There is however a so-called **virtual** or **expected dimension**

$$\operatorname{vdim} \bar{M}_{g,n}(X,\beta) := -K_X \cdot \beta + (\dim X - 3)(1-g) + n$$

from the deformation theory of stable maps. The local dimension of the moduli space is at least equal to this expected dimension at any point $(C,x_1,\dots,x_n,f) \in \bar{M}_{g,n}(X,\beta)$. If the deformation theory is unobstructed at a point then the local dimension is equal to the expected dimension, and the moduli space is smooth at this point. This is e.g. the case if the cohomology group $H^1(C,f^*T_X)$ vanishes.

In any case there is a naturally defined **virtual fundamental class** $[\bar{M}_{g,n}(X,\beta)]^{\text{virt}} \in A_*(\bar{M}_{g,n}(X,\beta))$ in the Chow group of the moduli space whose dimension is equal to the virtual dimension (for intersection theory on Deligne-Mumford stacks we refer to [**Vi**]). The idea is that in intersection-theoretic computations (that usually evaluate some product of cohomology classes on the fundamental class of the moduli space) the fundamental class should be replaced by this virtual one. If the deformation theory is unobstructed for all stable maps in the moduli space then the virtual fundamental class is equal to the usual one. This is e.g. the case for rational curves in projective (or more generally homogeneous) spaces, but not very often for other varieties and almost never for curves of positive genus. For details of the construction and properties of virtual fundamental classes we refer to [**B**], [**BF**], [**LT1**].

CONSTRUCTION 1.2.2. Let $\bar{M}_{g,n}(X,\beta)$ be a moduli space of stable maps. Fix an integer $1 \le i \le n$. We consider the line bundle

$$L_i = \sigma_i^* \omega_{\bar{M}_{g,n+1}(X,\beta)/\bar{M}_{g,n}(X,\beta)}$$

where $\omega_{\bar{M}_{g,n+1}(X,\beta)/\bar{M}_{g,n}(X,\beta)}$ denotes the relative dualizing sheaf of the universal curve, and $\sigma_i : \bar{M}_{g,n}(X,\beta) \to \bar{M}_{g,n+1}(X,\beta)$ is the section corresponding to the i-th marked point (see example 1.1.16). Geometrically, L_i can be thought of as the line bundle whose fiber at a stable curve $(C, x_1, \dots, x_n, f)$ is the cotangent space $T^{\vee}_{C,x_i}$ to C at x_i.

The first Chern class $c_1(L_i) \in A^1(M)$ of this line bundle is called the i-th **cotangent line class** or **psi class**, denoted ψ_i.

DEFINITION 1.2.3. Let $\bar{M}_{g,n}(X,\beta)$ be a moduli space of stable maps. For any cohomology classes $\gamma_1, \dots, \gamma_n \in A^*(X)$ and non-negative integers $m_1, \dots, m_n$ we define the **Gromov-Witten invariant**

$$\langle \tau_{m_1}(\gamma_1) \cdots \tau_{m_n}(\gamma_n) \rangle_{g,\beta} := \deg \left(\psi_1^{m_1} \cdot \mathrm{ev}_1^* \gamma_1 \cdots \psi_n^{m_n} \cdot \mathrm{ev}_n^* \gamma_n \cdot [\bar{M}_{g,n}(X,\beta)]^{\mathrm{virt}} \right) \in \mathbb{Q},$$

where $\deg(\alpha)$ is the degree of the dimension-0 part of the cycle $\alpha \in A_*(\bar{M}_{g,n}(X,\beta))$. If $m_i = 0$ for some i we abbreviate $\tau_{m_i}(\gamma_i)$ to γ_i within the brackets on the left hand side. If $g = 0$ we will sometimes leave out the index g in the notation. A Gromov-Witten invariant is called **primary** if it does not contain any cotangent line classes (i.e. all m_i are zero), and **descendant** otherwise. The Gromov-Witten invariants with n marked points are sometimes called ***n*-point invariants**. Sometimes the above invariants are called **absolute Gromov-Witten invariants** to distinguish them from the relative Gromov-Witten invariants that we will introduce in chapters 2 and 5.

REMARK 1.2.4. As the Gromov-Witten invariants are multilinear in the cohomology classes it suffices to pick the γ_i from among a fixed basis $\{T_a\}$ of the cohomology $A^*(X)$ modulo numerical equivalence. We denote the Poincaré-dual basis by $\{T^a\}$ and apply the summation convention (i.e. every variable occurring both as an upper and lower index is summed over) unless stated otherwise. The fundamental class of X and the class of a point will be denoted 1 and pt, respectively.

REMARK 1.2.5. In some cases the Gromov-Witten invariants have an enumerative interpretation. For example, let X be a projective (or more generally homogeneous) space, and let $V_1, \dots, V_n$ be subvarieties of X in general position. Then the Gromov-Witten invariant $\langle [V_1] \cdots [V_n] \rangle_{0,\beta}$ is equal to the number of rational curves of degree β in X that intersect each of the given V_i, provided that the dimensions of the V_i are chosen so that this number is finite (see **[FP]** lemma 14). For example, the invariant $\langle \mathrm{pt}^2 \rangle_{0,1}$ of $\mathbb{P}^N$ is 1 as there is one line in $\mathbb{P}^N$ through two points.

In general, a geometric interpretation of the invariants is not so easy however. For some more examples see section 1.6.

PROPOSITION 1.2.6. *Let $X = \mathbb{P}^r$, and denote by $T_a \in A^*(X)$ the class of a linear subspace of X of codimension a. The rational 1-point and 2-point invariants of X with at most one descendant class are given by the following generating functions:*

(i) $\langle\tau_m(T_a)\rangle_{0,d}$ *(with* $a+m = \operatorname{vdim}\bar{M}_{0,1}(\mathbb{P}^n,d)$*) is equal to the* z^{r-a}*-coefficient of the power series* $\prod_{i=1}^{d}\frac{1}{(z+i)^{r+1}}$.

(ii) $\langle T_b\tau_m(T_a)\rangle_{0,d}$ *(with* $a+b+m = \operatorname{vdim}\bar{M}_{0,2}(\mathbb{P}^n,d)$*) is equal to the* z^{r-a}*-coefficient of the power series* $\frac{1}{(z+d)^{r+1-b}}\prod_{i=1}^{d-1}\frac{1}{(z+i)^{r+1}}$.

PROOF. See [**P1**] section 1.4. □

REMARK 1.2.7. It is often convenient to encode the Gromov-Witten invariants as the coefficients of a generating function. So we introduce the so-called **correlation functions**

$$\langle\langle\tau_{m_1}(\gamma_1)\cdots\tau_{m_n}(\gamma_n)\rangle\rangle_g := \sum_{\beta}\left\langle\tau_{m_1}(\gamma_1)\cdots\tau_{m_n}(\gamma_n)\exp\left(\sum_m t_m^a\tau_m(T_a)\right)\right\rangle_{g,\beta} q^\beta$$

where the t_m^a and q^β are formal variables satisfying $q^{\beta_1}q^{\beta_2} = q^{\beta_1+\beta_2}$. The correlation functions are formal power series in the variables t_m^a and q^β whose coefficients describe all genus-g Gromov-Witten invariants containing at least the classes $\tau_{m_1}(\gamma_1)\cdots\tau_{m_n}(\gamma_n)$.

REMARK 1.2.8. Let M_1 and M_2 be Deligne-Mumford stacks over a smooth Deligne-Mumford stack S. Let $M = M_1\times_S M_2$, so that we have a Cartesian diagram

$$\begin{array}{ccc} M & \longrightarrow & M_1\times M_2 \\ \downarrow & & \downarrow \\ S & \xrightarrow{\Delta} & S\times S \end{array}$$

where Δ is the diagonal. Now assume that we are given classes $\gamma_1\in A_*(M_1)$ and $\gamma_2\in A_*(M_2)$ (usually thought of as virtual fundamental classes). Then the class $\Delta^!(\gamma_1\otimes\gamma_2)$ in M will be called induced by γ_1 and γ_2.

In particular, we see that the substacks $D(g_1,I_1,\beta_1\,|\,g_2,I_2,\beta_2)\subset\bar{M}_{g,n}(X,\beta)$ of remark 1.1.13 carry natural virtual fundamental classes that are induced by the ones of $\bar{M}_{g_1,\{0\}\cup I_1}(X,\beta_1)$ and $\bar{M}_{g_2,\{0\}\cup I_2}(X,\beta_2)$. Their dimensions are

$$\begin{aligned}&(-K_X\cdot\beta_1+(\dim X-3)(1-g_1)+1+\#I_1)\\ &+(-K_X\cdot\beta_2+(\dim X-3)(1-g_2)+1+\#I_2)-\dim X\\ &=\operatorname{vdim}\bar{M}_{g,n}(X,\beta)-1,\end{aligned}$$

so the substacks $D(g_1,I_1,\beta_1\,|\,g_2,I_2,\beta_2)$ have virtual codimension 1 in $\bar{M}_{g,n}(X,\beta)$. They are usually called **boundary divisors**.

REMARK 1.2.9. To every boundary divisor $D(g_1,I_1,\beta_1\,|\,g_2,I_2,\beta_2)$ we can construct an "associated Gromov-Witten invariant" by intersecting its virtual fundamental class with evaluation and cotangent line classes. Let us assume for the moment that all cohomology of X is algebraic. Then the class of the diagonal Δ_X in $X\times X$ is given by

$$[\Delta_X] = \sum_a T_a \otimes T^a \in A^*(X\times X)$$

by [**D**] exercise VIII.8.21.2. Therefore we have

$$\begin{aligned}\deg\Big(\psi_1^{m_1}\cdot \mathrm{ev}_1^*\gamma_1\cdots\psi_n^{m_n}\cdot\mathrm{ev}_n^*\gamma_n\cdot[D(g_1,I_1,\beta_1\,|\,g_2,I_2,\beta_2)]^{\mathrm{virt}}\Big)\\ = \Big\langle T_a\prod_{i\in I_1}\tau_{m_i}(\gamma_i)\Big\rangle_{g_1,\beta_1}\Big\langle T^a\prod_{i\in I_2}\tau_{m_i}(\gamma_i)\Big\rangle_{g_2,\beta_2}.\end{aligned}$$

This formula is usually referred to as the **diagonal splitting**.

1.3. Basic relations among Gromov-Witten invariants

In this section we will list the basic relations among Gromov-Witten invariants that we will need later. Although all of them are well-known, complete proofs in full generality (including virtual fundamental classes and descendant invariants) can often not be found or are scattered in the literature. For convenience we will therefore provide short proofs of these statements.

LEMMA 1.3.1. *Let $I\subset\{1,\dots,n\}$ and consider the forgetful morphism*

$$\pi:\bar M_{g,n}(X,\beta)\to\bar M_{g,I}(X,\beta)\qquad \textit{or}\qquad \pi:\bar M_{g,n}(X,\beta)\to\bar M_{g,I}$$

that forgets the points x_i with $i\notin I$, and maybe in addition the map. Then for all $i\in I$

$$\psi_i\cdot[\bar M_{g,n}(X,\beta)]^{\mathrm{virt}} = \pi^*\psi_i\cdot[\bar M_{g,n}(X,\beta)]^{\mathrm{virt}} + \sum[D(g,I_1,\beta_1\,|\,0,I_2,\beta_2)]^{\mathrm{virt}}$$

where the sum is taken over all splittings $I_1\cup I_2=\{1,\dots,n\}$ and $\beta_1+\beta_2=\beta$ with $i\in I_2$ such that the second component becomes unstable when applying the forgetful map, i.e. such that the following two conditions hold:

(i) $I_2\cap I=\{i\}$ *(i.e. the only marked point on the second component that is not forgotten is x_i),*
(ii) *if π does not forget the map then $\beta_2=0$.*

PROOF. If π does not forget the map then the result follows by induction on the number of forgotten points from **[Ge1]** proposition 11. The case when X is a point (and thus π only forgets some of the marked points) has been proven in **[Ge1]** theorem 8. The general statement is obtained by composition of these two results. □

COROLLARY 1.3.2. *Let $\pi : \bar{M}_{g,n+1}(X,\beta) \to \bar{M}_{g,n}$ be the universal curve over $\bar{M}_{g,n}$, i.e. the morphism that forgets the last marked point and stabilizes the result. Pick cohomology classes $\gamma_1,\dots,\gamma_n \in A^*(X)$ and non-negative integers $m_1,\dots,m_n$. Then for any cohomology class $\alpha \in A^*(\bar{M}_{g,n+1}(X,\beta))$ we have*

$$\begin{aligned}&\pi_*(\psi_1^{m_1}\cdot \mathrm{ev}_1^*\gamma_1\cdots\psi_n^{m_n}\cdot\mathrm{ev}_n^*\gamma_n\cdot\alpha\cdot[\bar{M}_{g,n+1}(X,\beta)]^{\mathrm{virt}})\\ &\quad=\psi_1^{m_1}\cdot \mathrm{ev}_1^*\gamma_1\cdots\psi_n^{m_n}\cdot\mathrm{ev}_n^*\gamma_n\cdot\pi_*(\alpha\cdot[\bar{M}_{g,n+1}(X,\beta)]^{\mathrm{virt}})\\ &\quad+\sum_{i:\,m_i>0}\psi_1^{m_1}\cdot \mathrm{ev}_1^*\gamma_1\cdots\psi_i^{m_i-1}\cdot\mathrm{ev}_i^*\gamma_i\cdots\psi_n^{m_n}\cdot\mathrm{ev}_n^*\gamma_n\cdot\pi_*(\alpha\cdot[D_i]^{\mathrm{virt}})\end{aligned}$$

in $A_*(\bar{M}_{g,n}(X,\beta))$, where $D_i \subset \bar{M}_{g,n+1}(X,\beta)$ denotes the divisor of example 1.1.16.

PROOF. Applying lemma 1.3.1 to the morphism π we get

$$\psi_i\cdot[\bar{M}_{g,n+1}(X,\beta)]^{\mathrm{virt}}=\pi^*\psi_i\cdot[\bar{M}_{g,n+1}(X,\beta)]^{\mathrm{virt}}+[D_i]^{\mathrm{virt}}$$

for $1 \le i \le n$. Note that ψ_i is zero when restricted to D_i, as the i-th marked point sits on a constant contracted rational 3-pointed component. Therefore it follows by induction that

$$\psi_i^{m_i}\cdot[\bar{M}_{g,n+1}(X,\beta)]^{\mathrm{virt}}=\pi^*\psi_i^{m_i}\cdot[\bar{M}_{g,n+1}(X,\beta)]^{\mathrm{virt}}+\pi^*\psi_i^{m_i-1}[D_i]^{\mathrm{virt}}$$

if $m_i > 0$. Next, ψ_j and $\pi^*\psi_j$ agree when restricted to D_i with $i \neq j$, as forgetting the $(n+1)$-st marked point only drops the contracted rational component but does not change the curve in a neighborhood of the j-th marked point. So we get

$$\begin{aligned}&\psi_1^{m_1}\cdots\psi_n^{m_n}\cdot[\bar{M}_{g,n+1}(X,\beta)]^{\mathrm{virt}}\\ &\quad=\pi^*(\psi_1^{m_1}\cdots\psi_n^{m_n})\cdot[\bar{M}_{g,n+1}(X,\beta)]^{\mathrm{virt}}\\ &\quad+\sum_{i:\,m_i>0}\pi^*(\psi_1^{m_1}\cdots\psi_i^{m_i-1}\cdots\psi_n^{m_n})[D_i]^{\mathrm{virt}}.\end{aligned}$$

The corollary now follows from the projection formula, taking into account that π commutes with the evaluation maps. □

COROLLARY 1.3.3 (The **fundamental class equation / string equation**). *Any Gromov-Witten invariant that has no conditions at one of the marked points is determined in terms of the others by*

$$\langle\tau_{m_1}(\gamma_1)\cdots\tau_{m_n}(\gamma_n)\;1\rangle_{g,\beta}=\sum_{i:\,m_i>0}\langle\tau_{m_1}(\gamma_1)\cdots\tau_{m_i-1}(\gamma_i)\cdots\tau_{m_n}(\gamma_n)\rangle_{g,\beta},$$

provided that $n > 2-2g$ *or* $\beta \neq 0$.

PROOF. Set $\alpha = 1$ in corollary 1.3.2. We have $\pi_*([\bar{M}_{g,n+1}(X,\beta)]^{\text{virt}}) = 0$ for dimensional reasons and $\pi_*([D_i]^{\text{virt}}) = [\bar{M}_{g,n}(X,\beta)]^{\text{virt}}$ by the construction of $[D_i]^{\text{virt}}$, so the result follows. (We need the condition that $n > 2-2g$ or $\beta \neq 0$ as otherwise there is no forgetful morphism to $\bar{M}_{g,n}(X,0) = \emptyset$.) □

COROLLARY 1.3.4 (The **divisor equation**). *Any Gromov-Witten invariant that has the class* γ *of a divisor at one of the marked points is determined in terms of the others by*

$$\langle \tau_{m_1}(\gamma_1)\cdots\tau_{m_n}(\gamma_n)\,\gamma\rangle_{g,\beta} = (\gamma\cdot\beta)\cdot\langle \tau_{m_1}(\gamma_1)\cdots\tau_{m_n}(\gamma_n)\rangle_{g,\beta} + \sum_{i:\,m_i>0} \langle \tau_{m_1}(\gamma_1)\cdots\tau_{m_i-1}(\gamma\cdot\gamma_i)\cdots\tau_{m_n}(\gamma_n)\rangle_{g,\beta},$$

provided that $n > 2-2g$ *or* $\beta \neq 0$.

PROOF. This time we set $\alpha = \text{ev}_{n+1}^*\gamma$ in corollary 1.3.2. To compute the expression $\pi_*(\text{ev}_{n+1}^*\gamma\cdot[\bar{M}_{g,n+1}(X,\beta)]^{\text{virt}})$ note that

(i) $[\bar{M}_{g,n+1}(X,\beta)]^{\text{virt}} = \pi^*[\bar{M}_{g,n}(X,\beta)]^{\text{virt}}$ by axiom IV of **[BM]** definition 7.1, proven in **[B]**.
(ii) $\pi_*(\text{ev}_{n+1}^*\gamma\cdot\pi^*[(C,x_1,\dots,x_n,f)]) = (\gamma\cdot\beta)\cdot[(C,x_1,\dots,x_n,f)]$ for any stable map $(C,x_1,\dots,x_n,f)$ as there are precisely $\gamma\cdot\beta$ choices for the point x_{n+1}.

Combining these results we see that

$$\pi_*(\text{ev}_{n+1}^*\gamma\cdot[\bar{M}_{g,n+1}(X,\beta)]^{\text{virt}}) = (\gamma\cdot\beta)\cdot[\bar{M}_{g,n}(X,\beta)]^{\text{virt}}$$

by the projection formula. On the other hand we have $\text{ev}_{n+1} = \text{ev}_i$ on D_i, so the claim now follows from corollary 1.3.2. □

COROLLARY 1.3.5 (The **cotangent line equation / dilaton equation**). *Any Gromov-Witten invariant that has a pure cotangent line class at one of the marked points is determined in terms of the others by*

$$\langle \tau_{m_1}(\gamma_1)\cdots\tau_{m_n}(\gamma_n)\tau_1(1)\rangle_{g,\beta} = (2g-2+n)\cdot\langle \tau_{m_1}(\gamma_1)\cdots\tau_{m_n}(\gamma_n)\rangle_{g,\beta},$$

provided that $n > 2-2g$ *or* $\beta \neq 0$.

PROOF. The proof is very similar to that of corollary 1.3.4, but now we set $\alpha = \psi_{n+1}$ in corollary 1.3.2. We claim that

$$\pi_*(\psi_{n+1}\cdot\pi^*[(C,x_1,\dots,x_n,f)]) = (2g-2+n)\cdot[(C,x_1,\dots,x_n,f)]$$

for all stable maps $(C, x_1, \dots, x_n, f)$. In fact, what we have to do is to compute the degree of the divisor ψ_{n+1} on the curve $\pi^{-1}(C, x_1, \dots, x_n, f) \cong C$. Any rational section of the line bundle ω_C gives rise to a rational section of $O(\psi_{n+1})$ on the curve $\pi^{-1}(C, x_1, \dots, x_n, f)$. A simple computation in local coordinates shows that this rational section has simple poles at the points x_i, together with zeros and poles at the same points as the section of ω_C. So the degree of ψ_{n+1} is $\deg(\omega_C) + n = 2g - 2 + n$.

In the same way as in the proof of corollary 1.3.4 it follows now that

$$\pi_*(\psi_{n+1} \cdot [\bar{M}_{g,n+1}(X, \beta)]^{\mathrm{virt}}) = (2g - 2 + n) \cdot [\bar{M}_{g,n}(X, \beta)]^{\mathrm{virt}}.$$

The claim now follows from corollary 1.3.2 and the observation that ψ_{n+1} is zero on D_i. □

EXAMPLE 1.3.6. Let X be a point. Then the only Gromov-Witten invariants of X are

$$\langle \tau_{m_1} \cdots \tau_{m_n} \rangle_g,$$

where we write τ_m for $\tau_m(1)$. These invariants are computable by Witten's conjecture [**W**], which has been proven by Kontsevich [**K1**]. The case of genus zero is simple: we claim that

$$\langle \tau_{m_1} \cdots \tau_{m_n} \rangle_g = \binom{n-3}{m_1, \dots, m_n} \cdot \delta_{n-3, m_1 + \cdots + m_n}.$$

In fact, the factor $\delta_{n-3, m_1 + \cdots + m_n}$ is clear by the dimension condition. But this dimension condition implies that every potentially non-zero invariant with more than three marked points has at least one τ_0 entry. Hence all rational Gromov-Witten invariants of a point are determined recursively by the fundamental class equation of corollary 1.3.3 and the trivial initial condition $\langle \tau_0^3 \rangle_0 = 1$. But it is checked immediately that the above expression for the invariants satisfies the fundamental class equation, so it must be the correct one.

EXAMPLE 1.3.7. The stable maps of class $\beta = 0$ are precisely the constant maps, so $\bar{M}_{g,n}(X, 0) \cong \bar{M}_{g,n} \times X$. If moreover $g = 0$ then the virtual fundamental class of $\bar{M}_{g,n}(X, 0)$ is equal to the usual one, and consequently the Gromov-Witten invariants reduce to ordinary integrals on $\bar{M}_{0,n}$ and X. Consequently, one has

$$\langle \tau_{m_1}(\gamma_1) \cdots \tau_{m_n}(\gamma_n) \rangle_{0,0} = \binom{n-3}{m_1, \dots, m_n} \cdot (\gamma_1 \cdots \gamma_n) \cdot \delta_{n-3, m_1 + \cdots + m_n},$$

where $\gamma_1 \cdots \gamma_n$ denotes the (zero-dimensional) intersection product on X.

PROPOSITION 1.3.8 (The **WDVV equations**). *Let $\gamma_1, \dots, \gamma_4 \in A^*(X)$ be cohomology classes, and let $m_1, \dots, m_4 \geq 0$. Then*

$$\begin{aligned} &\langle\langle \tau_{m_1}(\gamma_1) \tau_{m_2}(\gamma_2) T_a \rangle\rangle_0 \langle\langle \tau_{m_3}(\gamma_3) \tau_{m_4}(\gamma_4) T^a \rangle\rangle_0 \\ =&\langle\langle \tau_{m_1}(\gamma_1) \tau_{m_3}(\gamma_3) T_a \rangle\rangle_0 \langle\langle \tau_{m_2}(\gamma_2) \tau_{m_4}(\gamma_4) T^a \rangle\rangle_0. \end{aligned}$$

PROOF. The basic idea of this relation is the equation

$$D(0,\{1,2\}\,|\,0,\{3,4\}) = D(0,\{1,3\}\,|\,0,\{2,4\})$$

in $A_0(\bar{M}_{0,4}) = A_0(\mathbb{P}^1) \cong \mathbb{Z}$ (both sides are equal to the class of a point in $\mathbb{P}^1$). For any $n \geq 4$ we can now consider the pull-back of this equation by the forgetful maps $\pi : \bar{M}_{0,n}(X,\beta) \to \bar{M}_{0,4}$ that forget the map and all but the first four marked points. This yields

$$\sum_{\substack{\beta_1+\beta_2=\beta \\ I_1 \cup I_2 = \{1,\dots,n\} \\ 1,2 \in I_1, 3,4 \in I_2}} [D(0,I_1,\beta_1\,|\,0,I_2,\beta_2)]^{\mathrm{virt}} = \sum_{\substack{\beta_1+\beta_2=\beta \\ I_1 \cup I_2 = \{1,\dots,n\} \\ 1,3 \in I_1, 2,4 \in I_2}} [D(0,I_1,\beta_1\,|\,0,I_2,\beta_2)]^{\mathrm{virt}}$$

by [**BF**] axiom V of definition 7.1, proven in [**B**]. Intersecting these equations with evaluation and cotangent line classes then gives the desired result by the diagonal splitting of remark 1.2.9. □

PROPOSITION 1.3.9 (**Topological recursion relations** in genus 0). *Let $\gamma_1,\gamma_2,\gamma_3 \in A^*(X)$ be cohomology classes, and let $m_1,m_2,m_3 \geq 0$. Then*

$$\langle\langle \tau_{m_1+1}(\gamma_1)\tau_{m_2}(\gamma_2)\tau_{m_3}(\gamma_3)\rangle\rangle_0 = \langle\langle \tau_{m_2}(\gamma_2)\tau_{m_3}(\gamma_3)T^a\rangle\rangle_0 \langle\langle \tau_{m_1}(\gamma_1)T_a\rangle\rangle_0$$

PROOF. For $n \geq 3$ we apply lemma 1.3.1 to the forgetful map $\pi : \bar{M}_{g,n}(X,\beta) \to \bar{M}_{0,3}$ that forgets that map and all but the first three marked points. Note that the psi classes on $\bar{M}_{0,3}$ are trivial as $\bar{M}_{0,3}$ is just a point. So we get

$$\psi_1 \cdot [\bar{M}_{g,n}(X,\beta)]^{\mathrm{virt}} = \sum [D(0,I_1,\beta_1\,|\,0,I_2,\beta_2)]^{\mathrm{virt}}$$

where the sum is taken over all splittings $I_1 \cup I_2 = \{1,\dots,n\}$ and $\beta_1+\beta_2=\beta$ with $2,3 \in I_1$ and $1 \in I_2$. Intersecting these equations with evaluation and cotangent line classes then proves the proposition by the diagonal splitting of remark 1.2.9. □

PROPOSITION 1.3.10. *If the cohomology $A^*(X)$ is generated by divisors then there is an explicit algorithm to reconstruct all rational Gromov-Witten invariants from the rational 1-point invariants.*

PROOF. See [**LP**] theorem 2 (i). The computations work roughly as follows.

(i) Compute all primary 2-point invariants from the descendant 1-point invariants using the WDVV equations of proposition 1.3.8.
(ii) Compute all primary invariants from the primary 2-point invariants using the First Reconstruction Theorem of Kontsevich and Manin (see [**KM**] theorem 3.1).

(iii) Compute all descendant invariants from the primary ones using the topological recursion relations of proposition 1.3.9. If the descendant invariant has fewer than 3 marked points so that proposition 1.3.9 cannot be applied directly, add marked points first using the divisor equation of corollary 1.3.4.

□

COROLLARY 1.3.11. *There is an explicit algorithm to compute all rational Gromov-Witten invariants of projective spaces.*

PROOF. Combine proposition 1.3.10 with proposition 1.2.6 (i). □

PROPOSITION 1.3.12 (**Topological recursion relations** in any genus). *For all $g > 0$, $N \geq 3g-1$, $m \geq 0$, and $\gamma \in A^*(X)$ we have*

$$\langle\langle \tau_{N+m}(\gamma) \rangle\rangle_g = \sum_{i+j=N-1} \langle\langle \tau_i(T_a) \rangle\rangle_g \langle\langle \tau_m(\gamma) T^a \rangle\rangle^j,$$

where the auxiliary correlation functions $\langle\langle \cdots \rangle\rangle^i$ are defined recursively by

$$\langle\langle \tau_m(\gamma_1)\gamma_2 \rangle\rangle^i = \langle\langle \tau_{m+1}(\gamma_1)\gamma_2 \rangle\rangle^{i-1} - \langle\langle T^a \tau_m(\gamma_1) \rangle\rangle_0 \langle\langle T_a \gamma_2 \rangle\rangle^{i-1}$$

with the initial condition

$$\langle\langle \cdots \rangle\rangle^0 = \langle\langle \cdots \rangle\rangle_0.$$

PROOF. Consider the forgetful morphism $\pi : \bar{M}_{g,n}(X,\beta) \to \bar{M}_{g,1}$ that forgets the map and all marked points except the first one. Similarly to the Gromov-Witten invariants let us define

$$\langle \tau_{m_1,m_1'}(\gamma_1)\tau_{m_2}(\gamma_2)\cdots\tau_{m_n}(\gamma_n)\rangle_{g,\beta} := \deg\left(\psi_1^{m_1}\cdot\pi^*\psi_1^{m_1'}\cdot \mathrm{ev}_1^*\gamma_1\cdot\psi_2^{m_2}\cdot \mathrm{ev}_2^*\gamma_2\cdots\psi_n^{m_n}\cdot \mathrm{ev}_n^*\gamma_n\cdot[\bar{M}_{g,n}(X,\beta)]^{\mathrm{virt}}\right) \in \mathbb{Q}$$

and the corresponding correlation function

$$\langle\langle \tau_{m,m'}(\gamma) \rangle\rangle_g := \sum_\beta \left\langle \tau_{m,m'}(\gamma) \exp\left(\sum_m t_m^a \tau_m(T_a)\right)\right\rangle_{g,\beta} q^\beta.$$

Now apply lemma 1.3.1 to the morphism π. We get

$$\psi_1 = \pi^*\psi_1 + \sum [D(g,I_1,\beta_1\,|\,0,I_2,\beta_2)]^{\mathrm{virt}}$$

with the sum taken over all splittings $\beta_1+\beta_2=\beta$ and all $I_1 \cup I_2 = \{1,\ldots,n\}$ with $1 \in I_2$. Intersecting these equations with $\psi_1^m \cdot \pi^*\psi_1^{m'}$, as well as with evaluation and cotangent line classes at the other marked points, we obtain

$$\langle\langle \tau_{m+1,m'}(\gamma) \rangle\rangle_g = \langle\langle \tau_{m,m'+1}(\gamma) \rangle\rangle_g + \langle\langle \tau_{0,m'}(T_a) \rangle\rangle_g \langle\langle T^a \tau_m(\gamma) \rangle\rangle_0.$$

Iterating this equation N times we get

$$\langle\langle\tau_{m+N}(\gamma)\rangle\rangle_g = \langle\langle\tau_{m,N}(\gamma)\rangle\rangle_g + \sum_{i+j=N-1} \langle\langle\tau_{0,i}(T_a)\rangle\rangle_g \langle\langle T^a \tau_{m+j}(\gamma)\rangle\rangle_0 \qquad (1)$$

for all $N \geq 0$. Note that $\dim \bar{M}_{g,1} = 3g-2$ and therefore $\pi^*\psi_1^N = 0$ for dimensional reasons if $N \geq 3g-1$. Hence in this case we have $\langle\langle\tau_{m,N}(\gamma)\rangle\rangle_g = 0$ by definition. So to prove the proposition it only remains to be shown that

$$\sum_{i+j=N-1} \langle\langle\tau_{0,i}(T_a)\rangle\rangle_g \langle\langle T^a \tau_{m+j}(\gamma)\rangle\rangle_0 = \sum_{i+j=N-1} \langle\langle\tau_i(T_a)\rangle\rangle_g \langle\langle\tau_m(\gamma) T^a\rangle\rangle^j$$

for all $N \geq 0$.

We will prove this by induction on N. The equation is trivial for $N = 0$. Assuming that it is true for some given N we now check that

$$\begin{aligned}
&\textstyle\sum_{i+j=N} \langle\langle\tau_{0,i}(T_a)\rangle\rangle_g \langle\langle T^a \tau_{m+j}(\gamma)\rangle\rangle_0 \\
&= \langle\langle\tau_{0,N}(T_a)\rangle\rangle_g \langle\langle T^a \tau_m(\gamma)\rangle\rangle_0 \\
&\quad + \textstyle\sum_{i+j=N-1} \langle\langle\tau_i(T_a)\rangle\rangle_g \langle\langle\tau_{m+1}(\gamma) T^a\rangle\rangle^j && \text{by induction} \\
&= \langle\langle\tau_{0,N}(T_a)\rangle\rangle_g \langle\langle T^a \tau_m(\gamma)\rangle\rangle_0 \\
&\quad + \textstyle\sum_{i+j=N-1} \langle\langle\tau_i(T_a)\rangle\rangle_g \langle\langle T^a \tau_m(\gamma)\rangle\rangle^{j+1} \\
&\quad + \textstyle\sum_{i+j=N-1} \langle\langle\tau_i(T_a)\rangle\rangle_g \langle\langle T^b \tau_m(\gamma)\rangle\rangle_0 \langle\langle T_b T^a\rangle\rangle^j && \text{by definition of } \langle\langle\cdots\rangle\rangle^{j+1} \\
&= \textstyle\sum_{i+j=N-1} \langle\langle\tau_i(T_a)\rangle\rangle_g \langle\langle T^a \tau_m(\gamma)\rangle\rangle^{j+1} \\
&\quad + \langle\langle T^a \tau_m(\gamma)\rangle\rangle_0 \cdot \left(\langle\langle\tau_{0,N}(T_a)\rangle\rangle_g + \textstyle\sum_{i+j=N-1} \langle\langle\tau_i(T_b)\rangle\rangle_g \langle\langle T_a T^b\rangle\rangle^j \right) \\
&= \textstyle\sum_{i+j=N-1} \langle\langle\tau_i(T_a)\rangle\rangle_g \langle\langle T^a \tau_m(\gamma)\rangle\rangle^{j+1} \\
&\quad + \langle\langle T^a \tau_m(\gamma)\rangle\rangle_0 \cdot \left(\langle\langle\tau_{0,N}(T_a)\rangle\rangle_g + \textstyle\sum_{i+j=N-1} \langle\langle\tau_{0,i}(T_b)\rangle\rangle_g \langle\langle T^b \tau_j(T_a)\rangle\rangle_0 \right) \\
& && \text{by induction} \\
&= \textstyle\sum_{i+j=N-1} \langle\langle\tau_i(T_a)\rangle\rangle_g \langle\langle T^a \tau_m(\gamma)\rangle\rangle^{j+1} \\
&\quad + \langle\langle T^a \tau_m(\gamma)\rangle\rangle_0 \langle\langle\tau_N(T_a)\rangle\rangle_g && \text{by equation (1)} \\
&= \textstyle\sum_{i+j=N} \langle\langle\tau_i(T_a)\rangle\rangle_g \langle\langle T^a \tau_m(\gamma)\rangle\rangle^j .
\end{aligned}$$

This finishes the proof. □

REMARK 1.3.13. As in the case of the Gromov-Witten invariants we will expand the correlation functions $\langle\langle\cdots\rangle\rangle^i$ as a power series in q^β and t_m^a and call the resulting coefficients $\langle\cdots\rangle_\beta^i$ according to the formula

$$\langle\langle\tau_{m_1}(\gamma_1)\gamma_2\rangle\rangle^i = \sum_\beta \left\langle \tau_{m_1}(\gamma_1)\gamma_2 \exp\left(\sum_m t_m^a \tau_m(T_a)\right)\right\rangle_\beta^i q^\beta .$$

Note however that, in contrast to the Gromov-Witten numbers, the invariants $\langle\cdots\rangle^i$ must have at least two entries, of which the second one contains no cotangent line class.

1.4. The Virasoro relations

The Virasoro conditions are certain relations among Gromov-Witten invariants conjectured in [**EHX**] that have recently been proven for projective spaces by Givental [**Gi2**]. For the rest of this chapter we will therefore restrict ourselves to the case $X = \mathbb{P}^r$. It is expected that the same methods would work for other Fano varieties as well.

To state the Virasoro conditions we need some notation. We pick the obvious basis $\{T_a\}$ of $A^*(X)$ where T_a denotes the class of a linear subspace of codimension a for $a = 0, \dots, r$. Let $R : A^*(X) \to A^*(X)$ be the homomorphism of multiplication with the first Chern class $c_1(X)$. In our basis, the p-th power R^p of R is then given by $(R^p)_a{}^b = (r+1)^p \, \delta_{a+p,b}$.

For any $x \in \mathbb{Q}$, $k \in \mathbb{Z}_{\geq -1}$, and $0 \leq p \leq k+1$ denote by $[x]_p^k$ the z^p-coefficient of $\prod_{j=0}^k (z+x+j)$, or in other words the $(k+1-p)$-th elementary symmetric polynomial in $k+1$ variables evaluated at the numbers $x, \dots, x+k$.

Then the Virasoro conditions state that for any $k \geq 1$ and $g \geq 1$ we have an equation of power series in t_m^a and q^β (see e.g. [**EHX**])

$$0 = -\sum_{p=0}^{k+1} \left[\frac{3-r}{2}\right]_p^k (R^p)_0{}^b \langle\langle \tau_{k+1-p}(T_b) \rangle\rangle_g \tag{A}$$

$$+ \sum_{p=0}^{k+1} \sum_{m=0}^{\infty} \left[a+m+\frac{1-r}{2}\right]_p^k (R^p)_a{}^b t_m^a \langle\langle \tau_{k+m-p}(T_b) \rangle\rangle_g \tag{B}$$

$$+ \frac{1}{2} \sum_{p=0}^{k+1} \sum_{m=p-k}^{-1} (-1)^m \left[a+m+\frac{1-r}{2}\right]_p^k (R^p)_a{}^b \langle\langle \tau_{-m-1}(T^a) \tau_{k+m-p}(T_b) \rangle\rangle_{g-1} \tag{C}$$

$$+ \frac{1}{2} \sum_{p=0}^{k+1} \sum_{m=p-k}^{-1} \sum_{h=0}^{g} (-1)^m \left[a+m+\frac{1-r}{2}\right]_p^k (R^p)_a{}^b \langle\langle \tau_{-m-1}(T^a) \rangle\rangle_h \langle\langle \tau_{k+m-p}(T_b) \rangle\rangle_{g-h}. \tag{D}$$

We should mention that there are more general versions of these relations for all $k \geq -1$ and $g \geq 0$. The equations will then get additional correction terms that we have dropped here for the sake of simplicity.

For future computations it is convenient to construct a minor generalization of the topological recursion relations of proposition 1.3.12 that is mostly notational. Note that all genus-0 degree-0 invariants with fewer than 3 marked points are trivially zero, as the moduli spaces of stable maps are empty in this case. It is an important and interesting fact that many formulas concerning Gromov-Witten invariants get easier if we assign “virtual values” to these invariants in the unstable range:

CONVENTION 1.4.1. For the rest of this chapter we will from now on allow formal negative powers of the cotangent line classes (i.e. the index m in the $\tau_m(\gamma)$ can be any integer). Invariants $\langle\cdots\rangle_{g,\beta}$ and $\langle\cdots\rangle^i_\beta$ are simply defined to be zero if they contain a negative power of a cotangent line class at any point, except for the following cases of genus-0 degree-0 invariants with fewer than 3 marked points:

(i) $\langle\tau_{-2}(\mathrm{pt})\rangle_{0,0} = 1$,
(ii) $\langle\tau_{m_1}(\gamma_1)\tau_{m_2}(\gamma_2)\rangle_{0,0} = (-1)^{\max(m_1,m_2)}(\gamma_1\cdot\gamma_2)\delta_{m_1+m_2,-1}$,
(iii) $\langle\tau_{-i-1}(\gamma_1)\gamma_2\rangle^i_0 = (\gamma_1\cdot\gamma_2)$ for all $i \geq 0$.

The correlation functions $\langle\langle\cdots\rangle\rangle$ are changed accordingly so that the equations of remarks 1.2.7 and 1.3.13 remain true. In particular, these functions will now depend additionally on the variables t^a_m for $m < 0$.

REMARK 1.4.2. Note that convention 1.4.1 is consistent with the general formula for genus-0 degree-0 invariants

$$\langle\tau_{m_1}(\gamma_1)\cdots\tau_{m_n}(\gamma_n)\rangle_{0,0} = \binom{n-3}{m_1,\ldots,m_n}(\gamma_1\cdots\gamma_n)\delta_{m_1+\cdots+m_n,n-3},$$

as well as with the recursion relations for the $\langle\langle\cdots\rangle\rangle^i$ of proposition 1.3.12.

Using this convention we can now restate our topological recursion relations as follows:

COROLLARY 1.4.3 (**Topological recursion relations**). *For all $g > 0$, $N \geq 3g-1$, $m \in \mathbb{Z}$, and $\gamma \in A^*(X)$ we have*

$$\langle\langle\tau_{N+m}(\gamma)\rangle\rangle_g = \sum_{i+j=N-1}\langle\langle\tau_m(\gamma)T_a\rangle\rangle^i\langle\langle\tau_j(T^a)\rangle\rangle_g,$$

where the auxiliary correlation functions $\langle\langle\cdots\rangle\rangle^i$ are defined recursively by the formulas given in proposition 1.3.12, together with convention 1.4.1.

PROOF. The equations in the corollary are the same as in proposition 1.3.12 if $m \geq 0$. For $m < 0$ they reduce to the trivial equations $\langle\langle\tau_{N+m}(\gamma)\rangle\rangle_g = \langle\langle\tau_{N+m}(\gamma)\rangle\rangle_g$ by convention 1.4.1. □

Let us now apply convention 1.4.1 to the Virasoro relations. It is checked immediately that this realizes the (A) and (B) terms as part of the (D) terms via the conventions (i) and (ii), respectively. So by applying our convention we can drop the (A) and (B) terms if we allow arbitrary integers in the sum over m. We are thus left with the equations

$$0 = \frac{1}{2}\sum_{p=0}^{k+1}\sum_{m}(-1)^m \left[a+m+\frac{1-r}{2}\right]_p^k (R^p)_a{}^b \langle\langle \tau_{-m-1}(T^a)\tau_{k+m-p}(T_b)\rangle\rangle_{g-1} \tag{C}$$

$$+\frac{1}{2}\sum_{p=0}^{k+1}\sum_{m}\sum_{h=0}^{g}(-1)^m \left[a+m+\frac{1-r}{2}\right]_p^k (R^p)_a{}^b \langle\langle \tau_{-m-1}(T^a)\rangle\rangle_h \langle\langle \tau_{k+m-p}(T_b)\rangle\rangle_{g-h}. \tag{D}$$

Let us now analyze how these equations can be used to compute Gromov-Witten invariants. First of all we will compute the invariants recursively over the genus of the curves. The genus-0 invariants of $\mathbb{P}^r$ are known by corollary 1.3.11. So let us assume that we want to compute the invariants of some genus $g > 0$, and that we already know all invariants of smaller genus. In the Virasoro equations above this means that we know all of (C), as well as the terms of (D) where $h \neq 0$ and $h \neq g$. Noting that the (D) terms are symmetric under $h \mapsto g-h$ we can therefore rewrite the Virasoro conditions as

$$\sum_{p=0}^{k+1}\sum_{m}(-1)^m \left[a+m+\frac{1-r}{2}\right]_p^k (R^p)_a{}^b \langle\langle \tau_{-m-1}(T^a)\rangle\rangle_0 \langle\langle \tau_{k+m-p}(T_b)\rangle\rangle_g$$
$$= \text{(recursively known terms)}.$$

Next, we will compute the invariants of genus g recursively over the number of marked points. So let us assume that we want to compute the n-point genus-g invariants, and that we already know all invariants of genus g with fewer marked points. In the equations above this means that we fix a degree $d \geq 0$, integers $m_2,\dots,m_n$, and $n-1$ cohomology classes $T_{a_2},\dots,T_{a_n}$, and compare the $\left(q^d \cdot \prod_{i=2}^n t_{m_i}^{a_i}\right)$-coefficients of the equations. By the recursion process we then know all the invariants in which at least one of the marked points $x_2,\dots,x_n$ is on the genus-0 invariant. So we can write

$$\sum_{p=0}^{k+1}\sum_{m}\sum_{d_1+d_2=d}(-1)^m \left[a+m+\frac{1-r}{2}\right]_p^k (R^p)_a{}^b \cdot$$
$$\cdot \langle \tau_{-m-1}(T^a)\rangle_{0,d_1} \langle \tau_{k+m-p}(T_b)\tau_{m_2}(T_{a_2})\cdots\tau_{m_n}(T_{a_n})\rangle_{g,d_2}$$
$$= \text{(recursively known terms)}.$$

These are linear equations for the unknown numbers $\langle \tau_j(T_b)\tau_{m_2}(T_{a_2})\cdots\tau_{m_n}(T_{a_n})\rangle_{g,e}$, where j, b, and e vary. Note that for a given $j \geq 0$ there is exactly one such invariant $\langle \tau_j(T_{b_j})\tau_{m_2}(T_{a_2})\cdots\tau_{m_n}(T_{a_n})\rangle_{g,e_j}$: the values of b_j and e_j are determined uniquely by the dimension condition

$$(r+1)e_j + (r-3)(1-g) + n = j + b_j + \sum_{i=2}^n (m_i + a_i) \tag{2}$$

as we must have $0 \le b_j \le r$. Let us denote this invariant by x_j. Of course it may happen that $e_j < 0$, in which case we set $x_j = 0$. Our equations now read

$$\sum_{p=0}^{k+1} \sum_m (-1)^{m+p-k} \left[a+m+p-k+\frac{1-r}{2} \right]_p^k (R^p)_a{}^{b_m} \langle \tau_{-m-p+k-1}(T^a) \rangle_{0,d-e_m} \cdot x_m$$
$$= \text{(recursively known terms)}.$$

Let us now check how many non-trivial equations of this sort we get. Together with (2) the dimension conditions

$$(d-e_m)(r+1)+r-3+1 = -m-p+k-1+r-a$$

(for the genus-0 invariant) and $a+p=b_m$ (from the R^p factor) give

$$k = d(r+1)+(r-3)(1-g)+n-1-\sum_{i=2}^{n}(m_i+a_i), \tag{3}$$

which means that the value of k is determined by d. To avoid overly complicated notation we will denote the number k determined by (3) by $k(d)$. Moreover, let δ be the smallest value of d for which $k(d)$ is positive. We are then getting one equation for every degree $d \ge \delta$. As there are $r+1$ unknown invariants x_j in every degree however it is clear that our equations alone are not sufficient to determine the x_j.

Let us now apply our topological recursion relations. In terms of the recursion at hand, these relations can express every invariant x_m as a linear combination of invariants of the same form with $m < 3g-1$, plus some terms that are known recursively because they contain only invariants with fewer than n marked points. More precisely, we have

$$x_m = \sum_{i+j=N-1} \langle \tau_{m-N+2}(T_{b_m}) T^{b_j} \rangle^i_{e_m-e_j} x_j + \text{(recursively known terms)}$$

for all $N \ge 3g-1$ by corollary 1.4.3. Inserting this into the Virasoro conditions we get

$$\sum_{p=0}^{k(d)+1} \sum_m \sum_{i+j=N-1} (-1)^{m+p-k(d)} \left[a+m+p-k(d)+\frac{1-r}{2} \right]_p^{k(d)} (R^p)_a{}^{b_m} \cdot$$
$$\cdot \langle \tau_{-m-p+k(d)-1}(T^a) \rangle_{0,d-e_m} \cdot \langle \tau_{m-N+2}(T_{b_m}) T^{b_j} \rangle^i_{e_m-e_j} \cdot x_j$$
$$= \text{(recursively known terms)}.$$

Using the dimension conditions again, and noting that the sum over m is equivalent to independent sums over b_m and e_m, we can rewrite this as

$$\sum_{p=0}^{k(d)+1} \sum_{e} \sum_{i+j=N-1} (-1)^{1-a-(d-e)(r+1)} \left[\frac{3-r}{2} - (d-e)(r+1)\right]_p^{k(d)} (R^p)_a{}^b \cdot$$

$$\cdot \langle \tau_\bullet(T^a) \rangle_{0,d-e} \cdot \langle \tau_\bullet(T_b) T^{b_j} \rangle^i_{e-e_j} \cdot x_j$$

$$= \text{(recursively known terms)},$$

where the dots in the τ functions denote the uniquely determined numbers so that the invariants satisfy the dimension condition.

We are thus left with infinitely many equations (one for every $d \geq \delta$) for finitely many variables $x_0, \dots, x_{N-1}$. It is of course strongly expected that this system of equations should be solvable, i.e. that the matrix $V^{(N)} = (V^{(N)}_{d,j})_{d\geq\delta, 0\leq j<N}$ with

$$V^{(N)}_{d,j} := \sum_{p=0}^{k(d)+1} \sum_{e} (-1)^{1-a-(d-e)(r+1)} \left[\frac{3-r}{2} - (d-e)(r+1)\right]_p^{k(d)} (R^p)_a{}^b \cdot$$

$$\cdot \langle \tau_\bullet(T^a) \rangle_{0,d-e} \cdot \langle \tau_\bullet(T_b) T^{b_j} \rangle^{N-1-j}_{e-e_j} \tag{4}$$

has maximal rank N. This is what we will show in the next section. In fact, we will prove that *every* $N \times N$ submatrix of V is invertible. So we have shown

THEOREM 1.4.4. *The Virasoro conditions together with the topological recursion relations of corollary 1.4.3 give a constructive way to determine all Gromov-Witten invariants of projective spaces.*

It should be noted that the calculation of some genus-g degree-d invariant usually requires the recursive calculation of invariants of smaller genus with *bigger* degree and *more* marked points. This is the main factor for slowing down the algorithm as the genus grows.

Some numbers that have been computed using this algorithm can be found in section 1.6.

1.5. Proof of the Virasoro algorithm

The goal of this section is to prove the technical result needed for theorem 1.4.4:

PROPOSITION 1.5.1. *Fix any $N \geq 1$, and let $V^{(N)} = (V^{(N)}_{d,j})_{d\geq\delta,0\leq j<N}$ be the matrix defined in equation (4). Then any $N \times N$ submatrix of $V^{(N)}$, obtained by picking N distinct values of d, has non-zero determinant.*

We will prove this statement in several steps. In a first step, we will make the entries of the matrix independent of N and reduce the invariants $\langle\cdots\rangle^i$ to ordinary rational Gromov-Witten invariants:

LEMMA 1.5.2. *Let $W = (W_{d,j})_{d\geq\delta,j\geq 0}$ be the matrix with entries $W_{d,j} = V_{d,j}^{(j+1)}$. Then:*

(i) *For all $d \geq \delta$, $N \geq 1$, and $0 \leq j < N$ we have*
$$V_{d,j}^{(N+1)} = V_{d,j}^{(N)} - \langle T_{b_N} T^{b_j}\rangle_{e_N-e_j}^{N-1-j} \cdot W_{d,N}.$$

(ii) *For any $N \geq 1$ and any $N \times N$ submatrix of W obtained by taking the* first N *columns of* any N *rows, the determinant of this submatrix is the same as the corresponding submatrix of $V^{(N)}$.*

PROOF. (i): Comparing the q^{e-e_j}-terms of the recursive relations of proposition 1.3.12 we find that
$$\langle\tau_\bullet(T_b)T^{b_j}\rangle_{e-e_j}^{N-j} = \langle\tau_\bullet(T_b)T^{b_j}\rangle_{e-e_j}^{N-1-j} - \langle\tau_\bullet(T_b)T^{b_N}\rangle_{0,e-e_N}\langle T_{b_N}T^{b_j}\rangle_{e_N-e_j}^{N-1-j},$$
from which the claim follows.

(ii): We prove the statement by induction on N. There is nothing to show for $N = 1$. Now assume that we know the statement for some value of N, i.e. any two corresponding $N \times N$ submatrices of the matrices with columns
$$(W_{\cdot,0},\dots,W_{\cdot,N-1}) \quad \text{and} \quad (V_{\cdot,0}^{(N)},\dots,V_{\cdot,N-1}^{(N)})$$
have the same determinant. Of course, the same is then also true for any corresponding $(N+1)\times(N+1)$ submatrices of
$$(W_{\cdot,0},\dots,W_{\cdot,N-1},W_{\cdot,N}) \quad \text{and} \quad (V_{\cdot,0}^{(N)},\dots,V_{\cdot,N-1}^{(N)},W_{\cdot,N}).$$
But by (i), the latter matrix is obtained from
$$(V_{\cdot,0}^{(N+1)},\dots,V_{\cdot,N-1}^{(N+1)},W_{\cdot,N}) = (V_{\cdot,0}^{(N+1)},\dots,V_{\cdot,N}^{(N+1)})$$
by an elementary column operation, so the result follows. □

So by the lemma, it suffices to consider the matrix W. Let us now evaluate the genus-0 Gromov-Witten invariants contained in the definition of W.

CONVENTION 1.5.3. For the rest of this section, we will make the usual convention that a product $\prod_{i=i_1}^{i_2} A_i$ is defined to be $\prod_{i=i_2+1}^{i_1-1} A_i^{-1}$ if $i_1 > i_2$.

LEMMA 1.5.4. *For all $d \geq \delta$ and $j \geq 0$ the matrix entry $W_{d,j}$ is equal to the z^j-coefficient of*

$$-\frac{\prod_{i=0}^{k(d)} \left(r+1+\left(\frac{3-r}{2}+i\right)z\right)}{(1+(d-e_j)z)^{b_j+1}\prod_{i=0}^{d-e_j-1}(1+iz)^{r+1}}.$$

PROOF. Recall that by equation (4) the matrix entries $W_{d,j}$ are given by

$$\sum_{e,p} \underbrace{\left[\frac{3-r}{2}-(d-e)(r+1)\right]_p^{k(d)} (R^p)_a{}^b}_{\text{(A)}} \underbrace{(-1)^{1-a-(d-e)(r+1)}\langle\tau_\bullet(T^a)\rangle_{0,d-e}}_{\text{(B)}} \underbrace{\langle\tau_\bullet(T_b)T^{b_j}\rangle_{0,e-e_j}}_{\text{(C)}}.$$

The three terms in this expression can all be expressed easily in terms of generating functions. Recalling that $(R^p)_a{}^b = (r+1)^p\delta_{a+p,b}$, the (A) term is by definition equal to the z^p-coefficient of

$$\delta_{a+p,b}\prod_{i=0}^{k(d)}\left((r+1)z+\frac{3-r}{2}-(d-e)(r+1)+i\right).$$

The (B) and (C) terms are rational 2-point invariants of $\mathbb{P}^r$ which are determined by proposition 1.2.6: the Gromov-Witten invariant in (B) (without the sign) is equal to the z^a-coefficient of $\prod_{i=1}^{d-e}\frac{1}{(z+i)^{r+1}}$. So including the sign factor we get the z^a-coefficient of $-\prod_{i=1}^{d-e}\frac{1}{(z-i)^{r+1}}$. The (C) term is again by proposition 1.2.6 equal to the z^{r-b}-coefficient of $\frac{1}{(z+e-e_j)^{b_j+1}}\prod_{i=1}^{e-e_j-1}\frac{1}{(z+i)^{r+1}}$.

Multiplying these expressions and performing the sums over a, b, and p, we find that $W_{d,j}$ is the z^r-coefficient of

$$-\sum_e \frac{\prod_{i=0}^{k(d)}\left((r+1)z+\frac{3-r}{2}-(d-e)(r+1)+i\right)}{(z+e-e_j)^{b_j+1}\prod_{i=1}^{d-e}(z-i)^{r+1}\cdot\prod_{i=1}^{e-e_j-1}(z+i)^{r+1}},$$

which can be rewritten as the sum of residues

$$-\sum_e \operatorname{res}_{z=0}\frac{\prod_{i=0}^{k(d)}\left((r+1)z+\frac{3-r}{2}-(d-e)(r+1)+i\right)}{(z+e-e_j)^{b_j+1}\prod_{i=e-d}^{e-e_j-1}(z+i)^{r+1}}\,dz.$$

Note that this fraction depends on z and e only in the combination $z+e$. Consequently, instead of summing the above residues at 0 over all e we can as well set $e=d$ and sum over all poles $z\in\mathbb{C}$ of the rational function. So we see that $W_{d,j}$ is equal to

$$-\sum_{z_0\in\mathbb{C}}\operatorname{res}_{z=z_0}\frac{\prod_{i=0}^{k(d)}\left((r+1)z+\frac{3-r}{2}+i\right)}{(z+d-e_j)^{b_j+1}\prod_{i=0}^{d-e_j-1}(z+i)^{r+1}}\,dz.$$

By the residue theorem this is nothing but the residue at infinity of our rational function. So we conclude that

$$W_{d,j} = \mathrm{res}_{z=0} \frac{\prod_{i=0}^{k(d)} \left(\frac{r+1}{z} + \frac{3-r}{2} + i\right)}{\left(\frac{1}{z} + d - e_j\right)^{b_j+1} \prod_{i=0}^{d-e_j-1} \left(\frac{1}{z} + i\right)^{r+1}} \, d\left(\frac{1}{z}\right).$$

Finally note that by equations (2) and (3) we have the dimension condition

$$k(d) + 1 = j + b_j + (d - e_j)(r+1),$$

so multiplying our expression with $z^{k(d)+1}$ in the numerator and denominator we get

$$W_{d,j} = -\,\mathrm{res}_{z=0} \frac{\prod_{i=0}^{k(d)} \left(r + 1 + \left(\frac{3-r}{2} + i\right) z\right)}{z^{j+1} (1 + (d - e_j)z)^{b_j+1} \prod_{i=0}^{d-e_j-1} (1 + iz)^{r+1}} \, dz.$$

This proves the lemma. □

To avoid unnecessary factors in the determinants, let us divide row d of W by the non-zero number $-(r+1)^{k(d)+1}$ and call the resulting matrix $\tilde{W}$. So we will now consider $N \times N$ submatrices of $\tilde{W} = (\tilde{W}_{d,j})$, obtained by picking the first N columns of any N rows, where $\tilde{W}_{d,j}$ is the z^j-coefficient of

$$\frac{\prod_{i=0}^{k(d)} \left(1 + \left(\frac{3-r}{2r+2} + \frac{i}{r+1}\right) z\right)}{(1 + (d - e_j)z)^{b_j+1} \prod_{i=0}^{d-e_j-1} (1 + iz)^{r+1}}. \tag{5}$$

The following technical lemma is the main step in computing their determinants.

LEMMA 1.5.5. *Assume that we are given $N, n \in \mathbb{N}$, $M \in \mathbb{Z}$, $q, c \in \mathbb{R}$, and distinct integers $a_0, \dots, a_N$. Set*

$$f(z) = \sum_{k=0}^{N} \left(\prod_{i \neq k} \frac{1}{a_k - a_i} \cdot \prod_{i=M}^{na_k} \left(1 + \frac{c+i}{n} z\right) \cdot (1 + a_k z)^q \cdot \prod_{i=a_k}^{-1} (1 + iz)^n \right)$$

as a formal power series in z.

(i) *For any $i \geq 0$ the z^i-coefficient of the power series $f(z)$ is a symmetric polynomial in $a_0, \dots, a_N$ of degree at most $i - N$. (In particular, it is zero for $i < N$.)*

(ii) *The z^N-coefficient of $f(z)$ is equal to*

$$\frac{1}{N!} \prod_{i=1}^{N} \left(c + q - N + \frac{n+1}{2} + i\right).$$

PROOF. In the following proof, we will slightly abuse notation and vary the arguments given explicitly for the function f. So if we e.g. want to study how $f(z)$ changes if we vary c, we will write $f(z)$ also as $f(z,c)$, and denote by $f(z,c+1)$ the function obtained from $f(z)$ when substituting c by $c+1$.

(i): It is obvious by definition that $f(z)$ is symmetric in the a_i. We will prove the polynomiality and degree statements by induction on N.

"$N = 0$": In this case we have

$$f(z) = \prod_{i=M}^{na_0} \left(1 + \frac{c+i}{n} z\right) \cdot (1 + a_0 z)^q \cdot \prod_{i=a_0}^{-1} (1 + iz)^n.$$

We have to show that the z^i-coefficient of $f(z)$ is a polynomial in a_0 of degree at most i. Note that this property is stable under taking products, so if we write $f(z) = \prod_{j=0}^n f^{(j)}(z)$ with

$$f^{(0)}(z) = \prod_{i=M}^{0} \left(1 + \frac{c+i}{n} z\right) \cdot (1 + a_0 z)^q$$

$$\text{and} \quad f^{(j)}(z) = \prod_{i=0}^{a_0-1} \left(1 + \frac{c+j}{n} \frac{z}{1+iz}\right) \quad \text{for } 1 \le j \le n$$

then it suffices to prove the statements for the $f^{(j)}$ separately. But the statement is obvious for $f^{(0)}$, so let us focus on $f^{(j)}$ for $j > 0$. Note that

$$f^{(j)}(z, a_0 + 1) = f^{(j)}(z, a_0) \cdot \left(1 + \frac{c+j}{n} \frac{z}{1 + a_0 z}\right).$$

So if f_i denotes the z^i-coefficient of $f(z)$ we get

$$f_i^{(j)}(a_0 + 1) - f_i^{(j)}(a_0) = \frac{c+j}{n} \sum_{k=0}^{i-1} (-a_0)^k f_{i-1-k}^{(j)}(a_0). \tag{6}$$

The statement now follows by induction on i: it is obvious that the constant z-term of $f^{(j)}(z)$ is 1. For the induction step, assume that we know that $f_i^{(j)}$ is polynomial of degree at most i in a_0 for $i = 0, \dots, i_0 - 1$. Then the right hand side of (6) is polynomial of degree at most $i_0 - 1$ in a_0, so $f_{i_0}^{(j)}$ is polynomial of degree at most i_0. This completes the proof of the $N = 0$ part of (i).

"$N \to N+1$": Note that

$$f(z, N+1, a_0, \dots, a_{N+1}) = \frac{f(z, N, a_0, \dots, a_N) - f(z, N, a_1, \dots, a_{N+1})}{a_0 - a_{N+1}}. \tag{7}$$

By symmetry we have $f(z, N, a_0, \dots, a_N) = f(z, N, a_1, \dots, a_{N+1})$ if $a_0 = a_{N+1}$. So every z^i-coefficient of this expression is a polynomial in the a_k. Its degree is at most $(i - N) - 1$ by the induction hypothesis. This proves (i).

(ii): By (i) the z^N-coefficient of $f(z, N)$ does not depend on the choice of a_k, so we can set $a_k = a + k$ for all k and keep only a as a variable. It does not depend on M either, as a shift $M \mapsto M \pm 1$ corresponds to multiplication of $f(z)$ with $(1 + \alpha z)^{\mp 1}$

for some α, which does not affect the leading coefficient of $f(z)$. So we can set $M = 1$ without loss of generality.

The recursion relation (7) now reads

$$f(z,N+1,a) = \frac{f(z,N,a+1) - f(z,N,a)}{N+1}. \tag{8}$$

By (i) the z^i-coefficient of $f(z)$ has degree at most $i-N$ in a. So if we denote by f_i the a^{i-N}-coefficient of the z^i-coefficient of $f(z,a)$, comparing the z^i-coefficients in (8) yields $f_i(N+1) = \frac{i-N}{N+1} f_i(N)$ and therefore

$$f_N(N) = \frac{1}{n} \cdot \frac{2}{n-1} \cdots \frac{n}{1} \cdot f_N(0) = f_N(0).$$

In other words, instead of computing the z^N-coefficient of $f(z,N)$ we can as well set $N = 0$ and compute the a^N-coefficient (i.e. the leading coefficient in a) of the z^N-coefficient of $f(z,N=0)$. So let us set $N = 0$ to obtain

$$f(z) = \prod_{i=1}^{na} \left(1 + \frac{c+i}{n} z\right) \cdot (1+az)^q \cdot \prod_{i=a}^{-1} (1+iz)^n,$$

and denote by g_N the a^N-coefficient of the z^N-coefficient of $f(z,a)$. Moreover, set $g(z) = \sum_{N \geq 0} g_N z^N$. Our goal is then to compute $g(z)$.

We will do this by analyzing how $f(z)$ (and thus $g(z)$) varies when we vary q, c, or n. To start, it is obvious that

$$g(z,q+\alpha) = (1+z)^\alpha g(z,q) \tag{9}$$

for all $\alpha \in \mathbb{R}$. Next, note that

$$f(z,c+1) = f(z,c) \cdot \frac{1 + \frac{c+1+na}{n} z}{1 + \frac{c+1}{n} z}.$$

For $g(z)$ we can drop all terms in which the degree in a is smaller than the degree in z. So we conclude

$$g(z,c+1) = (1+z) g(z,c).$$

Combining this with (9) we see that $g(z)$ will depend on q and c only through their sum $q+c$. So in what follows we can set $c = 0$, and replace q by $q+c$ in the final result.

Varying n is more complicated. We have

$$f(z,n+1) = f(z,n) \cdot \underbrace{\prod_{j=0}^{n} \prod_{i=1}^{a} \left(1 + \frac{j}{n(n+1)} \cdot \frac{z}{1 + (i - \frac{j}{n})z}\right)}_{=: \tilde{f}(z)}.$$

Recall that for $g(z)$ we only need the summands in $\tilde{f}(z)$ in which the degree in a is equal to the degree in z. So let us denote the a^N-coefficient of the z^N-coefficient of $\tilde{f}(z,a)$ by $\tilde{g}_N$, and assemble the $\tilde{g}_N$ into a generating function $\tilde{g}(z) = \sum_{N\geq 0} \tilde{g}_N z^N$, so that $g(z,n+1) = g(z,n)\cdot\tilde{g}(z)$. To determine $\tilde{g}(z)$ compare the a^N-coefficient of the z^N-coefficient in the recursive equation

$$\frac{\tilde{f}(z,a)-\tilde{f}(z,a-1)}{z} = \tilde{f}(z,a-1)\cdot\frac{1}{z}\left(\prod_{j=0}^{n}\left(1+\frac{j}{n(n+1)}\cdot\frac{z}{1+(a-\frac{j}{n})z}\right)-1\right).$$

On the left hand side this coefficient is $(N+1)\tilde{g}_{N+1}$. On the right hand side it is the z^N-coefficient of

$$\tilde{g}(z)\cdot\left(\sum_{j=0}^{N}\frac{j}{n(n+1)}\right)\cdot\frac{1}{1+z} = \tilde{g}(z)\cdot\frac{1}{2}\frac{1}{1+z}.$$

So we see that

$$\frac{d\tilde{g}(z)}{dz} = \frac{1}{2}\frac{1}{1+z}\tilde{g}(z).$$

Together with the obvious initial condition $\tilde{g}(0)=1$ we conclude that $\tilde{g}(z)=\sqrt{1+z}$, and therefore

$$g(z,n+1) = g(z,n)\cdot\sqrt{1+z}.$$

Comparing this with (9) we see that $g(z)$ depends on n and q only through the sum $q+\frac{n}{2}$. We can therefore set $n=1$ and then replace q by $q+\frac{n-1}{2}$ in the final result.

But setting n to 1 (and c to 0) we are simply left with

$$f(z) = \prod_{i=1}^{a}(1+iz)\cdot(1+az)^q\cdot\prod_{i=a}^{-1}(1+iz) = (1+az)^{q+1}.$$

So it follows that $g(z) = (1+z)^{q+1}$ and therefore

$$g_N = \binom{q+1}{N} = \frac{1}{N!}\prod_{i=1}^{N}(q+1-N+i).$$

Setting back in the c and n dependence, i.e. replacing q by $q+c+\frac{n-1}{2}$, we get the desired result. □

We are now ready to compute our determinant.

PROPOSITION 1.5.6. *Let $(\tilde{W})_{d\geq\delta,j\geq 0}$ be the matrix defined in equation (5). Pick N distinct integers $d_0,\dots,d_{N-1}$ with $d_i\geq\delta$ for all i. Then the determinant of the $N\times N$ submatrix of $\tilde{W}$ obtained by picking the first N columns of rows $d_0,\dots,d_{N-1}$ is equal to*

$$\frac{\prod_{i>j}(d_i-d_j)}{\prod_{i=1}^{N-1} i!}\cdot\prod_{i=1}^{N-1}\left(i+\frac{1}{2}\right)^{N-i}.$$

In particular, this determinant is never zero.

PROOF. We prove the statement by induction on N. The result is obvious for $N = 1$ as every entry in the first column (i.e. $j = 0$) of $\tilde{W}$ is equal to 1. So let us assume that we know the statement for a given value N. We will prove it for $N+1$.

Denote by $\Delta(d_0,\dots,d_{N-1})$ the determinant of the $N \times N$ submatrix of $\tilde{W}$ obtained by picking the first N columns of rows $d_0,\dots,d_{N-1}$. Then by expansion along the last column and the induction assumption we get

$$\begin{aligned}\Delta(d_0,\dots,d_N) &= \sum_{k=0}^{N}(-1)^{k+N}\tilde{W}_{d_k,N}\cdot\Delta(d_0,\dots,d_{k-1},d_{k+1},\dots,d_N)\\ &= \frac{\prod_{i>j}(d_i-d_j)}{\prod_{i=1}^{N-1} i!}\cdot\prod_{i=1}^{N-1}\left(i+\frac{1}{2}\right)^{N-i}\cdot\sum_{k=0}^{N}\left(\prod_{i\neq k}\frac{1}{d_k-d_i}\cdot\tilde{W}_{d_k,N}\right).\end{aligned}$$

But by lemma 1.5.5, applied to the values $n = r+1$, $a_k = d_k - e_N$, $q = -b_N - 1$, $M = (r+1)(d_k - e_N) - k(d_k) = 1 - N - b_N$, and $c = \frac{3-r}{2} - M = \frac{1-r}{2} + N + b_N$, the sum in this expression is equal to

$$\frac{1}{N!}\prod_{i=1}^{N}\left(i+\frac{1}{2}\right).$$

Inserting this into the expression for the determinant, we obtain

$$\Delta(d_0,\dots,d_N) = \frac{\prod_{i>j}(d_i-d_j)}{\prod_{i=1}^{N} i!}\cdot\prod_{i=1}^{N}\left(i+\frac{1}{2}\right)^{N+1-i},$$

as desired. □

REMARK 1.5.7. It should be remarked that the expression for the determinant in proposition 1.5.6 is surprisingly simple, given the complicated structure of the Virasoro conditions and the topological recursion relations. It would be interesting to see if there is a deeper relation between these two sets of equations that is not yet understood and explains the simplicity of our results.

Combining the arguments of this section, we see that the systems of linear equations obtained from the Virasoro conditions and the topological recursion relations in section 1.4 are always solvable. This completes the proof of theorem 1.4.4.

1.6. Numerical examples and applications

In this section we will give some examples of invariants that can be computed using the method of section 1.4. The computations have been performed using the C++ program GROWI [**Ga5**].

EXAMPLE 1.6.1 (Hurwitz numbers). For $g,d \geq 0$ let $H_{g,d}$ be the number of degree-d coverings of $\mathbb{P}^1$ by a curve of genus g with simple ramification over $2d+2g-2$ fixed points in $\mathbb{P}^1$. These so-called **Hurwitz numbers** have been studied and computed first in [**Hu**]. They have an easy interpretation as Gromov-Witten invariants of $\mathbb{P}^1$, namely

$$H_{g,d} = \langle \tau_1(\mathrm{pt})^{2d+2g-2} \rangle_{g,d}.$$

To see this, note that the condition $\mathrm{ev}_i^* \mathrm{pt}$ ensures that the i-th marked point is mapped to a given point in $\mathbb{P}^1$, and the additional condition ψ_i requires this marked point to be a ramification point (as this is precisely the degeneracy locus of the induced morphism $T_{C,x_i} \to T_{\mathbb{P}^1, f(x_i)}$ on the tangent spaces). It is easily checked that the presence of virtual fundamental classes on boundary components of $\bar{M}_{g,n}(\mathbb{P}^1,d)$ does not give rise to a contribution to the Gromov-Witten invariants.

The following table shows the Hurwitz numbers, i.e. the Gromov-Witten invariants mentioned above, in the cases when $g \leq 4$ and $d \leq 6$.

	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$
$g=0$	1	$\frac{1}{2}$	4	120	8400	1088640
$g=1$	0	$\frac{1}{2}$	40	5460	1189440	382536000
$g=2$	0	$\frac{1}{2}$	364	206640	131670000	100557737280
$g=3$	0	$\frac{1}{2}$	3280	7528620	13626893280	24109381296000
$g=4$	0	$\frac{1}{2}$	29524	271831560	1379375197200	5576183206513920

Recently a closed formula has been found that expresses all Hurwitz numbers $H_{g,d}$ in terms of certain integrals over the moduli spaces $\bar{M}_{g,n}$ of stable curves. See [**ELSV**], [**FaP**] for details.

EXAMPLE 1.6.2 (Plane curves through given points). For any $g \geq 0, d > 0$ consider the invariant $\langle \mathrm{pt}^{3d-1+g} \rangle_{g,d}$ of $\mathbb{P}^2$. The first few numbers are given in the following table.

	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
$g=0$	1	1	12	620	87304	26312976	14616808192
$g=1$	0	0	1	225	87192	57435240	60478511040
$g=2$	0	0	0	27	36855	58444767	122824720116
$g=3$	0	0	0	1	7915	34435125	153796445095
$g=4$	0	0	0	0	882	12587820	128618514477

It can be shown (see [**Va2**] section 4) that these numbers are all enumerative, i.e. they are the numbers of genus-g degree-d plane curves through $3d-1+g$ points in general position. The numbers for genus zero have been found by Kontsevich (see [**KM**] claim 5.2.1), the numbers for general g some time later by Caporaso and Harris [**CH**].

EXAMPLE 1.6.3 (1-point invariants). Let $X = \mathbb{P}^r$, and denote by $T_a \in A^*(X)$ the class of a linear subspace of X of codimension a. We have seen in proposition 1.2.6 (i) that the rational 1-point functions $\langle \tau_m(T_a) \rangle_{0,d}$ of $\mathbb{P}^r$ (satisfying the dimension condition $a + m = \operatorname{vdim} \bar{M}_{0,1}(\mathbb{P}^r, d)$) are given as the z^{r-a}-coefficient of the generating function

$$\prod_{i=1}^{d} \frac{1}{(z+i)^{r+1}}.$$

Similarly, it follows from the elliptic topological recursion relations (see **[Ge1]** equation 3) that the elliptic 1-point functions $\langle \tau_m(T_a) \rangle_{1,d}$ of $\mathbb{P}^r$ (satisfying the dimension condition $a + m = \operatorname{vdim} \bar{M}_{1,1}(\mathbb{P}^r, d)$) are the z^{r-a}-coefficients of the generating function

$$\frac{(r+1)(2z+2d-r)}{48(z+d)^2} \prod_{i=1}^{d-1} \frac{1}{(z+i)^{r+1}}.$$

For genus bigger than 1 no such generating functions for the 1-point invariants are known. As a numerical example, we list in the following table some 1-point invariants of $\mathbb{P}^2$, i.e. invariants of the form $\langle \tau_m(\gamma) \rangle_{g,d}$, where m is determined by the dimension condition $3d + g = m + \deg \gamma$.

	$\gamma = \mathrm{pt}$			$\gamma = H$			$\gamma = 1$		
	$d=0$	$d=1$	$d=2$	$d=0$	$d=1$	$d=2$	$d=0$	$d=1$	$d=2$
$g=0$	–	1	$\frac{1}{8}$	–	-3	$-\frac{9}{16}$	–	6	$\frac{3}{2}$
$g=1$	–	0	$\frac{1}{32}$	$-\frac{1}{8}$	$\frac{1}{8}$	$-\frac{3}{32}$	$\frac{1}{8}$	$-\frac{1}{4}$	$\frac{23}{128}$
$g=2$	0	$-\frac{1}{240}$	$-\frac{1}{960}$	$-\frac{1}{960}$	$-\frac{1}{960}$	$\frac{13}{1536}$	$\frac{7}{640}$	$\frac{1}{128}$	$-\frac{27}{1280}$
$g=3$	0	$\frac{1}{3360}$	$-\frac{1}{16128}$	$-\frac{1}{40320}$	0	$-\frac{163}{645120}$	$\frac{41}{161280}$	$-\frac{97}{161280}$	$\frac{43}{36864}$
$g=4$	0	$-\frac{1}{80640}$	$\frac{11}{1075200}$	$-\frac{1}{1075200}$	$-\frac{1}{153600}$	$-\frac{1}{147456}$	$\frac{127}{12902400}$	$\frac{173}{4300800}$	$-\frac{4567}{103219200}$

EXAMPLE 1.6.4 (Invariants of $\mathbb{P}^3$). The expected dimension of genus-g degree-d curves in $\mathbb{P}^3$ is $4d$ for all g. So given integers $a, b \geq 0$ with $a + 2b = 4d$ we can ask for the number of degree-d space curves of genus g that intersect a given lines and b given points in general position. Of course the naïve expectation is that this number corresponds to the Gromov-Witten invariant

$$n_{g,d}(a,b) := \langle L^a \mathrm{pt}^b \rangle_{g,d}$$

where L denotes the class of a line.

It follows from remark 1.2.5 that this interpretation is indeed correct for rational curves. In higher genus however this is no longer true for two reasons:

(i) In general the number of genus-g degree-d space curves through a general lines and b general points (with $a + 2b = 4d$) is *not* finite. For example, consider curves of genus 3 and degree 4 through 10 lines $L_1, \dots, L_{10}$ and 3 points P_1, P_2, P_3. The three given points span a plane $\mathbb{P}^2 \subset \mathbb{P}^3$. In this plane we have a family of genus-3 degree-4 curves of dimension $\binom{4+2}{2} - 1 = 14$. Hence there is a one-dimensional family of such plane curves that intersect the 13 points $L_1 \cap \mathbb{P}^2, \dots, L_{10} \cap \mathbb{P}^2, P_1, P_2, P_3$. The answer to our enumerative problem is therefore not finite.

(ii) Even if the enumerative number is finite it will in general not be equal to the Gromov-Witten invariant $n_{g,d}(a,b)$. The reason for this is that there are genus-g degree-d stable maps that consist of a degree-d component of some genus $g' < g$ and various contracted components whose genera add up to the difference $g - g'$. These stable maps (that correspond geometrically to space curves of genus g') will give a non-enumerative contribution to the Gromov-Witten invariants. Consequently, the invariants $n_{g,d}(a,b)$ should be some linear combinations of the corresponding enumerative numbers for all genera $g' \leq g$ with the same incidence conditions.

As a generalization of the Gopakumar-Vafa conjecture [**GoV1, GoV2**] Pandharipande has conjectured a general formula that should compute the enumerative numbers (if they are finite) in terms of the Gromov-Witten invariants. More precisely, let us fix $d, a, b \geq 0$ with $a + 2b = 4d$ and set $n_g := n_{g,d}(a,b)$. Define numbers $N_g := N_{g,d}(a,b)$ by the equation of formal power series in z

$$\sum_{g \geq 0} n_g z^{2g-2} = \sum_{g \geq 0} N_g z^{2g-2} \left(\frac{\sin \frac{z}{2}}{\frac{z}{2}} \right)^{4d+2g-2}.$$

Then the numbers N_g are conjectured to be integers for all g. Moreover, if the number of genus-g degree-d curves through a lines and b points is finite then this number should be equal to $N_{g,d}(a,b)$. The conjecture has been proven for $g \leq 2$ in [**P2**] theorem 3, with the invariants being enumerative in all cases. For more details about the conjecture see [**P3**] section 3 and the references therein.

The following table lists the numbers $N_{g,d}(a,b)$ for $g \leq 4$ and $d \leq 6$. The numbers are all integers and thus support the conjecture. Note however that some of them are negative.

$d = 1$	$n_0 = N_0$	n_1	N_1	n_2	N_2	n_3	N_3	n_4	N_4
$L^0\mathrm{pt}^2$	1	$-\frac{1}{12}$	0	$\frac{1}{360}$	0	$-\frac{1}{20160}$	0	$\frac{1}{1814400}$	0
$L^2\mathrm{pt}^1$	1	$-\frac{1}{12}$	0	$\frac{1}{360}$	0	$-\frac{1}{20160}$	0	$\frac{1}{1814400}$	0
$L^4\mathrm{pt}^0$	2	$-\frac{1}{6}$	0	$\frac{1}{180}$	0	$-\frac{1}{10080}$	0	$\frac{1}{907200}$	0

$d=2$	$n_0=N_0$	n_1	N_1	n_2	N_2	n_3	N_3	n_4	N_4
$L^0\mathrm{pt}^4$	0	0	0	0	0	0	0	0	0
$L^2\mathrm{pt}^3$	1	$-\frac{1}{4}$	0	$\frac{7}{240}$	0	$-\frac{2}{945}$	0	$\frac{13}{120960}$	0
$L^4\mathrm{pt}^2$	4	-1	0	$\frac{7}{60}$	0	$-\frac{8}{945}$	0	$\frac{13}{30240}$	0
$L^6\mathrm{pt}^1$	18	$-\frac{9}{2}$	0	$\frac{21}{40}$	0	$-\frac{4}{105}$	0	$\frac{13}{6720}$	0
$L^8\mathrm{pt}^0$	92	-23	0	$\frac{161}{60}$	0	$-\frac{184}{945}$	0	$\frac{299}{30240}$	0

$d=3$	$n_0=N_0$	n_1	N_1	n_2	N_2	n_3	N_3	n_4	N_4
$L^0\mathrm{pt}^6$	1	$-\frac{5}{12}$	0	$\frac{1}{12}$	0	$-\frac{43}{4032}$	0	$\frac{713}{725760}$	0
$L^2\mathrm{pt}^5$	5	$-\frac{25}{12}$	0	$\frac{5}{12}$	0	$-\frac{215}{4032}$	0	$\frac{713}{145152}$	0
$L^4\mathrm{pt}^4$	30	$-\frac{25}{2}$	0	$\frac{5}{2}$	0	$-\frac{215}{672}$	0	$\frac{713}{24192}$	0
$L^6\mathrm{pt}^3$	190	$-\frac{469}{6}$	1	$\frac{46}{3}$	0	$-\frac{19207}{10080}$	0	$\frac{8701}{51840}$	0
$L^8\mathrm{pt}^2$	1312	$-\frac{1598}{3}$	14	$\frac{307}{3}$	0	$-\frac{30997}{2520}$	0	$\frac{46517}{45360}$	0
$L^{10}\mathrm{pt}^1$	9864	-3960	150	747	0	$-\frac{1219}{14}$	0	$\frac{23077}{3360}$	0
$L^{12}\mathrm{pt}^0$	80160	-31900	1500	5930	0	$-\frac{56585}{84}$	0	$\frac{19099}{378}$	0

$d=4$	$n_0=N_0$	n_1	N_1	n_2	N_2	n_3	N_3	n_4	N_4
$L^0\mathrm{pt}^8$	4	$-\frac{4}{3}$	1	$-\frac{1}{180}$	0	$\frac{103}{1080}$	0	$-\frac{26813}{907200}$	0
$L^2\mathrm{pt}^7$	58	$-\frac{179}{6}$	4	$\frac{2491}{360}$	0	$-\frac{1927}{2160}$	0	$\frac{94343}{1814400}$	0
$L^4\mathrm{pt}^6$	480	-248	32	58	0	$-\frac{343}{45}$	0	$\frac{727}{1512}$	0
$L^6\mathrm{pt}^5$	4000	$-\frac{6070}{3}$	310	$\frac{4090}{9}$	0	$-\frac{2923}{54}$	0	$\frac{9179}{4536}$	0
$L^8\mathrm{pt}^4$	35104	$-\frac{51772}{3}$	3220	$\frac{164486}{45}$	0	$-\frac{49522}{135}$	0	$-\frac{82603}{16200}$	0
$L^{10}\mathrm{pt}^3$	327888	-156594	34674	$\frac{155518}{5}$	27	$-\frac{49029}{20}$	-1	$-\frac{6294731}{25200}$	0
$L^{12}\mathrm{pt}^2$	3259680	-1515824	385656	$\frac{847322}{3}$	792	$-\frac{715949}{45}$	-28	$-\frac{31738771}{7560}$	0
$L^{14}\mathrm{pt}^1$	34382544	-15620216	4436268	$\frac{41109847}{15}$	15498	$-\frac{4210961}{45}$	-526	$-\frac{4459477639}{75600}$	0
$L^{16}\mathrm{pt}^0$	383306880	-170763640	52832040	$\frac{85167148}{3}$	258300	$-\frac{3408023}{9}$	-8460	$-\frac{743152381}{945}$	0

$d=5$	$n_0=N_0$	N_1	N_2	N_3	N_4
$L^0\mathrm{pt}^{10}$	105	42	0	0	0
$L^2\mathrm{pt}^9$	1265	354	8	0	0
$L^4\mathrm{pt}^8$	13354	3492	128	0	0
$L^6\mathrm{pt}^7$	139098	38049	1776	0	0
$L^8\mathrm{pt}^6$	1492616	441654	24252	0	0
$L^{10}\mathrm{pt}^5$	16744080	5378454	335580	10	0
$L^{12}\mathrm{pt}^4$	197240400	68292324	4742064	648	0
$L^{14}\mathrm{pt}^3$	2440235712	901654884	68549424	28951	-636
$L^{16}\mathrm{pt}^2$	31658432256	12358163808	1014183168	913930	-30624
$L^{18}\mathrm{pt}^1$	429750191232	175599635328	15361183296	23427930	-920208
$L^{20}\mathrm{pt}^0$	6089786376960	2583319387968	238229466240	530660500	-22421040

$d=6$	$n_0 = N_0$	N_1	N_2	N_3	N_4
$L^0\mathrm{pt}^{12}$	2576	2860	312	11	0
$L^2\mathrm{pt}^{11}$	44416	30470	4367	171	0
$L^4\mathrm{pt}^{10}$	573312	366120	59004	2832	0
$L^6\mathrm{pt}^{9}$	7200416	4647564	828798	45166	64
$L^8\mathrm{pt}^{8}$	91797312	61777480	12040292	724268	2048
$L^{10}\mathrm{pt}^{7}$	1207360512	855953100	180487314	11810592	48384
$L^{12}\mathrm{pt}^{6}$	16492503552	12328963680	2788102824	196520672	1031760
$L^{14}\mathrm{pt}^{5}$	234526910784	184285049520	44340020688	3340214784	21101700
$L^{16}\mathrm{pt}^{4}$	3472451647488	2854482435648	725356675584	58003817136	424528824
$L^{18}\mathrm{pt}^{3}$	53486265350784	45759236016480	12197239929576	1028899986730	8512493205
$L^{20}\mathrm{pt}^{2}$	855909223176192	758233413373440	210683581713696	18637079002808	171326389452
$L^{22}\mathrm{pt}^{1}$	14207926965714432	12970894985136000	3735671208730416	344593423079268	3473410465074
$L^{24}\mathrm{pt}^{0}$	244274488980962304	228804132309160320	67948997430660192	6501236425429816	71052031752988

CHAPTER 2

Relative Gromov-Witten invariants in genus zero

Having computed the Gromov-Witten invariants of projective spaces we will now move on to the study of hypersurfaces. Much work has in fact been done recently on Gromov-Witten invariants related to hypersurfaces. There are essentially two different problems that have been studied. The first one is simply the question how to compute the Gromov-Witten invariants of a hypersurface from those of the ambient space [**Be, Gi1, K2, Ki, L, LLY1**]. The second one is the theory of relative Gromov-Witten invariants, i.e. the enumeration of curves in a manifold with given local orders of contact to a fixed hypersurface [**IP1, IP2, LR, Li1, Li2, R, Va1**]. The goal of this thesis is to show that these two problems that have been studied almost independently so far are in fact very closely related. In this chapter we will restrict ourselves to the case of very ample hypersurfaces and curves of genus zero. We will discuss some generalizations in chapter 5.

Let us start by giving a very short description of the ideas of this chapter, skipping all technical details. Let Y be a smooth very ample hypersurface in a complex projective manifold X. Our goal is to compute the Gromov-Witten invariants of Y from those of X. To do so, fix $n \geq 1$ and $\beta \in H_2^+(X)$. For $m \geq 0$ we let $\bar{M}_{(m)}$ (the official notation will be $\bar{M}^Y_{0,(m,0,\dots,0)}(X,\beta)$) be a suitable compactification of the moduli space of all irreducible stable maps $(\mathbb{P}^1, x_1, \dots, x_n, f)$ to X such that f has multiplicity at least m to Y at the point x_1. Obviously, $\bar{M}_{(0)}$ should be just the ordinary moduli space of stable maps to X. On the other hand, $\bar{M}_{(Y\cdot\beta+1)}$ should correspond to the moduli space of stable maps to Y, as all irreducible curves in X having multiplicity $Y\cdot\beta+1$ to Y must actually lie inside Y. Moreover, $\bar{M}_{(m+1)}$ is a subspace of $\bar{M}_{(m)}$ of (expected) codimension one for all m.

The strategy is now obvious: if we can describe the (virtual) divisor $\bar{M}_{(m+1)}$ in $\bar{M}_{(m)}$ intersection-theoretically in terms of known classes (and our main theorem 2.2.6 does precisely this) then we can compute intersection products on $\bar{M}_{(m+1)}$ if we can compute them on $\bar{M}_{(m)}$. Iterating this procedure for m from 0 to $Y\cdot\beta$ this means that we can compute the Gromov-Witten invariants of Y if we can compute the Gromov-Witten invariants of X.

Let us make the step from multiplicity m to $m+1$ a bit more precise. It is easily seen that there is a section of a line bundle $L_{(m)}$ on $\bar{M}_{(m)}$ whose zero locus describes exactly the condition that f vanishes to order at least $m+1$ along Y at x_1. Hence one would naïvely expect that $\bar{M}_{(m+1)}$ is just the first Chern class of $L_{(m)}$, which turns out to be $m\psi_1 + \mathrm{ev}_1^* Y$. However, this intuition breaks down for those stable maps where x_1 lies on a component that is completely mapped to Y by f (see the picture in construction 2.2.1), as f actually has infinite multiplicity to Y at x_1 in this case. Thus we get correction terms from reducible curves of that kind in our final equation. These correction terms are quite complicated, but they can be recursively computed as they are made up of invariants of smaller degree.

In this chapter we will define more general spaces than the $\bar{M}_{(m)}$ mentioned above. Namely, we will allow the specification of multiplicities to Y not only at the point x_1 but at all marked points. We call those moduli spaces the spaces of stable relative maps, and equip them with virtual fundamental classes. Intersection products on them are then called relative Gromov-Witten invariants. Of course, they have the obvious (possibly virtual) geometric interpretation as numbers of curves having given multiplicities to Y and satisfying some additional incidence conditions.

A few remarks seem in order how this work is related to the existing literature. The original ideas and motivation for our work came from the work of Vakil [**Va1**] who proved our main theorem under the following restrictions: $Y \subset X$ is a hyperplane in $\mathbb{P}^N$, the sum of the prescribed multiplicities is equal to the degree of the curves, and one of the multiplicities is raised from zero to one. It is interesting to note that he used the main theorem in the opposite direction, namely to compute the invariants of X from those of Y. But the algorithm used there is very specific to the case of a hyperplane in $\mathbb{P}^N$; it does not work for general $Y \subset X$.

Other methods to compute rational Gromov-Witten invariants of hypersurfaces do exist in the literature [**Be, L, LLY1, LLY4, Gi1**]. All of them have two properties in common though that are not shared by our methods:

- They use the technique of equivariant cohomology and fixed point localization for torus actions. The first papers therefore required X to be a manifold with a suitable torus action or even a projective space [**Be, LLY1, Gi1**]. Our method does not use localization techniques and therefore does not need any torus action. After the work of this chapter had been published the localization methods were generalized to the cases when X can be embedded in a manifold with a torus action, i.e. to all projective manifolds X [**L, LLY4**].
- They are only applicable if the anticanonical bundle of Y is non-negative. Our methods work without any restriction on the canonical bundle of Y (although

we will see in chapter 3 that they do not give rise to a nice "mirror formula" in the case of a positive anticanonical bundle).

The construction of the moduli spaces of stable relative maps (or rather their virtual fundamental classes) that we give in this chapter is only applicable to very ample hypersurfaces and curves of genus 0. Very recently Li has generalized the construction to arbitrary hypersurfaces and any genus of the curves [**Li1, Li2**]. We will present and use this generalized construction in chapter 5.

Relative Gromov-Witten invariants have also been considered in symplectic geometry by Li and Ruan [**LR**] as well as Ionel and Parker [**IP1, IP2**]. They have been defined for any codimension two symplectic submanifold Y of a symplectic manifold X.

Let us briefly sketch the outline of this chapter. In section 2.1 we define the moduli spaces of stable relative maps and define their virtual fundamental classes. The construction of the line bundles $L_{(m)}$ and the moduli spaces for the correction terms mentioned above is given in section 2.2. At the end of this section we state the main theorem 2.2.6 of this chapter that describes how the moduli spaces of relative invariants change if one of the multiplicities is raised by one. The proof of this theorem is done in two steps. In the first step in section 2.3 we look at the special case when $Y \subset X$ is a hyperplane in projective space. In this case no virtual fundamental classes are needed, and the main theorem is established by a purely geometric analysis. The ideas for the main proofs of this section are taken from [**Va1**]. In the second step in section 2.4 we prove the general case by "pulling back" the result for hyperplanes in $\mathbb{P}^N$ along the morphism $\bar{M}_{0,n}(X,\beta) \to \bar{M}_{0,n}(\mathbb{P}^N,d)$ induced by the complete linear system $|Y|$. Finally, in section 2.5 we prove that the main theorem can be used to compute the Gromov-Witten invariants of Y (as well as the relative invariants) in terms of the Gromov-Witten invariants of X. We will also discuss some numerical examples. The computer program GROWI can be used to compute all absolute and relative Gromov-Witten invariants of projective spaces and their hypersurfaces using the methods of this chapter [**Ga5**].

2.1. Moduli spaces of stable relative maps

We begin with the description of the set-up and the definition of the moduli spaces of stable relative maps. Let X be a complex projective manifold and $Y \subset X$ a smooth very ample hypersurface.

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be an n-tuple of non-negative integers. As usual, for such an n-tuple we define $|\alpha| := n$ and $\sum \alpha := \sum_{i=1}^n \alpha_i$. If $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\alpha' =$

$(\alpha'_1,\dots,\alpha'_m)$, we write $\alpha \cup \alpha'$ for $(\alpha_1,\dots,\alpha_n,\alpha'_1,\dots,\alpha'_m)$. For $1 \le k \le n$, we write $\alpha \pm e_k$ for $(\alpha_1,\dots,\alpha_k \pm 1,\dots,\alpha_n)$.

Let $\beta \in H_2^+(X)$ be a non-zero class, and let g be a non-negative integer. The moduli space $\bar{M}^Y_{g,\alpha}(X,\beta)$ that we want to construct should be thought of as a compactification of the space of all irreducible stable maps $(C,x_1,\dots,x_n,f)$ to X of genus g and class β that meet Y in the points x_i with multiplicity α_i for all i. We define it first as a subset of the set of geometric points of $\bar{M}_{g,n}(X,\beta)$, but we will see later that it has the structure of a closed substack of $\bar{M}_{g,n}(X,\beta)$.

DEFINITION 2.1.1. With notations as above, we define $\bar{M}^Y_{g,\alpha}(X,\beta)$ to be the locus in $\bar{M}_{g,n}(X,\beta)$ of all stable maps $(C,x_1,\dots,x_n,f)$ such that

(i) $f(x_i) \in Y$ for all i with $\alpha_i > 0$,
(ii) $f^*Y - \sum_i \alpha_i x_i \in A_0(f^{-1}(Y))$ is effective.

REMARK 2.1.2. Condition (i) is obviously necessary for (ii) to make sense. The cycle class $f^*Y \in A_0(f^{-1}(Y))$ is well-defined by [**F**] chapter 6 as the refined intersection product $Y \cdot C$ in $Y \times_X C = f^{-1}(Y)$. Note that the Chow groups of a scheme are equal to the Chow groups of its underlying reduced scheme (see [**F**] example 1.3.1 (a)), so we may replace $f^{-1}(Y)$ by its underlying reduced scheme above. So, by abuse of notation, if we talk about connected (resp. irreducible) components of $f^{-1}(Y)$ in the sequel we will always mean connected (resp. irreducible) components of the underlying reduced scheme of $f^{-1}(Y)$.

REMARK 2.1.3. In this chapter we will only be concerned with curves of genus zero. Hence we will assume $g = 0$ from now on and abbreviate $\bar{M}^Y_{0,\alpha}(X,\beta)$ as $\bar{M}^Y_{\alpha}(X,\beta)$ or $\bar{M}_{\alpha}(X,\beta)$ if there is no risk of confusion. We will also write $\bar{M}_n(X,\beta)$ for the space $\bar{M}_{0,n}(X,\beta)$ of rational stable maps.

The case of curves of higher genus will be discussed in chapter 5.

REMARK 2.1.4. For degree reasons, the space $\bar{M}_{\alpha}(X,\beta)$ is obviously empty if $\sum \alpha > Y \cdot \beta$. So we will tacitly assume from now on that $\sum \alpha \le Y \cdot \beta$.

REMARK 2.1.5. The Chow group A_0 of a point as well as of (connected but not necessarily irreducible) genus-zero curves is just $\mathbb{Z}$. So condition (ii) in definition 2.1.1 can be reformulated as follows: for any connected component Z of $f^{-1}(Y)$ we must have

(i) if Z is a point it is either unmarked or a marked point x_i such that the multiplicity of f at x_i along Y is at least α_i;

(ii) if Z is one-dimensional let $C^{(i)}$ for $1 \leq i \leq r$ be the irreducible components of C not in Z but intersecting Z, and let $m^{(i)}$ be the multiplicity of $f|_{C^{(i)}}$ at $Z \cap C^{(i)}$ along Y. Then we must have

$$Y \cdot f_* Z + \sum_{i=1}^{r} m^{(i)} \geq \sum_{x_i \in Z} \alpha_i.$$

EXAMPLE 2.1.6. Let $X = \mathbb{P}^3$, $Y = H$ a plane, $\beta = 5 \cdot [\text{line}]$, and $\alpha = (1,2)$. In the following picture the curve on the left is in $\bar{M}_{(1,2)}(X,\beta)$, whereas the one on the right is not (condition (ii) of remark 2.1.5 is violated for the line marked Z since $1+1 \not\geq 2+1$).

The first thing we will do is to study the space $\bar{M}_\alpha(X,\beta)$ in the special case where $X = \mathbb{P}^N$ and $Y = H$ is a hyperplane. In this case we will write $\bar{M}_\alpha(X,\beta)$ as $\bar{M}_\alpha(\mathbb{P}^N,d)$ with $d = H \cdot \beta$. The main result of this section is that the general element of $\bar{M}_\alpha(\mathbb{P}^N,d)$ corresponds to an irreducible stable map whose image is not contained in H, i.e. that the curves in $\bar{M}_\alpha(\mathbb{P}^N,d)$ are exactly those that can be deformed to an *irreducible* curve that still satisfies the given multiplicity conditions and that is not contained in H. (Here and in the following, by "the curve $\mathcal{C}$ can be deformed to a curve satisfying a property P" we mean that there is a family of stable maps such that the central fiber is $\mathcal{C}$ and the general fiber has P.)

DEFINITION 2.1.7. We define $M_\alpha(\mathbb{P}^N,d)$ to be the subset of $\bar{M}_\alpha(\mathbb{P}^N,d)$ of all stable maps $(C,x_1,\dots,x_n,f)$ with $C \cong \mathbb{P}^1$ and $f(C) \not\subset H$.

REMARK 2.1.8. We will often consider first the easier case of the spaces $\bar{M}_\alpha(\mathbb{P}^N,d)$ with the additional condition that $\sum \alpha = d$. (This is the situation that has been studied in [**Va1**].) In this case condition (ii) in definition 2.1.1 actually means that $f^*H - \sum_i \alpha_i x_i = 0 \in A_0(f^{-1}(H))$. Correspondingly, the conditions in remark 2.1.5 read as follows: for any connected component Z of $f^{-1}(H)$ we must have

(i) if Z is a point it is a marked point x_i with α_i being equal to the multiplicity of f at x_i along H;

(ii) if Z is one-dimensional let $C^{(i)}$ for $1 \leq i \leq r$ be the irreducible components of C not in Z but intersecting Z, and let $m^{(i)}$ be the multiplicity of $f|_{C^{(i)}}$ at $Z \cap C^{(i)}$ along H. Then we must have

$$\deg f|_Z + \sum_{i=1}^{r} m^{(i)} = \sum_{x_i \in Z} \alpha_i.$$

LEMMA 2.1.9. *The space $M_\alpha(\mathbb{P}^N, d)$ has the structure of an irreducible and locally closed substack of $\bar{M}_n(\mathbb{P}^N, d)$.*

PROOF. The locus of irreducible stable maps $(\mathbb{P}^1, x_1, \dots, x_n, f) \in \bar{M}_n(\mathbb{P}^N, d)$ such that $f(\mathbb{P}^1) \not\subset H$ can be written as $M_n(\mathbb{P}^N, d) \backslash \bar{M}_n(H, d)$, so it is open in $\bar{M}_n(\mathbb{P}^N, d)$. On the other hand, the condition that f vanishes to order at least α_i along H at x_i is closed, so $M_\alpha(\mathbb{P}^N, d)$ is the intersection of a closed subset with an open subset in $\bar{M}_n(\mathbb{P}^N, d)$. It is irreducible as there is a surjective rational map

$$\begin{array}{ccc} \mathbb{C}^{2n} \times H^0(\mathbb{P}^1, O(d - \sum \alpha)) \times H^0(\mathbb{P}^1, O(d))^N & \dashrightarrow & M_\alpha(\mathbb{P}^n, d) \\ (a_1, b_1, \dots, a_n, b_n, f_0, f_1, \dots, f_N) & \mapsto & (\mathbb{P}^1, (a_1 : b_1), \dots, (a_n : b_n), f) \end{array}$$

where

$$f(z) = f(z_0 : z_1) = (f_0(z) \cdot \prod_{i=1}^{n} (z_1 a_i - z_0 b_i)^{\alpha_i} : f_1(z) : \cdots : f_N(z))$$

whose domain space is irreducible. □

LEMMA 2.1.10. *The closure of $M_\alpha(\mathbb{P}^N, d)$ in $\bar{M}_n(\mathbb{P}^N, d)$ is contained in $\bar{M}_\alpha(\mathbb{P}^N, d)$.*

PROOF. This follows from the continuity of intersection products. To be more precise, let C be a point in the closure of $M_\alpha(\mathbb{P}^N, d)$. By lemma 2.1.9 there is a family $\phi : T \to \bar{M}_n(\mathbb{P}^N, d)$ of stable maps over a smooth curve T with a distinguished point $0 \in T$ such that $\phi(0) = C$ and $\phi(t) \in M_\alpha(\mathbb{P}^N, d)$ for $t \neq 0$. We have to prove that $\phi(0) \in \bar{M}_\alpha(\mathbb{P}^N, d)$. As it is obvious that $\phi(0)$ satisfies condition (i) of definition 2.1.1 it remains to show (ii).

The family ϕ is given by the data $(C, x_1, \dots, x_n, f)$ where $\pi : C \to T$ is a curve over T, the $x_i : T \to C$ are sections of π, and $f : C \to \mathbb{P}^N$ is a morphism. Set $C_H = f^{-1}(H)$ and consider the 1-cycles f^*H and $\sum_i \alpha_i x_i(T)$ in $A_1(C_H)$. By assumption the cycle $\gamma := f^*H - \sum_i \alpha_i x_i(T)$ is effective (it might however have components over $0 \in T$ coming from f^*H). Applying [**F**] proposition 11.1 (b) to the cycles f^*H and $\gamma + \sum_i \alpha_i x_i(T)$ we see that the specialization of f^*H at $t = 0$ is equal to the limit cycle of $\gamma + \sum_i \alpha_i x_i(T)$ as $t \to 0$. As the limit cycle of γ for $t \to 0$ is effective we have shown that $\phi(0)$ satisfies (ii). This shows the lemma. □

DEFINITION 2.1.11. Let $\mathcal{C} = (C, x_1, \dots, x_n, f) \in \bar{M}_\alpha(\mathbb{P}^N, d)$ be a stable map. An irreducible component Z of C is called an *internal component* of $\mathcal{C}$ if $f(C) \subset H$, and an *external component* otherwise. A *subcurve* of $\mathcal{C}$ is a stable map $\mathcal{C}' = (C', x'_1, \dots, x'_m, f') \in \bar{M}_{\alpha'}(\mathbb{P}^N, d')$ constructed from $\mathcal{C}$ as follows. Let C' be any proper connected subcurve of C, and let $f' = f|_{C'}$. The marked points $x'_1, \dots, x'_m$ are the marked points x_i contained in C', together with all the intersection points of C' with the other irreducible components of C. We assign multiplicities $\alpha' = (\alpha'_1, \dots, \alpha'_m)$ to the points $x'_1, \dots, x'_m$ as follows: The points x_i on C' will have their given multiplicity α_i. The intersection points with other irreducible components of C will be assigned the multiplicity of f' along H at that point if the point lies on an external component of C', and 0 otherwise. Let d' be the degree of f' on C'. The following picture shows an example of this construction (the marked points are labeled with their multiplicities).

LEMMA 2.1.12. *Let $\mathcal{C} \in \bar{M}_\alpha(\mathbb{P}^N, d)$ be a stable map and assume that $\sum \alpha = d$. Let $\mathcal{C}' = (C', x'_1, \dots, x'_n, f')$ be a subcurve of $\mathcal{C}$ with the following property: if Z is an internal irreducible component of C contained in C' then any adjacent irreducible component of Z in C is also contained in C'. (For example, the subcurve in the picture above satisfies this property.) Then $\sum \alpha' = d'$.*

PROOF. The condition $\sum \alpha = d$ means that $f^*H - \sum \alpha_i x_i = 0 \in A_0(f^{-1}(H))$. We claim that also $f'^*H - \sum \alpha'_i x'_i = 0 \in A_0(f'^{-1}(H))$, which then implies that $\sum \alpha' = d'$. In fact, this can be checked on the connected components of $f'^{-1}(H)$. Let Z be a connected component of $f'^{-1}(H)$. By assumption, there are only two possibilities:

- C and C' are locally isomorphic in a neighborhood of Z, i.e. Z is also a connected component of $f^{-1}(H)$. Therefore $(f'^*H - \sum \alpha'_i x'_i)|_Z = 0 \in A_0(Z)$.
- Z is an intersection point of C' with $\overline{C \backslash C'}$ that lies on an external component of C'. Then by definition of a subcurve Z is a marked point of $\mathcal{C}'$ with multiplicity equal to the multiplicity of f' along H at Z. In particular, we have again that $(f'^*H - \sum \alpha'_i x'_i)|_Z = 0 \in A_0(Z)$.

This proves the lemma. □

LEMMA 2.1.13. *A stable map* $\mathcal{C} = (C, x_1, \dots, x_n, f) \in \bar{M}_\alpha(\mathbb{P}^N, d)$ *can be deformed to an irreducible curve in* $\bar{M}_\alpha(\mathbb{P}^N, d)$ *if one of the following conditions is satisfied:*

(i) *C has only internal components.*
(ii) *$\sum \alpha = d$, and C consists exactly of one internal component $C^{(0)}$ and r external components $C^{(1)}, \dots, C^{(r)}$ intersecting $C^{(0)}$ for some $r \geq 0$ (i.e. C is a "comb" with the central component being internal and the teeth external; see the picture in construction 2.2.1). Moreover, in this case $\mathcal{C}$ can even be deformed to an irreducible curve that is not contained in H (which is then obvious unless $r = 0$).*
(iii) *$\sum \alpha = d$, and C has exactly two irreducible components $C^{(1)}$ and $C^{(2)}$, both being external.*

PROOF. To show (i) note that by definition every curve with $f(C) \subset H$ lies in $\bar{M}_\alpha(\mathbb{P}^N, d)$, so $\bar{M}_n(H, d) \subset \bar{M}_\alpha(\mathbb{P}^N, d)$. But it is well-known that the space of irreducible curves inside $\bar{M}_n(H, d)$ is dense. So $\mathcal{C}$ can be deformed to an irreducible curve in $\bar{M}_\alpha(\mathbb{P}^N, d)$.

(ii) has been shown in [**Va1**] theorem 6.1. (In fact, in the notations used in [**Va1**], our curve $\mathcal{C}$ is an element of a space $\mathcal{Y}$ with suitable decorations as introduced in [**Va1**] definition 3.7.)

Finally, in the case (iii) it is easy to construct an explicit deformation. Choose homogeneous coordinates $z_0, \dots, z_N$ on $\mathbb{P}^N$ such that H is given by the equation $z_0 = 0$. The map $f : C \to \mathbb{P}^N$ is then given by sections $s_0, \dots, s_N$ of a suitable line bundle $\mathcal{L}$ on C. We may assume that the coordinates are chosen such that the s_i do not vanish at $C^{(1)} \cap C^{(2)}$ (as for s_0 note that $s_0(C^{(1)} \cap C^{(2)}) = 0$ would mean that the intersection point lies on H, so it must be a marked point by remark 2.1.8 (i), hence it must be non-singular, which is a contradiction). Let $D_i = (s_i)$ be the associated divisors, in particular $D_0 = \sum \alpha_i x_i$.

Now let W be the blow-up of $\mathbb{C} \times \mathbb{P}^1$ at the point $(0,0)$, considered as a one-dimensional family of curves by the projection map $\pi : W \to \mathbb{C}$. We can identify the fiber $\pi^{-1}(0)$ with $C^{(1)} \cup C^{(2)}$. The points $x_i \in \pi^{-1}(0)$ can be extended to sections $\tilde{x}_i$ of π, giving rise to an extended divisor $\tilde{D}_0 = \sum \alpha_i \tilde{x}_i$. In the same way one can find divisors $\tilde{D}_i$ on W such that $\tilde{D}_i|_{\pi^{-1}(0)} = D_i$ for all i. As $\operatorname{Pic} W = \operatorname{Pic} C$, these divisors will be linearly equivalent and define a line bundle $\tilde{\mathcal{L}}$ on W such that $\tilde{\mathcal{L}}|_{\pi^{-1}(0)} = \mathcal{L}$. Moreover, after possibly restricting the base $\mathbb{C}$ to a smaller open neighborhood of 0 we can assume that the $\tilde{D}_i$ are base-point free. Finally, we can choose sections $\tilde{s}_i$ of $\tilde{\mathcal{L}}$ such that $(\tilde{s}_i) = \tilde{D}_i$ and $\tilde{s}_i|\pi^{-1}(0) = s_i$. Then $(W, \tilde{x}_0, \dots, \tilde{x}_n, (\tilde{s}_0 : \dots : \tilde{s}_N))$ is a family of stable maps whose central fiber is $\mathcal{C}$ and whose general element is in $M_\alpha(\mathbb{P}^N, d)$. □

LEMMA 2.1.14. *Let* $\mathcal{C} = (C, x_1, \dots, x_n, f) \in \bar{M}_\alpha(\mathbb{P}^N, d)$ *be a reducible stable map and assume that* $\sum \alpha = d$. *Then* $\mathcal{C}$ *can be deformed to a stable map in* $\bar{M}_\alpha(\mathbb{P}^N, d)$ *with fewer nodes.*

PROOF. This is essentially obtained from lemma 2.1.13 by gluing. Pick a node $P \in C$ and a subcurve $\mathcal{C}^{(0)} = (C^{(0)}, x_1^{(0)}, \dots, x_{n^{(0)}}^{(0)}, f^{(0)}) \in \bar{M}_{\alpha^{(0)}}(\mathbb{P}^N, d^{(0)})$ of $\mathcal{C}$ as follows:

(i) If C has a node connecting two internal components of C, let P be this node and let $C^{(0)}$ be the connected component of $f^{-1}(H)$ containing P.
(ii) Otherwise, if C has a node connecting an internal component Z to an external component of C, let P be this node and let $C^{(0)}$ be Z together with all adjacent (necessarily external) components of C.
(iii) Otherwise, let P be any node of C (necessarily connecting two external components of C) and let $C^{(0)}$ be the two irreducible components of C meeting at P.

Let $C^{(1)}, \dots, C^{(r)}$ with $r \geq 0$ be the connected components of $\overline{C \backslash C^{(0)}}$.

In any case, we can deform $\mathcal{C}^{(0)}$ to an irreducible map in $\bar{M}_{\alpha^{(0)}}(\mathbb{P}^N, d^{(0)})$ by lemma 2.1.13 (in the cases (ii) and (iii) it follows from lemma 2.1.12 that $\sum \alpha^{(0)} = d^{(0)}$). So let $\phi : T \to \bar{M}_{\alpha^{(0)}}(\mathbb{P}^N, d^{(0)})$ be a deformation of $\mathcal{C}^{(0)}$ for some smooth pointed curve $(T, 0)$, i.e. $\phi(0) = \mathcal{C}^{(0)}$ and for all $0 \neq t \in T$ the curve $\phi(t)$ is irreducible. This deformation is given by a family $\pi : \tilde{C} \to T$ of curves, sections $\tilde{x}_1, \dots, \tilde{x}_n$ of π and a map $\tilde{f} : \tilde{C} \to \mathbb{P}^N$. For all $1 \leq i \leq r$ the intersection point of $C^{(0)}$ and $C^{(i)}$ is one of the marked points of $C^{(0)}$, hence corresponds to a marked point of ϕ, say $\tilde{x}_i$. Note that in all cases (i) to (iii) above the deformation ϕ has the property that $\tilde{f}(\tilde{x}_i(t)) \in H$ for all $t \in T$ if this is true for $t = 0$. In particular, there are T-valued projective automorphisms $\psi_i : T \to \mathrm{PGL}(N)$ keeping H fixed such that $\psi_i(t)(\tilde{f}(\tilde{x}_i(0))) = \tilde{f}(\tilde{x}_i(t))$. The induced action of $\mathrm{PGL}(N)$ on the moduli spaces $\bar{M}_{\alpha^{(i)}}(\mathbb{P}^N, d^{(i)})$ makes ψ_i into a deformation of $\mathcal{C}^{(i)}$ over T such that for all $t \in T$ the marked point corresponding to $C^{(0)} \cap C^{(i)}$ is mapped to the same point in $\mathbb{P}^N$ by the families ϕ and ψ_i. This means that the families ϕ and ψ_i can actually be glued to give a deformation of the original curve $\mathcal{C}$. This deformation smoothes the node P. □

PROPOSITION 2.1.15. *The closure of* $M_\alpha(\mathbb{P}^N, d)$ *in* $\bar{M}_n(\mathbb{P}^N, d)$ *is* $\bar{M}_\alpha(\mathbb{P}^N, d)$. *In particular,* $\bar{M}_\alpha(\mathbb{P}^N, d)$ *has the structure of an irreducible, proper, reduced substack of* $\bar{M}_n(\mathbb{P}^N, d)$.

PROOF. "$\subset$" has been shown in lemma 2.1.10, so it remains to show "$\supset$". Let $\mathcal{C} \in \bar{M}_\alpha(\mathbb{P}^N, d)$ be a stable map. Assume first that $\sum \alpha = d$. Using lemma 2.1.14

inductively we can deform C to an irreducible curve in $\bar{M}_\alpha(\mathbb{P}^N,d)$. If this irreducible curve does not lie inside H then we are done, otherwise use the $r=0$ case of lemma 2.1.13 (ii).

If $k = d - \sum\alpha > 0$ let $\alpha' = \alpha \cup (1,\dots,1)$ such that $\sum\alpha' = d$. By adding marked points (and possibly introducing new contracted components) it is easy to find a stable map $C' \in \bar{M}_{\alpha'}$ that maps to C under the forgetful morphism $\bar{M}_{n+k}(\mathbb{P}^N,d) \to \bar{M}_n(\mathbb{P}^N,d)$. By the above C' can be deformed to an irreducible curve in $M_{\alpha'}(\mathbb{P}^N,d)$. This induces a deformation of C to an irreducible curve in $M_\alpha(\mathbb{P}^N,d)$.

Hence we have finally shown that $\bar{M}_\alpha(\mathbb{P}^N,d)$ is closed. So by giving it the reduced substack structure we get a proper, reduced substack of $\bar{M}_n(\mathbb{P}^N,d)$ which is irreducible by lemma 2.1.9. □

LEMMA 2.1.16. *The moduli space $\bar{M}_\alpha(\mathbb{P}^N,d)$ has the following properties:*

(i) *If $k = d-\sum\alpha > 0$ and we let $\alpha' = \alpha\cup(1,\dots,1)$ such that $\sum\alpha' = d$, then there is a degree-$k!$ generically finite cover $\bar{M}_{\alpha'}(\mathbb{P}^N,d) \to \bar{M}_\alpha(\mathbb{P}^N,d)$, given by forgetting the last k marked points and stabilizing.*
(ii) *$\bar{M}_{\alpha\cup(0)}(\mathbb{P}^N,d)$ is the universal curve over $\bar{M}_\alpha(\mathbb{P}^N,d)$. In particular, if $\alpha = (0,\dots,0)$ then $\bar{M}_\alpha(\mathbb{P}^N,d) = \bar{M}_{|\alpha|}(\mathbb{P}^N,d)$.*
(iii) *The moduli space $\bar{M}_\alpha(\mathbb{P}^N,d)$ is purely of the expected dimension, which is $\dim \bar{M}_{|\alpha|}(\mathbb{P}^N,d) - \sum\alpha = d(N+1)+N-3+|\alpha|-\sum\alpha$.*

PROOF. To show (i) note that from the parametrization of $M_\alpha(\mathbb{P}^N,d)$ given in the proof of lemma 2.1.9 one can see that the general element of $M_\alpha(\mathbb{P}^N,d)$ corresponds to a stable map $(\mathbb{P}^1,x_1,\dots,x_n,f)$ such that f^*H is equal to $\sum_i \alpha_i x_i$ plus a union of $k = d-\sum\alpha_i$ distinct unmarked points with multiplicity one. Obviously, the map $\bar{M}_{\alpha'}(\mathbb{P}^N,d) \to \bar{M}_\alpha(\mathbb{P}^n,d)$ is finite over these elements, and it has degree $k!$, corresponding to the choice of order of the k added marked points.

As in the proof of (i), the statement of (ii) is obvious on the dense open subset of $\bar{M}_\alpha(\mathbb{P}^N,d)$ described above, and it extends to the closures because of the flatness of the map $\bar{M}_{n+1}(\mathbb{P}^N,d) \to \bar{M}_n(\mathbb{P}^N,d)$.

Finally, (iii) has been shown in [**Va1**] proposition 5.7 if $\sum\alpha = d$. Otherwise use (i) first. Alternatively, (iii) can be read off from the parametrization given in the proof of lemma 2.1.9. □

REMARK 2.1.17. The stack $\bar{M}_\alpha(\mathbb{P}^N,d)$ is in general singular, even in codimension 1 (see [**Va1**] corollary 4.16). However, it is smooth at all points $(\mathbb{P}^1,x_1,\dots,x_n,f) \in$

$M_\alpha(\mathbb{P}^N,d)$. In fact, for these curves the obstruction space for deformations inside $\bar{M}_\alpha(\mathbb{P}^N,d)$ is $H^1(\mathbb{P}^1,f^*T'_{\mathbb{P}^N})$, where $f^*T'_{\mathbb{P}^N}$ is the kernel of the composite map

$$f^*T_{\mathbb{P}^N} \to f^*N_{H/\mathbb{P}^N} \to (f^*N_{H/\mathbb{P}^N})|_Z$$

with Z being the zero-dimensional subscheme of $\mathbb{P}^1$ having length α_i at the point x_i for all i. But as both these maps are surjective on global sections (for the second one note that $f^*N_{H/\mathbb{P}^N} = O(d)$ and $\sum\alpha \le d$) it follows that $H^1(\mathbb{P}^1,f^*T'_{\mathbb{P}^N}) = 0$.

However, we will not need any smoothness results here.

Now we return to the general case of the moduli space $\bar{M}^Y_n(X,\beta)$ where X is any smooth projective variety and $Y \subset X$ a smooth very ample hypersurface. One of the main problems is that these spaces will in general not have the expected dimension. This means in particular that we need virtual fundamental classes, which cannot be obtained using the techniques above. To overcome this problem we use the linear system $|Y|$ to get a map $X \to \mathbb{P}^N$ and consider the space $\bar{M}^Y_\alpha(X,\beta)$ as the "intersection" of two problems we already know: (a) stable maps in X and (b) stable maps in $\mathbb{P}^N$ with given multiplicities to the hyperplane $H \subset \mathbb{P}^N$ induced by Y.

We fix the following notation: let $\varphi : X \to \mathbb{P}^N$ be the morphism determined by $|Y|$, and let $H \subset \mathbb{P}^N$ the hyperplane such that $Y = \varphi^{-1}(H)$. As $d := Y \cdot \beta > 0$ the map φ induces a morphism $\phi : \bar{M}_n(X,\beta) \to \bar{M}_n(\mathbb{P}^N,d)$.

REMARK 2.1.18. Let $C \in \bar{M}_n(X,\beta)$. As the conditions (i) and (ii) of definition 2.1.1 pull back nicely, it is obvious that $C \in \bar{M}^Y_\alpha(X,\beta)$ if and only if $\phi(C) \in \bar{M}^H_\alpha(\mathbb{P}^N,d)$.

DEFINITION 2.1.19. By the previous remark the space $\bar{M}^Y_\alpha(X,\beta)$ has the structure of a proper closed substack of $\bar{M}_n(X,\beta)$ by requiring the diagram of inclusions

$$\begin{array}{ccc} \bar{M}^Y_\alpha(X,\beta) & \longrightarrow & \bar{M}^H_\alpha(\mathbb{P}^N,d) \\ \downarrow & & \downarrow \\ \bar{M}_n(X,\beta) & \xrightarrow{\phi} & \bar{M}_n(\mathbb{P}^N,d) \end{array}$$

to be Cartesian. We define the virtual fundamental class $[\bar{M}^Y_\alpha(X,\beta)]^{\text{virt}}$ to be the one induced by the virtual fundamental class of $\bar{M}_n(X,\beta)$ and the usual fundamental class of $\bar{M}^H_\alpha(\mathbb{P}^N,d)$, in the sense of remark 1.2.8.

By lemma 2.1.16 (iii) the virtual fundamental class of $\bar{M}^Y_\alpha(X,\beta)$ defined above has dimension $\dim \bar{M}_n(X,\beta) - \sum\alpha$, which is the expected dimension of $\bar{M}^Y_\alpha(X,\beta)$. If X is a projective space and $Y \subset X$ a hyperplane it is obvious by definition that the virtual fundamental class of $\bar{M}^Y_\alpha(X,\beta)$ is equal to the usual one.

2.2. Raising the multiplicities

By construction, the space $\bar{M}_{\alpha+e_k}(X,\beta)$ is a closed substack of $\bar{M}_\alpha(X,\beta)$ of expected codimension one. The main goal of this chapter is to compute $[\bar{M}_{\alpha+e_k}(X,\beta)]^{\mathrm{virt}}$ as a cycle in the Chow group of $\bar{M}_\alpha(X,\beta)$. We start with the following naïve approach describing the transition from multiplicity α_k to α_k+1 at the point x_k.

CONSTRUCTION 2.2.1. Consider a moduli space $M=\bar{M}_n(X,\beta)$ and let $\mathcal{C}\to M$ be the universal curve, with evaluation map $\mathrm{ev}:\mathcal{C}\to X$. Fix k with $1\le k\le n$ and let $s_k : M\to\mathcal{C}$ denote the section corresponding to the marked point x_k. Let $y\in H^0(\mathcal{O}_X(Y))$ be the equation of Y. Choose an integer $m\ge 0$. We pull y back to $\mathcal{C}$ by ev, take the m-jet relative to M of it, and pull this back to M by s_k to get a section

$$\sigma_k^m := s_k^* d^m_{C/M}\,\mathrm{ev}^*y \in H^0(M, s_k^*\mathcal{P}^m_{C/M}(\mathrm{ev}^*\mathcal{O}_X(Y))),$$

where $\mathcal{P}^m_{C/M}(\mathrm{ev}^*\mathcal{O}_X(Y))$ denotes relative principal parts of order m (or m-jets) of the line bundle $\mathrm{ev}^*\mathcal{O}_X(Y)$, and $d^m_{C/M}$ is the derivative up to order m (see **[EGA4]** 16.3, 16.7.2.1 for precise definitions). Geometrically, σ_k^m vanishes precisely on the stable maps that have multiplicity at least $m+1$ to Y at the point x_k. By **[EGA4]** 16.10.1, 16.7.3 there is an exact sequence

$$0\to L_k^{\otimes m}\otimes\mathrm{ev}_k^*\mathcal{O}_X(Y)\to s_k^*\mathcal{P}^m_{C/M}(\mathrm{ev}^*\mathcal{O}_X(Y))\to s_k^*\mathcal{P}^{m-1}_{C/M}(\mathrm{ev}^*\mathcal{O}_X(Y))\to 0$$

where we set $\mathcal{P}^{-1}_{C/M}(\mathrm{ev}^*\mathcal{O}_X(Y))=0$, and where $L_k=s_k^*\omega_{C/M}$ is the k-th cotangent line, i.e. the line bundle on M whose fiber at a point $(C,x_1,\dots,x_n,f)$ is $T^\vee_{C,x_k}$. Note that the last map in this sequence sends σ_k^m to σ_k^{m-1} for $m>0$. Now restrict these bundles and sections to $\bar{M}_\alpha(X,\beta)$. As all stable maps in $\bar{M}_\alpha(X,\beta)$ have multiplicity (at least) α_k at x_k, the restriction of $\sigma_k^{\alpha_k}$ to $\bar{M}_\alpha(X,\beta)$ defines a section

$$\sigma_k := \sigma_k^{\alpha_k}|_{\bar{M}_\alpha(X,\beta)} \in H^0(L_k^{\otimes\alpha_k}\otimes\mathrm{ev}_k^*\mathcal{O}_X(Y)) = H^0(\mathcal{O}(\alpha_k\psi_k+\mathrm{ev}_k^*Y))$$

on $\bar{M}_\alpha(X,\beta)$.

The vanishing of this section describes exactly the condition that a stable map in $\bar{M}_\alpha(X,\beta)$ vanishes up to order α_k+1 at x_k. Hence naïvely one would expect that $\bar{M}_{\alpha+e_k}(X,\beta)$ is described inside $\bar{M}_\alpha(X,\beta)$ by the vanishing of this section, and that $[\bar{M}_{\alpha+e_k}(X,\beta)]^{\mathrm{virt}}$ is given by

$$(\alpha_k\,\psi_k+\mathrm{ev}_k^*Y)\cdot[\bar{M}_\alpha(X,\beta)]^{\mathrm{virt}}. \tag{10}$$

This is not true however because of the presence of stable maps with the property that the component on which x_k lies is mapped entirely into Y. Of course the section σ_k vanishes on those stable maps, but they are in general not in $\bar{M}_{\alpha+e_k}(X,\beta)$. Hence these stable maps will also contribute to the expression (10). We will now introduce the moduli spaces of the stable maps occurring in these correction terms. Informally speaking, generic stable maps in these correction terms have $r+1$ irreducible components $C^{(0)},\dots,C^{(r)}$ for some $r \geq 0$, where $C^{(0)}$ (called the internal component) is mapped into Y, and all $C^{(i)}$ for $i > 0$ (called the external components) intersect $C^{(0)}$ and have a prescribed multiplicity $m^{(i)}$ to Y at this intersection point (see the picture below, where $m^{(1)} = 1$ and $m^{(2)} = 2$). The point x_k has to lie on $C^{(0)}$. The initial multiplicity conditions α as well as the homology class β get distributed in all possible ways to the components $C^{(i)}$.

We now describe this more formally.

DEFINITION 2.2.2. Consider a moduli space $\bar{M}_\alpha(X,\beta)$ and $1 \leq k \leq n$ as above. Let r be a non-negative integer. Choose a partition $A = (\alpha^{(0)},\dots,\alpha^{(r)})$ of α such that $\alpha_k \in \alpha^{(0)}$. Let $B = (\beta^{(0)},\dots,\beta^{(r)})$ be an $(r+1)$-tuple of homology classes with $\beta^{(0)} \in H_2^+(Y)$ and $\beta^{(i)} \in H_2^+(X)\backslash\{0\}$ for $i > 0$ such that $i_*\beta^{(0)} + \beta^{(1)} + \cdots + \beta^{(r)} = \beta$, where $i : Y \to X$ is the inclusion. Finally, choose an r-tuple $M = (m^{(1)},\dots,m^{(r)})$ of positive integers. With these notations we define the moduli space $D_k(X,A,B,M)$ to be the fiber product

$$D_k(X,A,B,M) := \bar{M}_{|\alpha^{(0)}|+r}(Y,\beta^{(0)}) \times_{Y^r} \prod_{i=1}^{r} \bar{M}_{\alpha^{(i)}\cup(m^{(i)})}(X,\beta^{(i)})$$

where the map from the first factor to Y^r is the evaluation at the last r marked points, and the map from the second factor to Y^r is the evaluation at the last marked point of each of its factors. We define the virtual fundamental class of $D_k(X,A,B,M)$ to be $\frac{m^{(1)}\cdots m^{(r)}}{r!}$ times the class induced by the virtual fundamental classes of its factors, in the sense of remark 1.2.8. The reason for the unusual multiplicity will become clear in the proof of proposition 2.3.3.

DEFINITION 2.2.3. With notations as in the previous definition, let $D_{\alpha,k}(X,\beta)$ be the disjoint union of the $D_k(X,A,B,M)$ for all possible A, B, and M satisfying

$$Y \cdot i_*\beta^{(0)} + \sum_i m^{(i)} = \sum \alpha^{(0)} \tag{11}$$

(the reason for this condition will become clear in the following lemma). The virtual fundamental class of $D_{\alpha,k}(X,\beta)$ is defined to be the sum of the virtual fundamental classes of its components $D_k(X,A,B,M)$.

LEMMA 2.2.4. *In the case where $X = \mathbb{P}^N$ and $Y = H$ is a hyperplane the moduli spaces $D_k(\mathbb{P}^N,A,B,M)$ satisfying equation (11) of definition 2.2.3 are proper irreducible substacks of $\bar{M}_\alpha(\mathbb{P}^N,d)$ of codimension one.*

PROOF. Considering the definition of the space $D_k(X,A,B,M)$, the fact that it is irreducible follows from the following three observations:

(i) $\bar{M}_{|\alpha^{(0)}|+r}(H,d^{(0)})$ is irreducible,
(ii) the evaluation maps $\bar{M}_{\alpha^{(i)}\cup(m^{(i)})}(\mathbb{P}^N,d^{(i)}) \to H$ at the last marked point are flat and surjective (this follows from the action of the group of automorphisms of $\mathbb{P}^N$ keeping H fixed on the space $\bar{M}_{\alpha^{(i)}\cup(m^{(i)})}(\mathbb{P}^N,d^{(i)})$),
(iii) the fibers of the maps in (ii) are irreducible (by the Bertini theorem, as the spaces $\bar{M}_{\alpha^{(i)}\cup(m^{(i)})}(\mathbb{P}^N,d^{(i)})$ itself are irreducible by proposition 2.1.15).

Moreover, these arguments show that the dimension of $D_k(\mathbb{P}^N,A,B,M)$ is equal to

$$\dim \bar{M}_{|\alpha^{(0)}|+r}(H,d^{(0)}) + \sum_{i=1}^{r} \dim \bar{M}_{\alpha^{(i)}\cup(m^{(i)})}(\mathbb{P}^N,d^{(i)}) - r\cdot(N-1).$$

By a quick computation using lemma 2.1.16 (iii) this is equal to

$$\dim \bar{M}_\alpha(\mathbb{P}^N,d) + \sum \alpha^{(0)} - d^{(0)} - \sum_i m^{(i)} - 1,$$

so the dimension statement follows from equation (11) of definition 2.2.3.

The stack $D_k(\mathbb{P}^N,A,B,M)$ is visibly a closed substack of

$$\bar{M}_{|\alpha^{(0)}|+r}(\mathbb{P}^N,d^{(0)}) \times_{(\mathbb{P}^N)^r} \prod_{i=1}^{r} \bar{M}_{|\alpha^{(i)}|+1}(\mathbb{P}^N,d^{(i)}),$$

which in turn is a closed substack of $\bar{M}_n(\mathbb{P}^N,d)$. To prove that it is contained in $\bar{M}_\alpha(\mathbb{P}^N,d)$ it suffices to show that a general element $\mathcal{C} = (C,x_1,\dots,x_n,f) \in D_k(\mathbb{P}^N,A,B,M)$ satisfies the conditions of remark 2.1.5. As $\mathcal{C}$ is general, we have $C = C^{(0)} \cup \dots \cup C^{(r)}$ where $C^{(0)} \in M_{r+|\alpha^{(0)}|}(H,d^{(0)})$ and $C^{(i)} \in M_{\alpha^{(i)}\cup(m^{(i)})}(\mathbb{P}^N,d^{(i)})$. The condition of remark 2.1.5 is obvious for all connected components of $f^{-1}(H)$ besides $C^{(0)}$. As for $C^{(0)}$, the condition is exactly the "$\geq$" part of equation (11) of definition 2.2.3. □

REMARK 2.2.5. We will see in proposition 2.4.4 that even for general X the moduli spaces $D_k(X,A,B,M)$ satisfying equation (11) of definition 2.2.3 are proper substacks of $\bar{M}_\alpha(X,\beta)$ of expected codimension one. Thus we can view the virtual fundamental class of the $D_k(X,A,B,M)$ as well as of $D_{\alpha,k}(X,\beta)$ as cycles in the Chow group of $\bar{M}_\alpha(X,\beta)$ whose dimension is equal to the expected dimension of $\bar{M}_\alpha(X,\beta)$ minus one.

We can now state the main theorem of this chapter.

THEOREM 2.2.6. *With notations as above, we have*

$$(\alpha_k \psi_k + \mathrm{ev}_k^* Y) \cdot [\bar{M}_\alpha(X,\beta)]^{\mathrm{virt}} = [\bar{M}_{\alpha+e_k}(X,\beta)]^{\mathrm{virt}} + [D_{\alpha,k}(X,\beta)]^{\mathrm{virt}}$$

in the Chow group of $\bar{M}_\alpha(X,\beta)$ *for all* $1 \le k \le n$.

The proof will be given at the end of section 2.4.

2.3. Proof of the main theorem for hyperplanes in $\mathbb{P}^N$

In this section we will prove the main theorem 2.2.6 in the case where $X = \mathbb{P}^N$ and $Y = H$ is a hyperplane. Most of the proofs are generalized versions from those in [**Va1**], where the generalizations are quite straightforward. Recall that in construction 2.2.1 we defined a section σ_k of a suitable line bundle on $\bar{M}_\alpha(\mathbb{P}^N,d)$ such that the zero locus of σ_k has class $\alpha_k \psi_k + \mathrm{ev}_k^* H$ and describes exactly those stable maps $(C,x_1,\dots,x_n,f)$ where f vanishes to order at least $\alpha_k + 1$ along H at x_k. For simplicity we will restrict ourselves first to the case $\sum\alpha = d$ (note that the term $[\bar{M}_{\alpha+e_k}(\mathbb{P}^N,d)]^{\mathrm{virt}}$ in the main theorem is then absent for degree reasons). We begin by proving a set-theoretic version of the main theorem.

LEMMA 2.3.1. *Assume that* $\sum\alpha = d$. *Then the zero locus of the section* σ_k *on* $\bar{M}_\alpha(\mathbb{P}^N,d)$ *is equal to* $D_{\alpha,k}(\mathbb{P}^N,d)$.

PROOF. By construction it is obvious that σ_k vanishes on $D_{\alpha,k}(\mathbb{P}^N,d)$, so let us prove the converse. Let $\mathcal{C} = (C,x_1,\dots,x_n,f) \in \bar{M}_\alpha(\mathbb{P}^N,d)$ be a stable map with $\sigma_k(\mathcal{C}) = 0$.

Assume first that x_n is an isolated point of $f^{-1}(H)$. As f vanishes to order at least $\alpha_k + 1$ along H at x_k this is a contradiction to remark 2.1.8 (i).

So x_n is not an isolated point of $f^{-1}(H)$. Let $C^{(0)}$ be the connected component of $f^{-1}(H)$ containing x_k, and let $C^{(1)},\dots,C^{(r)}$ be the connected components of $\overline{C\backslash C^{(0)}}$. Let $m^{(i)}$ be the multiplicity of $f|_{C^{(i)}}$ at $C^{(0)} \cap C^{(i)}$ along H, let $d^{(i)}$ be the degree of

f on $C^{(i)}$, and let $\alpha^{(i)}$ be the collection of the multiplicities α_j such that $x_j \in C^{(i)}$. Then it is obvious that $C \in D_k(\mathbb{P}^N, A, B, M)$ with A, B, M as in definition 2.2.2. Moreover, equation (11) of definition 2.2.3 is satisfied by remark 2.1.8 (ii) applied to $C^{(0)}$, hence it follows that $C \in D_{\alpha,k}(\mathbb{P}^N, d)$. □

REMARK 2.3.2. As the spaces $D_k(\mathbb{P}^N, A, B, M)$ are irreducible and of codimension one by lemma 2.2.4, lemma 2.3.1 tells us that in the case $\sum\alpha = d$ we must have

$$(\alpha_k \psi_k + \mathrm{ev}_k^* H) \cdot [\bar{M}_\alpha(\mathbb{P}^N, d)] = \sum \lambda_{A,B,M} \, [D_k(\mathbb{P}^N, A, B, M)]^{\mathrm{virt}}$$

for some $\lambda_{A,B,M}$, where the sum is taken over all A, B, M for which $D_k(\mathbb{P}^N, A, B, M)$ occurs in $D_{\alpha,k}(\mathbb{P}^N, d)$. Note that the virtual fundamental class of $D_k(\mathbb{P}^N, A, B, M)$ was defined to be $\frac{m^{(1)} \cdots m^{(r)}}{r!}$ times the usual one (where $r = |M|$), but that on the other hand every irreducible component of the zero locus of σ_k (which is of the form $D_k(\mathbb{P}^N, A, B, M)$ for some A, B, M) gets counted $r!$ times in the above sum, corresponding to the choice of order of the external components $C^{(1)}, \dots, C^{(r)}$. Hence, to prove the main theorem for hyperplanes in $\mathbb{P}^N$ in the case $\sum\alpha = d$, we have to show that σ_k vanishes along $D_k(\mathbb{P}^N, A, B, M)$ with multiplicity $m^{(1)} \cdots m^{(r)}$.

We will now prove the main theorem for $X = \mathbb{P}^1$ and $Y = H$ a point, in the case where $\sum\alpha = d$. The proof is very similar to the proof of [**Va1**] proposition 4.8, in fact (modulo notations) identical up to the end where the section σ_k comes into play. So we will only sketch these identical parts and refer to [**Va1**] for details.

PROPOSITION 2.3.3 (Main theorem for $H \subset \mathbb{P}^1, \sum\alpha = d$). *If $\sum\alpha = d$ then*

$$(\alpha_k \psi_k + \mathrm{ev}_k^* H) \cdot [\bar{M}_\alpha(\mathbb{P}^1, d)] = [D_{\alpha,k}(\mathbb{P}^1, d)]^{\mathrm{virt}}$$

for all $1 \le k \le n$ in the Chow group of $\bar{M}_\alpha(\mathbb{P}^1, d)$.

PROOF. Let $D_k(\mathbb{P}^1, A, B, M)$ be a component of $D_{\alpha,k}(\mathbb{P}^1, d)$. By equation (11) of definition 2.2.3 we know that $\sum\alpha^{(0)} = \sum_i m^{(i)}$. Call this number d'. Moreover, we must obviously have $r > 0$.

We start by defining two easier moduli spaces that model locally the situation at hand (in a sense that is made precise later). Fix a point $P \in \mathbb{P}^1$ distinct from H. Let $M \subset \bar{M}_{|\alpha^{(0)}|+r}(\mathbb{P}^1, d')$ be the closure of all degree-d' irreducible stable maps

$$(\mathbb{P}^1, (x_i)_{1 \le i \le |\alpha^{(0)}|}, (y_i)_{1 \le i \le r}, f)$$

such that

$$f^* H = \sum_i \alpha_i^{(0)} x_i \quad \text{and} \quad f^* P = \sum_i m^{(i)} y_i.$$

Let $D \subset \bar{M}_{|\alpha^{(0)}|+r}(\mathbb{P}^1, d')$ be the closure of all degree-d' reducible stable maps

$$(C^{(0)} \cup \cdots \cup C^{(r)}, (x_i)_{1 \leq i \leq |\alpha^{(0)}|}, (y_i)_{1 \leq i \leq r}, f)$$

with $r+1$ components such that

- f contracts $C^{(0)}$ to H, and $C^{(i)} \cap C^{(0)} \neq \emptyset$ for all $1 \leq i \leq r$,
- $x_i \in C^{(0)}$ for all $1 \leq i \leq |\alpha^{(0)}|$,
- $(f|_{C^{(i)}})^* H = m^{(i)} (C^{(i)} \cap C^{(0)})$ and $(f|_{C^{(i)}})^* P = m^{(i)} y_i$ for all $1 \leq i \leq r$.

General elements of these moduli spaces look as follows (the picture represents the case $\alpha = (0,4,1)$ and $M = (2,3)$):

In short, in addition to our usual multiplicity requirements for $f^* H$ we require multiplicities $m^{(i)}$ over the point P (so that the curves $C^{(i)}$ in D are ramified completely over H and P for $i > 0$).

We are now ready to compute the multiplicity of σ_k to $D_k(\mathbb{P}^1, A, B, M)$ at a general element $\mathcal{C}' = (C', x'_1, \dots, x'_n, f')$. Let $\mathcal{C} = (C, (x_i), (y_i), f)$ be the unique stable map in D whose internal component $C^{(0)}$ is equal to the internal component of $\mathcal{C}'$, viewed as a marked curve whose marked points are the x_i and the points $C^{(0)} \cap C^{(i)}$.

By construction, the stable maps $\mathcal{C}$ and $\mathcal{C}'$ are étale locally isomorphic around $C^{(0)}$, so let $(U, (x_i), f|_U)$ be a sufficiently small common étale neighborhood of $C^{(0)}$. By [**Va1**] proposition 4.3 the deformation spaces of $\mathcal{C}$ in M and $\mathcal{C}'$ in $\bar{M}_\alpha(\mathbb{P}^1, d)$ are products one of whose factors is the deformation space of $(U, (x_i), f|_U)$, viewed as a map from U to $\mathbb{P}^1$ satisfying the given multiplicity conditions at the points x_i. As the section σ_k is defined on this common factor, the order of vanishing of σ_k along $D_k(\mathbb{P}^1, A, B, M)$ in $\bar{M}_\alpha(\mathbb{P}^1, d)$ at the point $\mathcal{C}'$ is equal to its order of vanishing along D in M at the point $\mathcal{C}$.

To simplify the calculations even further let us fix the marked curve $(C, (x_i), (y_i))$. Consider the morphism $\pi : M \to \bar{M}_{|\alpha(0)|+r}$ given by forgetting the map f and stabilizing if necessary. Note that π will contract all external components of $\mathcal{C}$ as they only have two special points, so π maps C to a general point of $\bar{M}_{|\alpha(0)|+r}$. Denote by $M' \subset M$ and $D' \subset D$ the fibers of this morphism over $\pi(C)$. Then the multiplicity we seek is equal to the multiplicity of σ_k along D' in M' in the point C.

But general elements in M' are actually easy to describe explicitly: choose $g_1, g_2 \in O_{\mathbb{P}^1}(d')$ with associated divisors

$$(g_1) = \sum_i \alpha_i^{(0)} x_i \quad \text{and} \quad (g_2) = \sum_i m^{(i)} y_i$$

where x_i and y_i are now fixed points in $\mathbb{P}^1$, determined by the element $\pi(C) \in \bar{M}_{|\alpha(0)|+r}$. Then a general stable map in M' is of the form

$$C_\lambda = (\mathbb{P}^1, (x_i), (y_i), f) \quad \text{where} \quad f : \mathbb{P}^1 \to \mathbb{P}^1, x \mapsto (\lambda g_1(x) : g_2(x))$$

for $\lambda \in \mathbb{C}^*$. (Here we have chosen coordinates on the target $\mathbb{P}^1$ such that $H = (0:1)$ and $P = (1:0)$.) The locus $D' \subset M'$, which is set-theoretically the zero locus of σ_k, corresponds to the degeneration $\lambda \to 0$.

After a finite base change we can extend the family $\{C_\lambda\}$ to $\lambda = 0$. The central fiber C_0 of this extended family is equal to C.

Let z be a local coordinate around $x_k \in \mathbb{P}^1$. This means that z is a local coordinate around x_k on all C_λ with $\lambda \neq 0$, and in fact it extends to a local coordinate around x_k for $\lambda = 0$. Consider the local trivialization of the line bundle $L_k^{\otimes \alpha_k} \otimes \mathrm{ev}_k^* O(H)$ given by $dz(x_k)^{\otimes \alpha_k} \otimes h(x_k) \mapsto 1$ (where $h \in H^0(\mathbb{P}^1, O(H))$ is the section vanishing at H that is used to define σ_k). Then by construction the section σ_k on the family C_λ is given by $\lambda \mapsto \frac{\partial^{\alpha_k}}{\partial z^{\alpha_k}} \lambda g_1(z)|_{z=x_k}$ in this local trivialization. In particular, this has a zero of first order in λ at $\lambda = 0$. This means that the class of the zero locus of σ_k on M' is

$$(\alpha_k \psi_k + \mathrm{ev}_k^* H) \cdot [M'] = 1 \cdot [C_\lambda]$$

for general λ.

Finally, as the automorphism group of a general C_λ is trivial, whereas the automorphism group of C is $\mathbb{Z}_{m^{(1)}} \times \cdots \times \mathbb{Z}_{m^{(r)}}$, we conclude that

$$(\alpha_k \psi_k + \mathrm{ev}_k^* H) \cdot [M'] = m^{(1)} \cdots m^{(r)} \cdot [C].$$

Hence the statement of the proposition follows from remark 2.3.2. □

COROLLARY 2.3.4 (Main theorem for $H \subset \mathbb{P}^N, \sum \alpha = d$). *If $\sum \alpha = d$, then*

$$(\alpha_k \psi_k + \mathrm{ev}_k^* H) \cdot [\bar{M}_\alpha(\mathbb{P}^N, d)] = [D_{\alpha,k}(\mathbb{P}^N, d)]^{\mathrm{virt}}$$

for all $1 \le k \le n$ in the Chow group of $\bar{M}_\alpha(\mathbb{P}^N, d)$.

PROOF. (Compare to **[Va1]** theorem 6.1.) By the previous proposition we can assume that $N \geq 2$. Consider a general element $\mathcal{C} = (C, x_1, \dots, x_n, f)$ of a component $D_k(\mathbb{P}^N, A, B, M)$ of $D_{\alpha,k}(\mathbb{P}^N, d)$. Let $A \subset H$ be a general $(N-2)$-plane. The projection from A in $\mathbb{P}^N$ induces a rational map $\rho_A : \bar{M}_n(\mathbb{P}^N, d) \dashrightarrow \bar{M}_n(\mathbb{P}^1, d)$. By **[Va1]** proposition 5.5 the map ρ_A is defined and smooth at $\mathcal{C}$. Moreover, ρ_A maps $D_k(\mathbb{P}^N, A, B, M)$ to $D_k(\mathbb{P}^1, A, B, M)$ at the points of $D_k(\mathbb{P}^N, A, B, M)$ where it is defined, and the section σ_k on $\bar{M}_\alpha(\mathbb{P}^1, d)$ pulls back along ρ_A to the section σ_k on $\bar{M}_\alpha(\mathbb{P}^N, d)$. Hence the multiplicity of σ_k on $\bar{M}_\alpha(\mathbb{P}^N, d)$ along $D_k(\mathbb{P}^N, A, B, M)$ is the same as the multiplicity of σ_k on $\bar{M}_\alpha(\mathbb{P}^1, d)$ along $D_k(\mathbb{P}^1, A, B, M)$. The corollary then follows from proposition 2.3.3 and remark 2.3.2. □

COROLLARY 2.3.5 (Main theorem for $H \subset \mathbb{P}^N$). *We have*

$$(\alpha_k \psi_k + \mathrm{ev}_k^* H) \cdot [\bar{M}_\alpha(\mathbb{P}^N, d)] = [\bar{M}_{\alpha+e_k}(\mathbb{P}^N, d)] + [D_{\alpha,k}(\mathbb{P}^N, d)]^{\mathrm{virt}}$$

for $1 \leq k \leq n$ in the Chow group of $\bar{M}_\alpha(\mathbb{P}^N, d)$.

PROOF. Let $s = d - \sum \alpha$, and let $\alpha' = \alpha \cup (1, \dots, 1)$ such that $\sum \alpha' = d$. By corollary 2.3.4 we know that

$$(\alpha'_k \psi'_k + \mathrm{ev}_k'^* H) \cdot [\bar{M}_{\alpha'}(\mathbb{P}^N, d)] = [D_{\alpha',k}(\mathbb{P}^N, d)]^{\mathrm{virt}} \qquad (12)$$

for $1 \leq k \leq n$, where ψ'_k is the k-th cotangent line class on $\bar{M}_{n+s}(\mathbb{P}^N, d)$, and ev'_k the evaluation map $\bar{M}_{n+s}(\mathbb{P}^N, d) \to \mathbb{P}^N$ at the k-th marked point. We will show that the push-forward of this equation along the morphism $\phi : \bar{M}_{\alpha'}(\mathbb{P}^N, d) \to \bar{M}_\alpha(\mathbb{P}^N, d)$ that forgets the additional s marked points is exactly the statement of the corollary.

First note that $\alpha'_k = \alpha_k$ and $\mathrm{ev}'_k = \mathrm{ev}_k \circ \phi$. For the computation of the push-forward of ψ'_k we may assume that $\alpha_k > 0$, as otherwise there is no ψ'_k-term in (12). It is well-known that $\psi'_k = \phi^* \psi_k + \gamma$, where the correction term γ is the class of the locus of those stable maps $\mathcal{C} = (C, x_1, \dots, x_{n+s}, f)$ where ϕ contracts the irreducible component Z of C on which x_k lies, i.e. where Z is an unstable component of the prestable map $(C, x_1, \dots, x_n, f)$. This can only happen if Z is contracted by f, in particular $\sigma_k(\mathcal{C}) = 0$, so by lemma 2.3.1 the cycle γ must be a union of some of the components of $D_k(\mathbb{P}^N, A, B, M)$ of $D_{\alpha',k}(\mathbb{P}^N, d)$. To determine which of them occur in γ, we can assume that $\mathcal{C}$ is a generic element of some $D_k(\mathbb{P}^N, A, B, M)$. It is easy to see that ϕ contracts $Z = C^{(0)}$ if and only if $r = |M| = 1$, $d^{(0)} = 0$, and the marked points on Z are x_k and at least one of the points $x_{n+1}, \dots, x_{n+s}$. If there is more than one of these points on Z, the map ϕ has positive-dimensional fibers on $D_k(\mathbb{P}^N, A, B, M)$, and hence $\phi_*[D_k(\mathbb{P}^N, A, B, M)]$ vanishes, hence we can assume that the marked points on Z are exactly x_k and one of the forgotten points. Then $\phi(\mathcal{C})$ contracts Z, so by remark 2.1.8 the stable map $\phi(\mathcal{C})$ will be irreducible with multiplicity $\alpha_k + 1$ at x_k to H. This means that $\phi(D_k(\mathbb{P}^N, A, B, M)) = \bar{M}_{\alpha+e_k}(\mathbb{P}^N, d)$.

As there is an $s!$-fold choice of order of the forgotten marked points, we have shown that

$$\phi_* \gamma \cdot [M_{\alpha'}(\mathbb{P}^N, d)] = s! \cdot [\bar{M}_{\alpha+e_k}(\mathbb{P}^N, d)]$$

and that therefore the left hand side of the push-forward of (12) by ϕ is equal to

$$s! \cdot (\alpha_k \psi_k + \mathrm{ev}_k^* H) \cdot [\bar{M}_\alpha(\mathbb{P}^N, d)] + \alpha_k s! \cdot [\bar{M}_{\alpha+e_k}(\mathbb{P}^N, d)]. \tag{13}$$

Now we look at the right hand side of the push-forward of (12) by ϕ. Consider a component $D_k(\mathbb{P}^N, A, B, M)$ of $D_{\alpha',k}(\mathbb{P}^N, d)$ and let $\mathcal{C} = (C, x_1, \ldots, x_{n+s}, f)$ be a generic element of this component. For the push-forward of this component by ϕ to be non-zero the fibers of ϕ have to be zero-dimensional, i.e. there must not be a deformation of $\mathcal{C}$ inside $D_k(\mathbb{P}^N, A, B, M)$ that changes nothing but the position of the points $x_{n+1}, \ldots, x_{n+s}$. In particular this means that we must have one of the following two cases:

- $C^{(0)}$ contains none of the points $x_{n+1}, \ldots, x_{n+s}$, i.e. the points $x_{n+1}, \ldots, x_{n+s}$ are just the s unmarked transverse points of intersection of $\phi(\mathcal{C})$ with H. In this case the map ϕ does not contract any components of C, and it changes no multiplicities to H. Hence the push-forward by ϕ of all these components together is just $s! \cdot [D_{\alpha,k}(\mathbb{P}^N, d)]^{\mathrm{virt}}$.
- $C^{(0)}$ is a contracted component, i.e. $d^{(0)} = 0$, $r = |M| = 1$, and the marked points on $C^{(0)}$ are exactly x_k and one of the points $x_{n+1}, \ldots, x_{n+s}$. As above, the push-forward of such a component yields $\bar{M}_{\alpha+e_k}(\mathbb{P}^N, d)$, and it occurs with multiplicity $(\alpha_k + 1)\, s!$, where the factor $\alpha_k + 1$ comes from the definition of the virtual fundamental class of $D_k(\mathbb{P}^N, A, B, M)$.

Putting everything together we have shown that the push-forward of the right hand side of (12) by ϕ is equal to

$$s! \cdot [D_{\alpha,k}(\mathbb{P}^N, d)]^{\mathrm{virt}} + (\alpha_k + 1)\, s! \cdot [\bar{M}_{\alpha+e_k}(\mathbb{P}^N, d)].$$

Combining this with (13) we get the desired result. □

2.4. Proof of the main theorem for very ample hypersurfaces

Let X be a complex projective manifold and Y a smooth very ample hypersurface. We fix the following notation. Let $i : Y \to X$ be the inclusion map. For $\beta \in H_2^+(X)$ we denote by $\bar{M}_n(Y, \beta)$ the disjoint union of all moduli spaces $\bar{M}_n(Y, \beta')$ for $\beta' \in H_2^+(Y)$ such that $i_* \beta' = \beta$. Consider the embedding $\varphi : X \to \mathbb{P}^N$ given by the complete linear system $|Y|$ and let $H \subset \mathbb{P}^N$ be the hyperplane such that $\varphi^{-1}(H) = Y$. There is an induced morphism $\phi : \bar{M}_n(X, \beta) \to \bar{M}_n(\mathbb{P}^N, d)$, where $d = Y \cdot \beta$. In this section we will show that the "pull-back" of the main theorem for $H \subset \mathbb{P}^N$ by ϕ

yields the main theorem for $Y \subset X$. The most difficult part of the proof is to show that the spaces $D_{\alpha,k}(\mathbb{P}^N, d)$ pull back to $D_{\alpha,k}(X, \beta)$ (proposition 2.4.4). Recall that curves in $D_{\alpha,k}(X, \beta)$ are reducible curves with one component in Y (and some multiplicity conditions). Hence we will show first that the moduli spaces of curves in Y (lemma 2.4.2) and those of reducible curves in X (lemma 2.4.3) pull back nicely under ϕ.

CONVENTION 2.4.1. In this section all occurring spaces are equipped with virtual fundamental classes as follows.

- The moduli spaces $\bar{M}_\alpha(\cdot,\cdot)$, $D_k(\dots)$, and $D_{\alpha,k}(\dots)$ have virtual fundamental classes constructed in definitions 2.1.19, 2.2.2, and 2.2.3, respectively.
- The varieties Y, X, H, and $\mathbb{P}^N$ are equipped with their usual fundamental class.
- The virtual fundamental class of a disjoint union of spaces is the sum of the virtual fundamental classes of its components.
- In any fiber product $V_1 \times_V V_2$ occurring in this section V will always be smooth and equipped with the usual fundamental class. The virtual fundamental class of the fiber product is then taken to be the one induced by the virtual fundamental classes of V_1 and V_2 in the sense of remark 1.2.8.

When we say that two spaces V_1 and V_2 are equal we will always mean that V_1 and V_2 are isomorphic and that $[V_1]^{\text{virt}} = [V_2]^{\text{virt}}$ under this isomorphism. We will write this as $V_1 \equiv V_2$.

LEMMA 2.4.2. *For any $n \geq 0$ and $\beta \in H_2^+(X)$ we have*

$$\bar{M}_n(Y,\beta) \equiv \bar{M}_n(H,d) \times_{\bar{M}_n(\mathbb{P}^N,d)} \bar{M}_n(X,\beta).$$

PROOF. As $Y = H \cap X \subset \mathbb{P}^N$ it follows from the definitions that the diagram of inclusions

$$\begin{array}{ccc} \bar{M}_n(Y,\beta) & \longrightarrow & \bar{M}_n(X,\beta) \\ \downarrow & & \downarrow \\ \bar{M}_n(H,d) & \xrightarrow{\psi} & \bar{M}_n(\mathbb{P}^N,d) \end{array} \tag{14}$$

is Cartesian. We denote by $\pi_X : \bar{M}_{n+1}(X,\beta) \to \bar{M}_n(X,\beta)$ the universal curve and by $f_X : \bar{M}_{n+1}(X,\beta) \to X$ its evaluation map, and similarly for the moduli spaces of maps to Y, H, and $\mathbb{P}^N$. Applying the functor $R\pi_{Y*}f_Y^*$ to the distinguished triangle

$$L_X|_Y \to L_Y \to L_{Y/X} \to L_X|_Y[1] \tag{15}$$

on Y, we get the distinguished triangle

$$\begin{aligned} &R\pi_{Y*}(f_X^* L_X)|_{\bar{M}_{n+1}(Y,\beta)} \to R\pi_{Y*} f_Y^* L_Y \to R\pi_{Y*}(f_H^* L_{H/\mathbb{P}^N})|_{\bar{M}_{n+1}(Y,\beta)} \\ &\quad \to R\pi_{Y*}(f_X^* L_X)|_{\bar{M}_{n+1}(Y,\beta)}[1] \end{aligned}$$

on $\bar{M}_n(Y,\beta)$. By [**B**] proposition 5 the vector bundle $f_X^* L_X$ is quasi-isomorphic to a complex K of vector bundles on $\bar{M}_{n+1}(X,\beta)$ such that $R\pi_{X*}K$ is also a complex of vector bundles. As π_X is flat it follows from the theorem on cohomology and base change that $(R\pi_{X*}K)_{\bar{M}_n(Y,\beta)} = R\pi_{Y*}(K|_{\bar{M}_{n+1}(Y,\beta)})$. The same argument applies to $f_H^* L_{H/\mathbb{P}^N}$ instead of $f_X^* L_X$, so we arrive at the distinguished triangle

$$\begin{aligned} &(R\pi_{X*} f_X^* L_X)|_{\bar{M}_n(Y,\beta)} \to R\pi_{Y*} f_Y^* L_Y \to (R\pi_{H*} f_H^* L_{H/\mathbb{P}^N})|_{\bar{M}_n(Y,\beta)} \\ &\quad \to (R\pi_{X*} f_X^* L_X)|_{\bar{M}_n(Y,\beta)}[1]. \end{aligned} \tag{16}$$

Starting with the distinguished triangle of $L_{H/\mathbb{P}^N}$ instead of $L_{Y/X}$ in (15), the same calculation as above shows that we also have a distinguished triangle on $\bar{M}_n(H,d)$

$$\begin{aligned} &(R\pi_{\mathbb{P}^N *} f_{\mathbb{P}^N}^* L_{\mathbb{P}^N})|_{\bar{M}_n(H,d)} \to R\pi_{H*} f_H^* L_H \to R\pi_{H*} f_H^* L_{H/\mathbb{P}^N} \\ &\quad \to (R\pi_{\mathbb{P}^N *} f_{\mathbb{P}^N}^* L_{\mathbb{P}^N})|_{\bar{M}_n(H,d)}[1]. \end{aligned}$$

But the first and second term in this sequence are just $L_{\bar{M}_n(\mathbb{P}^N,d)/\mathfrak{M}_n}|_{\bar{M}_n(H,d)}$ and $L_{\bar{M}_n(H,d)/\mathfrak{M}_n}$, where $\mathfrak{M}_n$ denotes the stack of pre-stable n-pointed rational curves. Hence we see that $R\pi_{H*} f_H^* L_{H/\mathbb{P}^N} = L_{\bar{M}_n(H,d)/\bar{M}_n(\mathbb{P}^N,d)}$. So (16) becomes

$$\begin{aligned} &(R\pi_{X*} f_X^* L_X)|_{\bar{M}_n(Y,\beta)} \to R\pi_{Y*} f_Y^* L_Y \to L_{\bar{M}_n(H,d)/\bar{M}_n(\mathbb{P}^N,d)}|_{\bar{M}_n(Y,\beta)} \\ &\quad \to (R\pi_{X*} f_X^* L_X)|_{\bar{M}_n(Y,\beta)}[1]. \end{aligned}$$

Note that the first two terms in this sequence are the relative obstruction theories of $\bar{M}_n(X,\beta)$ and $\bar{M}_n(Y,\beta)$ over $\mathfrak{M}_n$, respectively. So we get a homomorphism of this distinguished triangle to

$$\begin{aligned} &L_{\bar{M}_n(X,\beta)/\mathfrak{M}_n}|_{\bar{M}_n(Y,\beta)} \to L_{\bar{M}_n(Y,\beta)/\mathfrak{M}_n} \to L_{\bar{M}_n(Y,\beta)/\bar{M}_n(X,\beta)} \\ &\quad \to L_{\bar{M}_n(X,\beta)/\mathfrak{M}_n}|_{\bar{M}_n(Y,\beta)}[1]. \end{aligned}$$

Hence, by [**BF**] proposition 7.5 it follows that $\psi^![\bar{M}_n(X,\beta)]^{\text{virt}} = [\bar{M}_n(Y,\beta)]^{\text{virt}}$ in (14). This proves the lemma. □

LEMMA 2.4.3. *Let $n^{(i)} \geq 0$ and $d^{(i)} \geq 0$ such that $\sum_i n^{(i)} = n$ and $\sum_i d^{(i)} = d$. Then*

$$\begin{aligned} &\dot{\bigcup_{(\beta^{(i)})}} \left(\bar{M}_{n^{(0)}+r}(X,\beta^{(0)}) \times_{X^r} \prod_{i=1}^r \bar{M}_{n^{(i)}+1}(X,\beta^{(i)}) \right) \equiv \\ &\quad \left(\bar{M}_{n^{(0)}+r}(\mathbb{P}^N,d^{(0)}) \times_{(\mathbb{P}^N)^r} \prod_{i=1}^r \bar{M}_{n^{(i)}+1}(\mathbb{P}^N,d^{(i)}) \right) \times_{\bar{M}_n(\mathbb{P}^N,d)} \bar{M}_n(X,\beta), \end{aligned}$$

where the union is taken over all $(\beta^{(i)})$ *with* $Y \cdot \beta^{(i)} = d^{(i)}$ *for all i, and where the maps to* X^r *and* $(\mathbb{P}^N)^r$ *are given in the same way as in definition 2.2.2.*

PROOF. In the language of [**BM**] let τ be the graph corresponding to rational curves with components $C^{(0)}, \dots, C^{(r)}$ such that $C^{(0)} \cap C^{(i)} \neq \emptyset$ for all $i > 0$ and $C^{(i)}$ has $n^{(i)}$ marked points for $i \geq 0$. Let $\mathcal{M}_n$ be the stack of pre-stable n-pointed rational curves, and let $\mathcal{M}_\tau \subset \mathcal{M}_n$ be the substack of τ-marked pre-stable curves as defined in [**BM**] definition 2.6. Moreover, we will abbreviate the moduli spaces in the large brackets in the statement of the lemma as $\bar{M}_\tau(X, (\beta^{(i)}))$ and $\bar{M}_\tau(\mathbb{P}^N, (d^{(i)}))$, respectively.

Consider the commutative diagram

$$\begin{array}{ccccc} \bar{M}_\tau(X,(\beta^{(i)})) & \longrightarrow & \bar{M}_\tau(\mathbb{P}^N,(d^{(i)})) & \longrightarrow & \mathcal{M}_\tau \\ \downarrow & & \downarrow & & \downarrow \psi \\ \bar{M}_n(X,\beta) & \longrightarrow & \bar{M}_n(\mathbb{P}^N,d) & \longrightarrow & \mathcal{M}_n \end{array}$$

where none of the maps involves stabilization of the underlying pre-stable curves. By [**B**] lemma 10 the right square and the big square are Cartesian, so the left one is also Cartesian. Moreover, by the same lemma we have $\psi^![\bar{M}_n(X,\beta)]^{\mathrm{virt}} = [\bar{M}_\tau(X,(\beta^{(i)}))]^{\mathrm{virt}}$. □

PROPOSITION 2.4.4. *For any* $1 \leq k \leq n$ *we have*

$$D_{\alpha,k}(X,\beta) \equiv D_{\alpha,k}(\mathbb{P}^N,d) \times_{\bar{M}_n(\mathbb{P}^N,d)} \bar{M}_n(X,\beta).$$

In particular, the moduli spaces $D_k(X,A,B,M)$ *satisfying equation (11) of definition 2.2.3 are proper substacks of* $\bar{M}_\alpha(X,\beta)$ *of virtual codimension one.*

PROOF. We consider a component $D_k(\mathbb{P}^N,A,(d^{(i)}),M)$ of $D_{\alpha,k}(\mathbb{P}^N,d)$ and show that the fiber product of this component with $\bar{M}_n(X,\beta)$ over $\bar{M}_n(\mathbb{P}^N,d)$ is the union of all $D_k(X,A,(\beta^{(i)}),M)$ such that $Y \cdot \beta^{(i)} = d^{(i)}$.

We start with the pull-back compatibility statement for general curves of the form $C^{(0)} \cup \cdots \cup C^{(r)}$ with $C^{(0)} \cap C^{(i)} \neq \emptyset$, as given in lemma 2.4.3. Taking the fiber product of this equation with $\bar{M}_{n^{(0)}+r}(H,d^{(0)})$ over $\bar{M}_{n^{(0)}+r}(\mathbb{P}^N,d^{(0)})$ (i.e. requiring the central component $C^{(0)}$ to lie in H) and using lemma 2.4.2 on the left hand side yields

$$\dot{\bigcup_{(\beta^{(i)})}} \left(\bar{M}_{n^{(0)}+r}(Y,\beta^{(0)}) \times_{X^r} \prod_{i=1}^{r} \bar{M}_{n^{(i)}+1}(X,\beta^{(i)}) \right) \equiv$$

$$\left(\bar{M}_{n^{(0)}+r}(H,d^{(0)}) \times_{(\mathbb{P}^N)^r} \prod_{i=1}^{r} \bar{M}_{n^{(i)}+1}(\mathbb{P}^N,d^{(i)}) \right) \times_{\bar{M}_n(\mathbb{P}^N,d)} \bar{M}_n(X,\beta).$$

This can obviously be written in a more complicated way as

$$\dot{\bigcup_{(\beta^{(i)})}} \left(\bar{M}_{n^{(0)}+r}(Y,\beta^{(0)}) \times_{Y^r} \left(H^r \times_{(\mathbb{P}^N)^r} \prod_{i=1}^{r} \bar{M}_{n^{(i)}+1}(X,\beta^{(i)}) \right) \right) \equiv$$

$$\left(\bar{M}_{n^{(0)}+r}(H,d^{(0)}) \times_{H^r} \left(H^r \times_{(\mathbb{P}^N)^r} \prod_{i=1}^{r} \bar{M}_{n^{(i)}+1}(\mathbb{P}^N,d^{(i)}) \right) \right) \times_{\bar{M}_n(\mathbb{P}^N,d)} \bar{M}_n(X,\beta).$$

Note that

$$H \times_{\mathbb{P}^N} \bar{M}_{n^{(i)}+1}(\mathbb{P}^N,d^{(i)}) \equiv \bar{M}_{\tilde{\alpha}^{(i)}}(\mathbb{P}^N,d^{(i)})$$

for all $i > 0$, where $\tilde{\alpha}^{(i)} = (0,\dots,0,1)$. So we get

$$\dot{\bigcup_{(\beta^{(i)})}} \left(\bar{M}_{n^{(0)}+r}(Y,\beta^{(0)}) \times_{Y^r} \prod_{i=1}^{r} \bar{M}_{\tilde{\alpha}^{(i)}}(\mathbb{P}^N,d^{(i)}) \times_{\bar{M}_{n^{(i)}+1}(\mathbb{P}^N,d^{(i)})} \bar{M}_{n^{(i)}+1}(X,\beta^{(i)}) \right) \equiv$$

$$\left(\bar{M}_{n^{(0)}+r}(H,d^{(0)}) \times_{H^r} \prod_{i=1}^{r} \bar{M}_{\tilde{\alpha}^{(i)}}(\mathbb{P}^N,d^{(i)}) \right) \times_{\bar{M}_n(\mathbb{P}^N,d)} \bar{M}_n(X,\beta).$$

Finally, we take the fiber product of this equation with $\bar{M}_{\alpha^{(i)}\cup(m^{(i)})}(\mathbb{P}^N,d)$ over $\bar{M}_{\tilde{\alpha}^{(i)}}(\mathbb{P}^N,d)$ for all $i > 0$, yielding the same equation with the $\tilde{\alpha}^{(i)}$ replaced by $\alpha^{(i)} \cup (m^{(i)})$. By definition this is then exactly the equation stated in the proposition. □

We are now ready to give the proof of our main theorem.

PROOF OF THEOREM 2.2.6. Consider the Cartesian diagram

$$\begin{array}{ccc} \bar{M}_\alpha(X,\beta) & \longrightarrow & \bar{M}_\alpha(\mathbb{P}^N,d) \\ \downarrow & & \downarrow \\ \bar{M}_n(X,\beta) & \xrightarrow{\phi} & \bar{M}_n(\mathbb{P}^N,d). \end{array}$$

The main theorem for $H \subset \mathbb{P}^N$ (see corollary 2.3.5) gives an equation in the Chow group of $\bar{M}_\alpha(\mathbb{P}^N,d)$. We pull this equation back by ϕ to get an equation in the Chow group of $\bar{M}_\alpha(X,\beta)$. As the morphism ϕ does not involve any contractions of the underlying pre-stable curves, the cotangent line class ψ_k on $\bar{M}_n(\mathbb{P}^N,d)$ pulls back to the cotangent line class ψ_k on $\bar{M}_n(X,\beta)$. So by definition the left hand side of corollary 2.3.5 pulls back to $(\alpha_k\psi_k + \mathrm{ev}^* Y) \cdot [\bar{M}_\alpha(X,\beta)]^{\mathrm{virt}}$. In the same way $[\bar{M}_{\alpha+e_k}(\mathbb{P}^N,d)]$ pulls back to $[\bar{M}_{\alpha+e_k}(X,\beta)]^{\mathrm{virt}}$. Finally, proposition 2.4.4 shows that $[D_{\alpha,k}(\mathbb{P}^N,d)]^{\mathrm{virt}}$ pulls back to $[D_{\alpha,k}(X,\beta)]^{\mathrm{virt}}$. □

2.5. Enumerative applications

As usual, the first thing to do to get enumerative results from moduli spaces of maps is to define invariants by intersecting the virtual fundamental class of the moduli space with various cotangent line classes and pull-backs of classes via evaluation maps. For simplicity let us assume first that all evaluation maps are taken to map to X. The corresponding invariants will then be called the restricted invariants. For a generalization see remark 2.5.2 and proposition 2.5.9.

DEFINITION 2.5.1. Let $\beta \in H_2^+(X)$, $n \geq 0$, $k_1, \dots, k_n \geq 0$, and $\gamma_1, \dots, \gamma_n \in A^*(X)$. Then the **restricted Gromov-Witten invariants** of Y are defined as

$$\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle_\beta^Y = \deg \left(\mathrm{ev}_1^* \gamma_1 \cdot \psi_1^{k_1} \cdots \mathrm{ev}_n^* \gamma_n \cdot \psi_n^{k_n} \cdot [\bar{M}_n(Y, \beta)]^{\mathrm{virt}} \right) \in \mathbb{Q}$$

if $\sum_i (\operatorname{codim} \gamma_i + k_i) = \operatorname{vdim} \bar{M}_n(Y, \beta)$, and where the ev_i denote the evaluation maps to X. Similarly, for any $\alpha = (\alpha_1, \dots, \alpha_n)$ the **restricted relative Gromov-Witten invariants** of X relative Y are defined as

$$\langle \tau_{k_1}^{\alpha_1}(\gamma_1) \cdots \tau_{k_n}^{\alpha_n}(\gamma_n) \rangle_\beta = \deg \left(\mathrm{ev}_1^* \gamma_1 \cdot \psi_1^{k_1} \cdots \mathrm{ev}_n^* \gamma_n \cdot \psi_n^{k_n} \cdot [\bar{M}_\alpha(X, \beta)]^{\mathrm{virt}} \right) \in \mathbb{Q}$$

if $\sum_i (\operatorname{codim} \gamma_i + k_i) = \operatorname{vdim} \bar{M}_\alpha(X, \beta)$. We will often leave out the α_i exponents and k_i indices that are zero.

REMARK 2.5.2. This definition can obviously be generalized in the following two ways:

(i) We can take cohomology classes $\tilde{\gamma}_k \in A^*(Y)$ and the evaluation maps $\tilde{\mathrm{ev}}_k$ to Y instead of $\gamma_k \in A^*(X)$ and ev_k (provided that $\alpha_k > 0$ in the case of the relative invariants). We will apply the same notation in this case and just mark the cohomology classes that are pulled back from Y by a tilde.

(ii) For the absolute invariants we could use a homology class on Y instead of summing over all homology classes on Y that push forward to a given class on X.

The invariants obtained in this way are called the **(unrestricted) Gromov-Witten invariants** of Y, or relative Gromov-Witten invariants of X relative Y, respectively. In most cases we can compute these unrestricted (relative) Gromov-Witten invariants as well; see proposition 2.5.9.

REMARK 2.5.3. If we intersect the main theorem 2.2.6

$$(\alpha_k \psi_k + \mathrm{ev}_k^* Y) \cdot [\bar{M}_\alpha(X, \beta)]^{\mathrm{virt}} = [\bar{M}_{\alpha+e_k}(X, \beta)]^{\mathrm{virt}} + [D_{\alpha,k}(X, \beta)]^{\mathrm{virt}}$$

with suitably many cotangent line classes or pull-backs from classes on X or Y by the evaluation maps, we obviously get many relations among the relative Gromov-Witten invariants of X relative Y, the Gromov-Witten invariants of X (for $\alpha = (0,\dots,0)$), and the Gromov-Witten invariants of Y (as the moduli spaces of stable maps to Y are included as factors in the spaces $D_{\alpha,k}(X,\beta)$). As for $D_{\alpha,k}(X,\beta)$ one uses the usual "diagonal splitting" of remark 1.2.9 to express a component

$$D_k(X,A,B,M) = \bar{M}_{|\alpha^{(0)}|+r}(Y,\beta^{(0)}) \times_{Y^r} \prod_{i=1}^{r} \bar{M}_{\alpha^{(i)}\cup(m^{(i)})}(X,\beta^{(i)})$$

(and its virtual fundamental class) by the Cartesian diagram

$$\begin{array}{ccc} D_k(X,A,B,M) & \longrightarrow & \bar{M}_{|\alpha^{(0)}|+r}(Y,\beta^{(0)}) \times \prod_{i=1}^{r} \bar{M}_{\alpha^{(i)}\cup(m^{(i)})}(X,\beta^{(i)}) \\ \downarrow & & \downarrow \mathrm{ev} \\ Y^r & \xrightarrow{\Delta^r} & Y^r \times Y^r, \end{array}$$

i.e. intersection products on $D_k(X,A,B,M)$ become intersection products of the same classes on products of moduli spaces of stable (absolute and relative) maps, with additional classes coming from the diagonal. So the term $[D_{\alpha,k}(X,\beta)]^{\mathrm{virt}}$ in the main theorem will turn into a sum of products of Gromov-Witten invariants of Y and relative Gromov-Witten invariants of X relative Y.

For simplicity we first want to look only at the restricted (relative) Gromov-Witten invariants. It is not obvious that this is possible, as even if we only use pull-backs of classes from X at the marked points $x_1,\dots,x_n$ the classes from the diagonal splitting in the terms $D_{\alpha,k}(X,\beta)$ (see above) will throw in classes from Y. To see that these do not do any harm we will first show in the next two lemmas that absolute as well as relative invariants vanish if they contain exactly one class from Y and this class lies in the orthogonal complement $A^*(X)^\perp$ of $i^*A^*(X)$ in $A^*(Y)$.

LEMMA 2.5.4. *Let $\tilde{\gamma}_1 \in A^*(X)^\perp$ and $\gamma_2,\dots,\gamma_n \in A^*(X)$. Then for any $\beta \in H_2^+(X)$ we have $\langle \tau_{k_1}(\tilde{\gamma}_1)\tau_{k_2}(\gamma_2)\cdots\tau_{k_n}(\gamma_n)\rangle_\beta^Y = 0$.*

PROOF. (This is a variant of proposition 4 in [**P1**].) Consider the Cartesian diagram (see lemma 2.4.2)

$$\begin{array}{ccccc} & & Y & \xrightarrow{i} & X \\ & & \uparrow{\scriptstyle \tilde{\mathrm{ev}}_1} & & \uparrow{\scriptstyle \mathrm{ev}_1} \\ \bar{M}_n(Y,\beta) & \longrightarrow & \bar{M}_{\tilde{\alpha}}(X,\beta) & \longrightarrow & \bar{M}_n(X,\beta) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \phi} \\ \bar{M}_n(H,d) & \longrightarrow & \bar{M}_{\tilde{\alpha}}(H,d) & \xrightarrow{j} & \bar{M}_n(\mathbb{P}^N,d) \end{array}$$

where $\tilde{\alpha} = (1,0,\ldots,0)$. Let $\pi : \bar{M}_{n+1}(\mathbb{P}^N,d) \to \bar{M}_n(\mathbb{P}^N,d)$ be the universal map and $f : \bar{M}_{n+1}(\mathbb{P}^N,d) \to \mathbb{P}^N$ its evaluation map. Let E be the kernel of the surjective bundle morphism $\pi_* f^* O(H) \to \mathrm{ev}_1^* O(H)$ given by evaluation. By [**P1**] construction 2.1 and proposition 4 we have that $[\bar{M}_n(H,d)] = j^*(c_{top}(E) \cdot [\bar{M}_n(\mathbb{P}^N,d)])$. Intersecting with $[\bar{M}_n(X,\beta)]^{\mathrm{virt}}$ yields by lemma 2.4.2

$$[\bar{M}_n(Y,\beta)]^{\mathrm{virt}} = i^!(\phi^* c_{top}(E) \cdot [\bar{M}_n(X,\beta)]^{\mathrm{virt}})$$

on $\bar{M}_{\tilde{\alpha}}(X,\beta)$. Moreover, the class $\gamma = \psi^{k_1} \cdot \mathrm{ev}_2^* \gamma_2 \cdot \psi^{k_2} \cdots \mathrm{ev}_n^* \gamma_n \cdot \psi^{k_n}$ is actually defined on $\bar{M}_n(X,\beta)$. Therefore we get

$$\begin{aligned} I^Y_{n,\beta}(\tilde{\gamma}_1\psi^{k_1}, \gamma_2\psi^{k_2}, \ldots, \gamma_n\psi^{k_n}) &= \tilde{\gamma}_1 \cdot \tilde{\mathrm{ev}}_{1*} i^!(\gamma \cdot \phi^* c_{top}(E) \cdot [\bar{M}_n(X,\beta)]^{\mathrm{virt}}) \\ &= \tilde{\gamma}_1 \cdot i^* \mathrm{ev}_{1*}(\gamma \cdot \phi^* c_{top}(E) \cdot [\bar{M}_n(X,\beta)]^{\mathrm{virt}}) \\ &= 0 \end{aligned}$$

as $\tilde{\gamma}_1 \in A^*(X)^{\perp}$. □

LEMMA 2.5.5. *Assume that* $\alpha_1 > 0$. *Let* $\tilde{\gamma}_1 \in A^*(X)^{\perp}$ *and* $\gamma_2, \ldots, \gamma_n \in A^*(X)$. *Then* $\langle \tau^{\alpha_1}_{k_1}(\tilde{\gamma}_1) \tau^{\alpha_2}_{k_2}(\gamma_2) \cdots \tau^{\alpha_n}_{k_n}(\gamma_n) \rangle_\beta = 0$.

PROOF. We prove the statement by induction on $d = Y \cdot \beta$, n, and $\sum \alpha$, in that order. This means: if we want to prove the statement for an invariant with certain values of d, n, and $\sum \alpha$, we assume that it is true for all invariants having

(i) smaller d, or
(ii) the same d and smaller n, or
(iii) the same d, the same n, and smaller $\sum \alpha$.

For $\sum \alpha = 1$, i.e. $\alpha = (1,0,\ldots,0)$, the statement follows by exactly the same calculation as in the proof of lemma 2.5.4, just leaving out the factor $c_{top}(E)$. So we can assume that $\sum \alpha > 1$. If $\alpha_1 > 1$ set $k = 1$, otherwise choose any $k > 1$ with $\alpha_k > 0$. By the main theorem 2.2.6 we have

$$((\alpha_k - 1)\,\psi_k + \mathrm{ev}_k^* Y) \cdot [\bar{M}_{\alpha - e_k}(X,\beta)]^{\mathrm{virt}} = [\bar{M}_\alpha(X,\beta)]^{\mathrm{virt}} + [D_{\alpha - e_k,k}(X,\beta)]^{\mathrm{virt}}.$$

Intersect this equation with $\tilde{\mathrm{ev}}_1^*\tilde{\gamma}_1 \cdot \psi^{k_1} \cdot \mathrm{ev}_2^*\gamma_2 \cdot \psi^{k_2} \cdots \mathrm{ev}_n^*\gamma_n \cdot \psi^{k_n}$. The first term on the right hand side is then exactly the desired invariant. We will show that all other terms vanish.

The term on the left hand side has the same d and n, and smaller $\sum \alpha$. The invariant coming from the ψ_k-summand has exactly one class in $A^*(X)^\perp$ and hence vanishes by the induction hypothesis. The same is true for the invariant coming from the $\mathrm{ev}_k^* Y$-term if $k > 1$. If $k = 1$ all classes in the invariant come from X, but the invariant contains the class $\mathrm{ev}_1^* Y \cdot \tilde{\mathrm{ev}}_1^*\tilde{\gamma}_1 = \tilde{\mathrm{ev}}_1^*(\tilde{\gamma}_1 \cdot i^*Y)$, which is zero as $\tilde{\gamma}_1 \in A^*(X)^\perp$. Hence the left hand side of the equation vanishes.

Now we look at the terms $D_k(X,A,B,M)$ on the right hand side that give products of (relative) invariants by the diagonal trick as described in remark 2.5.3. Note that the class of the diagonal in $Y \times Y$ is $\sum_i T_i \otimes T_i^\vee$, where $\{T_i\}$ is a basis of $A^*(Y)$. If we choose this basis such that it respects the orthogonal decomposition $A^*(Y) = i^*A^*(X) \oplus A^*(X)^\perp$, then $T_i \in A^*(X)^\perp$ if and only if $T_i^\vee \in A^*(X)^\perp$. Hence the i-th diagonal (where $1 \le i \le r$) will contribute one class each to the invariants for $C^{(0)}$ and $C^{(i)}$, and either both of them are in $A^*(X)^\perp$ or none of them.

For a given term $D_k(X,A,B,M)$ the components $C^{(i)}$ for $i > 0$ all have either smaller d, or the same d and smaller n (the latter happens only if $r = 1$ and $\beta^{(0)} = 0$). Hence by induction hypothesis ($i > 0$) or lemma 2.5.4 ($i = 0$) we know for any $i \ge 0$ that the invariant for $C^{(i)}$ vanishes if it contains exactly one class from $A^*(X)^\perp$. We show that this has always to be the case for at least one i. Assume that this is not true. We distinguish two cases:

(i) $x_1 \in C^{(0)}$. Then the external components $C^{(i)}$ can have at most one class from $A^*(X)^\perp$, namely the class from the diagonal. Hence by our assumption they have no such class, i.e. the diagonal contributes a class from $i^*A^*(X)$ to $C^{(i)}$ and hence also to $C^{(0)}$. But then the invariant for $C^{(0)}$ has exactly one class from $A^*(X)^\perp$, namely $\tilde{\gamma}_1$, which is a contradiction.

(ii) $x_1 \in C^{(i)}$ for some $i > 0$. Then by our assumption, the diagonals must contribute a class from $A^*(X)^\perp$ to $C^{(i)}$, and a class from $i^*A^*(X)$ to all other $C^{(j)}$ with $j > 0$. But then we have again exactly one class from $A^*(X)^\perp$ in $C^{(0)}$, namely the one from the i-th diagonal. This is again a contradiction.

This shows the lemma. □

COROLLARY 2.5.6. *Let X be a smooth projective variety and $Y \subset X$ a smooth very ample hypersurface. Assume that the Gromov-Witten invariants of X are known. Then there is an explicit algorithm to compute the restricted Gromov-Witten invariants of Y as well as the restricted relative Gromov-Witten invariants of X relative Y.*

PROOF. This is now straightforward. We will compute the absolute and relative invariants at the same time, and we will use recursion on the same variables as in the previous lemma.

Assume that we want to compute a relative invariant $\langle \tau_{k_1}^{\alpha_1}(\gamma_1)\cdots\tau_{k_n}^{\alpha_n}(\gamma_n)\rangle_\beta$. If $\sum\alpha = 0$ then this is a Gromov-Witten invariant on X and therefore assumed to be known. So we can assume that $\sum\alpha > 0$. On the other hand we can also assume that $\sum\alpha \leq Y\cdot\beta = d$, as otherwise the invariant is zero by definition.

Choose k such that $\alpha_k > 0$ and intersect the main theorem 2.2.6

$$((\alpha_k - 1)\,\psi_k + \mathrm{ev}_k^* Y)\cdot[\bar{M}_{\alpha-e_k}(X,\beta)]^{\mathrm{virt}} = [\bar{M}_\alpha(X,\beta)]^{\mathrm{virt}} + [D_{\alpha-e_k,k}(X,\beta)]^{\mathrm{virt}} \quad (17)$$

with $\mathrm{ev}_1^*\gamma_1\cdot\psi^{k_1}\cdots\mathrm{ev}_n^*\gamma_n\cdot\psi^{k_n}$. Then the first term on the right hand side is the invariant that we want to compute. We will show that all other terms in the equation are recursively known.

This is obvious for the invariants on the left hand side, since they have the same d, same n, and smaller $\sum\alpha$. Now look at a term coming from $D_k(X,A,B,M)$ on the right hand side, it is a product of invariants for the components $C^{(i)}$ for $i = 0,\ldots,r$. First we will show that we only get products of *restricted* invariants. The invariant for the components $C^{(i)}$ for $i > 0$ can have at most one class from $A^*(X)^\perp$, namely from the diagonal. But if it has exactly one it vanishes by lemma 2.5.5, so it has none. This means that it is a restricted invariant, and moreover that the diagonal contributes only classes from $A^*(X)$ to the invariant for $C^{(0)}$. This means that the invariant for $C^{(0)}$ is also a restricted one.

Now, as in the previous lemma, the invariants for the components $C^{(i)}$ for $i > 0$ all have either smaller d, or the same d and smaller n, and are therefore recursively known. The Gromov-Witten invariant for the component $C^{(0)}$ can certainly have no bigger d. We will show now that it cannot have the same d either. Assume the contrary, then we must have $r = 0$. But then the dimension condition says

$$\begin{aligned} &\mathrm{vdim}\,\bar{M}_\alpha(X,\beta) = \mathrm{vdim}\,\bar{M}_n(Y,\beta) \\ \iff\ &\mathrm{vdim}\,\bar{M}_n(X,\beta) - \sum\alpha = \mathrm{vdim}\,\bar{M}_n(X,\beta) - d - 1, \end{aligned} \quad (18)$$

i.e. $\sum \alpha = d+1 > d$, which is a contradiction. Hence also the invariant for $C^{(0)}$ has smaller d. In summary, we have seen that we can compute the desired relative Gromov-Witten invariant.

Now we compute the absolute Gromov-Witten invariants for the same values of d and n. Assume that there is such an invariant $\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle^Y_\beta$. Without loss of generality we may assume that $n > 0$ (if $n = 0$ we can just add one marked point and require it to be on Y, which changes the invariant only by a factor of d according to the divisor equation). Set $\alpha = (d+1, 0, \dots, 0)$. Now consider exactly the same equation (17) as above and intersect it again with $\mathrm{ev}_1^*\gamma_1 \cdot \psi^{k_1} \cdots \mathrm{ev}_n^*\gamma_n \cdot \psi^{k_n}$. The dimension calculation (18) above then shows that the term $[\bar{M}_n(Y,\beta)]^{\mathrm{virt}}$ and hence the desired Gromov-Witten invariant will appear on the right hand side of our equation as one term among the $D_k(X,A,B,M)$. The term coming from $\bar{M}_\alpha(X,\beta)$ will vanish as $\sum \alpha > d$, and all other terms are known recursively by exactly the same arguments as above for the relative invariants. □

REMARK 2.5.7. Although we have just shown that all restricted relative Gromov-Witten invariants of X relative Y can be computed from the Gromov-Witten invariants of X, only a very small subset of them is needed if one is only interested in the Gromov-Witten invariants of Y. First of all, analyzing the algorithm given above one sees that it is sufficient to consider relative invariants of the form $I_{(\alpha_1,0,\dots,0),\beta}(\gamma_1\psi_1^{k_1}, \gamma_2, \dots, \gamma_n)$, i.e. we need multiplicities and cotangent line classes at only one of the marked points. In fact, in many cases it will be sufficient to look at invariants with only one marked point at all as the other ones can then be reconstructed from these 1-point invariants by proposition 1.3.10.

EXAMPLE 2.5.8 (Rational curves in a quintic threefold). Let $Y \subset X = \mathbb{P}^4$ be a quintic threefold. Let us first consider lines. To compute the invariants in degree 1 we start with the moduli space $\bar{M}_1(\mathbb{P}^4, 1)$ of 1-pointed lines in $\mathbb{P}^4$ and raise the multiplicity from 0 to some value m by the equation of the main theorem 2.2.6. Note that there can be no correction terms from reducible curves: every external component must have positive degree, so in case of reducible curves the internal component would have to be contracted. But this internal component could only have two special points (the marked point and the node where it connects to the external line) and thus would not be stable.

Repeated application of the main theorem thus shows that the number of lines in Y is given by raising the multiplicity from 0 to 6:

$$\begin{aligned}\langle H\rangle_1^Y &= \deg\left(\mathrm{ev}_1^* H\cdot\prod_{i=0}^{5}(i\psi_1+\mathrm{ev}_1^* 5H)\cdot[\bar{M}_1(\mathbb{P}^4,1)]^{\mathrm{virt}}\right)\\ &= 28125\cdot\langle\tau_3(H^4)\rangle_1^{\mathbb{P}^4}+6850\cdot\langle\tau_4(H^3)\rangle_1^{\mathbb{P}^4}+600\cdot\langle\tau_5(H^2)\rangle_1^{\mathbb{P}^4}\\ &= 28125\cdot 1+6850\cdot(-5)+600\cdot 15\\ &= 2875,\end{aligned}$$

where H denotes the class of a hyperplane. In the same way we get the relative Gromov-Witten invariants without correction terms, namely

$$\begin{aligned}\langle\tau^5(H^2)\rangle_1 &= \deg\left(\mathrm{ev}_1^* H^2\cdot\prod_{i=0}^{4}(i\psi_1+\mathrm{ev}_1^* 5H)\cdot[\bar{M}_1(\mathbb{P}^4,1)]^{\mathrm{virt}}\right)\\ &= 1250\cdot\langle\tau_3(H^4)\rangle_1^{\mathbb{P}^4}+120\cdot\langle\tau_4(H^3)\rangle_1^{\mathbb{P}^4}\\ &= 650\end{aligned}$$

and

$$\begin{aligned}\langle\tau^4(H^3)\rangle_1 &= \deg\left(\mathrm{ev}_1^* H^3\cdot\prod_{i=0}^{3}(i\psi_1+\mathrm{ev}_1^* 5H)\cdot[\bar{M}_1(\mathbb{P}^4,1)]^{\mathrm{virt}}\right)\\ &= 30\cdot\langle\tau_3(H^4)\rangle_1^{\mathbb{P}^4}\\ &= 30.\end{aligned}$$

These are the only primary invariants in degree 1 with only one marked point.

Let us now turn to conics in Y. We start with the moduli space $\bar{M}_1(\mathbb{P}^4,2)$ of 1-pointed conics in $\mathbb{P}^4$ and raise the multiplicity to Y at the marked point from 0 to 11. So we evaluate the Gromov-Witten invariant of $\mathbb{P}^4$

$$\deg\left(\mathrm{ev}_1^* H\cdot\prod_{i=0}^{10}(i\psi_1+\mathrm{ev}_1^* 5H)\cdot[\bar{M}_1(\mathbb{P}^4,1)]^{\mathrm{virt}}\right)=\cdots=\frac{21040875}{4}$$

using our main theorem 2.2.6. This time we get correction terms from reducible curves in the last three steps (the small numbers in the picture at the curves denote the local orders of contact to Y):

(In the pictures where we have drawn a marked point on a node there is in fact a contracted component with 3 special points: the marked point and two nodes that attach to the external components.)

As an example let us evaluate the correction term (A). The contribution from the external invariants is $\langle\tau^4(H^3)\rangle_1\cdot\langle\tau^4(H^3)\rangle_1$. The internal component is a moduli space $\bar{M}_3(Y,0)$ that receives $\frac{1}{5}$ times the fundamental class of Y from the diagonal splitting at both node points. Consequently, the moduli space in (A) has dimension $\mathrm{vdim}\,\bar{M}_3(Y,0)=3$, and we have to intersect the internal component with the class $\mathrm{ev}^* H\cdot(\mathrm{ev}^*5H+9\psi)(\mathrm{ev}^*5H+10\psi)$. Hence the term (A) contributes

$$\frac{16}{2}\cdot\langle\tau^4(H^3)\rangle_1\cdot\langle\tau^4(H^3)\rangle_1\cdot\langle 1\ H^3\ 1\rangle_0^Y=\frac{16}{2}\cdot 30^2\cdot 5=36000.$$

In the same way we can compute the other correction terms. The result that we get for the Gromov-Witten invariant $\langle H\rangle_2^Y$ of the quintic is

$$\begin{aligned}\langle H\rangle_2^Y&=\frac{21040875}{4}-(\mathrm{A})-(\mathrm{B})-(\mathrm{C})-(\mathrm{D})-(\mathrm{E})\\&=\frac{21040875}{4}-36000-690000-390000-1868750-1056250\\&=\frac{4876875}{4}.\end{aligned}$$

Continuing this way we can compute all Gromov-Witten invariants of Y. In the following table we list the first few rational Gromov-Witten invariants $n_{0,d}:=\langle\ \rangle_d^Y$ of the quintic, together with the 1-point relative invariants of X relative Y. We also list the integers $N_{0,d}$ obtained from the $n_{0,d}$ by the multiple cover correction

$$n_{0,d} = \sum_{k|d} \frac{N_{0,d/k}}{k^3}$$

(see e.g. [**P3**] section 3). The numbers have been computed using the C++ program GROWI [**Ga5**].

	$n_{0,d}$	$N_{0,d}$
$d=1$	2875	2875
$d=2$	$\frac{4876875}{8}$	609250
$d=3$	$\frac{8564575000}{27}$	317206375
$d=4$	$\frac{15517926796875}{64}$	242467530000
$d=5$	229305888887648	229305888887625
$d=6$	248249742157695375	248249742118022000
$d=7$	$\frac{101216230345800061125625}{343}$	295091050570845659250

	$\langle \tau^{5d-1}(H^3) \rangle_d$	$\langle \tau^{5d}(H^2) \rangle_d$
$d=1$	30	650
$d=2$	4860	$\frac{583125}{2}$
$d=3$	2804480	$\frac{2512685000}{9}$
$d=4$	2638743330	$\frac{2943566903125}{8}$
$d=5$	3227732820000	579271009849776
$d=6$	4653354055079000	$\frac{3065135330975414950}{3}$
$d=7$	7509544856282388480	$\frac{95562541976247985920000}{49}$

A closed formula for these Gromov-Witten invariants (the "mirror formula") will be proven in example 3.2.1.

Let us now turn to the unrestricted Gromov-Witten invariants. In fact, we can compute these invariants in most cases as well:

PROPOSITION 2.5.9. *Let Y be a smooth very ample hypersurface in a projective manifold X. Assume that the Gromov-Witten invariants of X are known. Let s be the minimum intersection product of a curve in X with Y. Then there is an explicit algorithm to compute the unrestricted Gromov-Witten invariants of Y as well as the unrestricted relative Gromov-Witten invariants of X relative Y as long as they contain at most $s+1$ evaluation classes from Y.*

PROOF. As in the proof of corollary 2.5.6 we will reconstruct the invariants recursively on the same variables as in 2.5.5.

Let us consider a relative invariant

$$\langle \tau_{k_1}^{\alpha_1}(\tilde{\gamma}_1)\cdots\tau_{k_p}^{\alpha_p}(\tilde{\gamma}_p)\tau_{k_{p+1}}^{\alpha_{p+1}}(\gamma_{p+1})\cdots\tau_{k_n}^{\alpha_n}(\gamma_n)\rangle_\beta$$

first, i.e. an invariant in which the first p evaluation classes come from Y and the remaining ones from X. As usual we apply our main theorem 2.2.6 p to raise the multiplicities at the marked points from 0 to the values α_i:

$$\prod_{k=1}^{n}\prod_{i=0}^{\alpha_k-1}(\mathrm{ev}_k^* Y + i\psi_k)\cdot[\bar{M}_n(X,\beta)]^{\mathrm{virt}} = [\bar{M}_\alpha(X,\beta)]^{\mathrm{virt}} + (\text{correction terms}), \quad (19)$$

where the correction terms are all certain intersection products on various moduli spaces $D_{\alpha',k'}(X,\beta)$. Note that in all terms of this equation the first p marked points are restricted to lie on Y, either by a multiplicity of at least 1 or by an evaluation condition $\mathrm{ev}^* Y$. So using refined intersection products in the proof of theorem 2.2.6 it is checked immediately that equation (19) does not only hold in the Chow group of $\bar{M}_n(X,\beta)$, but in fact also in the Chow group of the subspace Z of $\bar{M}_n(X,\beta)$ of all stable maps $(C,x_1,\dots,x_n,f)$ such that $f(x_i)\in Y$ for all $1\le i\le p$. But Z admits evaluation maps to Y at the first p marked points. Hence it makes sense to intersect equation (19) with evaluation classes from Y at these points. So we can compute our given relative invariant recursively using (19) as the left hand side is an invariant of X and the correction terms are known recursively in the same way as in corollary 2.5.6 (note that no component in a correction term can have more than p evaluation classes from Y).

Now assume that we want to compute an absolute invariant

$$\langle \tau_{k_1}(\tilde{\gamma}_1)\cdots\tau_{k_p}(\tilde{\gamma}_p)\tau_{k_{p+1}}(\gamma_{p+1})\cdots\tau_{k_n}(\gamma_n)\rangle_\beta^Y$$

with $p>0$ evaluation classes from Y. The number $m = Y\cdot\beta + 2 - p$ is positive by assumption. Now instead of equation (19) raise the multiplicities to m at the first marked point and to 1 at the points $x_2,\dots,x_p$ to get

$$\begin{aligned}\mathrm{ev}_2^* Y\cdots\mathrm{ev}_p^* Y\cdot\prod_{i=0}^{m-1}(\mathrm{ev}_1^* Y + i\psi_1)\cdot[\bar{M}_n(X,\beta)]^{\mathrm{virt}}&\\ = [\bar{M}_n(Y,\beta)]^{\mathrm{virt}} + (\text{reducible correction terms}).&\end{aligned}$$

The rest of the argument is now the same as for the relative invariants above. □

REMARK 2.5.10. Throughout this chapter we could have used homology groups instead of Chow groups with no significant changes in the statements or proofs of our propositions. In particular, if the hypersurface Y has non-algebraic cohomology classes (which must be in the middle dimension by the Lefschetz theorem) then proposition 2.5.9 can be used to compute Gromov-Witten invariants using these classes as well.

Even if our primary interest is only in algebraic classes, non-algebraic ones will in general appear if we consider diagonal splittings as in remark 1.2.9: by [**D**] exercise VIII.8.21.2 the class of the diagonal Δ_Y in $Y \times Y$ is

$$[\Delta_Y] = \sum_a (-1)^{\dim T_a} \cdot T_a \otimes T^a \in H^*(Y \times Y),$$

where the sum is taken over a basis $\{T_a\}$ of $H^*(Y)$, and $\{T^a\}$ denotes the Poincaré-dual basis. In this formula we are not allowed to leave out the non-algebraic classes. In other words, in Chow theory the class of the diagonal $[\Delta_Y] \in A^*(Y \times Y)$ is in general not in the image of the natural homomorphism $A^*(Y) \otimes A^*(Y) \to A^*(Y \times Y)$. This will become important when we compute the elliptic Gromov-Witten invariants of the quintic threefold in corollary 5.5.1; see also examples 2.5.11 and 5.5.2.

EXAMPLE 2.5.11. Let $Y \subset X = \mathbb{P}^4$ be a quintic threefold, and consider the moduli space $\bar{M}_{(1,4)}(\mathbb{P}^4, 1)$ of lines in $\mathbb{P}^4$ with two points of local multiplicities 1 and 4 to Y, respectively. The virtual dimension of this moduli space is $\operatorname{vdim} \bar{M}_2(\mathbb{P}^4, 1) - 1 - 4 = 3$. We want to compute the intersection product

$$I := \deg \left(\operatorname{ev}^*[\Delta_Y] \cdot [\bar{M}_{(1,4)}(\mathbb{P}^4, 1)]^{\text{virt}} \right),$$

where $\operatorname{ev} : \bar{M}_{(1,4)} \to Y \times Y$ denotes the evaluation morphism at the two marked points.

We use the diagonal splitting

$$[\Delta_Y] = \sum_{i=0}^{3} \frac{1}{5} \cdot \operatorname{ev}_1^* H^i \cdot \operatorname{ev}_2^* H^{3-i} - \sum_a \tilde{\operatorname{ev}}_1^* \tilde{\gamma}_a \cdot \tilde{\operatorname{ev}}_2^* \tilde{\gamma}^a$$

of remark 2.5.10, where the $\tilde{\gamma}_a$ run over a basis of the 204-dimensional space $H^3(Y)$.

The first terms involving evaluation classes from X are computed in the standard way using corollary 2.5.6. We get

$$\frac{1}{5} \cdot \sum_{i=0}^{3} \langle \tau^1(H^i) \tau^4(H^{3-i}) \rangle_1^Y = \frac{30 + 805 + 1330 + 480}{5} = 529.$$

For the second terms we apply the method of proposition 2.5.9. We raise the multiplicities in $\bar{M}_2(\mathbb{P}^4, 1)$ to 1 and 4, respectively: we get that

$$\operatorname{ev}_1^* Y \cdot \operatorname{ev}_2^* Y \cdot (\operatorname{ev}_2^* Y + \psi_2) \cdots (\operatorname{ev}_2^* Y + 3\psi_2) \cdot [\bar{M}_2(\mathbb{P}^4, 1)]^{\text{virt}} \qquad (20)$$

is equal to $\bar{M}_{(1,4)}(\mathbb{P}^4, 1)$ plus the following correction term:

We will now intersect this equation with $\tilde{\mathrm{ev}}_1^* \tilde{\gamma}_a \cdot \tilde{\mathrm{ev}}_2^* \tilde{\gamma}^a$. Note that (20) then vanishes because

$$\mathrm{ev}_1^* Y \cdot \tilde{\mathrm{ev}}_1^* \tilde{\gamma}_a = \mathrm{ev}_1^*(i_* \tilde{\gamma}_a) = 0,$$

where $i : Y \to X$ is the inclusion. So we see that $\langle \tau^1(\tilde{\gamma}_a)\tau^4(\tilde{\gamma}^a)\rangle_1^Y$ is equal to the negative of the correction term in the picture above. But this correction term is simply $\langle \tau^4(\frac{1}{5}H^3)\rangle_1^Y$ as $\tilde{\gamma}_a \cdot \tilde{\gamma}^a$ is the class of a point. So we get by example 2.5.8

$$\langle \tau^1(\tilde{\gamma}_a)\tau^4(\tilde{\gamma}^a)\rangle_1^Y = -4 \cdot \frac{1}{5} \cdot \langle \tau^4(H^3)\rangle_1^Y = -24,$$

and therefore

$$I = 529 - 204 \cdot (-24) = 5425.$$

CHAPTER 3

The mirror theorem

Let Y be a smooth very ample hypersurface of a projective manifold X. We have just seen in chapter 2 that the theory of relative Gromov-Witten invariants gives rise to an algorithm that allows one to compute the genus-zero Gromov-Witten invariants of Y from those of X. We now want to show that in the case when $-K_Y$ is nef this algorithm can be "solved" explicitly to obtain a formula that expresses the generating function of the 1-point Gromov-Witten invariants of Y in terms of that of X. This so-called "mirror formula" (also denoted "quantum Lefschetz hyperplane theorem" by some authors) has already been known for some time [**Be, Gi1, Ki, L, LLY1**]. Our approach is entirely different however and essentially "elementary" in the sense that it does not use any of the special techniques that have been used in the previous proofs, as e.g. special torus actions or moduli spaces other than the usual spaces of stable maps to X and their subspaces. This does not only make our proof much simpler than the previous ones but also hopefully easier to generalize. In fact, we hope that the generalizations of relative Gromov-Witten theory that we will present in chapter 5 will give rise to "mirror symmetry type formulas" at least in genus 1.

Let us briefly recall the ideas and results from chapter 2 in the form we will need them now. For $n \geq 0$ and a homology class $\beta \in H_2^+(X)$ we denote by $\bar{M}_n(X,\beta)$ the moduli space of n-pointed rational stable maps to X of class β. For any $m \geq 0$ there are closed subspaces $\bar{M}_{(m)}(X,\beta)$ of $\bar{M}_1(X,\beta)$ that can be thought of as parametrizing 1-pointed rational curves in X having multiplicity (at least) m to Y at the marked point. (For simplicity, we suppress in the notation the dependence of these spaces on Y.) These moduli spaces have expected codimension m in $\bar{M}_1(X,\beta)$. In fact, they come equipped with natural virtual fundamental classes $[\bar{M}_{(m)}(X,\beta)]^{\text{virt}}$ of this expected dimension. If X is a projective space and Y a hyperplane, then these moduli spaces do have the expected dimension, and their virtual fundamental classes are equal to the usual ones.

The idea is now to raise the multiplicity m of the curves from 0 up to $Y \cdot \beta + 1$ by one at a time. Curves with multiplicity (at least) 0 are just unrestricted curves in X, whereas a multiplicity of $Y \cdot \beta + 1$ forces at least the irreducible curves to lie inside Y. In other words, we consider the chain of inclusions

$$\bar{M}_1(Y,\beta) \subset \bar{M}_{(Y\cdot\beta)}(X,\beta) \subset \bar{M}_{(Y\cdot\beta-1)}(X,\beta) \subset \cdots \subset \bar{M}_{(0)}(X,\beta) = \bar{M}_1(X,\beta)$$

of "virtual codimension one". The main theorem 2.2.6 of chapter 2 describes each of these inclusions explicitly in terms of intersection theory. This gives us a way to describe $\bar{M}_1(Y,\beta)$ inside $\bar{M}_1(X,\beta)$, and hence to compute Gromov-Witten invariants of Y in terms of those of X.

It is easy to write down a naïve guess what these inclusions should look like. A stable map in X has multiplicity at least m to Y if and only if the $(m-1)$-jet of $\mathrm{ev}^* Y$ vanishes, where $\mathrm{ev} : \bar{M}_1(X,\beta) \to X$ denotes the evaluation map. Hence the cycle $\bar{M}_{(m+1)}(X,\beta)$ inside $\bar{M}_{(m)}(X,\beta)$ should just be the first Chern class of the line bundle of m-jets modulo $(m-1)$-jets of $\mathrm{ev}^* O(Y)$. This Chern class is easily computed to be $\mathrm{ev}^* Y + m\psi$, where ψ is as usual the cotangent line class, i.e. the first Chern class of the line bundle whose fiber at a stable map (C,x,f) is the cotangent space of C at the point x.

However, our above informal description of $\bar{M}_{(m)}(X,\beta)$ as the space of curves with multiplicity at least m to Y at the marked point breaks down at the "boundary", i.e. at those curves where the marked point lies on a component of the curve that lies completely inside Y, so that the multiplicity becomes "infinite". Hence the above calculation receives correction terms from these curves. Their explicit form is given by theorem 2.2.6. This theorem is in fact all we need from chapter 2 for the proof of the mirror formula. So for convenience we will restate it here together with the definitions of the notations used.

THEOREM 3.0.1. *For all $m \geq 0$ we have*

$$(\mathrm{ev}^* Y + m\psi) \cdot [\bar{M}_{(m)}(X,\beta)]^{\mathrm{virt}} = [\bar{M}_{(m+1)}(X,\beta)]^{\mathrm{virt}} + [D_{(m)}(X,\beta)]^{\mathrm{virt}}.$$

Here, the correction term $D_{(m)}(X,\beta) = \cup_r \cup_{B,M} D(X,B,M)$ is a disjoint union of individual terms

$$D(X,B,M) := \bar{M}_{1+r}(Y,\beta^{(0)}) \times_{Y^r} \prod_{i=1}^{r} \bar{M}_{(m^{(i)})}(X,\beta^{(i)})$$

where $r \geq 0$, $B = (\beta^{(0)},\dots,\beta^{(r)})$ with $\beta^{(i)} \in H_2(X)/torsion$ and $\beta^{(i)} \neq 0$ for $i > 0$, and $M = (m^{(1)},\dots,m^{(r)})$ with $m^{(i)} > 0$. The maps to Y^r are the evaluation maps for the last r marked points of $\bar{M}_{1+r}(Y,\beta^{(0)})$ and each of the marked points of $\bar{M}_{(m^{(i)})}(X,\beta^{(i)})$, respectively. The union in $D_{(m)}(X,\beta)$ is taken over all r, B, and M subject to the following three conditions:

$$\sum_{i=0}^{r} \beta^{(i)} = \beta \qquad \textit{(degree condition)},$$

$$Y \cdot \beta^{(0)} + \sum_{i=1}^{r} m^{(i)} = m \qquad \textit{(multiplicity condition)},$$

$$\textit{if } \beta^{(0)} = 0 \textit{ then } r \geq 2 \qquad \textit{(stability condition)}.$$

In the equation of the theorem, the virtual fundamental class of the summands $D(X,B,M)$ *is defined to be* $\frac{m^{(1)}\cdots m^{(r)}}{r!}$ *times the class induced by the virtual fundamental classes of the factors* $\bar{M}_{1+r}(Y,\beta^{(0)})$ *and* $\bar{M}_{(m^{(i)})}(X,\beta^{(i)})$*. We can consider the spaces* $D(X,B,M)$ *to be subspaces of* $\bar{M}_1(X,\beta)$ *(see below), so the equation of the theorem makes sense in the Chow group of* $\bar{M}_1(X,\beta)$*.*

Geometrically speaking, the moduli spaces $D(X,B,M)$ in the correction terms describe curves with $r+1$ irreducible components $C^{(0)},\dots,C^{(r)}$ with homology classes $\beta^{(0)},\dots,\beta^{(r)}$, such that $C^{(0)}$ lies inside Y, and the $C^{(i)}$ for $i>0$ intersect $C^{(0)}$ in a point where they have multiplicity $m^{(i)}$ to Y. The marked point is always on the component $C^{(0)}$. Using this description the spaces $D(X,B,M)$ can be considered as subspaces of $\bar{M}_1(X,\beta)$. The multiplicity condition ensures that they are actually subspaces of $\bar{M}_{(m)}(X,\beta)$ and have the correct expected dimension. The factor $\frac{1}{r!}$ in the definition of the virtual fundamental class of the correction terms is just combinatorial and corresponds to the choice of order of the components $C^{(1)},\dots,C^{(r)}$. In contrast, the factor $m^{(1)}\cdots m^{(r)}$ is of geometric nature and somewhat tricky to derive.

As an example of the theorem consider the case where $X=\mathbb{P}^3$, $Y=H$ is a hyperplane, and β is the class of cubic curves in X. Then the equations of the theorem for $m=0,\dots,3$ can be pictured as follows (where we set $\bar{M}_{(m)}:=\bar{M}_{(m)}(\mathbb{P}^3,3)$):

(Of course, in the pictures where we have drawn the marked point on a node of the curve, the corresponding stable maps have a contracted component, i.e. we have $\beta^{(0)} = 0$.)

So we see that $\bar{M}_1(H,3)$ is equal to $\prod_{i=0}^{3}(\mathrm{ev}^* H + i\psi) \cdot \bar{M}_1(\mathbb{P}^3,3)$ plus a bunch of correction terms coming from reducible curves as shown in the picture. This is an equation of 9-dimensional cycles in $\bar{M}_1(\mathbb{P}^3,3)$. To make this into equations for the Gromov-Witten invariants of H we have to intersect it with some cohomology class γ of codimension 9 that is a polynomial in $\mathrm{ev}^* H$ and ψ. Note that in the correction terms this will impose 9 conditions on the component $C^{(0)}$ contained in H. However, in all the terms where the degree of $C^{(0)}$ is at most 2 the moduli space for this component has dimension smaller than 9. Hence all these terms vanish, and it follows that the 1-point Gromov-Witten invariants of H (of degree 3 in this example) are expressible in terms of those of $\mathbb{P}^3$ as

$$\gamma \cdot [\bar{M}_1(H,3)]^{\mathrm{virt}} = \gamma \cdot \prod_{i=0}^{3}(\mathrm{ev}^* H + i\psi) \cdot [\bar{M}_1(\mathbb{P}^3,3)]^{\mathrm{virt}}.$$

The same argument works for higher degree of the curves.

Now let us come back to the case of general X and Y. Can we still hope that the correction terms vanish when we compute the Gromov-Witten invariants? Recall that the reason for the vanishing above was that the dimension of the moduli space of curves in Y quickly gets bigger when the degree of the curves goes up (in the example the 9 conditions that were needed for Gromov-Witten invariants for cubics in Y were "too many" for lines and conics in Y). Hence, as the (virtual) dimension of the moduli space of stable maps to Y is $\mathrm{vdim}\,\bar{M}_1(Y,\beta) = -K_Y \cdot \beta + \dim Y - 2$, we see that we need that $-K_Y$ is sufficiently positive.

If $-K_Y$ is negative basically all correction terms that could appear in the computation of the Gromov-Witten invariants will do so. The main nuisance about this is that the correction terms contain the full n-point Gromov-Witten invariants of Y (namely, $n = 1 + r$ in each of the correction terms), and not just the 1-point invariants that we originally wanted to compute. There would be two ways to proceed:

- Use the version of theorem 3.0.1 for n-point invariants as proven in chapter 2.
- Use the WDVV equations to compute the n-point invariants of Y in terms of 1-point invariants whenever they occur (see proposition 1.3.10).

Both methods can be used without problems to write down an algorithm to compute the Gromov-Witten invariants of Y in terms of those of X. However, we do not know at the moment how to express the result in a nice closed form.

Most interesting are the cases where $-K_Y$ is nef, but yet not "positive enough" to ensure the vanishing of all correction terms. We will show that whenever $-K_Y$ is nef the only n-point invariants of Y that might occur in the algorithm are those with fundamental or divisor classes at all but the first marked point. These invariants can of course be reduced immediately to 1-point invariants using the fundamental class and divisor equations for Gromov-Witten invariants. Thus we arrive at recursion formulas that involve only 1-point invariants. Solving them directly we obtain a nice expression for the invariants of Y: the "mirror formula".

The necessary computations to achieve this are performed in section 3.1. In section 3.2 we apply the results to two examples. First of all we rederive the expression for the genus zero Gromov-Witten invariants of the quintic threefold. Secondly, we prove a similar expression for the (virtual) numbers of plane rational curves of degree d having contact of order $3d$ to a smooth cubic. These numbers play a role in local mirror symmetry [**CKYZ, T**]. They are a by-product of our work, as they are just simple examples of relative Gromov-Witten invariants. The two main computational lemmas (that have nothing to do with algebraic geometry, but rather are formal statements about certain power series occurring in the calculation) are proved in section 3.3.

3.1. The mirror transformation

Let X be a smooth complex projective variety, and let Y be a smooth very ample hypersurface such that $-K_Y$ is nef. By abuse of notation we denote by $H^*(X)$ and $H_*(X)$ the groups of algebraic (co-)homology classes modulo torsion. For a class $\beta \in H_2(X)$ we write $\beta \geq 0$ if β is effective, and $\beta > 0$ if $\beta \geq 0$ and $\beta \neq 0$. To keep the notation as simple as possible we will assume in the following computations that the class of Y generates $H^2(X)$ over $\mathbb{Q}$ (see remark 3.1.14 for the changes needed in the general case).

For any $\beta > 0$ we denote by $\bar{M}_n(X,\beta)$ the space of n-pointed rational stable maps of class β to X. It is usual and convenient to encode all the 1-point invariants of class β in a single cohomology class

$$\begin{aligned} I^X_\beta &:= \mathrm{ev}_* \left(\frac{1}{1-\psi} \cdot [\bar{M}_1(X,\beta)]^{\mathrm{virt}} \right) \\ &= \sum_{i,j} \langle \tau_j(T^i) \rangle^X_\beta \cdot T_i \qquad\qquad \in H^*(X), \end{aligned}$$

where $\mathrm{ev} = \mathrm{ev}_1$, $\{T^i\}$ is a basis of $H^*(X) \otimes \mathbb{Q}$, and $\{T_i\}$ is the dual basis. Note that the dimension condition ensures that for each i at most one j contributes a non-zero term to the sum above, so all 1-point invariants of X of class β can be reconstructed from the cohomology class I^X_β.

We define the invariants I^Y_β of Y in the same way, replacing $\bar{M}_n(X,\beta)$ by $\bar{M}_n(Y,\beta)$, *but keeping the* ev_i *to denote the evaluation maps to* X. Note that β is still a homology class in X; so strictly speaking $\bar{M}_n(Y,\beta)$ is the space of stable maps to Y of all homology classes whose push-forward to X is β.

For $\beta = 0$, we set $I^X_0 := 1$ and $I^Y_0 := Y$.

Now consider the moduli spaces $\bar{M}_{(m)}(X,\beta)$ of 1-pointed stable relative maps to X with multiplicity m to Y at the marked point (see definition 2.1.1). In the same manner as above these spaces together with their virtual fundamental classes of definition 2.1.19 give rise to invariants $\langle \tau^m_k(\gamma)\rangle_\beta$ that can be assembled into a cohomology class

$$I_{\beta,(m)} = \mathrm{ev}_*\left(\frac{1}{1-\psi}\cdot[\bar{M}_{(m)}(X,\beta)]^{\mathrm{virt}}\right) \quad \in H^*(X).$$

REMARK 3.1.1. For future reference let us note that (as expected from geometry) $I_{\beta,(0)} = I^X_\beta$ and $I_{\beta,(m)} = 0$ for $m > Y\cdot\beta$.

Finally, let $D_{(m)}(X,\beta)$ be the correction terms defined in theorem 3.0.1, and set

$$J_{\beta,(m)} = \mathrm{ev}_*\left(\frac{1}{1-\psi}\cdot[D_{(m)}(X,\beta)]^{\mathrm{virt}}\right) + m\cdot \mathrm{ev}_*[\bar{M}_{(m)}(X,\beta)]^{\mathrm{virt}} \quad \in H^*(X). \quad (21)$$

The surprising additional term will appear in the proof of the following lemma. Geometrically, it corresponds to unstable maps that have two irreducible components $C^{(0)}$ and $C^{(1)}$, where $C^{(0)}$ is contracted to a point in Y and contains the marked point, and $C^{(1)}$ is a curve with multiplicity m to Y at this point (see the end of the proof of lemma 3.1.8).

The first thing to do is to rewrite theorem 3.0.1 in the new simplified notation.

LEMMA 3.1.2. *For all* $\beta > 0$ *and* $m \geq 0$ *we have*

$$(Y+m)\cdot I_{\beta,(m)} = I_{\beta,(m+1)} + J_{\beta,(m)} \quad \in H^*(X).$$

PROOF. Intersect the equation of theorem 3.0.1 with $\frac{1}{1-\psi}$ and push it forward by the evaluation map to get

$$\begin{aligned}\mathrm{ev}_*&\left((\mathrm{ev}^*Y + m\psi)\cdot\frac{1}{1-\psi}\cdot[\bar{M}_{(m)}(X,\beta)]^{\mathrm{virt}}\right)\\ &= \mathrm{ev}_*\left(\frac{1}{1-\psi}\cdot[\bar{M}_{(m+1)}(X,\beta)]^{\mathrm{virt}}\right) + \mathrm{ev}_*\left(\frac{1}{1-\psi}\cdot[D_{(m)}(X,\beta)]^{\mathrm{virt}}\right).\end{aligned}$$

As $\frac{\psi}{1-\psi} = \frac{1}{1-\psi} - 1$, the left hand side of this equation can be rewritten as

$$(Y+m)\cdot \mathrm{ev}_*\left(\frac{1}{1-\psi}\cdot[\bar{M}_{(m)}(X,\beta)]^{\mathrm{virt}}\right) - m\cdot \mathrm{ev}_*[\bar{M}_{(m)}(X,\beta)]^{\mathrm{virt}}.$$

Taking into account the definitions of $I_{\beta,(m)}$ and $J_{\beta,(m)}$, we arrive at the equation stated in the lemma. □

REMARK 3.1.3. In particular,

$$\prod_{i=0}^{Y\cdot\beta}(Y+i)\cdot I^X_\beta = \sum_{m=0}^{Y\cdot\beta}\prod_{i=m+1}^{Y\cdot\beta}(Y+i)\cdot J_{\beta,(m)}.$$

This follows from a recursive application of lemma 3.1.2, with the start and the end of the recursion given by remark 3.1.1.

The next thing to do is to evaluate the $J_{\beta,(m)}$ explicitly.

REMARK 3.1.4. Let us first consider the first summand $\mathrm{ev}_*\left(\frac{1}{1-\psi}\cdot[D_{(m)}(X,\beta)]^{\mathrm{virt}}\right)$ in the definition (21) of $J_{\beta,(m)}$. Using the definition of $D_{(m)}(X,\beta)$ and its virtual fundamental class given in theorem 3.0.1 we see that this first summand is a sum of individual terms, each of which has the form

$$\langle\tau_j(T^i)\gamma_1\cdots\gamma_r\rangle^Y_{\beta^{(0)}}\cdot\frac{1}{r!}\prod_{k=1}^{r}\left(m^{(k)}\cdot\langle\tau^{m^{(k)}}(\gamma_k^\vee)\rangle_{\beta^{(k)}}\right)\cdot T_i, \tag{22}$$

where $\gamma^\vee$ denotes the dual of a class γ *in* Y. These terms are summed over all i, $j\geq 0$, $r\geq 0$, $\beta^{(k)}$ (with $\beta^{(0)}\geq 0$ and $\beta^{(k)}>0$ if $k>0$), and $m^{(k)}>0$, subject to the conditions

(i) $\beta^{(0)}+\cdots+\beta^{(r)}=\beta$ (degree condition),
(ii) $Y\cdot\beta^{(0)}+m^{(1)}+\cdots+m^{(r)}=m$ (multiplicity condition),
(iii) if $\beta^{(0)}=0$ then $r\geq 2$ (stability condition).

Moreover, the γ_k have to run over a basis of $H^*(Y)\otimes\mathbb{Q}$ (actually it is sufficient to let them run over a basis of the part of $H^*(Y)\otimes\mathbb{Q}$ induced by X, see section 2.5).

The main simplification of this huge sum is due to the following lemma, which follows from a simple dimension count. It is the only point in our computations where we need that $-K_Y$ is nef.

LEMMA 3.1.5. *The above expression (22) can only be non-zero if all γ_k are fundamental or divisor classes. Moreover, for all k we must have*

$$m^{(k)} = Y\cdot\beta^{(k)} - K_Y\cdot\beta^{(k)} - 1 \qquad \text{if } \gamma_k \text{ is the fundamental class,}$$
$$m^{(k)} = Y\cdot\beta^{(k)} - K_Y\cdot\beta^{(k)} \qquad \text{if } \gamma_k \text{ is a divisor class.}$$

PROOF. As the invariants $\langle \tau^{m^{(k)}}(\gamma_k^\vee) \rangle_{\beta^{(k)}}$ must have dimension zero for all k it follows that

$$\begin{aligned}
\operatorname{codim} \gamma_k &= \dim Y - \operatorname{codim} \gamma_k^\vee \\
&= \dim Y - \dim \bar{M}_{(m^{(k)})}(X, \beta^{(k)}) \\
&= \dim Y - (-K_X \cdot \beta^{(k)} + \dim X - 2 - m^{(k)}) \\
&= -Y \cdot \beta^{(k)} + K_Y \cdot \beta^{(k)} + 1 + m^{(k)} \qquad \text{(by adjunction).}
\end{aligned}$$

This shows the equation for the $m^{(k)}$. Moreover, as $-K_Y$ is nef and we must have $m^{(k)} \leq Y \cdot \beta^{(k)}$ for the relative invariant to be non-zero (see remark 3.1.1), it follows that $\operatorname{codim} \gamma_k \leq 1$, as desired. □

REMARK 3.1.6. Obviously, in the same way one can show that:

- If $-K_Y \cdot \beta \geq 1$ for all $\beta > 0$ then all the γ_k have to be fundamental classes. (In the following computations this would mean that all $r_\beta = 0$, which greatly simplifies the calculation.) This is e.g. the case if Y is a hypersurface in $X = \mathbb{P}^n$ of degree at most n.
- If $-K_Y \cdot \beta \geq 2$ for all $\beta > 0$ then no γ_k can exist, i.e. we must always have $r = 0$. Hence in this case we conclude that there are no correction terms in the computation of the Gromov-Witten invariants. The only term on the right hand side of remark 3.1.3 is I_β^Y (for $r = 0$ and $m = Y \cdot \beta$), so it follows that the "naïve" formula
$$I_\beta^Y = \prod_{i=0}^{Y \cdot \beta} (Y + i) \cdot I_\beta^X$$
is true (as in the case considered in the introduction where $Y \subset X$ is a plane in $\mathbb{P}^3$). This is e.g. the case if Y is a hypersurface in $X = \mathbb{P}^n$ of degree at most $n - 1$.

REMARK 3.1.7. As we have assumed that the class of Y generates $H^2(X)$ over $\mathbb{Q}$, lemma 3.1.5 states that the only factors that can occur in the k-product in (22) are the numbers

$$\begin{aligned}
s_\beta &:= (Y \cdot \beta - K_Y \cdot \beta - 1) \cdot \langle \tau^{Y \cdot \beta - K_Y \cdot \beta - 1}(1^\vee) \rangle_\beta \\
\text{and} \quad r_\beta &:= (Y \cdot \beta - K_Y \cdot \beta) \cdot \langle \tau^{Y \cdot \beta - K_Y \cdot \beta}(Y^\vee) \rangle_\beta
\end{aligned}$$

for some $\beta > 0$. Thus we can then rewrite (22) using multi-index notation as follows. For a multi-index $\mu = (\mu_\beta)$ of non-negative integers indexed by the positive homology classes β of $H^2(X)$, we apply the usual notations

$$\Sigma\mu := \textstyle\sum_\beta \mu_\beta, \qquad s^\mu := \prod_\beta s_\beta^{\mu_\beta},$$
$$\mu! := \textstyle\prod_\beta \mu_\beta!, \qquad |\mu| := \sum_\beta \mu_\beta \cdot \beta.$$

Then we can rewrite (22) as

$$\langle \tau_j(T^i) \underbrace{1 \cdots 1}_{\sum\mu \text{ times}} \underbrace{Y \cdots Y}_{\sum\nu \text{ times}} \rangle^Y_{\beta^{(0)}} \cdot \frac{1}{r!} \cdot s^\mu r^\nu \cdot T_i, \tag{23}$$

where μ and ν are the multi-indices such that the factors s_β and r_β appear in (22) μ_β and ν_β times, respectively. In particular, $r = \sum\mu + \sum\nu$ is the number of nodes of the curves under consideration.

We are now ready to evaluate the $J_{\beta,(m)}$ explicitly in terms of the 1-point Gromov-Witten invariants I^Y_β of Y and the relative 1-point invariants s_β and r_β.

LEMMA 3.1.8. *With the notation of remark 3.1.7,*

$$J_{\beta,(m)} = \sum_{\mu,\nu} \left(Y + Y \cdot \beta^{(0)}\right)^{\sum\nu} \cdot \frac{s^\mu}{\mu!} \frac{r^\nu}{\nu!} \cdot I^Y_{\beta^{(0)}}$$

for all $\beta > 0$ *and* $m \geq 0$*, where the sum is taken over all multi-indices* μ *and* ν *such that* $\beta^{(0)} := \beta - |\mu| - |\nu| \geq 0$ *(degree condition) and* $m = Y \cdot \beta - K_Y \cdot (|\mu| + |\nu|) - \sum\mu$ *(multiplicity condition).*

PROOF. Inserting expression (23) for (22) in remark 3.1.4 we see that the first summand in the definition (21) of $J_{\beta,(m)}$ is

$$\mathrm{ev}_* \left(\frac{1}{1-\psi} \cdot [D_{(m)}(X,\beta)]^{\mathrm{virt}} \right)$$
$$= \sum_{i,j} \sum_{\mu,\nu} \langle \tau_j(T^i) \underbrace{1 \cdots 1}_{\sum\mu} \underbrace{Y \cdots Y}_{\sum\nu} \rangle^Y_{\beta^{(0)}} \cdot \frac{s^\mu}{\mu!} \frac{r^\nu}{\nu!} \cdot T_i,$$

where the sum is taken over all i, j, μ, ν such that

(i) $\beta^{(0)} := \beta - |\mu| - |\nu| \geq 0$ (degree condition),
(ii) $Y \cdot \beta - K_Y \cdot (|\mu| + |\nu|) - \sum\mu = m$ (multiplicity condition — here we inserted the expression of lemma 3.1.5 for the $m^{(i)}$),
(iii) if $\beta^{(0)} = 0$ then $\sum\mu + \sum\nu \geq 2$ (stability condition).

Now we compute the Gromov-Witten invariant $I^Y_{\beta^{(0)}}(\cdots)$ in terms of 1-point invariants of Y. We claim that for $\beta^{(0)} > 0$

$$\sum_{i,j} \langle \tau_j(T^i) \underbrace{1 \cdots 1}_{\sum\mu} \underbrace{Y \cdots Y}_{\sum\nu} \rangle^Y_{\beta^{(0)}} \cdot T_i = (Y + Y \cdot \beta^{(0)})^{\sum\nu} \cdot I^Y_{\beta^{(0)}}. \tag{24}$$

In fact, this follows from the fundamental class equation of corollary 1.3.3

$$\begin{aligned}\sum_{i,j}\langle\tau_j(T^i)\ 1\ \cdots\rangle^Y_{\beta^{(0)}}\cdot T_i &= \sum_{i,j\neq 0}\langle\tau_{j-1}(T^i)\ \cdots\rangle^Y_{\beta^{(0)}}\cdot T_i\\ &= \sum_{i,j}\langle\tau_j(T^i)\ \cdots\rangle^Y_{\beta^{(0)}}\cdot T_i\end{aligned}$$

and the divisor equation of corollary 1.3.4

$$\begin{aligned}\sum_{i,j}\langle\tau_j(T^i)\ Y\ \cdots\rangle^Y_{\beta^{(0)}}\cdot T_i &= \sum_{i,j}(Y\cdot\beta^{(0)})\cdot\langle\tau_j(T^i)\ \cdots\rangle^Y_{\beta^{(0)}}\cdot T_i\\ &\quad+\sum_{i,j\neq 0}\langle\tau_{j-1}(T^i\cdot Y)\ \cdots\rangle^Y_{\beta^{(0)}}\cdot T_i\\ &= \sum_{i,j}(Y\cdot\beta^{(0)})\cdot\langle\tau_j(T^i)\ \cdots\rangle^Y_{\beta^{(0)}}\cdot T_i\\ &\quad+\sum_{i,j\neq 0}\langle\tau_{j-1}(T^i)\ \cdots\rangle^Y_{\beta^{(0)}}\cdot(T_i\cdot Y)\\ &= (Y\cdot\beta^{(0)}+Y)\cdot\sum_{i,j}\langle\tau_j(T^i)\ \cdots\rangle^Y_{\beta^{(0)}}\cdot T_i,\end{aligned}$$

where the dots denote any cohomological entries (i.e. not including cotangent line classes). In fact, the same formula (24) is also true for $\beta^{(0)}=0$, as in this case

$$\begin{aligned}\sum_{i,j}\langle\tau_j(T^i)\underbrace{1\ \cdots\ 1}_{\Sigma\mu}\underbrace{Y\ \cdots\ Y}_{\Sigma\nu}\rangle^Y_0\cdot T_i &= (Y^{\Sigma\nu})\cdot Y\\ &= Y^{\Sigma\nu}\cdot I^Y_0\end{aligned}$$

by example 1.3.7. Hence the first summand in the definition (21) of $J_{\beta,(m)}$ is

$$\mathrm{ev}_*\left(\frac{1}{1-\psi}\cdot[D_{(m)}(X,\beta)]^{\mathrm{virt}}\right)=\sum_{\mu,\nu}\left(Y+Y\cdot\beta^{(0)}\right)^{\Sigma\nu}\cdot\frac{s^\mu}{\mu!}\frac{r^\nu}{\nu!}\cdot I^Y_{\beta^{(0)}}\tag{25}$$

with the sum taken over all μ,ν satisfying the degree, multiplicity, and stability conditions. The second summand is

$$\begin{aligned}m\cdot\mathrm{ev}_*[\bar{M}_{(m)}(X,\beta)]^{\mathrm{virt}} &= m\cdot\sum_i\langle\tau^m(T^i)\rangle_\beta\cdot T_i\\ &= s_\beta\cdot Y\cdot\delta_{m,Y\cdot\beta-K_Y\cdot\beta-1}\\ &\quad+r_\beta\cdot Y^2\cdot\delta_{m,Y\cdot\beta-K_Y\cdot\beta}\end{aligned}$$

by lemma 3.1.5. As we have defined $I^Y_0=Y$, this adds exactly the terms with $\beta^{(0)}=0$ and $\sum\mu+\sum\nu=1$ to the sum in (25) that were excluded because of the stability condition. It follows that

$$J_{\beta,(m)}=\sum_{\mu,\nu}\left(Y+Y\cdot\beta^{(0)}\right)^{\Sigma\nu}\cdot\frac{s^\mu}{\mu!}\frac{r^\nu}{\nu!}\cdot I^Y_{\beta^{(0)}},$$

with the sum taken over all μ, ν satisfying the degree and multiplicity conditions. □

REMARK 3.1.9. The multiplicity condition in lemma 3.1.8 can be replaced by

$$m = Y \cdot \beta - \varepsilon \sum \mu,$$

where $\varepsilon \in \{0,1\}$ depends only on Y. To see this recall that the multiplicity condition was obtained from the original one

$$m = Y \cdot \beta^{(0)} + \sum m^{(k)} \tag{26}$$

by inserting the expressions $m^{(k)} = Y \cdot \beta^{(k)} - K_Y \cdot \beta^{(k)}$ (for every $r_{\beta^{(k)}}$) or $m^{(k)} = Y \cdot \beta^{(k)} - K_Y \cdot \beta^{(k)} - 1$ (for every $s_{\beta^{(k)}}$), respectively. But by remark 3.1.1 we have $r_{\beta^{(k)}} = 0$ if $m^{(k)} = Y \cdot \beta^{(k)} - K_Y \cdot \beta^{(k)} > Y \cdot \beta^{(k)}$. So (as K_Y is nef) $r_{\beta^{(k)}}$ can only be non-zero if $m^{(k)} = Y \cdot \beta^{(k)}$. Hence we can insert this simplified expression for $m^{(k)}$ in (26).

In the same way $s_{\beta^{(k)}}$ can only be non-zero if $m^{(k)} = Y \cdot \beta^{(k)} - 1$ (in the case $K_Y = 0$) or $m^{(k)} = Y \cdot \beta^{(k)}$ (in the case $K_Y > 0$). In other words, $m^{(k)} = Y \cdot \beta^{(k)} - \varepsilon$ with $\varepsilon \in \{0,1\}$ depending only on Y.

If we now take the original multiplicity condition (26) and insert the new simplified expressions $m^{(k)} = Y \cdot \beta^{(k)}$ (for every $r_{\beta^{(k)}}$) and $m^{(k)} = Y \cdot \beta^{(k)} - \varepsilon$ (for every $s_{\beta^{(k)}}$), respectively, we arrive at the desired multiplicity condition $m = Y \cdot \beta - \varepsilon \sum \mu$.

REMARK 3.1.10. Now we can insert the expression of lemma 3.1.8 (with the multiplicity condition from remark 3.1.9) into the formula of remark 3.1.3. Thus we obtain

$$\begin{aligned}
\prod_{i=0}^{Y\cdot\beta}(Y+i)\cdot I_\beta^X &= \sum_{\mu,\nu} \prod_{i=Y\cdot\beta-\varepsilon\sum\mu+1}^{Y\cdot\beta} (Y+i)\cdot\left(Y+Y\cdot\beta^{(0)}\right)^{\Sigma\nu}\cdot\frac{s^\mu}{\mu!}\frac{r^\nu}{\nu!}\cdot I^Y_{\beta^{(0)}} \\
&= \sum_{\mu,\nu} \prod_{i=0}^{\varepsilon\sum\mu-1} (Y+Y\cdot\beta-i)\cdot\left(Y+Y\cdot\beta^{(0)}\right)^{\Sigma\nu}\cdot\frac{s^\mu}{\mu!}\frac{r^\nu}{\nu!}\cdot I^Y_{\beta^{(0)}},
\end{aligned}$$

where the sum is now taken over all μ, ν satisfying the degree condition $\beta^{(0)} := \beta - |\mu| - |\nu| \geq 0$. Note that this equation is trivially true in the case $\beta = 0$ as well (both sides are equal to Y in this case).

To get rid of the degree condition we multiply these equations with $q^{Y\cdot\beta}$ (where q is a formal variable) and add them up; so we get

$$\sum_\beta \prod_{i=0}^{Y\cdot\beta}(Y+i)\cdot I^X_\beta \cdot q^{Y\cdot\beta}$$

$$= \sum_{\beta^{(0)}}\sum_{\mu,\nu} \prod_{i=0}^{\varepsilon\sum\mu-1} (Y+Y\cdot\beta-i)\cdot\left(Y+Y\cdot\beta^{(0)}\right)^{\Sigma\nu}\cdot\frac{s^\mu}{\mu!}\frac{r^\nu}{\nu!}\cdot I^Y_{\beta^{(0)}}\cdot q^{Y\cdot\beta}, \qquad (27)$$

where the sum now runs over all multi-indices μ,ν (and $\beta=\beta^{(0)}+|\mu|+|\nu|$).

Although this equation looks quite complicated, note that all geometric ideas in its derivation are still visible: the left hand side is the "naïve" expression for the Gromov-Witten invariants of Y that we already encountered in the introduction and remark 3.1.6. The product $\prod_{i=0}^{Y\cdot\beta}(Y+i)$ here corresponds to the process of raising the multiplicity of the curves from 0 to $Y\cdot\beta+1$. The right hand side of the equation describes the correction terms. They correspond to reducible curves with one component in the hypersurface ($I^Y_{\beta^{(0)}}$) and various others in the ambient space with specified multiplicities to the hypersurface ($s^\mu r^\nu$). The factor $(Y+Y\cdot\beta^{(0)})^{\Sigma\nu}$ comes from the $(\sum\nu)$-fold application of the divisor equation that we used to describe the component in the hypersurface by a 1-point invariant instead of by a $(1+r)$-point invariant.

All that remains to be done to arrive at the "mirror formula" is to simplify the right hand side of equation (27). To do so define $P(t)$ to be "the right hand side with $Y\cdot\beta^{(0)}$ replaced by a formal variable t":

DEFINITION 3.1.11. Let

$$P(t) := \sum_{\mu,\nu}\prod_{i=0}^{\varepsilon\sum\mu-1}(Y+Y\cdot(|\mu|+|\nu|)+t-i)\cdot(Y+t)^{\Sigma\nu}\cdot\frac{s^\mu}{\mu!}\frac{r^\nu}{\nu!}\cdot q^{Y\cdot(|\mu|+|\nu|)},$$

so that (27) can be written as

$$\sum_\beta\prod_{i=0}^{Y\cdot\beta}(Y+i)\cdot I^X_\beta\cdot q^{Y\cdot\beta}=\sum_\beta P(Y\cdot\beta)\cdot I^Y_\beta\cdot q^{Y\cdot\beta}. \qquad (28)$$

LEMMA 3.1.12. *The power series $P(t)$ of definition 3.1.11 satisfies the differential equation $\frac{d^2}{dt^2}\ln P=0$. In particular, if $P(t)=P_0+P_1\cdot t+\cdots$ is the Taylor expansion of P then $P(t)=P_0\exp(\frac{P_1}{P_0}t)$.*

PROOF. This can be checked directly from the definition of $P(t)$. The statement does not depend on the special values of r_β and s_β; it is equally true if the r_β and s_β

are considered to be formal variables. We give a proof of the statement in appendix 3.3 (apply lemma 3.3.1 with the collection of variables x_i being the union of the r_β and s_β, $z = 0$, and t replaced by $t+Y$). □

COROLLARY 3.1.13 (The mirror formula). *If we formally set $\tilde{q} = q \cdot \exp \frac{P_1}{P_0}$ with P_0 and P_1 as in lemma 3.1.12 then*

$$\sum_\beta \prod_{i=0}^{Y\cdot\beta} (Y+i) \cdot I_\beta^X \cdot q^{Y\cdot\beta} = P_0 \cdot \sum_\beta I_\beta^Y \cdot \tilde{q}^{Y\cdot\beta},$$

i.e. the generating function $\sum_\beta I_\beta^Y \cdot q^{Y\cdot\beta}$ of the 1-point Gromov-Witten invariants of Y can be obtained from the "naïve" expression $\sum_\beta \prod_{i=0}^{Y\cdot\beta}(Y+i) \cdot I_\beta^X \cdot q^{Y\cdot\beta}$ by a formal change of variables ($q \mapsto \tilde{q}$) and a scaling factor ($\cdot P_0$).

PROOF. Immediately from (28) and lemma 3.1.12. □

REMARK 3.1.14. In the above computations we assumed that the class of Y generates $H^2(X)$ over $\mathbb{Q}$. In fact, this is not essential. All that happens for higher dimension of $H^2(X)$ is that the notation becomes more complicated at some steps of the calculation. Most importantly, in remark 3.1.7 there are now more factors that can occur in the k-product of (22). Namely, instead of the r_β we now have

$$r_{i,\beta} = (Y\cdot\beta - K_Y \cdot \beta) \cdot \langle \tau^{Y\cdot\beta - K_Y\cdot\beta}(\gamma_i^\vee) \rangle_\beta$$

for $i = 1, \dots, \dim H^2(X) \otimes \mathbb{Q}$, where the γ_i form a basis of $H^2(X)\otimes\mathbb{Q}$ chosen such that $\gamma_1 = Y$. Correspondingly, lemma 3.1.8 becomes

$$J_{\beta,(m)} = \sum_{\mu,\nu_i} \prod_i \left(\gamma_i + \gamma_i \cdot \beta^{(0)}\right)^{\Sigma\nu_i} \cdot \frac{s^\mu}{\mu!} \cdot \prod_i \frac{r_i^{\nu_i}}{\nu_i!} \cdot I^Y_{\beta^{(0)}}$$

where the ν_i are multi-indices. In the alternative multiplicity condition of remark 3.1.9 the number ε will now depend on β (it is 1 if $K_Y \cdot \beta = 0$ and 0 if $K_Y \cdot \beta > 0$). Hence the multiplicity condition is now $m = Y\cdot\beta - \varepsilon\mu$, where ε is a multi-index with entries 0 and 1. Finally, we need a formal variable q_i for each γ_i to replace the expression $q^{Y\cdot\beta}$ by $q^\beta := \prod_i q_i^{\gamma_i\cdot\beta}$. Definition 3.1.11 then becomes

$$P(\{t_i\}) := \sum_{\mu,\nu_i} \prod_{j=0}^{\varepsilon\mu-1} (Y + Y\cdot(|\mu| + \sum_i |\nu_i|) + t_1 - j) \cdot \prod_i (\gamma_i + t_i)^{\Sigma\nu_i}$$
$$\cdot \frac{s^\mu}{\mu!} \cdot \prod_i \frac{r_i^{\nu_i}}{\nu_i!} \cdot q^{|\mu| + \Sigma_i|\nu_i|},$$

with which we obtain the equation (compare to (28))

$$\sum_\beta \prod_{i=0}^{Y\cdot\beta}(Y+i)\cdot I^X_\beta \cdot q^\beta = \sum_\beta P(\{\gamma_i\cdot\beta\})\cdot I^Y_\beta\cdot q^\beta. \tag{29}$$

The same proof as for lemma 3.1.12 works to show that $\partial_{t_i}\partial_{t_j}\ln P = 0$ for all i, j, so it follows that $P(t) = P_0 \exp(\frac{\sum P_i t_i}{P_0})$, where $P(\{t_i\}) = P_0 + \sum_i P_i\cdot t_i + \cdots$ is the linear expansion of P. Hence the mirror formula of corollary 3.1.13 holds in the same way

$$\sum_\beta \prod_{i=0}^{Y\cdot\beta}(Y+i)\cdot I^X_\beta\cdot q^\beta = P_0\cdot\sum_\beta I^Y_\beta\cdot\tilde{q}^\beta,$$

where $\tilde{q}_i = q_i\cdot\exp\frac{P_i}{P_0}$.

3.2. Examples

EXAMPLE 3.2.1 (Application to the quintic threefold). Let $X = \mathbb{P}^4$, and let $Y\subset X$ be a smooth quintic hypersurface. Henve $Y = 5H \in H^*(X)$, where H is the class of a hyperplane. We are interested in the genus zero Gromov-Witten invariants of Y, i.e. in the numbers $n_d = \frac{1}{d}\langle H\rangle^Y_d$. As this is the H^3-coefficient of I^Y_d (up to a scaling factor), we consider the equation (28) modulo H^4. (This discards the invariants $\langle\tau_1(1)\rangle^Y_d$.)

Since the only Gromov-Witten invariants of Y are $\langle H\rangle^Y_d$ (and $\langle\tau_1(1)\rangle^Y_d$) the polynomials I^Y_d have no H^0, H^1, and H^2 terms for $d>0$. Hence as

$$I^X_d = \prod_{i=1}^d \frac{1}{(H+i)^5},$$

by proposition 1.2.6 (i) it follows from (28) that

$$\sum_{d\geq 0} 5H\cdot\frac{\prod_{i=1}^{5d}(5H+i)}{\prod_{i=1}^d (H+i)^5}\, q^{5d} = 5H\, P_0 \pmod{H^3}.$$

This is sufficient to reconstruct P: if we expand

$$\sum_{d\geq 0}\frac{\prod_{i=1}^{5d}(5H+i)}{\prod_{i=1}^d (H+i)^5}\, q^{5d} =: F_0 + F_1 H + F_2 H^2 + \cdots \tag{30}$$

then $P|_{t=H=0} = F_0$ and $\partial_H P|_{t=H=0} = F_1$. So as P is a function of $t+5H$ and satisfies $\partial_t^2 \ln P = 0$, it follows that $\partial_t P|_{t=H=0} = \frac{1}{5}F_1$, and hence

$$P = F_0\cdot\exp\left(\left(\frac{t}{5}+H\right)\cdot\frac{F_1}{F_0}\right).$$

In particular,

$$P_0 = F_0 \cdot \exp\left(H\frac{F_1}{F_0}\right) = F_0 + H F_1 + \frac{H^2}{2}\frac{F_1^2}{F_0} + \cdots .$$

So by comparing the H^3-coefficient of (28) we get

$$F_2 = \frac{1}{2}\frac{F_1^2}{F_0} + \frac{1}{5}\sum_{d>0} d n_d q^{5d} F_0 \exp\left(d\frac{F_1}{F_0}\right).$$

Together with (30), this equation determines the n_d recursively and gives the well-known numbers that we have listed already in example 2.5.8.

EXAMPLE 3.2.2 (Application to plane elliptic curves). We want to compute the (virtual) numbers of rational plane curves of degree d having multiplicity $3d$ to a smooth elliptic plane cubic, i.e. the relative Gromov-Witten invariants $\langle\tau^{3d}(1)\rangle_d = \frac{3}{d} r_d$ in the case when $X = \mathbb{P}^2$ and Y is a smooth elliptic cubic. According to [**T**] remark 1.11 these numbers are related to the local mirror symmetry of [**CKYZ**].

The computation of the numbers r_d is very similar (yet not identical) to that of the Gromov-Witten invariants of Y in section 3.1. This time we apply lemma 3.1.2 recursively only up to multiplicity $3d$ instead of $3d+1$, so we get

$$\prod_{i=0}^{3d-1}(3H+i)\, I_d^X = I_{d,(3d)} + \sum_{m=0}^{3d-1}\prod_{i=m+1}^{3d-1}(3H+i)\, J_{d,(m)}.$$

Note that $I_d^Y = 0$ for $d > 0$, as there are no rational curves in Y. So if we insert the expression for $J_{d,(m)}$ of lemma 3.1.8, we get in the same way as in remark 3.1.10

$$\sum_{d>0}\prod_{i=0}^{3d-1}(3H+i)\, I_d^X\, q^{3d} = \sum_{d>0}\frac{3H^2}{d} r_d q^{3d} + \sum_{\mu,\nu}\prod_{i=3d-\sum\mu+1}^{3d-1}(3H+i)\,(3H)^{\sum\nu}\cdot\frac{s^\mu}{\mu!}\frac{r^\nu}{\nu!}\cdot 3H\, q^{3d} \quad (31)$$

where we already inserted the expression $m = 3d - \sum\mu$ for Calabi-Yau hypersurfaces (see remark 3.1.9). Here, in the second line we set $d = |\mu| + |\nu|$, and we obviously only sum over those μ with $\sum\mu \geq 1$.

Similar to definition 3.1.11 let us set

$$Q(t) := \sum_{\mu}\prod_{i=1}^{\sum\mu-1}(3H\cdot|\mu| + t - i)\,\frac{s^\mu}{\mu!}\, q^{3H\cdot|\mu|}\, t,$$

where the sum is now taken over all μ — not only those with $\sum\mu \geq 1$. The $\mu = 0$ term contributes a 1 (together with the factor t). The definition of $Q(t)$ is so that $Q(3H)-1$ yields exactly the $\nu = 0$ terms in the second line of (31).

Similarly to lemma 3.1.12 the power series $Q(t)$ satisfies a differential equation: by lemma 3.3.2 $\ln Q(t)$ is linear in t, i.e. $Q(t) = \exp(c\cdot t)$. To compute c, we expand as in example 3.2.1 the left hand side of (31)

$$\sum_{d>0} 3H \cdot \frac{\prod_{i=1}^{3d-1}(3H+i)}{\prod_{i=1}^{d}(H+i)^3}\, q^{3d} =: F_1 H + F_2 H^2 + \cdots$$

(in [**T**] $F_1(q^3)$ is called $I_2^{(0)}(z)$, and $F_2(q^3)$ is called $I_3^{(0)}(z)$). As the t-expansion of $Q(t)$ is

$$Q(t) = 1 + ct + \frac{1}{2}c^2t^2 + \cdots,$$

comparison of the H^1 terms in (31) gives $F_1 =$ (the H^1 term of $Q(3H)) = 3c$; so $Q(t) = \exp(\frac{F_1\cdot t}{3})$.

Now compare the H^2 term in (31). Note that we must have $\sum\nu \leq 1$ because of the factor $(3H)^{\sum\nu+1}$. The $\nu = 0$ term is exactly the second coefficient of $Q(3H)$ as remarked above, i.e. $\frac{1}{2}F_1^2$. The terms with $\sum\nu = 1$ can be written as a sum over d, where d is the index of the one non-zero entry of ν. The contribution for a given d is exactly $9r_d q^{3d}\frac{Q(3d)}{3d} = \frac{3}{d}r_d q^{3d}\exp(dF_1)$, with the $\mu = 0$ term in $Q(3d)$ coming from the right hand side of the first line of (31). Thus we get the equation

$$F_2 = \frac{1}{2}F_1^2 + \sum_{d>0}\frac{3}{d}r_d\, q^{3d}\exp(dF_1),$$

which determines the numbers $\frac{3}{d}r_d = \langle\tau^{3d}(1)\rangle_d$. The first few numbers are given in the following table.

d	1	2	3	4	5	6	7	8
$\langle\tau^{3d}(1)\rangle_d$	9	$\frac{135}{4}$	244	$\frac{36999}{16}$	$\frac{635634}{25}$	307095	$\frac{193919175}{49}$	$\frac{3422490759}{64}$

This equation is equivalent to the conjecture of remark 1.11 in [**T**]. Together with [**T**] theorem 2.1 it proves that $\langle\tau^{3d}(1)\rangle_d = (-1)^d 3dK_d$, where K_d is the top Chern class of the rank-$(3d-1)$ bundle on $\bar{M}_0(\mathbb{P}^2,d)$ with fiber $H^1(C, f^*K_{\mathbb{P}^2})$ at the point $(C,f) \in \bar{M}_0(\mathbb{P}^2,d)$. At the moment we do not know of a geometric proof of this statement.

3.3. Proof of the main technical lemmas

The goal of this section is to show that the power series $P(t)$ and $Q(t)$ of definition 3.1.11 and example 3.2.2 satisfy certain differential equations.

LEMMA 3.3.1. *Let x_i be a collection of variables (possibly infinite), and let $a_i, b_i \in \mathbb{N}$, $c_i \in \mathbb{C}$. Define*

$$P(t,z) = \sum_k \frac{x^k}{k!} t^{ak} \prod_{i=0}^{bk-1} (ck+z+t-i),$$

where k is a multi-index, and where we used the usual multi-index notations $ak = \sum_i a_i k_i$, $x^k = \prod_i x_i^{k_i}$, $k! = \prod_i k_i!$. Assume that, for every i, the pair (a_i, b_i) is $(0,0)$, $(1,0)$, or $(0,1)$. Then

$$\partial_t^2 \ln P = \partial_z^2 \ln P = \partial_t \partial_z \ln P = 0.$$

PROOF. *Step 1.* We consider the c_i to be formal variables and show by induction on n that for every i and every $n \geq 0$

$$\begin{array}{ll} \text{if} & \partial_t^2 \ln P|_{c_i=0} = \partial_z^2 \ln P|_{c_i=0} = \partial_t\partial_z \ln P|_{c_i=0} = 0 \\ \text{then} & \partial_{c_i}^n \partial_t^2 \ln P|_{c_i=0} = \partial_{c_i}^n \partial_z^2 \ln P|_{c_i=0} = \partial_{c_i}^n \partial_t \partial_z \ln P|_{c_i=0} = 0. \end{array}$$

So assume that

$$\partial_{c_i}^j \partial_t^2 \ln P|_{c_i=0} = \partial_{c_i}^j \partial_z^2 \ln P|_{c_i=0} = \partial_{c_i}^j \partial_t \partial_z \ln P|_{c_i=0} = 0$$

for $j \leq n$. Note that by definition of P we have $\partial_{c_i} P = x_i \partial_{x_i} \partial_z P$. Let ∂_1 and ∂_2 denote either ∂_t or ∂_z. Then it follows that (everything in the following calculation is evaluated at $c_i = 0$):

$$\begin{aligned}
\partial_{c_i}^{n+1} \partial_1 \partial_2 \ln P &= \partial_{c_i}^n \partial_1 \partial_2 \frac{\partial_{c_i} P}{P} \\
&= x_i \partial_{c_i}^n \partial_1 \partial_2 \frac{\partial_{x_i} \partial_z P}{P} \\
&= x_i \partial_{c_i}^n \partial_1 \partial_2 \left(\partial_{x_i} \frac{\partial_z P}{P} - \partial_z P \cdot \partial_{x_i} \frac{1}{P} \right) \\
&= x_i \partial_{c_i}^n \partial_1 \partial_2 \left(\partial_{x_i} \frac{\partial_z P}{P} + \frac{\partial_z P}{P} \cdot \frac{\partial_{x_i} P}{P} \right) \\
&= x_i \partial_{x_i} \partial_z \underbrace{\partial_{c_i}^n \partial_1 \partial_2 \ln P}_{=0} + x_i \partial_{c_i}^n \partial_1 \partial_2 (\partial_z \ln P \cdot \partial_{x_i} \ln P) \\
&= x_i \partial_{c_i}^n (\partial_1 \partial_2 \partial_z \ln P \cdot \partial_{x_i} \ln P + \partial_1 \partial_z \ln P \cdot \partial_2 \partial_{x_i} \ln P \\
&\qquad + \partial_2 \partial_z \ln P \cdot \partial_1 \partial_{x_i} \ln P + \partial_z \ln P \cdot \partial_1 \partial_2 \partial_{x_i} \ln P) \\
&= 0
\end{aligned}$$

(for the last step note that every summand has a factor that contains a $\partial_t^2 \ln P$, $\partial_z^2 \ln P$, or $\partial_t \partial_z \ln P$ that gets at most n ∂_{c_i}'s, so it vanishes by the induction assumption).

Step 2. By step 1 it suffices to prove the lemma in the case $c = 0$. Note that then P becomes a product of two terms of the form

$$R = \sum_k \frac{x^k}{k!} t^{ak} \qquad \text{and} \qquad S = \sum_k \frac{x^k}{k!} \prod_{i=0}^{bk-1} (z+t-i)$$

where the first term contains all the x_i with $(a_i, b_i) = (0,0)$ or $(a_i, b_i) = (1,0)$, and the second term all the x_i with $(a_i, b_i) = (0,1)$. Obviously, it suffices to prove the lemma for R and S separately. But

$$R = \sum_k \prod_i \frac{(x_i t_i^a)^{k_i}}{k_i!} = \exp\Big(\sum_i x_i t^{a_i}\Big)$$

and

$$S = \sum_k \frac{x^k}{k!} \binom{z+t}{\sum k} (\sum k)! = \Big(1 + \sum_i x_i\Big)^{z+t},$$

and in both cases it is obvious that the lemma holds. □

LEMMA 3.3.2. *Let x_i be a collection of variables (possibly infinite), and let $c_i \in \mathbb{C}$. Define*

$$Q(t) = \sum_k \frac{x^k}{k!} t \prod_{i=1}^{\sum k-1} (ck+t-i)$$

in multi-index notation, where k is a multi-index. Then $\ln Q(t)$ is linear in t, i.e.

$$(t\partial_t - 1) \ln Q = 0.$$

PROOF. The proof is very similar to that of lemma 3.3.1.

Step 1. We consider the c_i to be formal variables and show by induction on n that for every i and every $n \geq 0$

$$\text{if} \quad (t\partial_t - 1) \ln Q|_{c_i=0} = 0 \quad \text{then} \quad \partial_{c_i}^n (t\partial_t - 1) \ln Q|_{c_i=0} = 0.$$

So assume that $\partial_{c_i}^j (t\partial_t - 1) \ln Q|_{c_i=0} = 0$ for $j \leq n$. By definition of Q we have $\partial_{c_i} Q = x_i \partial_{x_i} (\partial_t - \frac{1}{t}) Q$. Hence it follows that (everything in the following calculation is evaluated at $c_i = 0$):

$$\begin{aligned}
\partial_{c_i}^{n+1}(t\partial_t - 1)\ln Q &= \partial_{c_i}^{n}(t\partial_t - 1)\frac{x_i\partial_{x_i}(\partial_t - \frac{1}{t})Q}{Q} \\
&= x_i(t\partial_t - 1)\left(\underbrace{\partial_{c_i}^{n}\left(\partial_t - \frac{1}{t}\right)\partial_{x_i}\ln Q}_{=0} + \partial_{c_i}^{n}\partial_t \ln Q \cdot \partial_{x_i}\ln Q\right) \\
&= x_i\partial_{c_i}^{n}\left(\partial_t \ln Q \cdot \partial_{x_i}(t\partial_t - 1)\ln Q + \partial_t(t\partial_t - 1)\ln Q \cdot \partial_{x_i}\ln Q\right) \\
&= 0
\end{aligned}$$

(for the last step note that every summand has a factor that contains a $(t\partial_t - 1)\ln Q$ that gets at most n ∂_{c_i}'s, so it vanishes by the induction assumption).

Step 2. By step 1 it suffices to prove the lemma in the case $c = 0$. But then

$$Q(t) = \sum_k \frac{x^k}{k!} \prod_{i=0}^{\sum k-1}(t-i) = \left(1 + \sum_i x_i\right)^t,$$

which obviously satisfies the statement of the lemma. □

CHAPTER 4

The number of plane conics 5-fold tangent to a given curve

In the last chapters we have explained how the rational relative Gromov-Witten invariants of a projective manifold X relative a smooth very ample hypersurface Y can be computed. We now want to address a different question, namely that of the enumerative significance of the invariants. Recall that the relative Gromov-Witten invariants can be thought of as the numbers of curves in X having given local orders of contact to Y. But as the invariants are defined using a virtual fundamental class it may of course happen that the invariant is not the correct enumerative answer. In this chapter we will study this relation between the invariants and the enumerative numbers in a specific example.

The set-up of our example is as follows. Let $Y \subset \mathbb{P}^2$ be a generic plane curve of degree $d \geq 5$. We want to consider smooth plane conics that are 5-fold tangent to Y. As the space of all plane conics is 5-dimensional and each tangency imposes one condition on the curves, we expect a finite number of such 5-fold tangent conics. It will be easy to see that this number is indeed finite; let us call it n_d. The goal of this chapter is to compute it.

Of course this is a classical problem, and attempts have been made to solve it using classical methods of enumerative geometry. I. Vainsencher **[V]** tried to use various blow-ups of the ordinary $\mathbb{P}^5$ of conics as moduli spaces, but the intersection of the five tangency conditions in this moduli space always resulted in a scheme with many non-enumerative components, most of which were non-reduced with a multiplicity that could not be computed explicitly. Their geometry was so complicated that the problem could not be solved that way.

In this chapter we want use the moduli spaces of stable relative maps to solve the problem. We consider the moduli space $\bar{M}^Y_{(2,2,2,2,2)}(\mathbb{P}^2, 2) \subset \bar{M}_{0,5}(\mathbb{P}^2, 2)$ that parametrizes rational stable maps to $\mathbb{P}^2$ of degree 2 (i.e. conics) with 5 marked points such that the stable map is tangent to Y at all these points. It comes equipped with a 0-dimensional virtual fundamental class whose degree N_d can be computed explicitly using the methods of chapter 2.

We can interpret the number N_d as the "virtual number" of conics that are 5-fold tangent to Y. It is only virtual because it contains — just as in Vainsencher's classical computations — non-enumerative contributions from the "boundary" of the moduli space. These contributions are quite simple however. It is not hard to see that the only degree-2 rational stable maps $f : C \to \mathbb{P}^2$ that satisfy the tangency conditions at the 5 marked points are all double covers of a bitangent of Y, and have the marked points distributed in one of the following two ways:

There are only finitely many stable maps with marked points as in this picture on the left, so we just have to count them and subtract their number from the virtual invariant N_d. The picture on the right however shows a 1-dimensional family of stable maps (the unmarked ramification point of f can move). We will use equations from relative Gromov-Witten theory to compute the degree of the 0-dimensional virtual fundamental class of the moduli space on this 1-dimensional component. By subtracting both correction terms we finally arrive at the enumerative numbers n_d. They are

$$n_d = \frac{1}{5!}\, d\,(d-3)\,(d-4)\,(d^7+12d^6-18d^5-540d^4+251d^3+5712d^2-1458d-14580).$$

Our (as well as Vainsencher's) motivation for studying this problem came from a question concerning rational curves in K3 surfaces. If X is a K3 surface and $\beta \in H_2(X,\mathbb{Z})$ the homology class of a holomorphic curve in X then various authors [**BL, Bv, Gö, YZ**] have shown that the number of rational curves in X of class β is equal to the q^d coefficient of the series

$$G(q) = \prod_{i>0} \frac{1}{(1-q^i)^{24}} = 1 + 24q + 324q^2 + 3200q^3 + 25650q^4 + 176256q^5 + \cdots$$

with $d = \frac{1}{2}\beta^2 + 1$ *if the class* β *is primitive*, i.e. not a non-trivial multiple of a smaller integral homology class. There is a well-defined "K3 invariant" (using a modified obstruction theory on the moduli spaces of stable maps to X) for non-primitive β too [**BL**]; it is however not known yet how this invariant relates to the above series $G(q)$ or to the enumerative number.

The result of this chapter allows us to study this question in a non-trivial example: we let X be the double cover of $\mathbb{P}^2$ branched along a sextic curve Y, and take β to be the pull-back of the class of conics in $\mathbb{P}^2$. Our work allows us to compute the enumerative number of rational curves in X of class β, which we can then compare to the corresponding number of the series $G(q)$. The result is that the "K3 invariant" is equal to the corresponding term in the $G(q)$ series, plus a double cover correction term that is equal to $\frac{1}{8}$ times the number of rational curves in X of class $\frac{1}{2}\beta$. Note that this is the same sort of correction term as for multiple covers of rational curves in Calabi-Yau threefolds. We conjecture that this pattern continues for classes β of higher divisibility.

This chapter is organized as follows. In section 4.1 we show how to compute the relative Gromov-Witten invariant N_d. We analyze the moduli space $\bar{M}^Y_{(2,2,2,2,2)}(\mathbb{P}^2,2)$ and its virtual fundamental class in sections 4.2 and 4.3, respectively, leading to the final result for n_d in corollary 4.3.6. Section 4.4 contains the application to K3 surfaces mentioned above.

4.1. The Gromov-Witten approach

In this section we will show how to compute the relative Gromov-Witten invariant that corresponds to the number of conics that are 5-fold tangent to a given smooth plane curve. We will use the notations and results from chapter 2, to which we also refer for further details.

DEFINITION 4.1.1. Let $Y \subset \mathbb{P}^2$ be a smooth curve of degree d, and let $m_1,\dots,m_n$ be non-negative integers. We denote by $\bar{M}_{(m_1,\dots,m_n)} = \bar{M}^Y_{(m_1,\dots,m_n)}(\mathbb{P}^2,2)$ the moduli space of n-pointed stable relative maps of degree 2 to $\mathbb{P}^2$ relative to Y with multiplicities $m_1,\dots,m_n$ as of definition 2.1.1. Its virtual fundamental class defined in 2.1.19 is denoted $[\bar{M}_{(m_1,\dots,m_n)}]^{\text{virt}}$.

REMARK 4.1.2. The moduli space $\bar{M}_{(m_1,\dots,m_n)}$ can be thought of as a compactification of the space of irreducible plane conics together with n distinct marked points on them at which the conic has the prescribed local intersection multiplicities $m_1,\dots,m_n$ with Y. In particular, the moduli space $\bar{M}_{(2,2,2,2,2)}$ corresponds to conics 5-fold tangent to Y.

REMARK 4.1.3. For future reference let us unwind the construction from definition 2.1.19 in our specific example. Consider the degree-d Veronese embedding $i : \mathbb{P}^2 \to \mathbb{P}^D$ with $D = \binom{D+2}{2} - 1$. We have $i(Y) = i(\mathbb{P}^2) \cap H$ in $\mathbb{P}^D$ for a suitable hyperplane $H \subset \mathbb{P}^D$. The inclusion morphism i induces an inclusion of moduli spaces $\bar{M}_{0,n}(\mathbb{P}^2,2) \subset \bar{M}_{0,n}(\mathbb{P}^D,2d)$. Moreover, let $\bar{M}^H_{(m_1,\dots,m_n)}(\mathbb{P}^D,2d)$ be the closure in $\bar{M}_{0,n}(\mathbb{P}^D,2d)$ of all stable maps $(C,x_1,\dots,x_n,f)$ such that C is irreducible,

$f(C) \not\subset H$, and the divisor f^*Y on C contains the points x_i with multiplicities m_i. Then we define

$$\bar{M}_{(m_1,\dots,m_n)} := \bar{M}_{0,n}(\mathbb{P}^2,2) \cap \bar{M}^H_{(m_1,\dots,m_n)}(\mathbb{P}^D,2d)$$

(with the intersection taken in $\bar{M}_{0,n}(\mathbb{P}^D,2d)$). The virtual fundamental class is the corresponding refined intersection product

$$[\bar{M}_{(m_1,\dots,m_n)}]^{\text{virt}} := [\bar{M}_{0,n}(\mathbb{P}^2,2)] \cdot [\bar{M}^H_{(m_1,\dots,m_n)}(\mathbb{P}^D,2d)] \in A_*(\bar{M}_{(m_1,\dots,m_n)}).$$

The virtual dimension of $\bar{M}_{(m_1,\dots,m_n)}$ is $5-\sum_i(m_i-1)$. By abuse of notation we will always drop the superscript *virt* from the notation of the virtual fundamental class from now on, as we do not need the ordinary fundamental classes of these spaces.

REMARK 4.1.4. From remark 4.1.3 we get immediately the following statement: let $\mathcal{C} = (C,x_1,\dots,x_n,f) \in \bar{M}_{(m_1,\dots,m_n)}$ be an automorphism-free stable map such that $C \cong \mathbb{P}^1$ is irreducible and $f(C) \not\subset Y$. Then, locally around this point, $\bar{M}_{(m_1,\dots,m_n)}$ is scheme-theoretically the subscheme of $\bar{M}_{0,n}(\mathbb{P}^2,2)$ given by the $\sum_i m_i$ equations that describe the vanishing of the m_i-jets of $\mathrm{ev}_i^* Y$ at the points x_i. If moreover $\bar{M}_{(m_1,\dots,m_n)}$ has the expected dimension at this point $\mathcal{C}$, then this point lies on a unique irreducible component of $\bar{M}_{(m_1,\dots,m_n)}$, and the virtual fundamental class on this component is just the ordinary scheme-theoretic fundamental class, i.e. the length of the scheme $\bar{M}_{(m_1,\dots,m_n)}$ at this irreducible component.

REMARK 4.1.5. Recall from remark 2.1.5 that there is an easier description of $\bar{M}_{(m_1,\dots,m_n)}$ *as a set*. Namely, it is the subspace of $\bar{M}_{0,n}(\mathbb{P}^2,2)$ of all n-pointed rational stable maps $(C,x_1,\dots,x_n,f)$ of degree 2 to $\mathbb{P}^2$ such that the following two conditions are satisfied:

(i) $f(x_i) \in Y$ for all i such that $m_i > 0$,
(ii) $f^*Y - \sum_i m_i x_i \in A_0(f^{-1}(Y))$ is effective.

As mentioned above, the moduli space $\bar{M}_{(2,2,2,2,2)}$ has virtual dimension zero and corresponds to conics 5-fold tangent to Y (together with a labeling of the five tangency points). Hence we define:

DEFINITION 4.1.6. The number

$$N_d := \frac{1}{5!} \cdot \deg[\bar{M}_{(2,2,2,2,2)}] \in \mathbb{Q}$$

will be called the **virtual number of conics 5-fold tangent to** Y.

The number N_d is only virtual because it receives correction terms from double covers of lines (see sections 4.2 and 4.3). In the rest of this section we will show how to compute the number N_d. Obviously, we can assume that $d \geq 5$.

The computation is done using the main theorem 2.2.6 of chapter 2 that tells us "how to raise the multiplicities of the moduli spaces": it says that

$$(\mathrm{ev}_n^* Y + m_n \psi_n) \cdot [\bar{M}_{(m_1,\dots,m_n)}] = [\bar{M}_{(m_1,\dots,m_{n-1},m_n+1)}] + \text{correction terms,} \qquad (32)$$

where $\mathrm{ev}_n : \bar{M}_{(m_1,\dots,m_n)} \to \mathbb{P}^2$ is the evaluation map at the last marked point, and ψ_n is the first Chern class of the cotangent line bundle L_n, i.e. of the bundle whose fiber at a stable map $(C, x_1, \dots, x_n, f)$ is the cotangent space $T^\vee_{C,x_n}$. The correction terms are as follows. Every correction term corresponds to a moduli space of reducible curves with $r+1$ components $C_0, \dots, C_r$, where C_0 is contracted[1] to a point of Y, and the other components C_i intersect C_0 in a point where they have local intersection multiplicity μ_i to Y. We get such a correction term for every r, every choice of the μ_i, and every splitting of the total homology class and the marked points onto the components C_i, such that the following two conditions are satisfied:

(a) the last marked point x_n lies on the component C_0,
(b) the sum of all the μ_i is equal to the sum of those m_i such that $x_i \in C_0$.

These correction terms appear in the above equation with multiplicity $\prod_{i=1}^r \mu_i$.

EXAMPLE 4.1.7. Here is an example of equation (32). In the case

$$(\mathrm{ev}_5^* Y + \psi_5) \cdot [\bar{M}_{(2,2,2,2,1)}] = [\bar{M}_{(2,2,2,2,2)}] + \text{correction terms}$$

we want to figure out the correction terms. As this is an equation in (virtual) dimension 0, the contracted component C_0 must have exactly 3 special points (it would not be stable if it had fewer, and it would have moduli if it had more). Hence the correction terms fall into these two categories:

(i) $r = 1$ (in the above notation), C_1 is a conic, and C_0 is a contracted component with three special points x_5, the intersection point with C_1, and one other x_i for $i = 1, \dots, 4$,
(ii) $r = 2$, C_1 and C_2 two lines, and C_0 is a contracted component with three special points x_5 and the two intersection points with C_1 and C_2.

Actually, case (ii) cannot occur because condition (b) above cannot be satisfied: the sum $\mu_1 + \mu_2$ is at least 2, whereas m_5 is only 1. Hence the only correction terms are of type (i). We get four of them: one for each choice of the point x_i that is to lie on the contracted component C_0. We have $\mu_1 = m_i + m_5 = 3$ in each of these cases by condition (b). All four correction terms appear with multiplicity $\mu_1 = 3$. Pictorially, the equation reads

[1]This uses the fact that the curve Y has positive genus, and that therefore every rational stable map to Y must be constant. In general, C_0 can be a curve with any homology class in Y.

Here the dotted curve is the fixed curve Y, and the solid curve is the moving conic C. In the four correction terms the component with x_5 on it is meant to be contracted. Written down as an equation of virtual fundamental classes of moduli spaces, the formula reads

$$\begin{aligned}(\mathrm{ev}_5^* Y + \psi_5) \cdot [\bar{M}_{(2,2,2,2,1)}] =& [\bar{M}_{(2,2,2,2,2)}] + 3[\bar{M}_{(3,2,2,2)}] \\ &+ 3[\bar{M}_{(2,3,2,2)}] + 3[\bar{M}_{(2,2,3,2)}] + 3[\bar{M}_{(2,2,2,3)}].\end{aligned} \tag{33}$$

REMARK 4.1.8. The correction terms in equation (32) are themselves products of moduli spaces of stable relative maps of the form $\bar{M}^Y_{(m'_1,\dots,m'_k)}(\mathbb{P}^2, d')$ with $d' \leq 2$ and $m'_1 + \cdots m'_k \leq m_1 + \cdots m_n$. In other words, this equation expresses invariants (i.e. intersection products of $\mathrm{ev}_i^* H$ and ψ_i classes, where H is a line in $\mathbb{P}^2$) on the relative moduli space $\bar{M}_{(m_1,\dots,m_n+1)}$ in terms of other invariants on relative moduli spaces $\bar{M}^Y_{(m'_1,\dots,m'_k)}(\mathbb{P}^2, d')$ whose "total multiplicity" $\sum_i m'_i$ is smaller than the total multiplicity $1 + \sum_i m_i$ of $M_{(m_1,\dots,m_n+1)}$. Hence, applying equation (32) recursively $m_1 + \cdots + m_n$ times we can express every invariant on $\bar{M}_{(m_1,\dots,m_n)}$ in terms of invariants on $\bar{M}_{(0,\dots,0)}$, which are just ordinary Gromov-Witten invariants of $\mathbb{P}^2$. As the Gromov-Witten invariants of $\mathbb{P}^2$ are well-known we can thus compute all relative Gromov-Witten invariants recursively, in particular N_d. Example 4.1.7 is the first step in this recursion process; it expresses the invariant $\bar{M}_{(2,2,2,2,2)}$ (with total multiplicity 10) in terms of invariants with total multiplicity 9.

Without actually carrying out the recursion we can see the following.

LEMMA 4.1.9. *The function $d \mapsto N_d$ is a polynomial of degree 10 with leading coefficient $\frac{1}{5!}$.*

PROOF. Using equation (32) it is easy to show by induction that every invariant (i.e. intersection product of $\mathrm{ev}_i^* H$ and ψ_i classes) on a moduli space $\bar{M}_{(m_1,\dots,m_n)}$ is a polynomial in d of degree (at most) $m_1 + \cdots + m_n$. In fact, this is obvious for $m_1 + \cdots + m_n = 0$, as we then just have ordinary Gromov-Witten invariants of $\mathbb{P}^2$ (that do not depend on Y). Equation (32) reads

$$[\bar{M}_{(m_1,\dots,m_{n-1},m_n+1)}] = (d\ \mathrm{ev}_n^* H + m_n \psi_n) \cdot [\bar{M}_{(m_1,\dots,m_n)}] - \text{correction terms}.$$

All correction terms have total multiplicity at most $m_1 + \cdots + m_n$, so by induction hypothesis they contribute a polynomial in d of degree at most $m_1 + \cdots + m_n$. The same is true for the ψ_n summand on the right hand side. Hence, as every invariant on $\bar{M}_{(m_1,\dots,m_n)}$ is a polynomial in d of degree at most $m_1 + \cdots + m_n$ by assumption, it follows that every invariant on $\bar{M}_{(m_1,\dots,m_n+1)}$ is a polynomial in d of degree at most $m_1 + \cdots + m_n + 1$.

It can be seen from the same recursive formula that the d^{10} coefficient of the invariant $\deg[\bar{M}_{(2,2,2,2,2)}]$ is just

$$\prod_{i=1}^{5} \mathrm{ev}_i^* H^2 \cdot [\bar{M}_{0,5}(\mathbb{P}^2, 2)],$$

i.e. the number of conics through 5 general points in the plane. This number is 1, proving the statement of the lemma about the leading coefficient. $\square$

The precise form of the polynomial N_d is quite complicated and can only be obtained by carrying out the full recursion as described above. We only give the result here; a Maple program to compute it can be obtained from the author on request.

PROPOSITION 4.1.10. *For $d \geq 5$ the virtual number of conics 5-fold tangent to Y is*

$$N_d = \frac{1}{5!} d\,(d-3)\,(d-4)\,(d^7 + 12d^6 - 18d^5 - 540d^4 + 311d^3 + 5457d^2 - 2133d - 12690).$$

REMARK 4.1.11. The first few values of N_d are given in the following table.

d	5	6	7	8	9	10
N_d	1985	71442	687897	3893256	16180398	54679380

REMARK 4.1.12. In this section we have only used equation (32) in the Chow ring of the moduli space of stable maps $\bar{M}_{0,n}(\mathbb{P}^2, 2)$. In fact, there is a refined version of this equation that we will need in section 4.3. If $\mathcal{P}_k$ denotes the rank-$(k+1)$ bundle of (relative) k-jets of $\mathrm{ev}_n^* \mathcal{O}(Y)$, there is a section σ of the line bundle $\mathcal{P}_{m_n}/\mathcal{P}_{m_n-1}$ on $\bar{M}_{(m_1,\dots,m_n)}$ whose vanishing describes precisely the condition that the map f of a stable map $(C, x_1, \dots, x_n, f)$ has multiplicity (at least) $m_n + 1$ to Y at x_n. The first Chern class of this line bundle is $\mathrm{ev}_n^* Y + m_n \psi_n$. (In fact, this is the idea how equation (32) is proven.) The refined version of equation (32) now states that this equation also holds in the Chow group of the zero locus of the section σ on $\bar{M}_{(m_1,\dots,m_n)}$.

4.2. The components of $\bar{M}_{(2,2,2,2,2)}$

Having just computed the invariant N_d, we will now study its enumerative significance. To do this, we have to identify the components of $\bar{M}_{(2,2,2,2,2)}$ and compute their virtual fundamental classes. We will assume from now on that Y is generic of degree $d \geq 5$. The equation of Y is $F = \sum_I a_I z^I = 0$, where I runs over all multi-indices (i_0, i_1, i_2) with $i_0 + i_1 + i_2 = d$, and z_0, z_1, z_2 are the homogeneous coordinates on $\mathbb{P}^2$. We start with irreducible stable maps whose image is a smooth conic.

LEMMA 4.2.1. *Let $m_1, \dots, m_n$ be non-negative integers such that $\sum m_i \leq 10$, and let $C \in \bar{M}_{(m_1,\dots,m_n)}$ be an irreducible stable map that is not a double cover of a line. Then:*

(i) *The moduli space $\bar{M}_{(m_1,\dots,m_n)}$ is smooth of dimension $5 - \sum_i (m_i - 1)$ at C (which is the expected dimension).*
(ii) *If this expected dimension is negative, then there is no such point C.*

PROOF. The plane degree-d curves are parametrized by a projective space $\mathbb{P}^D$ with $D = \frac{1}{2} d(d+3)$, whose coordinates are the coefficients a_I of F. Let $Z \subset \bar{M}_{0,n}(\mathbb{P}^2, 2) \times \mathbb{P}^D$ be the closed substack of pairs $((C, x_1, \dots, x_n, f), Y)$ such that the pull-back by f of the equation of Y vanishes at the points x_i to order m_i for all i. We claim that Z is smooth of the expected dimension at every point $((C, x_1, \dots, x_n, f), Y)$ such that C is irreducible and f is not a double cover of a line.

To prove this, we have to show that the matrix of derivatives of the equations describing Z has maximal rank at the given point $((C, x_1, \dots, x_n, f), Y)$. By a projective coordinate transformation of $\mathbb{P}^2$ and choosing homogeneous coordinates on $C \cong \mathbb{P}^1$, we can assume that the map f is given by $(s : t) \mapsto (s^2 : st : t^2)$, and the n marked points are $(1 : \lambda_i)$ with pairwise distinct λ_i.

Let us now write down the derivatives of the multiplicity equations with respect to the first $m := \sum_i m_i$ of the variables $a_{(d,0,0)}, a_{(d-1,1,0)}, a_{(d-1,0,1)}, a_{(d-2,1,1)}, a_{(d-2,0,2)}, a_{(d-3,1,2)}, a_{(d-3,0,3)}, a_{(d-4,1,3)}, a_{(d-4,0,4)}, a_{(d-5,1,4)}$ (remember $m \leq 10$ and $d \geq 5$). These coordinates are chosen to be the coefficients of $s^{2d-i} t^i$ for $i = 0, \dots, 9$ when we substitute the map f into F.

Multiplicity m_i at the point $(s : t) = (1 : \lambda_i)$ means that $F|_{s=1, t=\lambda_i+\varepsilon}$ has no ε terms of order less than m_i. So the rows of the derivative matrix are just $(\binom{i}{k} \lambda_j^{i-k})_{i=0,\dots,m-1}$, for $0 \leq k < m_j$ and $1 \leq j \leq n$. For example, for $m_1 = m_2 = m_3 = m_4 = m_5 = 2$ we get the matrix

$$\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^9 \\ 0 & 1 & 2\lambda_1 & \cdots & 9\lambda_1^8 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \lambda_5 & \lambda_5^2 & \cdots & \lambda_5^9 \\ 0 & 1 & 2\lambda_5 & \cdots & 9\lambda_5^8 \end{pmatrix}.$$

By subtracting λ_1 times the i-th column from the $(i+1)$-st column for $1 \leq i < m$ and using induction, we see that the determinant is $\prod_{i<j}(\lambda_i - \lambda_j)^{m_i m_j}$. In particular, it is not zero, so Z is smooth of the expected dimension at $((C, x_1, \dots, x_n, f), Y)$.

By remark 4.1.4, the statement of the lemma is just that the fiber of Z over a general point of $\mathbb{P}^D$ is smooth of the expected dimension around a point considered above. This follows now from the Bertini theorem. □

Using remark 4.1.4 again, the following two corollaries are immediate.

COROLLARY 4.2.2. *Every irreducible stable map in $\bar{M}_{(m_1,\dots,m_n)}$ whose image in $\mathbb{P}^2$ is a smooth conic lies in a unique irreducible component of $\bar{M}_{(m_1,\dots,m_n)}$ of the expected dimension. The virtual fundamental class of this component is equal to the usual one.*

COROLLARY 4.2.3. *The number of smooth plane conics 5-fold tangent to Y is finite. We denote it by n_d.*

We will now study the additional non-enumerative contributions to the virtual invariant N_d.

LEMMA 4.2.4. *The moduli space $\bar{M}_{(2,2,2,2,2)}$ has the following connected components:*

(A) $5!$ *points for every smooth conic 5-fold tangent to Y.*
(B) $5! \cdot (d-4)(d-5)$ *points for every bitangent of Y, corresponding to double covers of the bitangent, with marked points as in the picture below on the left, i.e. the map is ramified over two transverse intersection points of the bitangent with Y, and the five marked points are the two ramification points, both inverse image points of one bitangency point, and one inverse image point of the other bitangency point.*

(C) $\frac{1}{4} \cdot 5! \cdot (d-4)$ *smooth rational curves for every bitangent of Y, corresponding to double covers of the bitangent, with marked points as in the picture above on the right, i.e. (for a general stable map in this smooth rational curve) the map is ramified over one transverse intersection point of the bitangent with Y and one other arbitrary point, and the five marked points are the first ramification point, and the four inverse image points of the two bitangency points.*

PROOF. Case 1: the image of the stable map is a smooth conic. If the five marked points are distinct in $\mathbb{P}^2$, we get the components (A) by lemma 4.2.1, with the 5! corresponding to the labeling of the marked points. If two of the points coincide in $\mathbb{P}^2$ (i.e. lie on a contracted component of the stable map), then by the description of $\bar{M}_{(2,2,2,2,2)}$ in remark 4.1.5 the conic must have contact of order (at least) 4 to Y at this point, i.e. it lies in $\bar{M}_{(4,2,2,2)}$. But this space is empty by lemma 4.2.1.

Case 2: the image of the stable map is a union of two (distinct) lines. It is easy to see that the conditions of remark 4.1.5 cannot be satisfied in this case.

Case 3: the stable map is a double cover of a line. There are six possible points of tangency to Y: the four inverse image points of the bitangency points, and the two ramification points if they are mapped to points of Y. For the stable map in $\bar{M}_{(2,2,2,2,2)}$ we can pick any five of these six points. If we leave out one of the points over the bitangency points, we arrive at the components (B), otherwise we get the components (C).

In case (B) we get a factor of 5! for the choice of labeling of the marked points, a factor of $\binom{d-4}{2}$ for the choice of two transverse intersection points of Y with the bitangent, and a factor of 2 for the choice of bitangency point over which we take only one inverse image point to be marked. In case (C) the second ramification point is not fixed, so we get one-dimensional families of such curves. Every such family has a 2:1 map to the bitangent given by the image of the moving ramification point; the two stable maps in a fiber of this map differ by exchanging the marked points over one bitangency point. The map is simply ramified over the two stable maps where the moving ramification point is one of the bitangency points. Hence every such family is a $\mathbb{P}^1$. The number of such families is $d-4$ (for the choice of transverse intersection point of the bitangent with Y) times $\frac{1}{4} \cdot 5!$ (for the labeling of the marked points, taking into account that exchanging the marked points over the bitangency points does not give us a new family). □

Of course, the virtual fundamental class of $\bar{M}_{(2,2,2,2,2)}$ splits naturally into a sum of virtual fundamental classes on each of the connected components that we have just identified. As it is well-known that the number of bitangents of Y is $\frac{1}{2}d(d+3)(d-2)(d-3)$ (see e.g. [**H**] exercise IV.2.3 f), we get the following corollary.

COROLLARY 4.2.5. *We have*

$$N_d = n_d + \frac{1}{2}d(d+3)(d-2)(d-3)(d-4)(d-5)b_d + \frac{1}{8}d(d+3)(d-2)(d-3)(d-4)c_d,$$

where b_d is the degree of the part of the virtual fundamental class of $\bar{M}_{(2,2,2,2,2)}$ supported on the point $\bar{M}_B \in \bar{M}_{(2,2,2,2,2)}$ below, and c_d is the corresponding degree supported on the smooth rational curve $\bar{M}_C \subset \bar{M}_{(2,2,2,2,2)}$ below.

4.3. Computation of the virtual fundamental classes

In this section we will do the necessary computations to determine the numbers b_d and c_d of corollary 4.2.5. Most of them are simple calculations in local coordinates, so we will only sketch these parts and leave the details to the reader.

The computation of b_d is quite simple, as the component $\bar{M}_B$ of $\bar{M}_{(2,2,2,2,2)}$ has the expected dimension.

LEMMA 4.3.1. $b_d = 1$ *for all d.*

PROOF. By remark 4.1.4 we just have to show that the 10 equations of vanishing of the 1-jets of $\mathrm{ev}_i^* F$ for $i = 1,\dots,5$ locally cut out the point $\bar{M}_B$ in $\bar{M}_{0,5}(\mathbb{P}^2, 2)$ scheme-theoretically with multiplicity 1. Let us start with the 1-jets at the points x_1, x_2, x_3, i.e. with the space $\bar{M}_{(2,2,2)}$. We can choose the coordinates on $\mathbb{P}^2$ such that the bitangent is $\{z_2 = 0\} \subset \mathbb{P}^2$ and the bitangency points are $(1:0:0)$ and $(0:1:0)$. This means that

$$\begin{aligned} &a_{(d,0,0)} = a_{(d-1,1,0)} = 0, \quad a_{(d-1,0,1)}, a_{(d-2,2,0)} \neq 0 \qquad \text{(tangency at } (1:0:0)),\\ &a_{(0,d,0)} = a_{(1,d-1,0)} = 0, \quad a_{(0,d-1,1)}, a_{(2,d-2,0)} \neq 0 \qquad \text{(tangency at } (0:1:0)). \end{aligned}$$

Moreover, we can choose coordinates on the source $\mathbb{P}^1$ such that the stable map $\bar{M}_B$ is given by $(s:t) \mapsto (s^2 - t^2 : st : 0)$, and the marked points are $x_1 = (1:0)$,

$x_2 = (0:1), x_3 = (1:1)$. Local coordinates of $\bar{M}_{0,3}(\mathbb{P}^2,2)$ around this point are then $\varepsilon_1,\dots,\varepsilon_8$, where the stable map is given by

$$(s:t) \mapsto (s^2-t^2+\varepsilon_1 s^2+\varepsilon_2 st+\varepsilon_3 t^2 : st+\varepsilon_4 s^2+\varepsilon_5 t^2 : \varepsilon_6 s^2+\varepsilon_7 st+\varepsilon_8 t^2).$$

The three tangency equations are that $F|_{s=1,t=\xi}$, $F|_{s=\xi,t=1}$, and $F|_{s=1,t=1+\xi}$ have no constant and linear ξ terms. It is an easy computation to see that these 6 equations, linearized in the ε_i, give

$$\varepsilon_1+\varepsilon_2+\varepsilon_3 = \varepsilon_4 = \varepsilon_5 = \varepsilon_6 = \varepsilon_7 = \varepsilon_8 = 0,$$

so $\bar{M}_{(2,2,2)}$ is smooth of dimension 2 at the point $\bar{M}_B$ (with the points x_4 and x_5 forgotten).

Now let us consider the two other tangency conditions at the points x_4 and x_5. As the four coordinates of $\bar{M}_{(2,2,2,0,0)}$ around $\bar{M}_B$ we can choose the images of the ramification points and a point in the domain of the stable map in the neighborhood of each ramification point. Considering only one ramification point for now, the two corresponding local coordinates are $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_2$, where the stable map is given locally in affine coordinates as $t \mapsto t^2+\tilde{\varepsilon}_1$, and the marked point is $t=\tilde{\varepsilon}_2$. Tangency means that the constant and linear ξ terms of $(\tilde{\varepsilon}_2+\xi)^2+\tilde{\varepsilon}_1$ vanish, so linearly in the $\tilde{\varepsilon}_i$ we get $\tilde{\varepsilon}_1=\tilde{\varepsilon}_2=0$. The same is true for the other ramification point, so we see that $\bar{M}_B$ is a smooth point of $\bar{M}_{(2,2,2,2,2)}$. □

To study the space $\bar{M}_C$, we need a lemma that tells us how the stable maps in $\bar{M}_C$ can be deformed if we relax some of the multiplicity conditions.

LEMMA 4.3.2. *Let H be a line in $\mathbb{P}^2$, and let $P \in H$ be a point where Y is simply tangent to H. Let $\mathcal{C} = (C,x_1,x_2,f) \in \bar{M}_{(2,2)}$ be a (possibly reducible) double cover of H, such that $f^{-1}(P) = \{x_1,x_2\}$. Then every stable map in $\bar{M}_{(2,2)}$ in a neighborhood of $\mathcal{C}$ is also a double cover of a (maybe different) line.*

PROOF. It is obvious that $\mathcal{C}$ cannot be deformed into a union of two distinct lines in $\bar{M}_{(2,2)}$. So we have to show that $\mathcal{C}$ cannot be deformed to an irreducible smooth conic in $\bar{M}_{(2,2)}$.

We use the classical space of complete conics (which is isomorphic to $\bar{M}_{0,0}(\mathbb{P}^2,2)$). Recall that this space is the closure in $\mathbb{P}^5 \times (\mathbb{P}^5)^\vee$ of the set $(C,C^\vee)$, where C is an irreducible conic and $C^\vee$ its dual. For our given point $\mathcal{C}$ (with the two marked points forgotten for the moment), C is the double line H, and $C^\vee$ is the union of the two lines in $\mathbb{P}^\vee$ that correspond to the two ramification points of f in $\mathbb{P}^2$. Assume that we can deform $(C,C^\vee)$ in the space of complete conics to an irreducible conic that is still tangent to Y at two points in the neighborhood of P. In particular, we would then deform $C^\vee$ to an irreducible conic that is tangent to the dual of Y (or more precisely if H is a bitangent of Y: to the branch of the dual of Y that corresponds to the point P) at two points in this neighborhood. By the continuity of intersection products this means that both lines of $C^\vee$ must actually be the line corresponding to the point P. Hence both ramification points of f would have to be P. This means that C must have a contracted rational component over P, in contradiction to the assumption $f^{-1}(P) = \{x_1, x_2\}$. □

We want to reduce the computation of c_d to spaces that have the expected dimension. To do this we use equation (33) from example 4.1.7. Note that by remark 4.1.12 this equation is true in the Chow group of the geometric zero locus of the section σ (that describes the tangency condition and whose zero locus has class $\mathrm{ev}_5^* Y + \psi_5$), so it makes sense to restrict the equation to a connected component of this zero locus. Using lemma 4.3.2 it is easy to see that $\bar{M}_C$ is such a connected component, so we will restrict equation (33) to $\bar{M}_C$ and denote this restriction by $|_{\bar{M}_C}$.

Note first that on $\bar{M}_C$ the point x_5 can only come close to x_4, but never to the other three marked points. Hence equation (33) restricted to $\bar{M}_C$ reads

$$\big((\mathrm{ev}_5^* Y + \psi_5) \cdot [\bar{M}_{(2,2,2,2,1)}]\big)\,|_{\bar{M}_C} = c_d + 3\,[\bar{M}_{(2,2,2,3)}]|_{\bar{M}_C}. \qquad (34)$$

Let us first compute the virtual fundamental classes occurring in this equation.

LEMMA 4.3.3.

(i) *The (one-dimensional) virtual fundamental class of $\bar{M}_{(2,2,2,2,1)}$ on $\bar{M}_C$ is twice the usual one.*
(ii) *The degree of the (zero-dimensional) virtual fundamental class of $\bar{M}_{(2,2,2,3)}$ on $\bar{M}_C$ is 3.*

PROOF. We will only sketch the computations.

(i): It is enough to do the computation at a general point of $\bar{M}_C$. We have seen in the proof of lemma 4.3.1 that $\bar{M}_{(2,2,2,2,0)}$ is smooth of dimension 2 at a general point of $\bar{M}_C$. But requiring multiplicity 1 at the point x_5 gives us a factor of 2 (i.e. $\mathrm{ev}_5^* Y$ cuts out $\bar{M}_C$ in $\bar{M}_{(2,2,2,2,0)}$ with multiplicity 2) because x_5 lies on a tangency point of the stable map with Y.

(ii): The only point of $\bar{M}_{(2,2,2,3)}$ in $\bar{M}_C$ is the stable map in the following picture on the left:

Its multiplicity can be computed using remark 4.1.4. Let us study the space $\bar{M}_{(2,2,2,2)}$ at this point first. By lemma 4.3.2 every stable map in $\bar{M}_{(2,2,2,2)}$ in a neighborhood of this point must also be a double cover of a line. It follows easily that, locally around this point, $\bar{M}_{(2,2,2,2)}$ is reducible, with two smooth one-dimensional components coming together: one of them is keeping the image line $f(C)$ to be the bitangent and moving the ramification point at x_4, while keeping x_4 on an inverse image point of the bitangency point (picture in the middle). A local coordinate for this component is ε, where the stable map is given locally in affine coordinates as $t \mapsto z = t(t+\varepsilon)$, and the marked point x_4 has coordinate $t = 0$. Now the equation F of Y vanishes on the bitangent with multiplicity 2 in z, so this equation pulled back to the curve is locally $t^2(t+\varepsilon)^2$. Its t^2 coefficient vanishes to order 2 in ε, so this component contributes 2 to the virtual fundamental class of $\bar{M}_{(2,2,2,3)}$. The other component is deforming the line in $\mathbb{P}^2$ away from the bitangent, with marked points as in the picture above on the right. Requiring multiplicity 3 at x_4 now restricts the line back to the bitangent with multiplicity 1. Hence the total degree of the virtual fundamental class of $\bar{M}_{(2,2,2,3)}$ on $\bar{M}_C$ is $2+1=3$. □

To evaluate the left hand side of equation (34) it is not enough to compute the integral of $\mathrm{ev}_5^* Y + \psi_5$ on $\bar{M}_C$. There may also be contributions from components of $\bar{M}_{(2,2,2,2,1)}$ that just intersect $\bar{M}_C$ if the section σ above (whose zero locus has class $\mathrm{ev}_5^* + \psi_5$) vanishes on them at a point of $\bar{M}_C$. Let us compute these contributions.

LEMMA 4.3.4. *The only component Z of $\bar{M}_{(2,2,2,2,1)}$ that meets $\bar{M}_C$ but is not $\bar{M}_C$ itself corresponds to double covers of simple tangent lines of Y, with ramification and marked points as in the picture on the left:*

Its virtual fundamental class is equal to the usual one. It intersects $\bar{M}_C$ in the point $\mathcal{C}$ in the picture on the right. The section σ on Z vanishes at $\mathcal{C}$ with multiplicity 4.

PROOF. By lemma 4.3.2 (applied to the two marked points x_1 and x_2) no stable map in $\bar{M}_C$ can be deformed into an element in $\bar{M}_{(2,2)}$ that is not itself a double cover of a line. So the only possible deformation is that we still have a double cover, however not of the bitangent but rather a nearby line. It is now easy to see that such a deformation is only possible for the stable map $\mathcal{C} \in \bar{M}_C$ in the picture above on the right, and that the deformation has to be the one in the picture on the left.

The computation of the virtual fundamental class on this component Z is completely analogous to similar calculations in previous lemmas and is therefore omitted. It remains to compute the order of vanishing of the section σ on Z at $\mathcal{C}$. Let us forget for a moment that the stable map splits off a contracted rational component at this point. Let $P = f(x_4) = f(x_5)$ be the bitangency point of $\mathcal{C}$. Choose local affine coordinates z_1, z_2 of $\mathbb{P}^2$ around P such that the local equation of Y is $z_2 = z_1^2 + O(z_1^3)$, and $z_2 = 0$ is the bitangent. As a local coordinate for Z around $\mathcal{C}$ we can choose ε, where the stable map is given locally around P by $t \mapsto (z_1 = t^2 - \frac{1}{2}\varepsilon^2, z_2 = \frac{1}{4}\varepsilon^4 + O(\varepsilon^5) + t^2 O(\varepsilon^4))$, and the marked points x_4 and x_5 are $t = 0$ and $t = \varepsilon$. (Note that this stable map is still a double cover of a line, $t = 0$ is a ramification point that maps to Y, and $t = \varepsilon$ another point that maps to Y.)

Now actually the stable map splits off a rational contracted component at $\mathcal{C}$, which corresponds to blowing up the point $(\varepsilon = 0, t = 0)$ in our family of stable maps. Hence the true local coordinates of this family of stable maps are the coordinates of this blow-up, i.e. t/ε and ε instead of t and ε. So the marked points x_4 and x_5 have coordinates $t = 0$ and $t/\varepsilon = 1$; in particular they do not coincide any more at $\varepsilon = 0$.

The vanishing of the section σ is the condition of tangency of f to Y at x_5. So to compute its order of vanishing at $\mathcal{C}$ we have to look at the linear ξ coefficient of the equation of Y evaluated at the point $t/\varepsilon = 1 + \xi$, i.e. of

$$z_2 - z_1^2 + O(z_1^3) = \frac{1}{4}\varepsilon^4 - (\frac{1}{2}\varepsilon^2 + \xi\varepsilon^2)^2 + O(\varepsilon^5) = -\xi\varepsilon^4 - \xi^2\varepsilon^4 + O(\varepsilon^5).$$

Hence the linear ξ coefficient vanishes with multiplicity 4 at $\varepsilon = 0$, which proves the lemma. □

We can now assemble the results of our local calculations to compute the number c_d.

LEMMA 4.3.5. *$c_d = -1$ for all d.*

PROOF. We will evaluate equation (34)

$$\left((\mathrm{ev}_5^* Y + \psi_5) \cdot [\bar{M}_{(2,2,2,2,1)}]\right)|_{\bar{M}_C} = c_d + 3\,[\bar{M}_{(2,2,2,3)}]|_{\bar{M}_C}.$$

The right hand side is $c_d + 9$ by lemma 4.3.3 (ii). The left hand side gets a contribution from the components of $\bar{M}_{(2,2,2,2,1)}$ that intersect $\bar{M}_C$, and a contribution from $\bar{M}_C \subset \bar{M}_{(2,2,2,2,1)}$ itself. The former is 4 by lemma 4.3.4. The latter is twice the degree of $\mathrm{ev}_5^* Y + \psi_5$ on $\bar{M}_C$ by lemma 4.3.3 (i). Note that the degree of $\mathrm{ev}_5^* Y$ on $\bar{M}_C$ is zero, as the image point of x_5 is fixed in $\bar{M}_C$.

To compute the integral of ψ_5 on $\bar{M}_C$ we will give a section of the cotangent line bundle L_5 and compute its zero locus. Let z be a local coordinate around the bitangency point $f(x_4) = f(x_5)$. Then $f^* dz$ defines a section of L_5. This section vanishes only at the point where the moving ramification point comes to x_4 and x_5, i.e. at the point C in the picture of lemma 4.3.4 on the right. The computation of the order of vanishing is very similar to the calculation in lemma 4.3.4. Ignoring the fact that C splits off a rational contracted component for x_4 and x_5, a local coordinate for $\bar{M}_C$ around C is ε, where the stable map is given locally by $t \mapsto z = t(t-\varepsilon)$, the points x_4 and x_5 are $t = 0$ and $t = \varepsilon$, and the moving ramification point is at $t = \frac{1}{2}\varepsilon$. Now, as in the proof of the previous lemma, taking into account the contracted rational component of C means that we have to blow up the point $(t = 0, \varepsilon = 0)$, and the coordinates are actually t/ε and ε. Now we see that

$$f^* dz = \frac{\partial}{\partial \frac{t}{\varepsilon}} \left(\varepsilon^2 \cdot \frac{t}{\varepsilon}\left(\frac{t}{\varepsilon} - 1\right)\right) d\frac{t}{\varepsilon} = \varepsilon^2 \left(2\frac{t}{\varepsilon} - 1\right) d\frac{t}{\varepsilon},$$

which vanishes with multiplicity 2 in ε around 0 at the point x_5. Hence the integral of ψ_5 over $\bar{M}_C$ is 2.

Putting everything together we get $4 + 2 \cdot 2 = c_d + 9$, and therefore $c_d = -1$. □

We can now insert the values for N_d, b_d and c_d from proposition 4.1.10, lemma 4.3.1, and lemma 4.3.5, respectively, into the equation from lemma 4.2.5, and get the following final result.

COROLLARY 4.3.6. *For $d \geq 5$ the enumerative number of conics 5-fold tangent to Y is*

$$n_d = \frac{1}{5!} d\,(d-3)\,(d-4)\,(d^7 + 12d^6 - 18d^5 - 540d^4 + 251d^3 + 5712d^2 - 1458d - 14580).$$

REMARK 4.3.7. The first few values of n_d are given in the following table.

d	5	6	7	8	9	10
n_d	2015	70956	684222	3878736	16137873	54575640

4.4. Application to rational curves on K3 surfaces

Let X be a K3 surface, and let $\beta \subset H_2(X,\mathbb{Z})$ be the class of a holomorphic curve in X. The moduli space of stable maps to X of class β has virtual dimension -1, so there is no corresponding Gromov-Witten invariant of X. However, we have chosen X such that it contains rational curves of class β, and we would like to count them. The reason for the mismatch in the virtual dimension is that, in the space of all K3 surfaces, only a 1-codimensional subset of K3 surfaces contains curves in the class β at all. So, if $\mathcal{X}$ is a general 1-dimensional family of K3 surfaces with X as central fiber, and $\tilde{\beta} \in H_2(\mathcal{X},\mathbb{Z})$ is the class induced by β via the inclusion $X \subset \mathcal{X}$, then the only rational curves in $\mathcal{X}$ of class $\tilde{\beta}$ are in fact curves of class β in X. But now the virtual dimension of the space of stable maps to $\mathcal{X}$ of class β is 0, hence there is a corresponding Gromov-Witten invariant. This invariant counts curves in $\mathcal{X}$ of class $\tilde{\beta}$ and therefore curves in X of class β; so we would like to call this number n_β "the number of rational curves in X of class β". A rigorous definition of the invariant n_β of X along these lines has been given in [**BL**]. The number n_β does not depend on a family $\mathcal{X}$ chosen to define it.

The numbers n_β have been computed in various papers ([**BL, Bv, Gö, YZ**]) under the assumption that the class β is primitive, i.e. not a non-trivial multiple of a smaller integral homology class. The result is that n_β is equal to the q^d coefficient in the series

$$G(q) = \prod_{i>0} \frac{1}{(1-q^i)^{24}} = 1 + 24q + 324q^2 + 3200q^3 + 25650q^4 + 176256q^5 + \cdots$$
$$=: \sum_{d\geq 0} G_d q^d.$$

where $d = \frac{1}{2}\beta^2 + 1$. It is not known yet what the numbers are if β is not primitive.

The results of this chapter allow us to compute the number n_β explicitly in a case where β is not primitive. Let $Y \subset \mathbb{P}^2$ be a general sextic curve, and let $\pi : X \to \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ branched along Y. It is well-known that X is a K3 surface. Let us start by considering curves on X in the (primitive) class $\beta = \pi^*\ell$, where ℓ is the class of a line. The pull-back of a general line in $\mathbb{P}^2$ will be a 2:1 cover of $\mathbb{P}^1$, branched along the 6 intersection points of Y with the line, hence it is a curve of genus 2. We get a rational curve (with 2 nodes) on Y as a pull-back of a line if and only if the line is a bitangent of Y: the pull-back is then again a 2:1 cover, but with 2 nodes (the bitangency points), and only 2 ramification points (the remaining 2 intersection points of Y with the line). So we see that $n_{\pi^*\ell}$ has to be the number of bitangents of Y. In fact, this number is 324 (see e.g. [**H**] exercise IV.2.3 f), which is equal to G_2 (note that $\frac{1}{2}(\pi^*\ell)^2 + 1 = 2$).

Now let us consider rational curves in X of class $2\pi^*\ell$, i.e. pull-backs of conics — this class is not primitive any more. The pull-back of a general conic will be a 2:1 cover of the conic ramified at 12 points, so it is a curve of genus 5. We can get rational curves in the following ways:

(i) Pull-backs of (smooth) conics that are 5-fold tangent to Y. These will be 2:1 covers of the conic, with 5 nodes over the tangency points, and only 2 ramification points (the remaining 2 intersection points of Y with the conic). By corollary 4.3.6, there are 70956 such curves.

(ii) Pull-backs of unions of two distinct lines: these give a rational curve only if the pull-backs of both lines are rational, i.e. they are both bitangents. The pull-back is then a union of two (2-nodal) rational curves on X that intersect in 2 points. To make this into a rational stable map we can glue these two components at either intersection point. Hence there are $2 \cdot \binom{324}{2} = 104652$ such stable maps.

(iii) Double covers of pull-backs of a line, necessarily again of a bitangent. The pull-back of such a bitangent is a 2-nodal rational curve C. There are two possible ways of double covers of such a curve:
  (a) Double covers that factor through the normalization of C. The space of these curves is the same as that of double covers of a smooth rational curve; it has dimension 2.
  (b) Double covers that do not factor through the normalization. They have two components that are both mapped to C with degree 1, and glued over one of the nodes of C in such a way that locally around this node the morphism of the stable map is an isomorphism onto C. There are $2 \cdot 324 = 648$ such curves.

Adding up just the numbers from (i), (ii), and (iii b), we get

$$70956 + 104652 + 648 = 176256,$$

which is exactly G_5 (and $\frac{1}{2}(2\pi^*\ell) + 1 = 5$). So we see that, for our non-primitive class β, the corresponding invariant from the series $G(q)$ does give the correct number, *except* for a correction (iii a) for double covers of curves of class $\frac{1}{2}\beta$ that factor through the normalization of these curves. Let us compute what this correction term is.

LEMMA 4.4.1. *With notations as above, the double covers of type (iii a) of the pull-back of a bitangent contribute* $\frac{1}{8}$ *to the invariant* $n_{2\pi^*\ell}$.

PROOF. Let $D \cong \mathbb{P}^1$ be the normalization of the nodal rational curve in X. The moduli space of the double covers that factor through the normalization is then just

$\bar{M}_{0,0}(D,2)$, which has dimension 2. As the normal bundle of the (local) immersion $D \to X$ is $O(-2)$, the rank-3 obstruction bundle for the corresponding Gromov-Witten invariant would be $R^1\pi_* f^* O(-2)$, where $\pi : \bar{M}_{0,1}(\mathbb{P}^1,2) \to \bar{M}_{0,0}(\mathbb{P}^1,2)$ is the forgetful map and $f : \bar{M}_{0,1}(\mathbb{P}^1,2) \to \mathbb{P}^1$ the evaluation.

As explained above, the K3-invariants of X are defined as the ordinary Gromov-Witten invariants of a 1-dimensional family $\mathcal{X}$ of K3 surfaces in which X is the only surface that contains rational curves in the given homology class. This means that the obstruction bundle for the K3 invariants is obtained from the usual Gromov-Witten obstruction bundle by taking the quotient with $\pi_* f^* N_{X/\mathcal{X}} = \pi_* f^* O = O$. So the integral that we want to compute is

$$c_{top}(R^1\pi_* f^* O(-2)/O) \cdot [\bar{M}_{0,0}(\mathbb{P}^1,2)].$$

This is easily done: from the two exact sequences on $\bar{M}_{0,1}(\mathbb{P}^2,2)$

$$0 \to f^* O(-1) \to f^* O \to f^* O_P \to 0$$
$$0 \to f^* O(-2) \to f^* O(-1) \to f^* O_P \to 0$$

(where $P \in \mathbb{P}^1$ is a point) we get the exact sequences of vector bundles on $\bar{M}_{0,0}(\mathbb{P}^2,2)$

$$0 \to O \to \pi_* f^* O_P \to R^1\pi_* f^* O(-1) \to 0$$
$$0 \to \pi_* f^* O_P \to R^1\pi_* f^* O(-2) \to R^1\pi_* f^* O(-1) \to 0$$

from which it follows that

$$c_2(R^1\pi_* f^* O(-2)/O) = c_2(R^1\pi_* f^*(O(-1) \oplus O(-1))),$$

i.e. the contribution of the double covers under consideration is the same as the double cover contribution for rational curves on Calabi-Yau threefolds with balanced normal bundle. This contribution is well-known to be $\frac{1}{8}$, see e.g. chapter 3. □

So we see that

$$n_{2\pi^*\ell} = G_5 + \frac{1}{8} \cdot G_2.$$

We conjecture that this pattern continues, i.e. that the numbers n_β receive multiple cover corrections similarly to the case of Gromov-Witten invariants of Calabi-Yau threefolds:

$$n_\beta = \sum_k \frac{1}{k^3} \cdot G_{\frac{1}{2}(\frac{\beta}{k})^2+1},$$

where the sum is taken over all $k > 0$ such that $\frac{\beta}{k}$ is an integral homology class.

CHAPTER 5

Relative Gromov-Witten invariants in higher genus

So far we have only considered Gromov-Witten invariants in genus zero. In this last chapter we want to generalize our earlier techniques to higher genus of the curves. As an example we will then apply these techniques to compute the elliptic Gromov-Witten invariants of a quintic threefold.

The first step in the generalization of our earlier techniques must of course be the construction of the moduli spaces of stable relative maps and their virtual fundamental classes in any genus. Recall that our construction of the virtual fundamental class for moduli spaces of rational stable relative maps in chapter 2 was quite ad hoc: we started from the case of $\mathbb{P}^n$ relative a hyperplane where the moduli spaces have the expected dimension, and then intersected this space with the virtual fundamental class of the moduli space of stable absolute maps to $X \subset \mathbb{P}^n$. But in higher genus the moduli spaces never have the expected dimension, not even for $\mathbb{P}^n$. Hence we cannot generalize our old trick to higher genus.

The solution to this problem has recently been given by Li [**Li1, Li2**]. He constructs moduli spaces of stable relative maps with virtual fundamental classes in any genus and for any smooth hypersurface Y in a projective manifold X. His construction uses certain blow-ups of the moduli spaces of definition 2.1.1 on which he can then define virtual fundamental classes using a suitable obstruction theory. Blowing the result down again one can then also get virtual fundamental classes on our old moduli spaces of definition 2.1.1. If we want to distinguish between our old spaces and the new blown-up ones we will call the former the moduli spaces of *collapsed* stable relative maps (see definition 5.1.10 and lemma 5.1.12).

The main property of the moduli spaces of stable relative maps proven in [**Li2**] is the so-called splitting formula. In our case at hand the idea of this formula is as follows. Let Z be the blow-up of $X \times \mathbb{P}^1$ in $Y \times \{0\}$. The natural projection morphism $Z \to \mathbb{P}^1$ has general fiber X and central fiber $X \cup_Y P$, where $P = \mathbb{P}(N_{Y/X}^\vee \oplus O_Y)$ is a $\mathbb{P}^1$-bundle over Y. We can therefore regard $X \cup_Y P$ as a degeneration of X.

The idea is now simply that the Gromov-Witten invariants of X and $X \cup_Y P$ should be the same (as Gromov-Witten invariants are constant in families), and that the invariants of $X \cup_Y P$ should be expressible in terms of invariants of the two factors X and P. More precisely, the splitting formula of [**Li2**] asserts that the (absolute) Gromov-Witten invariants of X are computable as a product of the relative Gromov-Witten invariants of X and P relative Y.

It is obvious how this picture is related to our degeneration methods of chapter 2: we simply have to push the splitting formula forward along the blow-down map $Z \to X \times \mathbb{P}^1$ (and then further to X). The stable relative maps to P are then projected down to (absolute) stable maps to Y. So we get relations between the absolute invariants of X, the absolute invariants of Y, and the relative invariants of X relative Y. In fact, we will see that in genus 0 the relations that we get are precisely the same as in chapter 2. In higher genus one gets similar relations, and one can hope that these relations can be used to compute Gromov-Witten invariants of Y from those of X in the same way as in section 2.5.

There is only one point in the above argument that needs some more explanation: we have to describe the projection map p from stable relative maps in P to stable absolute maps in Y explicitly, or in other words we must be able to compute the relative invariants of P relative Y in terms of the absolute invariants of Y. In fact, the major part of this chapter discusses this problem. We compute the push-forward by p of any intersection product of evaluation and cotangent line classes under the condition that this push-forward has (virtual) codimension 0 or 1 in the target space of stable absolute maps to Y. The main tool in the computation is the technique of virtual localization that has been established recently for moduli spaces of stable relative maps in [**GV**]. We show that these formulas together with the above ideas are sufficient to compute the elliptic Gromov-Witten invariants of the quintic three-fold from the invariants of $\mathbb{P}^4$. It is expected that similar reconstruction statements hold for other varieties as well, and that our methods can be extended to higher genus by a more careful analysis of the push-forward by p. We would also expect that the methods of chapter 3 can be extended at least to the elliptic case to arrive at a mirror type formula for the invariants of certain hypersurfaces.

Let us give a short outline of this chapter. In section 5.1 we review Li's construction of stable relative maps. We extend this construction to the so-called non-rigid stable maps to $\mathbb{P}^1$-bundles that have already been mentioned in [**GV**]. The most important technical property of relative invariants of $\mathbb{P}^1$-bundles that we need is the virtual push-forward property that we prove in section 5.2. It asserts that the push-forward by p of all relevant intersection products is zero if their dimension is bigger than the virtual dimension of the target, and a multiple of the virtual fundamental class if their dimension is equal to the virtual dimension of the target. We will then apply this theorem and its corollaries to compute the push-forwards by p explicitly in the cases when these push-forwards have codimension 0 (section 5.3) or 1 (section 5.4) in the target. As a simple consequence we reprove the algorithm of chapter 2 for the computation of Gromov-Witten invariants of very ample hypersurfaces in our new set-up. Finally, the results are applied to compute the elliptic Gromov-Witten invariants of the quintic threefold in section 5.5. To the best of our knowledge this is the first mathematical verification of these numbers. I have been informed by Li however that he has recently computed the same numbers using different methods [**Li3**].

5.1. Stable relative and non-rigid maps

Let us start by recalling Li's construction of stable relative maps [**Li1**]. The main idea of this new construction is that stable relative maps to X relative Y are in general *not* just stable maps to X with some properties, but rather stable maps to some degenerations of X. Let us introduce these degenerations first.

DEFINITION 5.1.1. Let Y be a smooth hypersurface in a smooth projective variety X. We denote by $P = \mathbb{P}(N_{Y/X}^\vee \oplus O_Y)$ the projective closure of the dual normal bundle of Y in X. It comes equipped with a natural $\mathbb{C}^*$ action that rescales the fibers by acting with weights 1 and 0 on $N_{Y/X}$ and O, respectively. The fixed point locus of this $\mathbb{C}^*$ action consists of two components: the zero section $Y_0 := \mathbb{P}(0 \oplus O) \cong Y \subset P$ and the infinity section $Y_\infty := \mathbb{P}(N_{Y/X}^\vee \oplus 0) \cong Y \subset P$. Note that the normal bundle of Y_0 (resp. Y_∞) in P is $N_{Y/X}^\vee$ (resp. $N_{Y/X}$).

For any $k \geq 0$ we define a normal crossing scheme X_k, called the **k-th degeneration of X**, as follows. It consists of $k+1$ irreducible components that will also be called **levels** and that are numbered from 0 to k. Level 0 is isomorphic to X, whereas all other levels are isomorphic to P. These components are glued transversally in the following way:

- we glue $Y \subset X$ in level 0 to $Y_0 \subset P$ in level 1,
- we glue $Y_\infty \subset P$ in level i to $Y_0 \subset P$ in level $i+1$ for $i = 1, \dots, k-1$.

The infinity section Y_∞ of X_k is defined to be the infinity section of the last level k.

Note that there is a projection morphism $\pi : X_k \to X$ by collapsing the fibers in all levels greater than 0. Moreover, the $\mathbb{C}^*$ action on the k copies of P makes the group $(\mathbb{C}^*)^k$ into a group of automorphisms of X_k. We will call these automorphisms the **allowed automorphisms** of X_k. When drawing pictures we will indicate their presence by two-sided arrows pointing along the fiber directions.

REMARK 5.1.2. Note that $X_0 = X$. Moreover, X_{k+1} is in fact a degeneration of X_k for all k in the following sense: if Z denotes the blow-up of $X_k \times \mathbb{P}^1$ in $Y_\infty \times \{0\}$ then $Z \to \mathbb{P}^1$ is a flat family with general fiber X_k and zero fiber X_{k+1}. The singular locus of X_k consists of k disjoint copies of Y.

We will now use these degenerations of X to construct stable relative maps. Following [**Li1**] we will only do this in the case when "all intersection points with Y are marked", i.e. if the sum of the prescribed local orders of contact is equal to the intersection product of Y with the homology class of the curve. The construction that we give here differs slightly from the one that we introduced in chapter 2 in the cases when both definitions are applicable. We will see in lemma 5.1.12 how the two constructions are related: in fact our old construction yields precisely the moduli spaces of *collapsed* stable relative maps that we will introduce in definition 5.1.10. So from now on we will refer to the construction of chapter 2 as *collapsed* stable relative maps.

DEFINITION 5.1.3. An **(n-pointed) pre-stable relative map** to X (relative Y) is an n-pointed pre-stable map $(C, x_1, \dots, x_n, f)$ to some degeneration X_k such that:

(i) No irreducible component of C maps entirely to $Y_\infty \subset X_k$ or to the singular locus of X_k.
(ii) Every point that maps to $Y_\infty \subset X_k$ is a marked point.

(iii) Every point of C that maps to the singular locus of X_k is a node with the property that the two local branches around the node map to the two different local components of X_k with the same orders of contact to the singular locus on both sides.

The integer k is then called the **level** of the pre-stable relative map. The level-i part of a pre-stable relative map is its restriction to the closed subset of C that maps to the level-i component of X_k.

A morphism $(C,x_1,\dots,x_n,f) \to (C',x'_1,\dots,x'_n,f')$ of n-pointed pre-stable relative maps of the same level k is a pair $(\varphi,\tilde{\varphi})$ where

- $\varphi : C \to C'$ is a morphism of the underlying curves,
- $\tilde{\varphi} : X_k \to X_k$ is an allowed automorphism in the sense of definition 5.1.1,

such that $\tilde{\varphi} \circ f \circ \varphi = f'$. In other words, *applying an allowed automorphism to X_k will result in an isomorphic pre-stable relative map*. Again we indicated this by the two-sided arrow in the picture above.

A pre-stable relative map is called **stable** if its group of automorphisms is finite. The **class** of a pre-stable relative map is defined to be the element $\pi_* f_* [C] \in H_2^+(X)$.

REMARK 5.1.4. A pre-stable relative map $(C,x_1,\dots,x_n,f)$ of level k is stable if and only if

(i) it is stable as a pre-stable map to X_k, i.e. every rational (resp. elliptic) component of C that is mapped to a point by f has at least three (resp. one) special points, and
(ii) for all $1 \le i \le k$ the level-i part is *not* invariant under the allowed isomorphisms of P, i.e. it is *not* a disjoint union of smooth rational components that are all multiple covers of fibers of P totally ramified over Y_0 and Y_∞ and with no special points away from Y_0 and Y_∞.

DEFINITION 5.1.5. Let $(C, x_1, \dots, x_n, f)$ be a pre-stable relative map of level k to X relative Y. For $i = 1, \dots, n$ we define the **multiplicity** α_i of the i-th marked point x_i to be the multiplicity of the point x_i in the divisor $f^* Y_\infty$. The collection of these multiplicities will be denoted $\alpha = (\alpha_1, \dots, \alpha_n)$.

EXAMPLE 5.1.6. The pre-stable relative map in the picture of definition 5.1.3 has multiplicities $\alpha = (\alpha_1, \dots, \alpha_5) = (0,0,1,1,1)$. Note that we must have $\sum_{i=1}^n \alpha_i = Y \cdot \beta$ for all pre-stable relative maps of class β.

DEFINITION 5.1.7. Fix integers $g, n \geq 0$, a homology class $\beta \in H_2^+(X)$, and a collection of non-negative integers $\alpha = (\alpha_1, \dots, \alpha_n)$ such that $\sum_i \alpha_i = Y \cdot \beta$. We denote by $\bar{\mathcal{M}}^Y_{g,\alpha}(X, \beta)$ the set of (isomorphism classes of) all stable relative maps (of any level) to X relative Y of genus g and class β whose multiplicities are α. If no confusion can result we will also denote this space by $\bar{\mathcal{M}}_{g,\alpha}(X, \beta)$.

THEOREM 5.1.8. *The moduli spaces $\bar{\mathcal{M}}^Y_{g,\alpha}(X, \beta)$ of stable relative maps are separated and proper Deligne-Mumford stacks of expected dimension*

$$\begin{aligned} \operatorname{vdim} \bar{\mathcal{M}}^Y_{g,\alpha}(X, \beta) &= \operatorname{vdim} \bar{M}_{g,n}(X, \beta) - \sum \alpha_i \\ &= -K_X \cdot \beta + (\dim X - 3)(1-g) + \sum_i (1 - \alpha_i). \end{aligned}$$

Moreover, there is a naturally defined virtual fundamental class $[\bar{\mathcal{M}}^Y_{g,\alpha}(X, \beta)]^{\text{virt}} \in A_*(\bar{\mathcal{M}}^Y_{g,\alpha}(X, \beta))$ of this dimension.

PROOF. See [**Li1**] and [**Li2**]. □

EXAMPLE 5.1.9. Note that the level is not fixed in the moduli spaces of stable relative maps. The idea is that a level is added whenever a component of the curve would otherwise lie in Y_∞ or the singular locus of X_k. The following picture illustrates this: if the marked point x_3 on the level-0 stable relative map on the left is moved towards the point x_2, the limit will be the level-1 stable relative map on the right. In this limit, x_2 and x_3 lie on a two-to-one cover of a fiber of P ramified over Y_0 and Y_∞. The position of the point x_3 on this cover is not fixed because of the allowed automorphisms of P. Note that the multiplicities $\alpha = (1,2,0)$ remain the same in the limit.

Let us now sketch how our new construction of stable relative maps is related to the old one from chapter 2.

DEFINITION 5.1.10. The projection maps $\pi : X_k \to X$ give rise to morphisms

$$\pi_* : \bar{\mathcal{M}}^Y_{g,\alpha}(X,\beta) \to \bar{M}_{g,n}(X,\beta)$$

that simply collapse all higher levels to the hypersurface Y. We denote the image of this morphism (with the reduced closed substack structure) by $\bar{M}^Y_{g,\alpha}(X,\beta)$ and call this the **moduli space of collapsed stable relative maps**. The virtual fundamental class of $\bar{M}_{g,\alpha}(X,\beta)$ is simply defined to be $\pi_*[\bar{\mathcal{M}}_{g,\alpha}(X,\beta)]^{\text{virt}}$.

REMARK 5.1.11. Note that $\bar{M}_{g,\alpha}(X,\beta)$ is not a "nice" moduli space in the usual sense: it does not represent a moduli functor, it does not have a naturally defined structure of stack (the choice of reduced substack structure in definition 5.1.10 is somewhat artificial), and there is no canonical obstruction theory on it that gives rise to its virtual fundamental class. Nevertheless, the spaces of collapsed stable relative maps are sometimes preferred for enumerative purposes for a number of reasons, e.g.

(i) they are subspaces of the well-known moduli spaces of stable maps that are reasonably easy to describe (see definition 2.1.1 and lemma 5.1.12),
(ii) they behave nicely for disconnected curves (see remark 5.1.13, proposition 5.2.8, and conjecture 5.2.9),
(iii) they are sufficient to apply the "splitting theorem" (see remark 5.1.14).

LEMMA 5.1.12. *Let $Y \subset X$ be a smooth hypersurface of a smooth projective variety X. Let $g,n \geq 0$ be non-negative integers, and let $\beta \in H_2^+(X)$ be a homology class. Pick non-negative integers $\alpha = (\alpha_1,\dots,\alpha_n)$ with $\sum_i \alpha_i = Y\cdot\beta$. Then the definitions 2.1.1 and 5.1.10 of $\bar{M}^Y_{g,\alpha}(X,\beta)$ agree. Moreover, if $g = 0$ and $Y \subset X$ is a very ample hypersurface then definitions 2.1.19 and 5.1.10 give rise to the same virtual fundamental class on $\bar{M}^Y_{g,\alpha}(X,\beta)$.*

PROOF. We will only sketch the argument and leave the details to the reader.

By definition, the statement about the spaces $\bar{M}^Y_{g,\alpha}(X,\beta)$ is purely set-theoretic. So let $(C,x_1,\dots,x_n,f)$ be a stable map to X of genus g and class β, and let $Z \subset C$ be a connected component of $f^{-1}(Y)$. By definition 2.1.1 the stable map will be in $\bar{M}^Y_{g,\alpha}(X,\beta)$ if and only if for all such Z we have

$$(f|_Z)^*Y = \sum_{i:x_i\in Z} \alpha_i x_i - \sum_j m^{(j)} y_j$$

in $A_0(Z)$, where $y_1,\dots,y_r$ are the intersection points of Z with the components of C not contained in Z, and $m^{(j)}$ are the multiplicities of f to Y at y_j at the branch of C that is not in Z. But by definition of rational equivalence this is precisely the condition that the morphism $f|_Z : Z \to Y$ lifts to a morphism $Z \to P$ with the required multiplicity conditions at Y_0 and Y_∞, which means that the stable map is in the image of the projection map $\pi : \bar{\mathcal{M}}_{g,\alpha}(X,\beta) \to \bar{M}_{g,\alpha}(X,\beta)$.

To prove the statement about the virtual fundamental classes, consider the embedding $i : X \hookrightarrow \mathbb{P}^N$ given by the complete linear system $|Y|$, and let $H \subset \mathbb{P}^N$ be the hyperplane such that $X \cap H = Y$ in $\mathbb{P}^N$. Note that we have a Cartesian diagram

$$\begin{array}{ccc} \bar{\mathcal{M}}^Y_{0,\alpha}(X,\beta) & \longrightarrow & \bar{\mathcal{M}}^H_{0,\alpha}(\mathbb{P}^N,d) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi'_*} \\ \bar{M}^Y_{0,\alpha}(X,\beta) & \longrightarrow & \bar{M}^H_{0,\alpha}(\mathbb{P}^N,d) \\ \downarrow & & \downarrow \\ \bar{M}_{0,n}(X,\beta) & \xrightarrow{i_*} & \bar{M}_{0,n}(\mathbb{P}^n,d) \end{array}$$

where $d = Y \cdot \beta$, and where all morphisms except π_* and π'_* are closed immersions. Let us compute the virtual fundamental class of $\bar{M}^Y_{0,\alpha}(X,\beta)$ using definition 5.1.10. By the explicit description of the obstruction theory of $\bar{\mathcal{M}}^Y_{g,\alpha}(X,\beta)$ in **[GV]** section 2.8 we see that $i^!_*[\bar{\mathcal{M}}^H_{g,\alpha}(\mathbb{P}^N,d)]^{\text{virt}} = [\bar{\mathcal{M}}^Y_{g,\alpha}(X,\beta)]^{\text{virt}}$. Moreover, $\pi'_*[\bar{M}^H_{0,\alpha}(\mathbb{P}^N,d)]^{\text{virt}}$ must be a multiple of $[\bar{M}^H_{0,\alpha}(\mathbb{P}^N,d)]^{\text{virt}}$ since the dimensions of these two cycles agree and the latter one is just the ordinary fundamental class of an irreducible space (see lemma 2.1.15 and lemma 2.1.16 (iii)). In fact, this multiple must be 1 since it is checked immediately (using again the description of the obstruction theory of $\bar{M}^H_{0,\alpha}(\mathbb{P}^N,d)$ in **[GV]** section 2.8) that the obstruction space vanishes at a general element in $\bar{M}^H_{0,\alpha}(\mathbb{P}^N,d)$. An application of **[F]** theorem 6.2a thus shows that

$$\begin{aligned} \pi_*[\bar{M}^Y_{0,\alpha}(X,\beta)]^{\text{virt}} &= \pi_* i^!_*[\bar{M}^H_{0,\alpha}(\mathbb{P}^N,d)]^{\text{virt}} \\ &= i^!_* p'_*[\bar{M}^H_{0,\alpha}(\mathbb{P}^N,d)]^{\text{virt}} \\ &= i^!_*[\bar{M}^H_{\alpha}(\mathbb{P}^N,d)]^{\text{virt}}, \end{aligned}$$

which is definition 2.1.19 of the virtual fundamental class of $\bar{M}^Y_{g,\alpha}(X,\beta)$. □

REMARK 5.1.13. Moduli spaces for stable curves, stable maps, and stable relative maps can of course be defined in the very same way for disconnected curves. In this case we have to fix the number of connected components as well as the data of the curves (number of marked points, genus, class, and maybe multiplicities) for every such component. To avoid overly clumsy notation we will usually denote such a moduli space simply by $\bar{M}_\Gamma$, $\bar{M}_\Gamma(X)$, or $\bar{M}^Y_\Gamma(X)$, where Γ denotes the collection of the data mentioned above.

For stable curves and stable maps it is obvious by definition that these moduli spaces of disconnected curves (and their virtual fundamental classes) are simply the products of the moduli spaces (resp. their virtual fundamental classes) for the individual connected components. The situation for stable relative maps is a little more subtle however, because the allowed automorphisms in the higher levels in definition 5.1.3 act globally on the curve and not on each connected component separately. The moduli spaces for disconnected stable relative maps are thus *not* just the products of the moduli spaces for the factors. This difference occurs only in higher levels though, so we would expect that the product property does hold for *collapsed* stable relative maps. In fact, we will prove this in proposition 5.2.8, at least in genus 1.

REMARK 5.1.14. The main property of stable relative maps is the "splitting theorem" proven in [**Li2**]. The idea of this theorem is to compute Gromov-Witten invariants of a smooth projective variety by degenerating it into a normal crossing scheme with two components intersecting transversally in a divisor. We will state it here only in the case that we will need later.

Let X be a smooth projective variety, and let $Y \subset X$ be a smooth hypersurface. Let Z be the blow-up of $X \times \mathbb{P}^1$ in $Y \times \{0\}$, so that the general fiber of the projection $Z \to \mathbb{P}^1$ is isomorphic to X, whereas the fiber over zero is the normal crossing scheme $X_1 = X \cup_Y P$. Now fix non-negative integers $g, n \geq 0$ and fix a homology class $\beta \in H_2^+(X)$. Let M be the moduli space of n-pointed stable maps of genus g to Z whose class is β in a (general) fiber of the morphism $Z \to \mathbb{P}^1$. Then M has a projection morphism to the base $\mathbb{P}^1$ as well, and the general fiber of this morphism is simply $\bar{M}_{g,n}(X,\beta)$. We can therefore think of the virtual fundamental class $[\bar{M}_{g,n}(X,\beta)]^{\text{virt}}$ as being a cycle in $A_*(M)$.

Now we consider the fiber of M over the zero point in $\mathbb{P}^1$. Roughly speaking, this is the space of stable maps to $X_1 = X \cup_Y P$, which should be expressible as a product of moduli spaces of stable maps to X and P. The precise formula is

$$[\bar{M}_{g,n}(X,\beta)]^{\text{virt}} = \sum_{\Gamma_1,\Gamma_2} m(\Gamma_1,\Gamma_2) \cdot [\bar{M}^Y_{\Gamma_1}(X)]^{\text{virt}} \boxtimes [\bar{M}^Y_{\Gamma_2}(P)]^{\text{virt}}$$

in $A_*(M)$, where we have used the following notation. The spaces $\bar{M}^Y_{\Gamma_1}(X)$ (resp. $\bar{M}^Y_{\Gamma_2}(P)$) are moduli spaces of (possibly disconnected) stable *relative* maps to X (resp. P) relative Y, where Γ_1 (resp. Γ_2) denotes the collection of the following data:

(i) the number of connected components of the stable relative maps,
(ii) the genus and homology class of all connected components (where every homology class must be non-zero),
(iii) for every connected component the subset of $\{1,\dots,n\}$ of the marked points lying on it, where all these points have multiplicity 0 (i.e. do not lie on Y),
(iv) for every connected component a collection of additional marked points $\{y_i\}$ lying on Y with associated positive multiplicities α_i.

The sum in the above formula is taken over all pairs of data (Γ_1,Γ_2) such that

- the glued stable map is connected and has the correct genus and homology class, and
- the additional marked points y_i are labeled on both the X and the P side by the same index set $\{1,\dots,r\}$ for some r, and the multiplicities α_i associated to these points agree on both sides.

To explain the rest of the notation in the above formula, the coefficient $m(\Gamma_1,\Gamma_2)$ is defined to be $\frac{\alpha_1\cdots\alpha_r}{r!}$ divided by the order of the automorphism group of the data (Γ_1,Γ_2). The notation $\boxtimes$ means that we take the moduli spaces of *collapsed* stable relative maps on both sides and take their fiber product over the r-fold evaluation map to Y at the points y_i.

We should mention that the same formula also holds if the starting curve (i.e. the left hand side in the above equation) is already disconnected. The only difference is then that we have to sum over all pairs (Γ_1,Γ_2) of data such that gluing the parts along Y reproduces the combinatorics of all these components correctly.

For our applications we will need another variant of stable relative maps. Intuitively speaking it corresponds to stable relative maps without the level-0 part, so that we consider maps to a chain of $\mathbb{P}^1$-bundles where we then fix multiplicities both to Y_0 and to Y_∞.

DEFINITION 5.1.15. Let L be a line bundle on a smooth projective variety Y, and denote by $X=\mathbb{P}(L\oplus O_Y)$ the projective closure of L. We can consider $Y\cong Y_\infty=\mathbb{P}(L\oplus 0)$ as a divisor in X. In definition 5.1.1 we then have $P=X$, so that P_k is a chain of $k+1$ copies of P. We denote by $Y_0 \subset P_k$ the zero section in the first copy of P.

An **(n-pointed) pre-stable non-rigid map** to P is an n-pointed pre-stable map $(C, x_1, \ldots, x_n, f)$ to some degeneration P_k such that:

(i) No irreducible component of C maps entirely to Y_0, Y_∞, or to the singular locus of P_k.
(ii) Every point that maps to Y_0 or Y_∞ is a marked point.
(iii) Every point of C that maps to the singular locus of X_k is a node with the property that the two local branches around the node map to the two different local components of X_k with the same orders of contact to the singular locus on both sides.

Morphisms of pre-stable non-rigid maps are defined in the same way as for pre-stable relative maps, however allowing automorphisms of P in *every* copy of P (including the first one). A pre-stable non-rigid map is called **stable** if its group of automorphisms is finite. For $i = 1, \ldots, n$ we define the **multiplicity** α_i of the i-th marked point x_i to be the multiplicity of the point x_i in the divisor $f^*Y_\infty - f^*Y_0$.

Fix integers $g, n \geq 0$, a homology class $\beta \in H_2^+(P)$, and a collection of integers $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that

$$\sum_{i:\alpha_i>0} \alpha_i = Y_\infty \cdot \beta \qquad \text{and} \qquad \sum_{i:\alpha_i<0} (-\alpha_i) = Y_0 \cdot \beta.$$

We denote by $\bar{\mathcal{M}}_{g,\alpha}(P,\beta)_\sim$ the set of (isomorphism classes of) all stable non-rigid maps (of any level) to P of genus g and class β whose multiplicities are α.

THEOREM 5.1.16. *The moduli spaces $\bar{\mathcal{M}}_{g,\alpha}(P,\beta)_\sim$ of stable non-rigid maps are separated and proper Deligne-Mumford stacks of expected dimension*

$$\operatorname{vdim} \bar{\mathcal{M}}_{g,\alpha}(P,\beta)_\sim = \operatorname{vdim} \bar{M}_{g,n}(Y, p_*\beta) - g,$$

where $p : P \to Y$ denotes the projection. Moreover, there is a naturally defined virtual fundamental class $[\bar{\mathcal{M}}_{g,\alpha}(P,\beta)_\sim]^{\mathrm{virt}} \in A_*(\bar{\mathcal{M}}_{g,\alpha}(P,\beta)_\sim)$ of this dimension.

PROOF. See [**GV**] section 2.4. □

CONSTRUCTION 5.1.17. Let $(C, x_1, \dots, x_n, f)$ be a stable non-rigid map to P of level k, given by a morphism $f : C \to P_k$. Let x_i be a marked point with positive multiplicity α_i. As f has multiplicity *exactly* α_i at x_i to Y_∞ there is an induced isomorphism $T_{C,x_i}^{\otimes \alpha_i} \to N_{Y_\infty/P,f(x_i)}$. Note however that this isomorphism is not preserved under isomorphisms of the stable non-rigid map, since the allowed isomorphisms of P act on the normal space $N_{Y_\infty/P,f(x_i)}$ of the target, but not on the tangent space T_{C,x_i}. To obtain a well-defined isomorphism that is invariant under automorphisms of the stable non-rigid map we have to take two marked points x_i and x_j with positive multiplicities and consider the "quotient" of the above constructions at these two points: the isomorphism

$$T_{C,x_i}^{\otimes \alpha_i} \otimes T_{C,x_j}^{\vee \otimes \alpha_j} \to N_{Y_\infty/P,f(x_i)} \otimes N_{Y_\infty/P,f(x_j)}^{\vee}$$

is well-defined on isomorphism classes of stable non-rigid maps. Consequently, there is an equality of cohomology classes

$$\alpha_j \psi_j - \alpha_i \psi_i = \mathrm{ev}_i^* c_1(L^\vee) - \mathrm{ev}_j^* c_1(L^\vee)$$

in $A^1(\bar{\mathcal{M}}_{g,\alpha}(P,\beta)_\sim)$. In other words, we conclude that the cohomology class

$$\Psi_\infty := \alpha_i \psi_i + \mathrm{ev}_i^* c_1(L^\vee) \in A^1(\bar{\mathcal{M}}_{g,\alpha}(P,\beta))$$

is independent of the choice of marked point x_i *as long as it is a point that maps to* Y_∞*, i.e. a point with positive multiplicity*. The class Ψ_∞ is called ψ in [**GV**] section 2.5.

Note that by symmetry the same construction can be made for the marked points that map to Y_0, leading to a cohomology class $\Psi_0 \in A^1(\bar{\mathcal{M}}_{g,\alpha}(P,\beta))$.

REMARK 5.1.18. Similarly to remark 5.1.13, moduli spaces of stable non-rigid maps can be defined for disconnected curves as well. In contrast to stable (ordinary or relative) maps there is no relation however between moduli spaces for disconnected curves and the moduli spaces for their connected components.

5.2. The virtual push-forward theorem

The goal of this section is to prove an important technical result about the behavior of virtual fundamental classes under certain push-forward maps. The results are not surprising — in fact they are trivial in the cases when the virtual fundamental classes are equal to the ordinary ones. But we cannot assume this here as the moduli spaces of stable maps are essentially never of the expected dimension for higher genus of the curves.

DEFINITION 5.2.1. Let $p : M \to M'$ be a morphism of moduli spaces of stable (absolute, relative, or non-rigid) maps. We say that p satisfies the **virtual push-forward property** if for every (homogeneous) cohomology class $\gamma \in A^*(M)$ that is made up from evaluation classes, cotangent line classes, and classes pulled back from M' by p, the following two conditions hold:

(i) If the dimension of the cycle $\gamma \cdot [M]^{\text{virt}}$ is bigger than the *virtual* dimension of M' then $p_*(\gamma \cdot [M]^{\text{virt}}) = 0$.
(ii) If the dimension of the cycle $\gamma \cdot [M]^{\text{virt}}$ is equal to the *virtual* dimension of M' then $p_*(\gamma \cdot [M]^{\text{virt}})$ is a scalar multiple of $[M']^{\text{virt}}$.

REMARK 5.2.2. Note that property (i) of definition 5.2.1 is trivial if the dimension of M' is equal to its virtual dimension, since then $p_*(\gamma \cdot [M]^{\text{virt}})$ is a cycle in a Chow group that is zero for dimensional reasons. In the same way, property (ii) is obvious if in addition M' is irreducible, since then $p_*(\gamma \cdot [M]^{\text{virt}})$ is a cycle in a Chow group that is generated by $[M']^{\text{virt}} = [M']$.

We now want to prove the virtual push-forward property for some morphisms that will occur later in this chapter. Actually we would expect that the virtual push-forward property holds for all "natural" morphisms of moduli spaces of stable maps, but we do not know a way to prove this statement (or even to formulate it in a rigorous way).

REMARK 5.2.3. To show the virtual push-forward property for a morphism p it is sufficient to check conditions (i) and (ii) of definition 5.2.1 for cohomology classes that are products of evaluation and cotangent line classes. In fact, let $\gamma = \gamma' \cdot p^*\delta$ be a cohomology class such that γ' is made up from evaluation and cotangent line classes, and δ is non-trivial. If the dimension of the cycle $\gamma \cdot [M]^{\text{virt}}$ is at least equal to the virtual dimension of M' then the dimension of the cycle $\gamma' \cdot [M]^{\text{virt}}$ is bigger than the virtual dimension of M'. So if we know that condition (i) of definition 5.2.1 holds for γ' it follows by the projection formula that

$$p_*(\gamma \cdot [M]^{\text{virt}}) = \delta \cdot p_*(\gamma' \cdot [M]^{\text{virt}}) = \delta \cdot 0 = 0,$$

i.e. conditions (i) and (ii) hold for the class γ as well. Moreover, this argument shows that a morphism satisfying the virtual push-forward property satisfies the following extension of condition (ii) as well:

(iii) If the dimension of the cycle $\gamma\cdot[M]^{\mathrm{virt}}$ is equal to the virtual dimension of M' and γ contains a non-trivial factor pulled back from M' by p then $p_*(\gamma\cdot[M]^{\mathrm{virt}})=0$.

LEMMA 5.2.4. *Every forgetful morphism $p:\bar{M}_{g,n}(X,\beta)\to\bar{M}_{g,n-m}(X,\beta)$ that forgets a given subset of the marked points of n-pointed stable maps to a smooth projective variety X satisfies the virtual push-forward property.*

PROOF. We will prove the statement by induction on the number m of forgotten marked points.

Let $\gamma\in A^*(\bar{M}_{g,n}(X,\beta))$ be a product of evaluation and cotangent line classes. Note that the morphism p forgets m points and has relative dimension m. So if the dimension of $\gamma\cdot[\bar{M}_{g,n}(X,\beta)]^{\mathrm{virt}}$ is to be at least equal to the virtual dimension of $\bar{M}_{g,n}(X,\beta)$ then there must be at least one marked point that is forgotten by p and at which we specify a class of codimension at most 1. We can assume without loss of generality that this is the point x_n. Factor p as $p=p'\circ p_n$, where p_n forgets the marked point x_n, and p' the other $m-1$ marked points forgotten by p. We then must have one of the following three cases:

- γ has no class at x_n: if $\gamma=\prod_{i=1}^{n-1}(\mathrm{ev}_i^*\gamma_i\cdot\psi_i^{k_i})$ then by the arguments of corollary 1.3.3 we have

$$p_{n*}(\gamma\cdot[\bar{M}_{g,n}(X,\beta)]^{\mathrm{virt}}) = \Big(\sum_{i:k_i>0}\mathrm{ev}_1^*\gamma_1\cdot\psi_1^{k_1}\cdots\mathrm{ev}_i^*\gamma_i\cdot\psi_i^{k_i-1}\cdots\mathrm{ev}_{n-1}^*\gamma_{n-1}\cdot\psi_{n-1}^{k_{n-1}}\Big)\cdot[\bar{M}_{g,n-1}(X,\beta)]^{\mathrm{virt}}.$$

- γ has an evaluation class at a divisor at x_n: if $\gamma=\mathrm{ev}_n^*\delta\cdot\prod_{i=1}^{n-1}(\mathrm{ev}_i^*\gamma_i\cdot\psi_i^{k_i})$ for a divisor δ then by the arguments of corollary 1.3.4 we have

$$\begin{aligned}p_{n*}&(\gamma\cdot[\bar{M}_{g,n}(X,\beta)]^{\mathrm{virt}})\\ &=\Big((\delta\cdot\beta)\cdot\mathrm{ev}_1^*\gamma_1\cdot\psi_1^{k_1}\cdots\mathrm{ev}_{n-1}^*\gamma_{n-1}\cdot\psi_{n-1}^{k_{n-1}}\\ &\quad+\sum_{i:k_i>0}\mathrm{ev}_1^*\gamma_1\cdot\psi_1^{k_1}\cdots\mathrm{ev}_i^*(\delta\cdot\gamma_i)\cdot\psi_i^{k_i-1}\cdots\mathrm{ev}_{n-1}^*\gamma_{n-1}\cdot\psi_{n-1}^{k_{n-1}}\Big)\\ &\quad\cdot[\bar{M}_{g,n-1}(X,\beta)]^{\mathrm{virt}}.\end{aligned}$$

- γ has a single cotangent line class at x_n: if $\gamma=\psi_n\cdot\prod_{i=1}^{n-1}(\mathrm{ev}_i^*\gamma_i\cdot\psi_i^{k_i})$ then by the arguments of corollary 1.3.5 we have

$$p_{n*}(\gamma\cdot[\bar{M}_{g,n}(X,\beta)]^{\mathrm{virt}}) = \Big((2g+n-3)\cdot\mathrm{ev}_1^*\gamma_1\cdot\psi_1^{k_1}\cdots\mathrm{ev}_{n-1}^*\gamma_{n-1}\cdot\psi_{n-1}^{k_{n-1}}\Big)\cdot[\bar{M}_{g,n-1}(X,\beta)]^{\mathrm{virt}}.$$

In all three cases the statement then follows from the induction hypothesis applied to p'. □

COROLLARY 5.2.5. *Let M be a moduli space of stable maps to a smooth projective variety X, possibly with several connected components. Let $p : M \to M'$ be a forgetful morphism that forgets a given subset of the marked points and / or connected components. (M' is thus also a moduli space of stable maps to X, with in general fewer marked points and connected components.)*

Then p satisfies the virtual push-forward property.

PROOF. This follows immediately from lemma 5.2.4, taking into account that the moduli spaces of disconnected stable maps are simply the products of the moduli spaces for the components. □

REMARK 5.2.6. For our main virtual push-forward theorem we will need the technique of virtual localization for moduli spaces of stable relative maps. This technique has been established in **[GV]** section 3. We will give a short review here.

Let L be a line bundle on a smooth projective variety Y, and denote by $P = \mathbb{P}(L \oplus \mathcal{O}_Y)$ its projective closure. Consider a moduli space $\bar{\mathcal{M}}_\Gamma(P)$ of stable relative maps to P (relative Y_∞). The $\mathbb{C}^*$ action in the fibers of P induces a $\mathbb{C}^*$ action on the moduli space $\bar{\mathcal{M}}_\Gamma(P)$. Then we have the virtual localization formula

$$[\bar{\mathcal{M}}_\Gamma(P)]^{\text{virt}} = \sum_F \frac{[F]^{\text{virt}}}{e(N^{\text{virt}}_{F/\bar{\mathcal{M}}_\Gamma(P)})}$$

in the equivariant Chow group of $\bar{\mathcal{M}}_\Gamma(P)$, where the sum is taken over all connected components $F \subset \bar{\mathcal{M}}_\Gamma(P)$ of the fixed point locus of the $\mathbb{C}^*$ action, and $e(N^{\text{virt}}_{F/\bar{\mathcal{M}}_\Gamma(P)})$ denotes the Euler class of the virtual normal bundle of F.

The fixed point locus of the $\mathbb{C}^*$ action on the moduli space is easy to describe. In level 0 (where the group acts in the fibers) all components of the curve must be of one of the following two types:

- curves contained in Y_0,
- rational curves that are multiple covers of a fiber of P totally ramified over Y_0 and Y_∞ and that have no marked points away from Y_0 and Y_∞.

In the higher levels (where the group acts trivially) there are no restrictions.

Every component F of the fixed point locus is therefore an étale quotient of the form $(M_0 \boxtimes \mathcal{M}_1)/G$, where

- M_0 is a moduli space of stable maps to Y (in Y_0 in level 0), including in the set of marked points the nodes y_i where the rational fibers in level 0 are attached;
- $\mathcal{M}_1$ is a moduli space of stable non-rigid maps to P (for the positive levels), including in the set of marked points the nodes y_i where the rational fibers in level 0 are attached;
- $\boxtimes$ denotes a fiber product over evaluation maps to Y at the gluing points y_i (in the intersection of the levels 0 and 1),
- G is the group of permutations of the points y_i that leaves the multiplicities at these points unchanged.

The virtual fundamental class of F is simply the one induced by this product structure.

To be able to apply the virtual localization formula it remains to describe the virtual normal bundle of the fixed point loci. Denote the generator of $H^*_{\mathbb{C}^*}(\mathrm{pt})$ by $\hbar$. Then $e(N^{\mathrm{virt}}_{F/\bar{\mathcal{M}}_\Gamma(P)})$ is a product of the following terms:

(i) moving the M_0 part out of Y_0 while keeping the structure of the singularities: the term that we get from these deformations is simply the equivariant Euler class of $H^0/H^1(f^*N_{Y_0/P}) = H^0/H^1(f^*L)$ on M_0.

(ii) deforming the m-th order ramification points at Y_0 in level 0: these terms have been evaluated in [**GP**] section 4.

(a) Every marked point x_i that lies over Y_0 on a multiple cover of degree m in level 0 contributes a product $\prod_{k=1}^{m} \frac{k(\hbar+\mathrm{ev}_{x_i}^* c_1(L))}{m}$.

(b) Every marked point y_i in M_0 that connects to a multiple cover of degree m in level 0 contributes a product $\prod_{k=1}^{m-1} \frac{k(\hbar+\mathrm{ev}_{y_i}^* c_1(L))}{m}$. (The $k = m$ term corresponds to simply moving the ramification point in the fiber direction without changing the ramification structure. We have already taken care of this for the points y_i in (i), so we have to leave the $k = m$ term out for them now.)

(c) Every smooth unmarked point over Y_0 on a multiple cover of degree m in level 0 contributes a product $\prod_{k=2}^{m} \frac{k(\hbar+\mathrm{ev}^* c_1(L))}{m}$. (The $k = 1$ term corresponds to moving the marked point on the curve without changing the curve or the map. As we do not have a marked point now we have to leave the $k = 1$ term out in this case.)

(iii) smoothing the nodes y_i at Y_0: As usual the contribution for such a deformation is the first Chern class of the tensor product of the tangent spaces of the two components that meet at the node. So every marked point y_i in M_0 gives rise to a factor $\frac{\hbar+\mathrm{ev}_{y_i}^* c_1(L)}{m} - \psi_{y_i}$.

(iv) deforming the target singularity to obtain level-0 maps: the contribution of this term has been computed in [**GV**] section 3.3 to be $-\hbar - \Psi_0$, where Ψ_0 is the class on $\mathcal{M}_1$ introduced in construction 5.1.17.

Note that these contributions respect the product structure of F: parts (i), (ii) for the y_i, and (iii) act on M_0 only, whereas (ii) for the x_i and (iv) act on $\mathcal{M}_1$. We will denote these parts by $e_0(N^{\mathrm{virt}}_{F/\bar{\mathcal{M}}_\Gamma(P)})$ and $e_1(N^{\mathrm{virt}}_{F/\bar{\mathcal{M}}_\Gamma(P)})$ respectively, so that the term in the virtual localization formula corresponding to a fixed point locus $F = (M_0 \boxtimes \mathcal{M}_1)/G$ reads

$$\frac{1}{|G|} \cdot \left(\frac{[M_0]^{\mathrm{virt}}}{e_0(N^{\mathrm{virt}}_{F/\bar{\mathcal{M}}_\Gamma(P)})} \boxtimes \frac{[\mathcal{M}_1]^{\mathrm{virt}}}{e_1(N^{\mathrm{virt}}_{F/\bar{\mathcal{M}}_\Gamma(P)})} \right).$$

For example, the contribution to the virtual localization formula of the fixed point locus shown in the picture above is

$$\frac{1}{2}\cdot\left(\frac{[M_0]^{\mathrm{virt}}}{(\frac{\hbar_1}{2}-\psi_{y_1})(\frac{2\hbar_1}{2})(\frac{\hbar_2}{2}-\psi_{y_2})(\frac{2\hbar_2}{2})(\hbar_3-\psi_{y_3})e(H^0/H^1(f^*L^\vee))}\right.$$
$$\left.\boxtimes\frac{[\mathcal{M}_1]^{\mathrm{virt}}}{(\hbar+\mathrm{ev}_{x_2}^*Y_0)(-\hbar-\Psi_0)}\right)$$

where $\hbar_i := \hbar + \mathrm{ev}_{y_i}^* c_1(L)$, and where the fiber product $\boxtimes$ is taken over the three diagonals $\Delta_Y \subset Y\times Y$ at the marked points y_1, y_2, y_3.

THEOREM 5.2.7. *Let L be a line bundle on a smooth projective variety Y, and denote by $P=\mathbb{P}(L\oplus\mathcal{O}_Y)$ its projective closure. Let $\mathcal{M}$ be a moduli space of stable relative maps (relative to $Y_\infty=\mathbb{P}(L\oplus 0)\subset P$) or stable non-rigid maps to P, possibly with several connected components.*

Let $p:\mathcal{M}\to M$ be the morphism that projects the curves in P down to the base Y, forgets a given subset of the marked points and / or connected components, and stabilizes the result. (M is thus a moduli space of stable absolute maps to Y, with the combinatorial data determined by $\mathcal{M}$.) We are assuming that p is well-defined, i.e. that every rational (resp. elliptic) connected component that is not forgotten by p and whose homology class is a multiple of a fiber of P has at least 3 (resp. 1) special points that are not forgotten by p.

Then p satisfies the virtual push-forward property.

PROOF. Let $\mathcal{M}$ be one of the moduli spaces mentioned in the the theorem, i.e.

(a) the moduli space of stable non-rigid maps to P with r connected components, where the i-th component has genus $g^{(i)}$, class $\beta^{(i)}$, and $n^{(i)}$ marked points with multiplicities $\alpha^{(i)}=(\alpha_1^{(i)},\dots,\alpha_{n^{(i)}}^{(i)})$ where $\alpha_j^{(i)}\in\mathbb{Z}$; or
(b) the moduli space of stable relative maps to P relative Y_∞ with r connected components, where the i-th component has genus $g^{(i)}$, class $\beta^{(i)}$, and $n^{(i)}$ marked points with multiplicities $\alpha^{(i)}=(\alpha_1^{(i)},\dots,\alpha_{n^{(i)}}^{(i)})$ where $\alpha_j^{(i)}\geq 0$.

Let $\gamma\in A^*(\mathcal{M})$ be a product of evaluation and cotangent line classes at the marked points such that $\dim(\gamma\cdot[\mathcal{M}]^{\mathrm{virt}})\geq\dim[M]^{\mathrm{virt}}$, and let $p:\mathcal{M}\to M$ be a morphism as in the statement of the theorem.

The proof of the theorem will be given by induction. So we assume that the statement is known already for all moduli spaces $\mathcal{M}$, cohomology classes γ, and morphisms p with source $\mathcal{M}$ where

(i) the sorted collection of classes $(\beta^{(i)})$ is smaller in the lexicographical ordering (here we are picking an ordering of homology classes such that an equality $\beta = \beta_1 + \beta_2$ of non-zero *effective* classes implies $\beta_1 < \beta$ and $\beta_2 < \beta$); or

(ii) the $\beta^{(i)}$ are the same, and the sorted collection of genera $(g^{(i)})$ is smaller in the lexicographical ordering; or

(iii) the $\beta^{(i)}$ and $g^{(i)}$ are the same, and the sorted collection of numbers $(n^{(i)})$ is smaller in the lexicographical ordering.

We will now treat the two cases (a) and (b) in turn.

Case (a): stable non-rigid maps. Let $\bar{\mathcal{M}}_\Gamma(P)$ be the moduli space of stable relative maps with the same number of connected components as $\mathcal{M}$, the same $g^{(i)}$ and $n^{(i)}$, and multiplicities $\max(\alpha_j^{(i)}, 0)$. There is a morphism $\bar{\mathcal{M}}_\Gamma(P) \to M$ that projects the curves to Y and forgets the same data as p. By abuse of notation we will denote this morphism also by p. Moreover, by construction 5.1.17 we can rewrite γ as $\Psi_0^k \cdot \gamma'$, where γ' contains no cotangent line classes at the points with $\alpha_j^{(i)} < 0$.

We want to use the virtual localization theorem of remark 5.2.6. Replace γ' by a $\mathbb{C}^*$-equivariant cohomology class that maps to the given one in ordinary cohomology. In addition, consider the $\mathbb{C}^*$-equivariant class

$$\tilde{\gamma} = \prod_{i,j:\alpha_j^{(i)}<0} \prod_{k=0}^{-\alpha_j^{(i)}-1} (k\psi_j^{(i)} + \mathrm{ev}_j^{(i)\,*} Y_0)$$

on $\bar{\mathcal{M}}_\Gamma(P)$. Now intersect the virtual localization formula of remark 5.2.6 with $\tilde{\gamma}$ and γ', push the result forward to M by p, restrict to ordinary cohomology, and collect the terms of dimension $\dim(\gamma \cdot [\mathcal{M}]^{\mathrm{virt}})$. First of all note that the result must be zero for dimensional reasons, since

$$\begin{aligned} \dim(\tilde{\gamma} \cdot \gamma' \cdot [\bar{\mathcal{M}}_\Gamma(P)]^{\mathrm{virt}}) &= \dim(\gamma \cdot [\mathcal{M}]^{\mathrm{virt}}) + k + 1 \\ &\neq \dim(\gamma \cdot [\mathcal{M}]^{\mathrm{virt}}). \end{aligned}$$

Now evaluate the right hand side of the virtual localization formula. Consider a fixed point locus $F = (M_0 \boxtimes \mathcal{M}_1)/G$ as in remark 5.2.6. Note that the classes $\tilde{\gamma}$ and γ' can be written as products $\tilde{\gamma}_0 \cdot \tilde{\gamma}_1$ and $\gamma_0' \cdot \gamma_1'$ where $\tilde{\gamma}_0$ and γ_0' (resp. $\tilde{\gamma}_1$ and γ_1') act on M_0 (resp. $\mathcal{M}_1$) only. In the same way the push-forward p acts on the two factors separately, so that the contribution of F can be written as

$$\frac{1}{|G|} \cdot p_{0*}\left(\tilde{\gamma}_0 \cdot \gamma_0' \cdot \frac{[M_0]^{\mathrm{virt}}}{e_0(N^{\mathrm{virt}}_{F/\bar{\mathcal{M}}_\Gamma(P)})}\right) \boxtimes p_{1*}\left(\tilde{\gamma}_1 \cdot \gamma_1' \cdot \frac{[\mathcal{M}_1]^{\mathrm{virt}}}{e_1(N^{\mathrm{virt}}_{F/\bar{\mathcal{M}}_\Gamma(P)})}\right) \qquad (35)$$

where p_0 (resp. p_1) project M_0 (resp. $\mathcal{M}_1$) down to the corresponding moduli space of stable maps to Y. (Note: If p forgets some marked points it may happen that some components of the elements in M_0 or $\mathcal{M}_1$ become unstable after forgetting these points. In this case p_0 or p_1 is taken to forget these entire components, and it may thus happen that in some diagonals of the fiber product $\boxtimes$ both evaluation maps come from the same factor. Our arguments that follow are not affected by this.)

We know by corollary 5.2.5 that p_0 satisfies the virtual push-forward property. Let us assume for a moment that p_1 does so too. Then the dimension of the cycle (35) can be at most the virtual dimension of the subspace of M consisting of stable maps with node structure given by the fiber product $\boxtimes$. If this fiber product contains at least one diagonal, i.e. there is at least one node, then the virtual dimension of this subspace is smaller than that of M, so (35) must be zero since $\dim(\gamma \cdot [\mathcal{M}]^{\text{virt}}) \geq \operatorname{vdim} M$. Otherwise there is only one component, and (35) must be a scalar multiple of $[M]^{\text{virt}}$. So all terms for which we know that p_1 satisfies the virtual push-forward property are zero or a multiple of $[M]^{\text{virt}}$.

Finally let us check for which terms (35) we do not know yet by induction that p_1 satisfies the virtual push-forward property. By induction assumptions (i) and (ii) there must then be exactly r components in level 1 of degrees $\beta^{(i)}$ and genera $g^{(i)}$. By (iii) all marked points $x_j^{(i)}$ with non-negative multiplicities must be in these level-1 components, and the remaining points must be over Y_0 in rational fibers in level 0, with at most one marked point on every such component. The following picture shows an example of these curves (in the case of only one connected component).

For the points $x_j^{(i)}$ over Y_0 denote by $m_j^{(i)}$ the degree of the cover of the fiber of P at $x_j^{(i)}$. As the class $\tilde{\gamma} = \tilde{\gamma}_1$ restricts to

$$\prod_{i,j:\alpha_j^{(i)}<0} \prod_{k=0}^{-\alpha_j^{(i)}-1} \left(1 - \frac{k}{m_j^{(i)}} \psi_j^{(i)}\right) \left(\hbar + \mathrm{ev}_j^{(i)\,*} c_1(L)\right)$$

on this fixed point locus, we must have $m_j^{(i)} \geq -\alpha_j^{(i)}$ at all these points. But the sum of all ramification orders at Y_0 must be equal to the intersection product $Y_0 \cdot \beta^{(i)}$, which in turn is equal to $\sum_{j:\alpha_j^{(i)}<0}(-\alpha_j^{(i)})$. So we conclude that all rational fibers in level 0 must have a marked point over Y_0, and $m_j^{(i)} = -\alpha_j^{(i)}$ for all i and j with $\alpha_j^{(i)} < 0$. In other words, the moduli space $\mathcal{M}_1$ is precisely $\mathcal{M}$, and the term (35) is

$$p_* \left(\tilde{\gamma} \cdot \gamma' \cdot \frac{[\mathcal{M}]^{\mathrm{virt}}}{e_1(N^{\mathrm{virt}}_{F/\bar{\mathcal{M}}_\Gamma(P)})} \right).$$

Now the term $\tilde{\gamma}_1$ cancels precisely the terms (ii) of $e_1(N^{\mathrm{virt}}_{F/\bar{\mathcal{M}}_\Gamma(P)})$ in remark 5.2.6. So we get

$$p_* \left(\gamma' \cdot \frac{[\mathcal{M}]^{\mathrm{virt}}}{-\hbar - \Psi_0} \right).$$

Finally, expanding the geometric series $\frac{1}{-\hbar-\Psi_0}$ and taking only the $\hbar$-independent terms of the correct dimension gives (up to sign) the desired term $p_*(\gamma \cdot [MM]^{\mathrm{virt}})$. So we have shown that this expression is either zero or a multiple of $[M]^{\mathrm{virt}}$. This finishes the proof of the theorem in case (a).

Case (b): stable relative maps. This time we set $\bar{\mathcal{M}}_\Gamma(P) = \mathcal{M}$. Another application of the virtual localization theorem expresses the class $p_*(\gamma \cdot \bar{\mathcal{M}}_\Gamma(P))$ in terms of contributions from fixed point loci

$$\frac{1}{|G|} \cdot p_{0*} \left(\gamma_0 \cdot \frac{[M_0]^{\mathrm{virt}}}{e_0(N^{\mathrm{virt}}_{F/\bar{\mathcal{M}}_\Gamma(P)})} \right) \boxtimes p_{1*} \left(\gamma_1 \cdot \frac{[\mathcal{M}_1]^{\mathrm{virt}}}{e_1(N^{\mathrm{virt}}_{F/\bar{\mathcal{M}}_\Gamma(P)})} \right)$$

similarly to (35) in case (a) above, where γ_0 (resp. γ_1) denotes the part of γ that acts on M_0 (resp. $\mathcal{M}_1$). By corollary 5.2.5 and case (a) we know that both p_0 and p_1 satisfy the virtual push-forward property. The conclusion that the above term is zero or a multiple of $[M]^{\mathrm{virt}}$ now works in the same way as in case (a). □

Recall from remark 5.1.13 that the moduli spaces of disconnected stable relative maps are not just the products of the individual moduli spaces for the connected components. Using our virtual push-forward theorems we now want to show however that this product property is satisfied for the moduli spaces of *collapsed* stable relative maps (at least in genus 1).

PROPOSITION 5.2.8 (**Product property for elliptic curves**). *Let $r > 0$ be an integer. Pick non-negative integers $g^{(1)}, \dots, g^{(r)}, n^{(1)}, \dots, n^{(r)}$ and classes $\beta^{(1)}, \dots, \beta^{(r)} \in H_2^+(X)$. Choose collections of multiplicities $\alpha^{(i)} = (\alpha_1^{(i)}, \dots, \alpha_{n^{(i)}}^{(i)})$ for $i = 1, \dots, r$. Denote by $\bar{M}_\Gamma^Y(X)$ the moduli space of collapsed stable relative maps with r connected components such that the i-th component has genus $g^{(i)}$, class $\beta^{(i)}$, and multiplicities $\alpha^{(i)}$.*

Assume that the total genus $\sum_i g^{(i)}$ is at most 1. Then we have

$$\bar{M}_\Gamma^Y(X) = \bar{M}^Y_{g^{(1)},\alpha^{(1)}}(X, \beta^{(1)}) \times \cdots \times \bar{M}^Y_{g^{(r)},\alpha^{(r)}}(X, \beta^{(r)}).$$

Moreover, there is an equality of cycles

$$[\bar{M}_\Gamma^Y(X)]^{\mathrm{virt}} = [\bar{M}^Y_{g^{(1)},\alpha^{(1)}}(X, \beta^{(1)})]^{\mathrm{virt}} \times \cdots \times [\bar{M}^Y_{g^{(r)},\alpha^{(r)}}(X, \beta^{(r)})]^{\mathrm{virt}}$$

in the Chow group of the subspace of $\prod_{i=1}^r \bar{M}_{g^{(i)},n^{(i)}}(X, \beta^{(i)})$ where all marked points with positive multiplicity map to Y.

PROOF. The statement about the spaces follows immediately from their description in definition 2.1.1. To prove the product property for the virtual fundamental classes in higher genus we will use the splitting theorem of remark 5.1.14 together with the virtual push-forward theorem 5.2.7. Let us assume for a moment that X itself is a $\mathbb{P}^1$-bundle (and thus equal to P). The moduli spaces occurring in the splitting formula are then the spaces of stable relative maps to X relative Y_0 or Y_∞. So we will prove the following statement: For X a $\mathbb{P}^1$-bundle, $Y = Y_0$ or $Y = Y_\infty$, and $g^{(i)}$, $n^{(i)}$, $\beta^{(i)}$, $\alpha^{(i)}$ as above, we have

$$[\bar{M}_\Gamma^Y(X)]^{\mathrm{virt}} = \prod_{i=1}^r [\bar{M}^Y_{g^{(i)},\alpha^{(i)}}(X, \beta^{(i)})]^{\mathrm{virt}}. \tag{36}$$

As in the proof of theorem 5.2.7 we will proceed by induction. So we assume that the statement is known already for any given data where

(i) the sorted collection of classes $(\beta^{(i)})$ is smaller in the lexicographical ordering; or
(ii) the $\beta^{(i)}$ are the same, and the sorted collection of genera $(g^{(i)})$ is smaller in the lexicographical ordering; or
(iii) the $\beta^{(i)}$ and $g^{(i)}$ are the same, and the sorted collection of numbers $(n^{(i)})$ is smaller in the lexicographical ordering.

Now let $M = \prod_{i=1}^{r} \bar{M}_{g^{(i)},n^{(i)}}(X,\beta^{(i)})$ be the moduli space of stable absolute maps to X with r connected components of the given classes and genera and with the given number of marked points. Consider the cohomology class

$$\tilde{\gamma} = \prod_{i=1}^{r} \prod_{j=1}^{n^{(i)}} \prod_{k=0}^{\alpha_j^{(i)}-1} (\mathrm{ev}_j^{(i)\,*} Y + k\psi_j^{(i)})$$

on M that describes the condition that the stable map has order at least $\alpha_j^{(i)}$ to Y at the marked points $x_j^{(i)}$. Applying the splitting theorem of remark 5.1.14 to M we obtain

$$\tilde{\gamma} \cdot [M]^{\mathrm{virt}} = \sum_{\Gamma_1,\Gamma_2} m(\Gamma_1,\Gamma_2) \cdot [\bar{M}^Y_{\Gamma_1}(X)]^{\mathrm{virt}} \boxtimes (\tilde{\gamma} \cdot [\bar{M}^Y_{\Gamma_2}(P)]^{\mathrm{virt}})$$

with the notation used there. (Note that the conditions in $\tilde{\gamma}$ restrict the marked points with positive multiplicities to lie over Y_∞ in the Γ_2 factor.) Recall that this is an equation in a moduli space of stable maps to the blow-up of $X \times \mathbb{P}^1$ in $Y \times \{0\}$. We can push it forward along the blow-up map to obtain

$$\tilde{\gamma} \cdot [M]^{\mathrm{virt}} = \sum_{\Gamma_1,\Gamma_2} m(\Gamma_1,\Gamma_2) \cdot [\bar{M}^Y_{\Gamma_1}(X)]^{\mathrm{virt}} \boxtimes p_*(\tilde{\gamma} \cdot [\bar{M}^Y_{\Gamma_2}(P)]^{\mathrm{virt}}) \qquad (37)$$

where p projects the stable relative maps to P down to stable absolute maps in Y (and forgets connected components in fibers that become unstable by the projection).

On the other hand we can equally well apply the splitting theorem to the individual factors $\bar{M}_{g^{(i)},n^{(i)}}(X,\beta^{(i)})$ of M and then take the product of the resulting equations. If we knew already that statement (36) is true for $\bar{M}^Y_{\Gamma_1}(X)$ and $\bar{M}^Y_{\Gamma_2}(P)$ then the result of this alternative application of the splitting theorem would be term-by-term the same as in (37). In other words, if we can show that (36) is true by induction hypothesis for all terms occurring in the right hand side of (37) except one, then statement (36) must hold in the remaining term as well. This would then complete the proof of the proposition in the case when X is a $\mathbb{P}^1$-bundle.

So let us examine the terms $\bar{M}^Y_{\Gamma_1}(X)$ and $\bar{M}^Y_{\Gamma_2}(P)$ in (37) for which statement (36) is not yet known by induction. By induction assumptions (i) and (ii) one of these spaces must then be a moduli space of stable relative maps with exactly r components of degrees $\beta^{(i)}$ and genera $g^{(i)}$. Let us assume that the moduli space $\bar{M}^Y_{\Gamma_2}(P)$ is of this form. Then the dimension of the cycle $\tilde{\gamma} \cdot [\bar{M}^Y_{\Gamma_2}(P)]^{\mathrm{virt}}$ is equal to the virtual dimension of a moduli space of stable relative maps to P relative Y_0 *and* Y_∞, which in turn is equal to the virtual dimension of the corresponding moduli space of stable non-rigid maps to P. So we get

$$\dim(\tilde{\gamma} \cdot [\bar{M}^Y_{\Gamma_2}(P)]^{\mathrm{virt}}) = \dim[M']^{\mathrm{virt}} + \sum_i (1 - g^{(i)})$$

where M' is the moduli space of stable absolute maps to Y to which p in (37) projects. But by our assumption on the genus we have

$$\sum_i (1-g^{(i)}) = r - \sum_i g^{(i)} \geq r-1.$$

As the statement of the proposition is trivial for $r=1$ we can assume that this number is positive. But then (37) is zero by the virtual push-forward theorem 5.2.7 applied to p. So we can neglect these terms and hence assume that it is the moduli space $\bar{M}^Y_{\Gamma_1}(X)$ that describes curves with exactly r components of degrees $\beta^{(i)}$ and genera $g^{(i)}$.

The connected components of the curves in the moduli space $\bar{M}^Y_{\Gamma_2}(P)$ must now all be multiple covers of fibers. By induction assumption (iii) all marked points whose multiplicities are zero must be in the Γ_1 component. The others have to be in the Γ_2 component because of the class $\tilde{\gamma}$, and by induction assumption (iii) there may be at most one such point on each multiple cover of a fiber. The class $\tilde{\gamma}$ forces the degree of the cover to be at least $\alpha^{(i)}_j$ at the component of the point $x^{(i)}_j$. As the sum of all these degrees must be the intersection product $Y_\infty \cdot \beta^{(i)}$ it follows that the degree of the cover at $x^{(i)}_j$ is exactly $\alpha^{(i)}_j$, and that all fibers in the Γ_2 space carry a marked point $x^{(i)}_j$. In other words, $\bar{M}^Y_{\Gamma_1}(X)$ is precisely the given moduli space $\bar{M}^Y_\Gamma(X)$. The Γ_2 part and the fiber product $\boxtimes$ are then trivial, so the corresponding contribution to (37) is just $\bar{M}^Y_\Gamma(X)$. This is the only term in (37) for which we do not yet know by induction the statement of the proposition. Hence, as remarked above, the splitting theorem applied to the r components of the curves in $\bar{M}^Y_\Gamma(X)$ implies that the proposition holds for $\bar{M}^Y_\Gamma(X)$ as well. □

CONJECTURE 5.2.9. We expect the following generalizations of proposition 5.2.8 to hold:

- (i) The equation that determines the virtual fundamental class $[\bar{M}^Y_\Gamma(X)]^{\text{virt}}$ is expected to hold in the Chow group of $\bar{M}^Y_\Gamma(X)$.
- (ii) The statement of the proposition should hold in any genus. Note that for disconnected curves the left hand side of the splitting theorem of remark 5.1.14 is manifestly the product of the corresponding terms for the individual components, whereas this would not be the case on the right hand side if proposition 5.2.8 did not hold in general.

Unfortunately we do not know how to prove these statements at the moment. In what follows we will only need the statement of proposition 5.2.8.

5.3. Push-forwards in codimension 0

Let $P = \mathbb{P}(L \oplus O)$ be a $\mathbb{P}^1$-bundle over a smooth projective variety Y. Recall that the relative Gromov-Witten invariants of P relative Y_∞ occur in the splitting formula of remark 5.1.14. We would like to express these invariants in terms of absolute invariants of Y, so that the splitting formula gives relations between the absolute invariants of X, the absolute invariants of Y, and the relative invariants of X relative Y, in the same way as in chapter 2. To do so we have to consider the projection $P \to Y$ and study the associated morphism that projects stable maps in P down to Y. Before we can do this for stable relative maps we need a result on stable non-rigid maps first.

LEMMA 5.3.1. *Let $P = \mathbb{P}(L \oplus O)$ be a $\mathbb{P}^1$-bundle over a smooth projective variety Y. Let $\mathcal{M}$ be the moduli space of stable non-rigid maps to P with r connected rational components such that for $i = 1, \dots, r$ the i-th component is of the following type:*

- *it has homology class $d^{(i)}$ times a fiber,*
- *it has $n^{(i)} + 1$ marked points $y^{(i)}, x_1^{(i)}, \dots, x_{n^{(i)}}^{(i)}$, where $y^{(i)}$ has multiplicity $d^{(i)}$ (and thus maps to Y_∞) and the $x_j^{(i)}$ have negative multiplicities (and thus map to Y_0) satisfying*

$$d^{(i)} + \sum_{j=1}^{n^{(i)}} \alpha_j^{(i)} = 0.$$

The moduli space $\mathcal{M}$ has virtual dimension

$$\operatorname{vdim} \mathcal{M} = -1 + \sum_{i=1}^{r} (\dim Y + n^{(i)} - 1).$$

Let $\Psi_0 \in A^1(\mathcal{M})$ be the class defined in construction 5.1.17, and let $\gamma^{(i)} \in A^*(Y)$. Denote by $\mathrm{ev}^{(i)} : \mathcal{M} \to Y$ the evaluation map in the i-th component.

(i) *If all $\gamma^{(i)}$ are the class of a point then*

$$\mathrm{ev}^{(1)*}\gamma^{(1)}\cdots\mathrm{ev}^{(r)*}\gamma^{(r)}\cdot\Psi_0^N\cdot[\mathcal{M}]^{\mathrm{virt}}=\prod_{i=1}^r\left(d^{(i)}\right)^{n^{(i)}-2}$$

where $N=\sum_{i=1}^r n^{(i)}-r-1$.

(ii) *If $\gamma^{(1)}$ is the class of a curve and all other $\gamma^{(i)}$ are the class of a point then*

$$\mathrm{ev}^{(1)*}\gamma^{(1)}\cdots\mathrm{ev}^{(r)*}\gamma^{(r)}\cdot\Psi_0^{N+1}\cdot[\mathcal{M}]^{\mathrm{virt}}$$
$$=(n^{(1)}-1)\cdot(c_1(L)\cdot\gamma^{(1)})\cdot\prod_{i=1}^r\left(d^{(i)}\right)^{n^{(i)}-2}.$$

PROOF. Let $\mathcal{M}_\Gamma^Y(P)$ be the moduli space of stable relative maps to P relative Y_∞ with r connected rational components such that for $i=1,\dots,r$ the i-th component is of the following type:

- it has homology class $d^{(i)}$ times a fiber,
- it has $n^{(i)}+1$ marked points $y^{(i)},x_1^{(i)},\dots,x_{n^{(i)}}^{(i)}$, where $y^{(i)}$ has multiplicity $d^{(i)}$ (and thus maps to Y_∞) and the $x_j^{(i)}$ have multiplicity 0.

This moduli space has virtual dimension

$$\mathrm{vdim}\,\mathcal{M}_\Gamma^Y(P)=\sum_{i=1}^r(\dim Y+d^{(i)}-1+n^{(i)}).$$

Now consider the equivariant cohomology class

$$\tilde{\gamma}=\prod_{i=1}^r\left(\mathrm{ev}^{(i)*}\gamma^{(i)}\cdot\prod_{j=2}^{n^{(i)}}(\alpha_j^{(i)}\psi_{x_j^{(i)}})\cdot\prod_{j=1}^{n^{(i)}}\prod_{k=0}^{-\alpha_j^{(i)}-1}(k\psi_{x_j^{(i)}}+\mathrm{ev}_{x_j^{(i)}}^*Y_0)\right)$$

on $\mathcal{M}_\Gamma^Y(P)$. We will compute the integral $\tilde{\gamma}\cdot[\mathcal{M}_\Gamma^Y(P)]^{\mathrm{virt}}$ using virtual localization. Note that the integral is of dimension $\sum_{i=1}^r\dim\gamma^{(i)}$, i.e. of dimension 0 in case (i) and of dimension 1 in case (ii).

Let us check which fixed point loci contribute to the integral. By the $k=0$ terms in the above product all points $x_j^{(i)}$ have to map to Y_0. We claim that there can be no component that maps to Y_0. In fact, if we had such a component then it would have to be a contracted rational component. It could have at most one node since there is only one marked point over Y_∞ in each connected component. So if such a contracted rational component has n special points then $n-1$ of them are marked points. As $\tilde{\gamma}$ contains a cotangent line class for every marked point but one it follows that this component has at least $n-2$ cotangent line classes on it. But the product of

$n-2$ cotangent line classes on $\bar{M}_{0,n}$ vanishes for dimensional reasons. Hence this cannot happen, and we conclude that there can be no components of the curves in the fixed point locus that map to Y_0.

The marked points $x_j^{(i)}$ must therefore all be on multiple covers of the fibers in level 0. As in the proof of theorem 5.2.7 the classes $k\psi_{x_j^{(i)}} + \mathrm{ev}^*_{x_j^{(i)}} Y_0$ in $\tilde{\gamma}$ force these multiple covers to have degree $-\alpha_j^{(i)}$ for all i, j. So there is only one fixed point locus which is isomorphic to our given moduli space $\mathcal{M}$ of stable non-rigid maps:

On this fixed point locus the $k\psi_{x_j^{(i)}} + \mathrm{ev}^*_{x_j^{(i)}} Y_0$ terms in $\tilde{\gamma}$ cancel exactly the virtual normal bundle terms (ii) of remark 5.2.6. The classes $\alpha_j^{(i)}\psi_{x_j^{(i)}}$ in $\tilde{\gamma}$ pull back to $\hbar + \mathrm{ev}^{(i)*}c_1(L)$. So we see that

$$\tilde{\gamma}\cdot[\mathcal{M}_\Gamma^Y(P)]^{\mathrm{virt}} = \prod_{i=1}^{r}\left(\mathrm{ev}^{(i)*}\gamma^{(i)}\cdot(\hbar+\mathrm{ev}^{(i)*}c_1(L))^{n^{(i)}-1}\right)\cdot\frac{[\mathcal{M}]^{\mathrm{virt}}}{-\hbar-\Psi_0}. \qquad (38)$$

Using this formula we can now prove the lemma:

(i): Note that we can drop the $\mathrm{ev}^{(i)*}c_1(L)$ terms on the right hand side of (38) because $\gamma^{(i)}\cdot c_1(L) = 0$ if $\gamma^{(i)}$ is the class of a point. The non-equivariant dimension-0 part of the right hand side is therefore (up to a factor of $(-1)^{N+1}$) the number that we want to compute. First of all note that this result must be symmetric under permutations of the marked points $x_1^{(i)},\dots,x_{n^{(i)}}^{(i)}$ of a connected component. The left hand side must therefore be symmetric too, which means that in the $\prod_{j=2}^{n^{(i)}}(\alpha_j^{(i)}\psi_{x_j^{(i)}})$ term in $\tilde{\gamma}$ it does not matter whether we leave out the first marked point or any other.

Combining this with the remark that

$$\begin{aligned}
&\mathrm{ev}^{(i)*}\gamma^{(i)}\cdot(\alpha_2^{(i)}\psi_{x_2^{(i)}})\cdot\prod_{k=0}^{-\alpha_1^{(i)}-1}(k\psi_{x_1^{(i)}}+\mathrm{ev}^*_{x_1^{(i)}}Y_0)\cdot\prod_{k=0}^{-\alpha_2^{(i)}-1}(k\psi_{x_2^{(i)}}+\mathrm{ev}^*_{x_2^{(i)}}Y_0)\\
&\quad=\mathrm{ev}^{(i)*}\gamma^{(i)}\cdot((\alpha_1^{(i)}+1)\psi_{x_1^{(i)}})\cdot\prod_{k=0}^{-\alpha_1^{(i)}-2}(k\psi_{x_1^{(i)}}+\mathrm{ev}^*_{x_1^{(i)}}Y_0)\cdot\prod_{k=0}^{-\alpha_2^{(i)}}(k\psi_{x_2^{(i)}}+\mathrm{ev}^*_{x_2^{(i)}}Y_0)
\end{aligned}$$

(if $-\alpha_1^{(i)}>1$) we conclude that the result we are looking for does not depend on the $\alpha^{(i)}$ (but only on the $d^{(i)}$ and $n^{(i)}$). We can therefore choose $\alpha^{(i)}$ so that $\alpha_1^{(i)}=-1$. Then the marked point $x_1^{(i)}$ occurs only with a single divisor evaluation class in $\tilde{\gamma}$. As the class $\tilde{\gamma}$ fixes the image points of all $x_j^{(i)}$ (and these points can be taken to be distinct and not on Y_0 or Y_∞) we can apply the divisor equation and conclude that the left hand side is multiplied by $d^{(i)}$ compared to the one for $n^{(i)}-1$ marked points and the class $\tilde{\gamma}$ that has

$$\begin{aligned}
&\mathrm{ev}^{(i)*}\gamma^{(i)}\cdot\prod_{j=2}^{n^{(i)}}(\alpha_j^{(i)}\psi_{x_j^{(i)}})\cdot\prod_{j=2}^{n^{(i)}}\prod_{k=0}^{-\alpha_j^{(i)}-1}(k\psi_{x_j^{(i)}}+\mathrm{ev}^*_{x_j^{(i)}}Y_0)\\
&\quad=-\mathrm{ev}^{(i)*}\gamma^{(i)}\cdot\prod_{j=3}^{n^{(i)}}(\alpha_j^{(i)}\psi_{x_j^{(i)}})\cdot\prod_{k=0}^{-\alpha_2^{(i)}}(k\psi_{x_2^{(i)}}+\mathrm{ev}^*_{x_2^{(i)}}Y_0)\cdot\prod_{j=3}^{n^{(i)}}\prod_{k=0}^{-\alpha_j^{(i)}-1}(k\psi_{x_j^{(i)}}+\mathrm{ev}^*_{x_j^{(i)}}Y_0)
\end{aligned}$$

in the i-th component. In other words, raising an $n^{(i)}$ by 1 multiplies the dimension-0 part of the left hand side by $-d^{(i)}$. But N is then raised by 1 as well, so (as the number we are looking for is $(-1)^{N+1}$ times the left hand side) we see that the desired number is multiplied by $d^{(i)}$ when we raise $n^{(i)}$ by 1. Part (i) of the lemma now follows by induction from the initial value $\frac{1}{d^{(2)}\cdots d^{(r)}}$ in the case when $n^{(1)}=2$ and $n^{(2)}=\cdots=n^{(r)}=1$ (and thus $N=0$).

(ii): The non-equivariant dimension-0 part of the left hand side of (38) is then zero for dimensional reasons. Collecting the same terms on the right hand side we get

$$0=\left(\prod_{i=1}^{r}\mathrm{ev}^{(i)*}\gamma^{(i)}\right)\cdot\left(\Psi_0^{N+1}-(n^{(i)}-1)\cdot\mathrm{ev}^{(1)*}c_1(L)\cdot\Psi_0^N\right)\cdot[\mathcal{M}]^{\mathrm{virt}}.$$

Hence the statement of (ii) follows from (i). □

We are now ready to study a morphism that projects stable relative maps to P down to stable absolute maps to Y. In our first proposition we will simply compute the push-forward of a specific integral that has codimension 0 in the target. We will see soon however that this single result is enough e.g. to reproduce our earlier results of chapter 2.

PROPOSITION 5.3.2 (Push-forwards in codimension 0). *Let $p : P = \mathbb{P}(L \oplus O) \to Y$ be a $\mathbb{P}^1$-bundle over a smooth projective variety Y. Let $\mathcal{M} = \mathcal{M}_{g,\alpha}(P,\beta)$ be a moduli space of stable relative maps to P relative Y_∞. Denote the marked points by $y_1,\dots,y_n,x_1,\dots,x_N$, where y_i are the points with positive multiplicity and x_i are the points with zero multiplicity (so that $\alpha = (\alpha_1,\dots,\alpha_n,0,\dots,0)$). Let $p_* : \mathcal{M} \to M := \bar{M}_{g,n+N}(Y, p_*\beta)$ be the morphism that projects the curves in P to stable maps to Y. Assume that p_* is well-defined, i.e. that $n+N \geq 3$ if $g = 0$ and β is a multiple of the class of a fiber. Pick non-negative integers $m_1,\dots,m_N$ with $\sum_i m_i = Y_0 \cdot \beta + 1 - g$. Then*

$$p_*\left(\prod_{i=1}^{N}\prod_{k=0}^{m_i-1}(k\psi_{x_i} + \mathrm{ev}_{x_i}^* Y_0)\cdot[\mathcal{M}]^{\mathrm{virt}}\right) = [M]^{\mathrm{virt}}.$$

PROOF. We will compute the expression using the virtual localization of remark 5.2.6. By the virtual push-forward theorem 5.2.7 for p we know already that the result must be a scalar multiple of $[M]^{\mathrm{virt}}$. So we only have to consider the fixed point loci $F = (M_0 \boxtimes \mathcal{M}_1)/G$ that give rise to a contribution that is a multiple of $[M]^{\mathrm{virt}}$. We then have to add up these contributions and show that the result is 1.

If the contribution of a fixed point locus is to be a multiple of $[M]^{\mathrm{virt}}$ then either M_0 or $\mathcal{M}_1$ must be a moduli space of connected curves of genus g and class β. We can thus distinguish the following two cases:

(a): $\mathcal{M}_1$ is a moduli space of stable relative maps of genus g and class β, with multiplicities $\alpha_1,\dots,\alpha_n$ to Y_∞. We must then have $F = \mathcal{M}_1$, i.e. in level 0 we only have multiple covers of fibers of P that have at most one marked point x_i over Y_0 on them.

As usual, the conditions $\prod_{k=0}^{m_i-1}(k\psi_{x_i} + \mathrm{ev}_{x_i}^* Y_0)$ in the cohomology class require the degree of the multiple cover to be at least m_i at the component that has the marked point x_i on it. As the m_i sum up to $Y_0 \cdot \beta + 1 - g$ there can be at most $g-1$ components in level 0 that do not have a marked point x_i on it. (If $g = 0$ then this is impossible, so there are no contributions from these fixed point loci.) The contribution of F is thus computed by

- projecting (some integral on) $\mathcal{M}_1$ down to the space of stable maps to Y, giving rise to a class of virtual codimension (at least) g; and then
- forgetting the marked points that had mapped to Y_0 but did not have a marked point x_i attached. As there are at most $g-1$ such points we are thus left with a class of virtual codimension at least $g-(g-1) = 1$ in M. In other words, these fixed point loci can not give rise to a multiple of $[M]^{\text{virt}}$.

So we conclude that the fixed point loci of type (a) do not give any contribution.

(b): $M_0 = M$, so we have a connected stable map of genus g and class β with $N+n$ marked points on it in Y_0 in level 0. The remaining parts of the curve can then only be rational multiple covers of fibers of P. The structure of these fibers is still a little bit complicated though. To be precise, at every marked point y_i in M_0 there is attached in level 0 a totally ramified rational cover of a fiber of P of some degree $M_i \le m_i$. In level 1 we then have a stable non-rigid map with

- one marked point of multiplicity m_i mapping to Y_∞;
- one marked point mapping to Y_0 with some multiplicity $m_i \le d_i$ and connecting through a cover of the fiber in level 0 to the curve in M_0;
- some marked points mapping to Y_0 whose multiplicities to Y_0 sum up to some number $m_i - d_i$. At every such point there is attached in level 0 a totally ramified rational cover of the fiber of P of the corresponding multiplicity.

In what follows we will call such a structure a **rational fiber tail** of type (m_i, d_i) attached to the marked point y_i in M_0. The additional marked points over Y_0 in level 1 whose multiplicities add up to $m_i - d_i$ will be called the **loose marked points**.

Let us now compute the contribution of these fixed point loci. As the result has virtual codimension 0 in M we can ignore cohomological terms in the equivariant integral and only need to compute the coefficients of the pure weight terms. The contribution of a fixed point locus is easily determined from the description of the virtual normal bundle in remark 5.2.6:

(i) The class given in the proposition is just $\prod_{i=1}^{N}\prod_{k=0}^{m_i-1}(k\psi_{x_i}+\hbar+\mathrm{ev}_{x_i}^* c_1(L))$ on the fixed point locus. The pure weight term of this is 1.
(ii) The inverse virtual normal bundle terms $H^1/H^0(f^*N_{Y_0/P})$ from remark 5.2.6 (i) have pure weight contribution 1 since the group $\mathbb{C}^*$ acts on $N_{Y_0/P}$ simply by multiplication.
(iii) The multiple cover of degree d that attaches to the M_0 part contributes a factor $\prod_{k=1}^{d-1}\frac{d}{k(\hbar+\mathrm{ev}_{y_i}^* c_1(L))}$ coming from the inverse virtual normal bundle terms of remark 5.2.6 (ii)(a). The pure weight term of this is $\frac{d^d}{d!}$.
(iv) Every loose marked point of multiplicity s contributes $\prod_{k=2}^{s}\frac{s}{k(\hbar+\mathrm{ev}_{y_i}^* c_1(L))}$ to the inverse virtual normal bundle terms by remark 5.2.6 (ii)(c). The pure weight term of this is $\frac{s^{s-1}}{s!}$. If several loose marked points have the same multiplicity we have to divide by the order of the group of permutations of the loose marked points that keeps the multiplicities unchanged.
(v) Smoothing the node where the rational fiber tail attaches to the M_0 part contributes a factor of $\frac{1}{\frac{\hbar+\mathrm{ev}_{y_i}^* c_1(L)}{d}-\psi}$ to the inverse virtual normal bundle by remark 5.2.6 (iii). The pure weight coefficient of this is just d.
(vi) Smoothing the target node contributes a factor of $\frac{1}{-\hbar-\Psi_0}$ to the inverse virtual normal bundle by remark 5.2.6 (iv). Note that this term is not a priori a product of contributions for the individual rational fiber tails. But every such component receives an evaluation at a point class from the diagonal splitting, so the $\mathcal{M}_1$ invariant is given by the result of lemma 5.3.1 (i), which does happen to be a product of terms for the individual components. So we can say that the inverse virtual normal bundle terms of remark 5.2.6 (iv) contribute a factor of $(-m)^{a-1}$ for every rational fiber tail, where a denotes the number of loose marked points. (The sign arises because we need the correct Ψ_0 coefficient of $\frac{1}{-\hbar-\Psi_0}$ given by the value of N in lemma 5.3.1 (i).)

We now have to multiply all these contributions, and then sum the result up over all fixed point loci. As we have just seen, all contributions are products of terms for the individual rational fiber tails. So it suffices to prove the proposition in the case of only one rational fiber tail (i.e. $n=1$). Let us first fix its type (m,d). We then get a fixed point locus for every partition of $m-d$, corresponding to the choice of multiplicities for the loose marked points. Of the above terms only (iv) and (vi) depend on this partition. Their sum over all partitions of $m-d$ is obviously the q^{m-d} coefficient of

$$\frac{1}{m}\cdot\exp\left(-m\sum_k \frac{k^{k-1}}{k!}q^k\right) = -\sum_{k=0}^{\infty}(k-m)^{k-1}\cdot\frac{q^k}{k!}.$$

As the remaining non-trivial factors (iii) and (v) are $\frac{d^{d+1}}{d!}$ and we have to sum over all $0 \le d \le m$ we conclude that the result is the q^m coefficient of

$$-\left(\sum_{k=0}^{\infty}(k-m)^{k-1}\cdot\frac{q^k}{k!}\right)\left(\sum_{k=0}^{\infty}k^{k+1}\cdot\frac{q^k}{k!}\right), \tag{39}$$

which is

$$\frac{(-1)^m}{m!}\sum_{k=0}^{m}(-1)^k\binom{m}{k}k^m = 1.$$

□

Let us now restrict to the case where we impose evaluation and cotangent line conditions at only one of the marked points. The following important corollary shows that in genus 0 we can then compute invariants in any codimension:

COROLLARY 5.3.3. *Let $p: P = \mathbb{P}(L\oplus O) \to Y$ be a $\mathbb{P}^1$-bundle over a smooth projective variety Y, and assume that L is nef. Let $\mathcal{M} = \mathcal{M}_{0,\alpha}(P,\beta)$ be a moduli space of rational stable relative maps to P relative Y_∞ such that the first marked point has multiplicity $\alpha_1 = 0$. Let $p_*: \mathcal{M} \to M := \bar{M}_{0,|\alpha|}(Y, p_*\beta)$ be the morphism that projects the curves in P to stable maps to Y. Assume that p_* is well defined, i.e. that $|\alpha| \ge 3$ if $g = 0$ and β is a multiple of the class of a fiber. Then we have for all $m \ge 0$*

$$p_*\left(\prod_{k=0}^{m}(k\psi_1 + \mathrm{ev}_1^* Y_0)\cdot[\mathcal{M}]^{\mathrm{virt}}\right)$$
$$= \begin{cases} 0 & \text{if } m < Y_0\cdot\beta, \\ \left(\prod_{k=Y_0\cdot\beta+1}^{m}(k\psi_1 + \mathrm{ev}_1^* c_1(L))\right)\cdot[M]^{\mathrm{virt}} & \text{if } m \ge Y_0\cdot\beta. \end{cases}$$

PROOF. The statement for $m < Y_0\cdot\beta$ follows immediately from the virtual push-forward theorem 5.2.7. For $m \ge Y_0\cdot\beta$ recall that the class $\prod_{k=0}^{Y_0\cdot\beta}(k\psi_1 + \mathrm{ev}_1^* Y_0)$ requires the curve to have contact of order at least $Y_0\cdot\beta+1$ to Y_0 at x_1. As L is nef this means that a neighborhood of the curve around x_1 has to map into $Y_0 \cong Y$. But then the projection $p: P \to Y$ restricted to the curve is trivial around x_1, which means that the cotangent line class at x_1 pulls back unchanged. The statement of the corollary then follows from proposition 5.3.2 and the projection formula. □

There is a simple extension of this corollary to the "unstable case", i.e. to the case when there morphism p_* would have to project to $\bar{M}_{0,2}$:

COROLLARY 5.3.4. *Let $p : P = \mathbb{P}(L \oplus O) \to Y$ be a $\mathbb{P}^1$-bundle over a smooth projective variety Y. Let $\mathcal{M} = \mathcal{M}_{0,(0,\alpha_2)}(P,\beta)$ be a moduli space of rational stable relative maps to P relative Y_∞ such that β is a multiple of the class of a fiber. Let $p_* : \mathcal{M} \to Y$ be the obvious projection morphism. Then we have for all $m \geq 0$*

$$p_*\left(\prod_{k=0}^{m}(k\psi_1 + \mathrm{ev}_1^* Y_0) \cdot [\mathcal{M}]^{\mathrm{virt}}\right)$$
$$= \begin{cases} 0 & \text{if } m \neq \alpha_2 - 1, \\ \frac{1}{\alpha_2} \cdot [Y] & \text{if } m = \alpha_2 - 1. \end{cases}$$

PROOF. Let us assume first that $m = \alpha_2 - 1$. Again we compute the integral using virtual localization. It is easy to see that there is only one relevant fixed point locus, namely a degree-α_2 multiple cover of a fiber of P that is totally ramified over Y_0 and Y_∞ in level 0 and has one marked point over each ramification point. Using a computation similar to that in the proof of proposition 5.3.2 it is easily checked that the pure weight coefficient of the resulting equivariant integral is $\frac{1}{\alpha_2}$.

If $m > \alpha_2 - 1$ then there are no relevant fixed point loci at all, so the result is 0. If $m < \alpha_2 - 1$ then the statement follows for dimensional reasons. □

REMARK 5.3.5. Corollaries 5.3.3 and 5.3.4 imply that all rational relative Gromov-Witten invariants of P relative Y_∞ such that

- there is a marked point of multiplicity 0,
- the invariant contains a class $\mathrm{ev}^* Y_0$ at this point,
- there are only evaluation classes from Y at the other marked points

are immediately computable in terms of rational Gromov-Witten invariants of Y.

In fact, this follows from the projection formula since all relative Gromov-Witten invariants with the above properties are linear combinations of the ones computed in the corollaries (with coefficients pulled back from the base Y).

REMARK 5.3.6. Let us now describe how the result of corollaries 5.3.3 and 5.3.4 can be used to reprove the algorithm of section 2.5 to compute the rational Gromov-Witten invariants of a hypersurface in terms of those of the ambient space.

Let Y be a smooth very ample hypersurface of a projective manifold X. Pick a homology class $\beta \in H_2^+(X)$. Recall that the method of section 2.5 was to evaluate the integral

$$\prod_{k=0}^{m}(k\psi_1 + \mathrm{ev}_1^* Y)\cdot[\bar{M}_n(X,\beta)]^{\mathrm{virt}} \qquad (40)$$

for all m by an $(m+1)$-fold application of the main theorem 2.2.6. The result is that this integral is equal to

$$[\bar{M}_{(m+1,0,\dots,0)}(X,\beta)]^{\mathrm{virt}} + \sum_{k=0}^{m}\prod_{i=k+1}^{m}(i\psi_1 + \mathrm{ev}_1^* Y)\cdot[D_{(k,0,\dots,0),k}(X,\beta)]^{\mathrm{virt}} \qquad (41)$$

with the $D_{(k,0,\dots,0),k}(X,\beta)$ as in definition 2.2.3. Roughly speaking they are (unions of) moduli spaces of reducible stable maps to X, with one "internal component" that lies in Y and contains the marked point x_1, and several "external components" that intersect the internal one and that have specified multiplicities at these intersection points. This equation can then be used recursively to compute the 1-point relative invariants (if $m < Y\cdot\beta$) and the invariants of Y (if $m = Y\cdot\beta$ and thus one of the correction terms is isomorphic to $\bar{M}_n(Y,\beta)$).

Now consider the new picture that we have developed in this chapter. We compute the same integral (40) using the splitting theorem of remark 5.1.14. The result is a sum of terms, each of which is a product of relative invariants of X and P relative Y. More precisely, we get

$$\sum_{\Gamma_1,\Gamma_2} m(\Gamma_1,\Gamma_2)\cdot[\bar{M}^Y_{\Gamma_1}(X)]^{\mathrm{virt}} \boxtimes \left(\prod_{k=0}^{m}(k\psi_1 + \mathrm{ev}_1^* Y_\infty)\cdot[\bar{M}^Y_{\Gamma_2}(P)]^{\mathrm{virt}}\right) \qquad (42)$$

with the notation of remark 5.1.14. Both moduli spaces $\bar{M}^Y_{\Gamma_1}(X)$ and $\bar{M}^Y_{\Gamma_2}(P)$ can a priori describe stable relative maps with several connected components, but the total glued curve must be connected and of genus 0. By the product property of proposition 5.2.8 we can think of them as products of the corresponding moduli spaces for the individual connected components.

We claim that the old expression (41) is simply the push-forward of the new expression (42) by the morphism p_* that projects stable maps to $X\cup_Y P$ down to X. This projection obviously forgets fibers in P without marked points, so let us ignore such components from now on. Note that any other component in P that does not have the marked point x_1 on it is projected to 0 by p_* for dimensional reasons. So there can be only one component in P. Let us distinguish two cases:

(i) The component in P is a D-fold cover of a fiber for some D, it meets the singular locus Y of $X \cup_Y P$ in exactly one point, and it has no other marked points except x_1. Then this component is forgotten by p_*, and there is only one component in X. The integral $\prod_{k=0}^{m}(k\psi_1 + \mathrm{ev}_1^* Y_\infty) \cdot [\bar{M}_{\Gamma_2}^Y(P)]^{\mathrm{virt}}$ can be evaluated using corollary 5.3.4: it is zero unless $m = D-1$, in which case it is $\frac{1}{D} \cdot [Y]$. The factor of D is canceled by the multiplicity factor in (42), and the one external component must have multiplicity $D = m+1$ to Y. So this term gives precisely the first summand $[\bar{M}_{(m+1,0,\dots,0)}(X,\beta)]^{\mathrm{virt}}$ of (41).

(ii) In any other case there is a well-defined projection morphism p_* that projects the component in P down to a stable absolute map to Y. As the resulting curve must be connected and of genus 0 the components in X must all be glued to the component in P in exactly one point. So the curves that we get from these terms have the same topological type as the curves in the correction terms in (41). In fact, corollary 5.3.3 shows that the push-forward $p_*(\prod_{k=0}^{m}(k\psi_1 + \mathrm{ev}_1^* Y_\infty) \cdot [\bar{M}_{\Gamma_2}^Y(P)]^{\mathrm{virt}})$ gives precisely the correction terms $\sum_{k=0}^{m}\prod_{i=k+1}^{m}(i\psi_1 + \mathrm{ev}_1^* Y) \cdot [D_{(k,0,\dots,0),k}(X,\beta)]^{\mathrm{virt}}$ of (41).

So we have seen that our old method of chapter 2 to compute rational Gromov-Witten invariants of very ample hypersurfaces is essentially the "collapsed version" of the new method based on corollaries 5.3.3 and 5.3.4.

REMARK 5.3.7. There is an easier proof of proposition 5.3.2 that works roughly as follows. We have seen that in the equivariant integral we can neglect all cohomological terms; the result is in fact just a combinatorial statement about the various possible fixed point loci. It is therefore sufficient to compute the coefficient of $[M]^{\mathrm{virt}}$ in the case when Y is a point (and hence L is trivial). The result can then be obtained by a simple local multiplicity computation similar to the proof of proposition 2.3.3.

The reason why we have preferred the more complicated explicit evaluation of the equivariant integral is that we will need these computations in the analogous proof for the codimension-1 case in proposition 5.4.1.

5.4. Push-forwards in codimension 1

We have seen in remark 5.3.6 that corollary 5.3.3 (together with corollary 5.3.4) is sufficient to compute the rational Gromov-Witten invariants of Y from those of X.

Recall the idea of the proof of corollary 5.3.3: the expression $\prod_{k=0}^{m}(k\psi_{x_1} + \mathrm{ev}_{x_1}^* Y_0)$ can be computed...

- for $m \leq Y_0 \cdot \beta - g$ by proposition 5.3.2;
- for $m > Y_0 \cdot \beta$ from the corresponding result for $m \leq Y_0 \cdot \beta$ as the condition $\prod_{k=0}^{Y_0 \cdot \beta}(k\psi_{x_1} + \mathrm{ev}_{x_1}^* Y_0)$ ensures that further ψ_1 classes are pulled back from the base.

For curves of genus 0 these two cases cover every possibility. For higher genus there is a gap however: for genus 1 for example we are not yet able to compute the above integral if $m = Y_0 \cdot \beta$, i.e. when the push-forward by p_* has codimension 1 in the base. In this section we want to fill this gap.

PROPOSITION 5.4.1 (Push-forwards in codimension 1). *Let the notations be the same as in proposition 5.3.2, except that $\sum_i m_i = Y_0 \cdot \beta + 2 - g$. Assume that $g \leq 1$ or that conjecture 5.2.9 (ii) holds. Then we have*

$$\begin{aligned} p_* \left(\prod_{i=1}^{N} \prod_{k=0}^{m_i - 1} (k\psi_{x_i} + \mathrm{ev}_{x_i}^* Y_0) \cdot [\mathcal{M}]^{\mathrm{virt}} \right) \\ = \left(\sum_{i=1}^{N} \left(\binom{m_i}{2} \psi_{x_i} + m_i \, \mathrm{ev}_{x_i}^* c_1(L) \right) \right. \\ \left. + \sum_{i=1}^{n} \left(\binom{\alpha_i + 1}{2} \psi_{y_i} - \alpha_i \, \mathrm{ev}_{y_i}^* c_1(L) \right) \right) \cdot [M]^{\mathrm{virt}} \\ + \frac{K_1 - K_2}{2} - \lambda + \sum_{\Delta} \binom{m_\Delta}{2} [\Delta]^{\mathrm{virt}} \end{aligned}$$

where we used the following notation:

- *$K_1 = \pi_*(\omega \cdot \mathrm{ev}^* c_1(L) \cdot [\mathcal{C}]^{\mathrm{virt}})$ and $K_2 = \pi_*(\mathrm{ev}^* c_1(L)^2 \cdot [\mathcal{C}]^{\mathrm{virt}})$, where $\pi : \mathcal{C} \to M$ is the universal stable map and ω its relative dualizing sheaf;*
- *$-\lambda$ is the first Chern class of the vector bundle on M whose fiber at a point $(C, y_1, \dots, y_n, x_1, \dots, x_N, f)$ is $H^1(C, \mathcal{O}_C)$;*
- *$\sum_\Delta$ denotes the sum over all non-looping codimension-1 boundary strata Δ of M (i.e. the locus of 1-nodal curves with two connected components, with the sum taken over all splittings of the genus, the homology class, and the marked points). For such a boundary stratum Δ we denote by m_Δ the absolute value of the integer*

$$c_1(L) \cdot \beta_1 + \sum_j \alpha_j - \sum_i m_i + 1 - g_1,$$

where β_1 and g_1 are the homology class and genus of one of the components, and the sums over i and j are taken over the marked points that lie on this component. (The resulting value for m_Δ does not depend on the chosen component.)

PROOF. Again we will compute the expression using virtual localization. We now have to consider fixed point loci though that push forward to M to classes of virtual codimension 0 or 1.

(a): Fixed point loci that push forward to M to classes of virtual codimension 0. We have considered all these fixed point loci already in the proof of proposition 5.3.2. The difference is that in the old proof we only had to collect the pure weight terms, whereas now we have to add up the terms whose cohomological part is of codimension 1. The contribution from these fixed point loci will then be the sum of all these cohomological terms, evaluated on $[M]^{\text{virt}}$.

Recall that in the proof of theorem 5.3.2 we had two types of fixed point loci, depending on whether the component of genus g and degree β sits in level 1 (type (a)) or in Y_0 in level 0 (type (b)). It is checked immediately that the proof that the type (a) fixed point loci do not contribute carries over to our new situation. So let us consider the fixed point loci of type (b). Their contribution was computed in the proof of proposition 5.3.2 as a product of six factors (i),...,(vi). The one cohomological term that we now need can come from any of these factors. Let us consider all possibilities in turn.

- *The terms from (i).* The given class is $\prod_{i=1}^{N}\prod_{k=0}^{m_i-1}(k\psi_{x_i}+\hbar+\mathrm{ev}_{x_i}^* c_1(L))$ on the fixed point loci. The terms in this that are of cohomological codimension 1 replace one $\hbar$ by
$$\sum_{i=1}^{N}\sum_{k=0}^{m_i-1}(k\psi_{x_i}+\mathrm{ev}_{x_i}^* c_1(L)) = \sum_{i=1}^{N}\left(\binom{m_i}{2}\psi_{x_i}+m_i\,\mathrm{ev}_{x_i}^* c_1(L)\right)$$
relative to the computation in the old proof. Hence from these terms we are getting this codimension-1 class evaluated on the old result (i.e. $[M]^{\text{virt}}$).
- *The terms from (ii).* The cohomological codimension-1 part of the expression $H^1/H^0(f^*N_{Y_0/P})$ replaces one $\hbar$ by $c_1(H^1/H^0(f^*c_1(L)))$, which can be computed by the Grothendieck-Riemann-Roch theorem applied to the universal curve over M. We get
$$c_1(H^1/H^0(f^*c_1(L))) = c_1(H^1/H^0(O))+\frac{K_1-K_2}{2} = -\lambda+\frac{K_1-K_2}{2}.$$

- *The evaluation terms from (iii),...,(vi).* The contributions (iii),...,(vi) contain a certain number of factors $\frac{1}{\hbar+\mathrm{ev}_{y_i}^* c_1(L)}$ that replace one $\hbar$ by $-c_1(L)$ in cohomological codimension 1. More precisely, the contributions (iii), (iv), (v), (vi) contain $d-1$, $m-d-a$, 1, a of these factors, respectively (with the notations used there). Here we have used lemma 5.3.1 (ii) for (vi). The sum of these numbers is just m, i.e. the multiplicity given at the point y_i. So the evaluation terms from (iii),...,(vi) give a total contribution of $-\sum_{i=1}^n \alpha_i \mathrm{ev}_{y_i}^* c_1(L)$.
- *The cotangent line terms from (v).* In addition, the contribution (v) replaces one $\hbar$ by $d\psi_{y_i}$ in cohomological codimension 1. So to compute the coefficient of ψ_{y_i} we have to redo the sum of proposition 5.3.2 (b) with an additional factor of d from (v). Hence instead of the q^m term of (39) we now need the q^m term of

$$-\left(\sum_{k=0}^{\infty}(k-m)^{k-1}\cdot\frac{q^k}{k!}\right)\left(\sum_{k=0}^{\infty}k^{k+2}\cdot\frac{q^k}{k!}\right),$$

which is

$$\frac{(-1)^m}{m!}\sum_{k=0}^{m}(-1)^k\binom{m}{k}k^{m+1}=\binom{m+1}{2}.$$

So the total contribution from these terms is $\sum_{i=1}^n \binom{\alpha_i+1}{2}\psi_{y_i}$.

(b): Fixed point loci that push forward to M to classes of virtual codimension 1, i.e. to loci of 1-nodal curves in M. Such a fixed point locus $F=(M_0\boxtimes\mathcal{M}_1)/G$ must then be of one of the following types:

(A) There are two connected components in M_0 that are projected non-trivially by p (drawn in bold in the picture above). These two components must then be connected through a component in $\mathcal{M}_1$. But this component in $\mathcal{M}_1$ must be projected non-trivially by p as well: if it does not have positive homology class in Y then it must be a multiple of a fiber, in which case it must have a marked point over Y_∞ (and thus a total of at least three marked points). So the resulting curve in M has at least three components. But the locus of such stable maps has virtual codimension 2, so we do not get a contribution in codimension 1 from these fixed point loci.

(B) There is one connected component in M_0 that is projected non-trivially by p, and this component has two marked points that are connected by a loop through $\mathcal{M}_1$. These fixed point loci do not contribute for the same reason as in (A).

(C) There are two connected components in $\mathcal{M}_1$ that are projected non-trivially by p. These components must then be connected through level 0, but this connection must become unstable when projected by p for the same reason as in (A). The connection can therefore only be a union of two multiple covers of a fiber without marked points that are glued over Y_0. Moreover, there must not be any component in M_0 that is projected non-trivially by p. The argument of the proof of proposition 5.3.2 (a) now shows that this cannot give rise to a non-zero contribution.

(D) There is one connected component in $\mathcal{M}_1$ that is projected non-trivially by p, and this component has two marked points over Y_0 that are connected by a loop through level 0. These fixed point loci do not contribute for the same reason as in (C).

(E) There is one component each in M_0 and $\mathcal{M}_1$ that is projected non-trivially by p. This is in fact the only case that gives a contribution, and that we will study now in detail.

We will call the two components the absolute and the non-rigid component, respectively. The absolute component can obviously have some of the marked points x_i on it. Moreover, it can connect to marked points y_i through rational fiber tails as in the proof of proposition 5.3.2 (b). The non-rigid component can obviously have marked points y_i on it, but it can also connect to some marked points x_i through multiple covers of a fiber in level 0. The homology class and the genus can obviously split in any way onto the two components. So in the result we will get every possible non-looping codimension-1 boundary stratum of M with some coefficient.

Let us fix such a stratum Δ now and compute its coefficient. For simplicity let us first assume that the two components of the curves in Δ are labeled, so that it makes sense to talk about the first and second component of the curves. We then just fix splittings $g = g_0 + g_1$, $p_*\beta = \beta_0 + \beta_1$, $I_0 \cup I_1 = \{1,\dots,N\}$, $J_0 \cup J_1 = \{1,\dots,n\}$ and require that the absolute (resp. non-rigid) component has genus g_0, homology class β_0, and marked points $\{x_i \,;\, i \in I_0\}$ and $\{y_j \,;\, j \in J_0\}$ (resp. genus g_1, homology class β_1 in Y, and marked points $\{x_i \,;\, i \in I_1\}$ and $\{y_j \,;\, j \in J_1\}$). The two components have to be connected by a multiple cover of a fiber in level 0 of some degree $D > 0$. We will call this the connecting fiber. Let us fix the value of D for a moment and only add up the contributions from the fixed point loci where D has this given value. We will later add up the contributions for all possible values of D.

Note that the projection p maps our fixed point loci to a codimension-1 stratum in M, and we are computing an integral in codimension 1. So we can ignore cohomological terms in the equivariant integral and only compute the pure weight terms. The sum of the pure weight terms over all fixed point loci as above will be the coefficient of $[\Delta]^{\text{virt}}$ in the final result.

The trick to compute this sum is to set up a different equivariant integral that yields exactly the same fixed point loci and the same pure weight terms, and that we have essentially computed already. To do so let $\mathcal{M}'$ be the moduli space of stable relative maps to P relative Y_∞ with two connected components, where

- the first component has genus g_0, marked points $\{y_j \,;\, j \in J_0\}$ with multiplicities α_j and z_0 with multiplicity 0, and homology class β'_0 that projects to β_0 in Y and intersects Y_∞ with multiplicity $\sum_{j \in J_0} \alpha_j$;
- the second component has genus g_1, marked points $\{x_i \,;\, i \in I_1\}$ with multiplicity 0, $\{y_i \,;\, i \in J_1\}$ with multiplicities α_j, and z_1 with multiplicity 0, and homology class β'_1 that projects to β_1 in Y and intersects Y_∞ with multiplicity $\sum_{j \in J_1} \alpha_j$.

For this moduli space we compute the equivariant integral $p_*(\gamma\cdot[\mathcal{M}']^{\mathrm{virt}})$ by localization, where

$$\gamma=\prod_{k=0}^{N_0-1}(k\psi_{z_0}+\mathrm{ev}_{z_0}^*Y_0)\cdot$$
$$D\psi_{z_1}\cdot\prod_{0\le k<N_1,k\ne D}(k\psi_{z_1}+\mathrm{ev}_{z_1}^*Y_0)\cdot\prod_{i\in I_1}\prod_{k=0}^{m_i-1}(k\psi_{x_i}+\mathrm{ev}_{x_i}^*Y_0)$$

with $N_0=Y_0\cdot\beta_0'+1-g_0$ and $N_1=Y_0\cdot\beta_1'-\sum_{i\in I_1}m_i+1-g_1$, and where $p:\mathcal{M}'\to M'$ denotes again the projection to the corresponding space of stable maps to Y. For dimensional reasons the result must be a multiple of $[M']^{\mathrm{virt}}$ by the virtual push-forward theorem 5.2.7. To compute this multiple we have to evaluate the pure weight terms of the relevant fixed point loci in the same way as we did in proposition 5.3.2. We claim that these relevant fixed point loci are in one-to-one correspondence with the ones for our original problem on $\mathcal{M}$, and that the coefficients of the pure weight terms agree up to a constant factor for every such fixed point locus. In fact, this is obvious by the description of the fixed point loci and their contributions in remark 5.2.6, except for the part concerning the connecting fiber:

- In the original integral on $\mathcal{M}$ the connecting fiber contributes pure weight coefficients of $\frac{D^D}{D!}$ for changing the ramification structure by remark 5.2.6 (ii) (b) and D for smoothing the node by remark 5.2.6 (iii).
- In the new integral on $\mathcal{M}'$ the product

$$D\psi_{z_1}\cdot\prod_{0\le k<N_1,k\ne D}(k\psi_{z_1}+\mathrm{ev}_{z_1}^*Y_0)\cdot\prod_{i\in I_1}\prod_{k=0}^{m_i-1}(k\psi_{x_i}+\mathrm{ev}_{x_i}^*Y_0)$$

in γ forces the marked point z_1 to lie over Y_0 on a multiple cover of a fiber in level 0 of degree D: first of all all marked points x_i and z_1 must lie on a multiple cover of a fiber of P as otherwise the fixed point locus would be projected by p to stable maps with a node. The condition above then forces the degrees of these covers to be at least m_i at x_i, and at least N_1 at z_1 unless it is equal to D. By the same dimension counting argument as in the proof of proposition 5.3.2 (a) we see that only the case of multiplicity D at z_1 is possible.

The pure weight coefficients are then

$$-\prod_{0\le k<N_1, k\neq D}(1-\frac{k}{D})$$

from the class γ, and $\frac{D^D}{D!}$ for changing the ramification structure by remark 5.2.6 (ii) (a).

Comparing these two computations we see that the coefficient of Δ in our original expression arising from fixed point loci with connecting fiber of degree D is equal to

$$-D\cdot\prod_{0\le k<N_1, k\neq D}\left(\frac{D}{D-k}\right)=(-1)^{D+N_1}\cdot\frac{D^{N_1}}{D!(N_1-D-1)!}$$

times the coefficient of $[M']^{\text{virt}}$ in $p_*(\gamma\cdot[\mathcal{M}']^{\text{virt}})$. But the latter is easily computed: simply note that the result does not change if we replace $D\psi_{z_1}$ by $D\psi_{z_1}+\text{ev}_{z_1}^* Y_0$ in γ since a factor of $\text{ev}_{z_1}^* Y_0^2$ would result in a class that is projected by p to codimension at least 1. It then follows immediately by the product property of proposition 5.2.8 (resp. conjecture 5.2.9 (ii)) and proposition 5.3.2 that $p_*(\gamma\cdot[\mathcal{M}']^{\text{virt}})=1\cdot[M']^{\text{virt}}$.

We therefore conclude that the total contribution of Δ in our original expression is equal to

$$\sum_{D>0}(-1)^{D+N_1}\cdot\frac{D^{N_1}}{D!(N_1-D-1)!}=-\max\left(\binom{N_1}{2},0\right)$$

where

$$\begin{aligned}N_1&=Y_0\cdot\beta_1'-\sum_{i\in I_1}m_i+1-g_1\\&=Y_\infty\cdot\beta_1'+c_1(L)\cdot\beta_1-\sum_{i\in I_1}m_i+1-g_1\\&=c_1(L)\cdot\beta_1+\sum_{j\in J_1}\alpha_j-\sum_{i\in I_1}m_i+1-g_1.\end{aligned}$$

Finally, recall that this is the result for a boundary stratum with two *labeled* components. So for simplicity we should add to this result the number that we get by exchanging the two components, and then only sum (as usual) over strata with unlabeled components. Exchanging the two factors replaces N_1 by

$$c_1(L)\cdot\beta_0+\sum_{j\in J_0}\alpha_j-\sum_{i\in I_0}m_i+1-g_0=(c_1(L)\cdot\beta+\sum_{j=1}^{n}\alpha_j-\sum_{i=1}^{N}m_i+2-g)-N_1$$
$$=-N_1.$$

So the coefficient of a boundary stratum Δ is just

$$-\max\left(\binom{N_1}{2},0\right)-\max\left(\binom{-N_1}{2},0\right)=-\binom{|N_1|}{2},$$

as we have claimed in the proposition. □

COROLLARY 5.4.2. *Let $p: P=\mathbb{P}(L\oplus O)\to Y$ be a $\mathbb{P}^1$-bundle over a smooth projective variety Y, and assume that L is nef. Let $\mathcal{M}=\mathcal{M}_{1,\alpha}(P,\beta)$ be a moduli space of elliptic stable relative maps to P relative Y_∞ such that the first marked point has multiplicity $\alpha_1=0$. Let $p_*:\mathcal{M}\to M:=\bar{M}_{1,|\alpha|}(Y,p_*\beta)$ be the morphism that projects the curves in P to stable maps to Y. Then we have for all $m\geq 0$*

$$p_*\left(\prod_{k=0}^{m}(k\psi_1+\mathrm{ev}_1^*Y_0)\cdot[\mathcal{M}]^{\mathrm{virt}}\right)$$
$$=\begin{cases}0 & \text{if } m<Y_0\cdot\beta-1,\\ [M]^{\mathrm{virt}} & \text{if } m=Y_0\cdot\beta-1,\\ \left(\prod_{k=Y_0\cdot\beta+1}^{m}(k\psi_1+\mathrm{ev}_1^*c_1(L))\right)\cdot\Gamma & \text{if } m\geq Y_0\cdot\beta,\end{cases}$$

where Γ denotes the expression of proposition 5.4.1. In particular, all elliptic relative Gromov-Witten invariants of P relative Y_∞ such that

- *there is a marked point of multiplicity 0,*
- *the invariant contains a class ev^*Y_0 at this point,*
- *there are only evaluation classes from Y at the other marked points*

are immediately computable in terms of elliptic (and rational) Gromov-Witten invariants of Y.

PROOF. The proof is the same as that of corollary 5.3.3, now using proposition 5.4.1 in addition to proposition 5.3.2. □

5.5. Elliptic Gromov-Witten invariants of the quintic threefold

COROLLARY 5.5.1. *Let $Y \subset X = \mathbb{P}^4$ be a smooth quintic threefold. There is an explicit algorithm to compute the rational and elliptic Gromov-Witten invariants of Y from those of $\mathbb{P}^4$.*

PROOF. By the results of section 2.5 we can assume that the rational Gromov-Witten invariants of Y are known, as well as the relative invariants of X relative Y with at most two marked points (including invariants with primitive cohomology classes). The elliptic Gromov-Witten invariants of $\mathbb{P}^4$ are known e.g. by theorem 1.4.4.

We will describe an algorithm that determines recursively the following four elliptic invariants in every degree d:

(i) the Gromov-Witten invariant $\langle H \rangle_{1,d}$ of Y;
(ii) the relative Gromov-Witten invariant $\langle \tau^{5d-2}(H^3) \rangle_{1,d}$;
(iii) the relative Gromov-Witten invariant $\langle \tau^{5d-1}(H^2) \rangle_{1,d}$;
(iv) the relative Gromov-Witten invariant $\langle \tau^{5d}(H) \rangle_{1,d}$.

In fact, these are the only primary 1-point absolute and relative elliptic invariants of the quintic threefold.

The strategy to compute these numbers is very similar to that of remark 5.3.6: on the moduli space of 1-pointed stable maps of degree d to X we compute the 0-dimensional integrals

$$\mathrm{ev}_1^* H^m \cdot \prod_{k=0}^{5d-m} (k\psi_1 + \mathrm{ev}_1^* Y) \cdot [\bar{M}_1(X,d)]^{\mathrm{virt}}$$

for $m = 0, \dots, 3$ using the splitting theorem of remark 5.1.14. We claim that the four resulting equations determine the four invariants listed above.

In fact, let us investigate the terms that occur in the splitting equation. It is clear that we only get invariants of genus 0 and 1. The invariants of P are projected down to Y by corollary 5.3.3 and 5.4.2, giving rise to absolute invariants of Y. All these invariants can be computed immediately in terms of the basic invariants $\langle \; \rangle_{1,d}$ as the moduli spaces of (0-pointed) curves in Y all have expected dimension 0. The invariants of X give rise to elliptic relative invariants with one marked point without cotangent line classes (and rational relative invariants which may have two marked points and primitive cohomology classes in case of a loop). So it is clear that the four invariants mentioned above are the only unknown invariants occurring in the four splitting equations. It is also clear that the unknown invariants must occur linearly in the equations.

It only remains to compute which of the four invariants occurs in which equations with non-zero coefficient. The case of the relative invariants is simple: they occur in the same way as in remark 5.3.6, so we get (ii), (iii), (iv) only in the equation for $m = 3$, $m = 2$, $m = 1$, respectively (and it occurs with coefficient 1). The absolute invariant (i) can occur in the equations for $m = 0$ and $m = 1$ only as higher powers of $\mathrm{ev}_1^* H$ are necessarily zero on moduli spaces of stable maps to Y. So it only remains to show that the coefficient of the absolute invariant (i) in the equation for $m = 0$ is non-zero. But this follows from proposition 5.4.1:

$$\begin{aligned} p_* &\left(\prod_{k=0}^{5d} (k\psi_1 + \mathrm{ev}_1^* Y) \cdot [\mathcal{M}]^{\mathrm{virt}} \right) \\ &\qquad = \left(\binom{5d+1}{2} \psi_1 + (5d+1)\, \mathrm{ev}_1^*(5H) \right) \cdot [\bar{M}_{1,1}(Y,d)]^{\mathrm{virt}} \\ &\qquad = 5d(5d+1) \langle\ \rangle_{1,d}, \end{aligned}$$

where $\mathcal{M}$ is the corresponding moduli space of stable relative maps to P.

Hence we have shown that the four equations from the splitting theorem for $m = 0, \ldots, 3$ are uniquely solvable for the four invariants (i),...,(iv). $\square$

EXAMPLE 5.5.2. As an example of the algorithm described in corollary 5.5.1 let us compute the first invariant $n_{1,1} = \langle\ \rangle_{1,1}$ of the quintic threefold explicitly. We compute the invariant

$$\prod_{k=0}^{5} (k\psi_1 + \mathrm{ev}_1^* 5H) \cdot [\bar{M}_{1,1}(\mathbb{P}^4, 1)]^{\mathrm{virt}} = \cdots = 250$$

(i.e. the $m = 0$ case in the proof of corollary 5.5.1) using the splitting theorem of remark 5.1.14. We get the following terms (the components in the pictures are labeled by their genus):

(A) We have a rational multiple cover of a fiber in P, glued to an elliptic component of degree 1 in X. These terms do not give any contribution as corollary 5.3.4 would require the degree of the multiple cover (and hence the multiplicity of the component in X to Y) to be 6, which is impossible.

(B) We have an elliptic multiple cover of a fiber in P, glued to a line in X. There are two non-zero degree-1 rational relative invariants $\langle\tau^4(H^3)\rangle_{0,1}$ and $\langle\tau^5(H^2)\rangle_{0,1}$, so the multiplicity at the gluing point can be 4 or 5. Let us consider the case of multiplicity 4, so that the invariant from X is $\langle\tau^4(H^3)\rangle_{0,1} = 30$ (see example 2.5.8). The elliptic component in P then gets an evaluation class $\frac{1}{5}\cdot \mathrm{ev}_{y_1}^* 1$ from the diagonal splitting. So the elliptic invariant from P is given by corollary 5.4.2 by

$$(5\psi_1 + \mathrm{ev}_1^* 5H)\cdot\left(\binom{5}{2}\psi_{x_1} + 5\,\mathrm{ev}_{x_1}^* 5H + \binom{5}{2}\psi_{y_1} - 4\,\mathrm{ev}_{y_1}^* 5H - \lambda\right) \cdot [\bar{M}_{1,2}(Y,0)]^{\mathrm{virt}}.$$

Note that $\bar{M}_{1,2}(Y,0) = \bar{M}_{1,2}\times Y$, and

$$[\bar{M}_{1,2}(Y,0)]^{\mathrm{virt}} = c_3(X) - \lambda c_2(X) = -40H^3 - 10\lambda H^2$$

using this decomposition. Inserting this in the above expression gives the result $-\frac{12625}{12}$. Together with a multiplicity factor of 4 from the splitting theorem we conclude that the total contribution from these curves is

$$-\frac{12625}{12}\cdot 4\cdot\frac{1}{5}\cdot 30 = -25250.$$

In the same way we find a contribution of -40625 for the case of multiplicity 5 at the gluing point. Altogether we therefore get a contribution of -65875 from the curves of type (B).

(C) Pushing this term forward to Y gives the invariant that we would like to compute. We get $30n_{1,1}$ by the calculation at the end of the proof of corollary 5.5.1.

(D) We have a rational multiple cover of a fiber in P, glued at two points (hence forming a loop and yielding genus 1) to a line in X. The two multiplicities at the gluing point can be any numbers m_1, m_2 with $m_1 + m_2 \leq 5$; we get a contribution for every such choice. As an example let us consider the choice $m_1 = 1$, $m_2 = 4$, so that the multiple cover in P has degree 5. We have computed the corresponding invariant in X already to be 5425 in example 2.5.11. The invariant in P is given by corollary 5.3.3 by $\frac{1}{5}\,\mathrm{ev}^* H^3$ (from the diagonal splitting) on $\bar{M}_{0,3}(Y,0)$, which is 1. Together with a multiplicity factor of $1 \cdot 4$ from the splitting theorem we thus get a contribution of 21700.
The contributions for the other multiplicities (m_1, m_2) are computed in a similar way. We list them in the following table.

$(1,1)$	$(1,2)$	$(1,3)$	$(2,2)$	$(1,4)$	$(2,3)$
$\frac{625}{2}$	1250	1875	1250	21700	32550

Their sum is $\frac{117875}{2}$, so this is the contribution that we get from the terms of type (D).

Altogether we now arrive at the equation

$$250 = -65875 + 30n_{1,1} + \frac{117875}{2},$$

from which we deduce that $n_{1,1} = \frac{2875}{12}$.

EXAMPLE 5.5.3. The following table lists the four primary elliptic 1-point invariants of the quintic threefold that occur in the proof of corollary 5.5.1. They have been computed with the C++ program GROWI [**Ga5**]. The numbers $n_{1,d} = \langle\ \rangle_{1,d}$ are the elliptic absolute invariant of the quintic. They agree numerically with the prediction of Bershadsky et al. in [**BCOV1**]. The numbers $N_{1,d}$ are the integral invariants obtained from the $n_{1,d}$ by the Gopakumar-Vafa correction of [**P3**] section 3.

	$\langle\tau^{5d-2}(H^3)\rangle_{1,d}$	$\langle\tau^{5d-1}(H^2)\rangle_{1,d}$	$\langle\tau^{5d}(H)\rangle_{1,d}$
$d=1$	$-\frac{55}{12}$	$-\frac{2425}{24}$	$-\frac{10375}{24}$
$d=2$	-2130	$-\frac{258525}{2}$	-1604750
$d=3$	-1739835	$-\frac{681816425}{4}$	$-\frac{96167766875}{24}$
$d=4$	-1104866000	-139084431500	$-\frac{96981145446875}{12}$
$d=5$	$\frac{2929112127165}{2}$	$\frac{1254446774084025}{4}$	$-\frac{254143882366065725}{24}$
$d=6$	10649810105988480	2492419792539577700	$\frac{70171494491795157875}{6}$
$d=7$	40406603028060820650	$\frac{32828999580140417044075}{3}$	$\frac{1350565740517460928228375}{8}$

	$n_{1,d}$	$N_{1,d}$
$d = 1$	$\frac{2875}{12}$	0
$d = 2$	$\frac{407125}{8}$	0
$d = 3$	$\frac{243388750}{9}$	609250
$d = 4$	$\frac{382833353125}{16}$	3721431625
$d = 5$	$\frac{93716201322650}{3}$	12129909700200
$d = 6$	$\frac{103669556513320375}{2}$	31147299733286500
$d = 7$	$\frac{8078223459917903604625}{84}$	71578406022880761750

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