THE NUMBER OF PLANE CONICS 5-FOLD TANGENT TO A GIVEN CURVE

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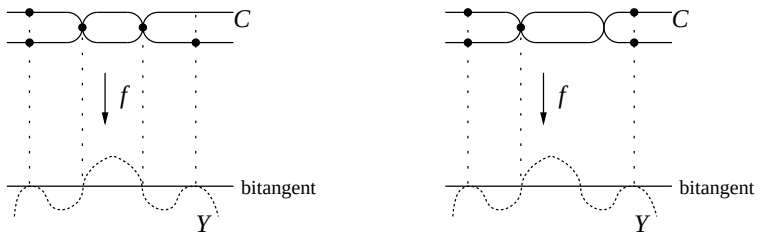
Abstract. Given a general plane curve Y of degree d, we compute the number n_d of irreducible plane conics that are 5-fold tangent to Y. This problem has been studied before by Vainsencher [V] using classical methods, but it could not be solved there because the calculations received too many non-enumerative correction terms that could not be analyzed. In our current approach, we express the number n_d in terms of relative Gromov-Witten invariants that can then be directly computed. As an application, we consider the K3 surface given as the double cover of $\mathbb{P}^2$ branched along a sextic curve. We compute the number of rational curves in this K3 surface in the homology class that is the pull-back of conics in $\mathbb{P}^2$, and compare this number to the corresponding Yau-Zaslow K3 invariant. This gives an example of such a K3 invariant for a non-primitive homology class.

Let $Y \subset \mathbb{P}^2$ be a generic plane curve of degree $d \geq 5$. We want to consider smooth plane conics that are 5-fold tangent to Y. As the space of all plane conics is 5-dimensional and each tangency imposes one condition on the curves, we expect a finite number of such 5-fold tangent conics. It will be easy to see that this number is indeed finite; let us call it n_d. The goal of this paper is to compute it.

Of course this is a classical problem, and attempts have been made to solve it using classical methods of enumerative geometry. I. Vainsencher [V] tried to use various blow-ups of the ordinary $\mathbb{P}^5$ of conics as moduli spaces, but the intersection of the five tangency conditions in this moduli space always resulted in a scheme with many non-enumerative and non-reduced components whose geometry was so complicated that the problem could not be solved that way.

In this paper we use different moduli spaces, namely moduli spaces of relative stable maps, to solve the problem. There is a well-defined compact moduli space $\bar{M}^Y_{(2,2,2,2,2)}(\mathbb{P}^2, 2) \subset \bar{M}_{0,5}(\mathbb{P}^2, 2)$ that parametrizes rational stable maps to $\mathbb{P}^2$ of degree 2 (i.e. conics) with 5 marked points such that the stable map is tangent to Y at all these points. It comes equipped with a 0-dimensional virtual fundamental class, whose degree N_d can be computed explicitly using the methods of [Ga1].

We can interpret the number N_d as the "virtual number" of conics that are 5-fold tangent to Y. It is only virtual because it contains — just as in Vainsencher's classical computations — non-enumerative contributions from the "boundary" of the moduli space. These contributions are quite simple however. It is not hard to see that the only degree-2 rational stable maps $f : C \to \mathbb{P}^2$ that satisfy the tangency conditions at the 5 marked points are all double covers of a bitangent of Y, and have the marked points distributed in one of the following two ways:



There are only finitely many stable maps with marked points as in this picture on the left, so we just have to count them and subtract their number from the virtual invariant N_d. The picture on the right however shows a 1-dimensional family of stable maps (the unmarked ramification point of f can move). We will use equations from relative Gromov-Witten theory to compute the degree of the 0-dimensional virtual fundamental class of the moduli space on this 1-dimensional component. By subtracting both correction terms, we finally arrive at the enumerative numbers n_d. They are

$$n_d = \frac{1}{5!}\, d\,(d-3)\,(d-4)\,(d^7+12d^6-18d^5-540d^4+251d^3+5712d^2-1458d-14580).$$

Our (as well as Vainsencher's) motivation for studying this problem came from a question concerning rational curves in K3 surfaces. If X is a K3 surface and $\beta \in H_2(X,\mathbb{Z})$ the homology class of a holomorphic curve in X, then various authors [YZ], [B], [G], [BL] have shown that the number of rational curves in X of class β is equal to the q^d coefficient of the series

$$G(q) = \prod_{i>0} \frac{1}{(1-q^i)^{24}} = 1 + 24q + 324q^2 + 3200q^3 + 25650q^4 + 176256q^5 + \cdots$$

with $d = \frac{1}{2}\beta^2 + 1$, *if the class β is primitive*, i.e. not a non-trivial multiple of a smaller integral homology class. There is a well-defined "K3 invariant" (using a modified obstruction theory on the moduli spaces of stable maps to X) for non-primitive β too [BL]; it is however not known yet how this invariant relates to the above series $G(q)$ or to the enumerative number.

The result of this paper allows us to study this question in a non-trivial example: we let X be the double cover of $\mathbb{P}^2$ branched along a sextic curve Y, and take β to be the pull-back of the class of conics in $\mathbb{P}^2$. Our work allows us to compute the enumerative number of rational curves in X of class β, which we can then compare to the corresponding number of the series $G(q)$. The result is that the "K3 invariant" is equal to the corresponding term in the $G(q)$ series, plus a double cover correction term that is equal to $\frac{1}{8}$ times the number of rational curves in X of class $\frac{1}{2}\beta$. Note that this is the same sort of correction term as for multiple covers of rational curves in Calabi-Yau threefolds. We conjecture that this pattern continues for classes β of higher divisibility.

The paper is organized as follows. In section 1 we will show how to compute the relative Gromov-Witten invariant N_d. We analyze the moduli space $\overline{M}^Y_{(2,2,2,2,2)}(\mathbb{P}^2,2)$ and its virtual fundamental class in sections 2 and 3, respectively, leading to the final result for n_d in corollary 3.6. Section 4 contains the application to K3 surfaces mentioned above.

We thank J. Kock for introducing us to the problem, as well as for a Maple program to compute the relative Gromov-Witten invariants using the algorithm

of [Ga1]. The author is also grateful to the Institute for Advanced Study for its hospitality and stimulating working atmosphere. This work has been funded by the NSF grant DMS 9729992.

1. Relative Gromov-Witten invariants

In this section we will show how to compute the relative Gromov-Witten invariant that corresponds to the number of conics that are 5-fold tangent to a given smooth plane curve. We will use the notations and results from [Ga1], to which we also refer for further details.

Definition 1.1. *Let $Y \subset \mathbb{P}^2$ be a smooth curve of degree d, and let $m_1, \ldots, m_n$ be non-negative integers. We denote by $\bar{M}_{(m_1,\ldots,m_n)} = \bar{M}^Y_{(m_1,\ldots,m_n)}(\mathbb{P}^2, 2)$ the moduli space of n-pointed relative stable maps of degree 2 to $\mathbb{P}^2$ relative to Y with multiplicities $m_1, \ldots, m_n$, as defined in* [Ga1] *definitions 1.1 and 1.18. Its virtual fundamental class is denoted $[\bar{M}_{(m_1,\ldots,m_n)}]^{virt}$.*

Remark 1.2. The moduli space $\bar{M}_{(m_1,\ldots,m_n)}$ can be thought of as a compactification of the space of irreducible plane conics together with n distinct marked points on them at which the conic has the prescribed local intersection multiplicities $m_1, \ldots, m_n$ with Y. In particular, the moduli space $\bar{M}_{(2,2,2,2,2)}$ corresponds to conics 5-fold tangent to Y.

Remark 1.3. For future reference let us recall the precise definition from [Ga1]. Consider the degree-d Veronese embedding $i : \mathbb{P}^2 \to \mathbb{P}^D$ with $D = \binom{d+2}{2} - 1$. We have $i(Y) = i(\mathbb{P}^2) \cap H$ in $\mathbb{P}^D$ for a suitable hyperplane $H \subset \mathbb{P}^D$. The inclusion morphism i induces an inclusion of moduli spaces $\bar{M}_{0,n}(\mathbb{P}^2, 2) \subset \bar{M}_{0,n}(\mathbb{P}^D, 2d)$. Moreover, let $\bar{M}^H_{(m_1,\ldots,m_n)}(\mathbb{P}^D, 2d)$ be the closure in $\bar{M}_{0,n}(\mathbb{P}^D, 2d)$ of all stable maps $(C, x_1, \ldots, x_n, f)$ such that C is irreducible, $f(C) \not\subset H$, and the divisor f^*Y on C contains the points x_i with multiplicities m_i. Then we define

$$\bar{M}_{(m_1,\ldots,m_n)} := \bar{M}_{0,n}(\mathbb{P}^2, 2) \cap \bar{M}^H_{(m_1,\ldots,m_n)}(\mathbb{P}^D, 2d)$$

(with the intersection taken in $\bar{M}_{0,n}(\mathbb{P}^D, 2d)$). The virtual fundamental class is the corresponding refined intersection product

$$[\bar{M}_{(m_1,\ldots,m_n)}]^{virt} := [\bar{M}_{0,n}(\mathbb{P}^2, 2)] \cdot [\bar{M}^H_{(m_1,\ldots,m_n)}(\mathbb{P}^D, 2d)] \in A_*(\bar{M}_{(m_1,\ldots,m_n)}).$$

The virtual dimension of $\bar{M}_{(m_1,\ldots,m_n)}$ is $5 - \sum_i (m_i - 1)$. By abuse of notation, we will always drop the superscript *virt* from the notation of the virtual fundamental class from now on, as we do not need the ordinary fundamental classes of these spaces.

Remark 1.4. From remark 1.3 we get immediately the following statement: let $\mathcal{C} = (C, x_1, \ldots, x_n, f) \in \bar{M}_{(m_1,\ldots,m_n)}$ be an automorphism-free stable map such that $C \cong \mathbb{P}^1$ is irreducible and $f(C) \not\subset Y$. Then, locally around this point, $\bar{M}_{(m_1,\ldots,m_n)}$ is scheme-theoretically the subscheme of $\bar{M}_{0,n}(\mathbb{P}^2, 2)$ given by the $\sum_i m_i$ equations that describe the vanishing of the m_i-jets of ev_i^*Y at the points x_i, where $ev_i : \bar{M}_{0,n}(\mathbb{P}^2, 2) \to \mathbb{P}^2$ are the evaluation maps. If moreover $\bar{M}_{(m_1,\ldots,m_n)}$ has the expected dimension at this point $\mathcal{C}$, then this point lies on a unique irreducible component of $\bar{M}_{(m_1,\ldots,m_n)}$, and the virtual fundamental class on this component is just the ordinary scheme-theoretic fundamental class, i.e. the length of the scheme $\bar{M}_{(m_1,\ldots,m_n)}$ at this irreducible component.

Remark 1.5. There is an easier description of $\bar{M}_{(m_1,\dots,m_n)}$ *as a set*. Namely, $\bar{M}_{(m_1,\dots,m_n)}$ is the subspace of $\bar{M}_{0,n}(\mathbb{P}^2,2)$ of all n-pointed rational stable maps $(C,x_1,\dots,x_n,f)$ of degree 2 to $\mathbb{P}^2$ such that the following two conditions are satisfied:

(i) $f(x_i)\in Y$ for all i such that $m_i>0$,
(ii) $f^*Y-\sum_i m_i x_i\in A_0(f^{-1}(Y))$ is effective.

As mentioned above, the moduli space $\bar{M}_{(2,2,2,2,2)}$ has virtual dimension zero and corresponds to conics 5-fold tangent to Y (together with a labeling of the five tangency points). Hence we define:

Definition 1.6. *The number*

$$N_d := \frac{1}{5!}\cdot\deg[\bar{M}_{(2,2,2,2,2)}]\in\mathbb{Q}$$

will be called the virtual number of conics 5-fold tangent to Y.

The number N_d is only virtual because it receives correction terms from double covers of lines (see sections 2 and 3). In the rest of this section we will show how to compute the number N_d. Obviously, we can assume that $d\geq 5$.

The computation is done using the main theorem 2.6 of [Ga1] that tells us "how to raise the multiplicities of the moduli spaces": it says that

$$(ev_n^*Y+m_n\psi_n)\cdot[\bar{M}_{(m_1,\dots,m_n)}]=[\bar{M}_{(m_1,\dots,m_{n-1},m_n+1)}]+\text{correction terms},\qquad(1)$$

where $ev_n:\bar{M}_{(m_1,\dots,m_n)}\to\mathbb{P}^2$ is the evaluation map at the last marked point, and ψ_n is the first Chern class of the cotangent line bundle L_n, i.e. of the bundle whose fiber at a stable map $(C,x_1,\dots,x_n,f)$ is the cotangent space $T^\vee_{C,x_n}$. The correction terms are as follows. Every correction term corresponds to a moduli space of reducible curves with $r+1$ components $C_0,\dots,C_r$, where C_0 is contracted[1] to a point of Y, and the other components C_i intersect C_0 in a point where they have local intersection multiplicity μ_i to Y. We get such a correction term for every r, every choice of the μ_i, and every splitting of the total homology class and the marked points onto the components C_i, such that the following two conditions are satisfied:

(a) the last marked point x_n lies on the component C_0,
(b) the sum of all the μ_i is equal to the sum of those m_i such that $x_i\in C_0$.

These correction terms appear in the above equation with multiplicity $\prod_{i=1}^r\mu_i$.

Example 1.7. Here is an example of equation (1). In the case

$$(ev_5^*Y+\psi_5)\cdot[\bar{M}_{(2,2,2,2,1)}]=[\bar{M}_{(2,2,2,2,2)}]+\text{correction terms}$$

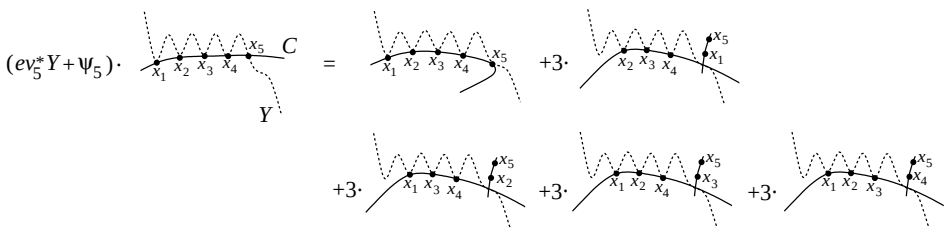
we want to figure out the correction terms. As this is an equation in (virtual) dimension 0, the contracted component C_0 must have exactly 3 special points (it would not be stable if it had fewer, and it would have moduli if it had more). Hence the correction terms fall into these two categories:

(i) $r=1$ (in the above notation), C_1 is a conic, and C_0 is a contracted component with three special points x_5, the intersection point with C_1, and one other x_i for $i=1,\dots,4$,

[1]This uses the fact that the curve Y has positive genus, and that therefore every rational stable map to Y must be constant. In general, C_0 can be a curve with any homology class in Y.

(ii) $r = 2$, C_1 and C_2 two lines, and C_0 is a contracted component with three special points x_5 and the two intersection points with C_1 and C_2.

Actually, case (ii) cannot occur, because condition (b) above cannot be satisfied: the sum $\mu_1 + \mu_2$ is at least 2, whereas m_5 is only 1. Hence the only correction terms are of type (i). We get four of them: one for each choice of the point x_i that is to lie on the contracted component C_0. We have $\mu_1 = m_i + m_5 = 3$ in each of these cases by condition (b). All four correction terms appear with multiplicity $\mu_1 = 3$. Pictorially, the equation reads



Here, the dotted curve is the fixed curve Y, and the solid curve is the moving conic C. In the four correction terms, the component with x_5 on it is meant to be contracted. Written down as an equation of virtual fundamental classes of moduli spaces, the formula reads

$$(ev_5^*Y + \psi_5) \cdot [\bar{M}_{(2,2,2,2,1)}] = [\bar{M}_{(2,2,2,2,2)}] + 3[\bar{M}_{(3,2,2,2)}] + 3[\bar{M}_{(2,3,2,2)}] + 3[\bar{M}_{(2,2,3,2)}] + 3[\bar{M}_{(2,2,2,3)}]. \tag{2}$$

Remark 1.8. The correction terms in equation (1) are themselves products of moduli spaces of relative stable maps of the form $\bar{M}^Y_{(m'_1,\dots,m'_k)}(\mathbb{P}^2, d')$ with $d' \leq 2$ and $m'_1 + \cdots m'_k \leq m_1 + \cdots m_n$. In other words, this equation expresses invariants (i.e. intersection products of ev_i^*H and ψ_i classes, where H is a line in $\mathbb{P}^2$) on the relative moduli space $\bar{M}_{(m_1,\dots,m_n+1)}$ in terms of other invariants on relative moduli spaces $\bar{M}^Y_{(m'_1,\dots,m'_k)}(\mathbb{P}^2, d')$ whose "total multiplicity" $\sum_i m'_i$ is smaller than the total multiplicity $1 + \sum_i m_i$ of $M_{(m_1,\dots,m_n+1)}$. Hence, applying equation (1) recursively $m_1 + \cdots + m_n$ times, we can express every invariant on $\bar{M}_{(m_1,\dots,m_n)}$ in terms of invariants on $\bar{M}_{(0,\dots,0)}$, which are just ordinary Gromov-Witten invariants of $\mathbb{P}^2$. As the Gromov-Witten invariants of $\mathbb{P}^2$ are well-known, we can thus compute all relative Gromov-Witten invariants recursively, in particular N_d. Example 1.7 is the first step in this recursion process; it expresses the invariant $\bar{M}_{(2,2,2,2,2)}$ (with total multiplicity 10) in terms of invariants with total multiplicity 9.

Without actually carrying out the recursion, we can see the following.

Lemma 1.9. *The function $d \mapsto N_d$ is a polynomial of degree 10 with leading coefficient $\frac{1}{5!}$.*

Proof. Using equation (1) it is easy to show by induction that every invariant (i.e. intersection product of ev_i^*H and ψ_i classes) on a moduli space $\bar{M}_{(m_1,\dots,m_n)}$ is a polynomial in d of degree (at most) $m_1 + \cdots + m_n$. In fact, this is obvious for $m_1 + \cdots + m_n = 0$, as we then just have ordinary Gromov-Witten invariants of $\mathbb{P}^2$ (that do not depend on Y). Equation (1) reads

$$[\bar{M}_{(m_1,\dots,m_{n-1},m_n+1)}] = (d\, ev_n^*H + m_n\psi_n) \cdot [\bar{M}_{(m_1,\dots,m_n)}] - \text{correction terms}.$$

All correction terms have total multiplicity at most $m_1 + \cdots + m_n$, so by induction hypothesis they contribute a polynomial in d of degree at most $m_1 + \cdots + m_n$. The same is true for the ψ_n summand on the right hand side. Hence, as every invariant on $\bar{M}_{(m_1,\ldots,m_n)}$ is a polynomial in d of degree at most $m_1 + \cdots + m_n$ by assumption, it follows that every invariant on $\bar{M}_{(m_1,\ldots,m_n+1)}$ is a polynomial in d of degree at most $m_1 + \cdots + m_n + 1$.

It can be seen from the same recursive formula that the d^{10} coefficient of the invariant $\deg[\bar{M}_{(2,2,2,2,2)}]$ is just

$$\prod_{i=1}^{5} ev_i^* H^2 \cdot [\bar{M}_{0,5}(\mathbb{P}^2, 2)],$$

i.e. the number of conics through 5 general points in the plane. This number is 1, proving the statement of the lemma about the leading coefficient. □

The precise form of the polynomial N_d is quite complicated and can only be obtained by carrying out the full recursion as described above. We only give the result here; a Maple program to compute it can be obtained from the author on request.

Proposition 1.10. *For $d \geq 5$, the virtual number of conics 5-fold tangent to Y is*

$$N_d = \frac{1}{5!}\, d\,(d-3)\,(d-4)\,(d^7 + 12d^6 - 18d^5 - 540d^4 + 311d^3 + 5457d^2 - 2133d - 12690).$$

Remark 1.11. The first few values of N_d are given in the following table.

d	5	6	7	8	9	10
N_d	1985	71442	687897	3893256	16180398	54679380

Remark 1.12. In this section we have only used equation (1) in the Chow ring of the moduli space of stable maps $\bar{M}_{0,n}(\mathbb{P}^2, 2)$. In fact, there is a refined version of this equation that we will need in section 3. If $\mathcal{P}_k$ denotes the rank-$(k+1)$ bundle of (relative) k-jets of $ev_n^*\mathcal{O}(Y)$, there is a section σ of the line bundle $\mathcal{P}_{m_n}/\mathcal{P}_{m_n-1}$ on $\bar{M}_{(m_1,\ldots,m_n)}$ whose vanishing describes precisely the condition that the map f of a stable map $(C, x_1, \ldots, x_n, f)$ has multiplicity (at least) $m_n + 1$ to Y at x_n. The first Chern class of this line bundle is $ev_n^* Y + m_n \psi_n$. (In fact, this is the idea how equation (1) is proven.) The refined version of equation (1) now states that this equation also holds in the Chow group of the zero locus of the section σ on $\bar{M}_{(m_1,\ldots,m_n)}$.

2. The components of $\bar{M}_{(2,2,2,2,2)}$

Having just computed the invariant N_d, we will now study its enumerative significance. To do this, we have to identify the components of $\bar{M}_{(2,2,2,2,2)}$ and compute their virtual fundamental classes. We will assume from now on that Y is generic of degree $d \geq 5$. The equation of Y is $F = \sum_I a_I z^I = 0$, where I runs over all multi-indices (i_0, i_1, i_2) with $i_0 + i_1 + i_2 = d$, and z_0, z_1, z_2 are the homogeneous coordinates on $\mathbb{P}^2$. We start with irreducible stable maps whose image is a smooth conic.

Lemma 2.1. *Let $m_1, \ldots, m_n$ be non-negative integers such that $\sum m_i \leq 10$, and let $\mathcal{C} \in \bar{M}_{(m_1,\ldots,m_n)}$ be an irreducible stable map that is not a double cover of a line. Then:*

(i) *The moduli space* $\bar{M}_{(m_1,\dots,m_n)}$ *is smooth of dimension* $5-\sum_i(m_i-1)$ *at* $\mathcal{C}$ *(which is the expected dimension).*
(ii) *If this expected dimension is negative, then there is no such point* $\mathcal{C}$*.*

Proof. The plane degree-d curves are parametrized by a projective space $\mathbb{P}^D$ with $D=\frac{1}{2}d(d+3)$, whose coordinates are the coefficients a_I of F. Let $Z\subset \bar{M}_{0,n}(\mathbb{P}^2,2)\times \mathbb{P}^D$ be the closed substack of pairs $((C,x_1,\dots,x_n,f),Y)$ such that the pull-back by f of the equation of Y vanishes at the points x_i to order m_i for all i. We claim that Z is smooth of the expected dimension at every point $((C,x_1,\dots,x_n,f),Y)$ such that C is irreducible and f is not a double cover of a line.

To prove this, we have to show that the matrix of derivatives of the equations describing Z has maximal rank at the given point $((C,x_1,\dots,x_n,f),Y)$. By a projective coordinate transformation of $\mathbb{P}^2$ and choosing homogeneous coordinates on $C\cong\mathbb{P}^1$, we can assume that the map f is given by $(s:t)\mapsto(s^2:st:t^2)$, and the n marked points are $(1:\lambda_i)$ with pairwise distinct λ_i.

Let us now write down the derivatives of the multiplicity equations with respect to the first $m:=\sum_i m_i$ of the variables $a_{(d,0,0)}$, $a_{(d-1,1,0)}$, $a_{(d-1,0,1)}$, $a_{(d-2,1,1)}$, $a_{(d-2,0,2)}$, $a_{(d-3,1,2)}$, $a_{(d-3,0,3)}$, $a_{(d-4,1,3)}$, $a_{(d-4,0,4)}$, $a_{(d-5,1,4)}$ (remember $m\le 10$ and $d\ge 5$). These coordinates are chosen to be the coefficients of $s^{2d-i}t^i$ for $i=0,\dots,9$ when we substitute the map f into F.

Multiplicity m_i at the point $(s:t)=(1:\lambda_i)$ means that $F|_{s=1,t=\lambda_i+\epsilon}$ has no ϵ terms of order less than m_i. So the rows of the derivative matrix are just $(\binom{i}{k}\lambda_j^{i-k})_{i=0,\dots,m-1}$, for $0\le k<m_j$ and $1\le j\le n$. For example, for $m_1=m_2=m_3=m_4=m_5=2$ we get the matrix

$$\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^9 \\ 0 & 1 & 2\lambda_1 & \cdots & 9\lambda_1^8 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \lambda_5 & \lambda_5^2 & \cdots & \lambda_5^9 \\ 0 & 1 & 2\lambda_5 & \cdots & 9\lambda_5^8 \end{pmatrix}.$$

By subtracting λ_1 times the i-th column from the $(i+1)$-st column for $1\le i<m$ and using induction, we see that the determinant is $\prod_{i<j}(\lambda_i-\lambda_j)^{m_im_j}$. In particular, it is not zero, so Z is smooth of the expected dimension at $((C,x_1,\dots,x_n,f),Y)$.

By remark 1.4, the statement of the lemma is just that the fiber of Z over a general point of $\mathbb{P}^D$ is smooth of the expected dimension around a point considered above. This follows now from the Bertini theorem. □

Using remark 1.4 again, the following two corollaries are immediate.

Corollary 2.2. *Every irreducible stable map in* $\bar{M}_{(m_1,\dots,m_n)}$ *whose image in* $\mathbb{P}^2$ *is a smooth conic lies in a unique irreducible component of* $\bar{M}_{(m_1,\dots,m_n)}$ *of the expected dimension. The virtual fundamental class of this component is equal to the usual one.*

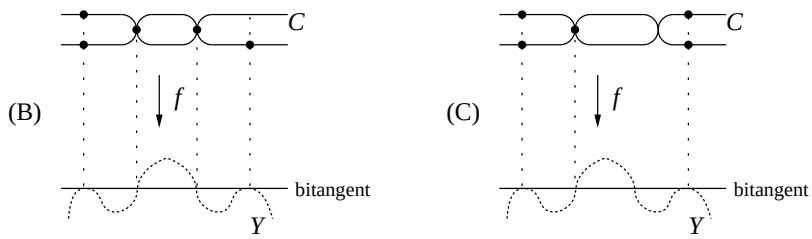
Corollary 2.3. *The number of smooth plane conics 5-fold tangent to* Y *is finite. We denote it by* n_d*.*

We will now study the additional non-enumerative contributions to the virtual invariant N_d.

Lemma 2.4. *The moduli space* $\bar{M}_{(2,2,2,2,2)}$ *has the following connected components:*

(A) $5!$ *points for every smooth conic 5-fold tangent to* Y.

(B) $5! \cdot (d-4)(d-5)$ *points for every bitangent of* Y*, corresponding to double covers of the bitangent, with marked points as in the picture below on the left, i.e. the map is ramified over two transverse intersection points of the bitangent with* Y*, and the five marked points are the two ramification points, both inverse image points of one bitangency point, and one inverse image point of the other bitangency point.*



(C) $\frac{1}{4} \cdot 5! \cdot (d-4)$ *smooth rational curves for every bitangent of* Y*, corresponding to double covers of the bitangent, with marked points as in the picture above on the right, i.e. (for a general stable map in this smooth rational curve) the map is ramified over one transverse intersection point of the bitangent with* Y *and one other arbitrary point, and the five marked points are the first ramification point, and the four inverse image points of the two bitangency points.*

Proof. Case 1: the image of the stable map is a smooth conic. If the five marked points are distinct in $\mathbb{P}^2$, we get the components (A) by lemma 2.1, with the $5!$ corresponding to the labeling of the marked points. If two of the points coincide in $\mathbb{P}^2$ (i.e. lie on a contracted component of the stable map), then by the description of $\bar{M}_{(2,2,2,2,2)}$ in remark 1.5 the conic must have contact of order (at least) 4 to Y at this point, i.e. it lies in $\bar{M}_{(4,2,2,2)}$. But this space is empty by lemma 2.1.

Case 2: the image of the stable map is a union of two (distinct) lines. It is easy to see that the conditions of remark 1.5 cannot be satisfied in this case.

Case 3: the stable map is a double cover of a line. There are six possible points of tangency to Y: the four inverse image points of the bitangency points, and the two ramification points if they are mapped to points of Y. For the stable map in $\bar{M}_{(2,2,2,2,2)}$ we can pick any five of these six points. If we leave out one of the points over the bitangency points, we arrive at the components (B), otherwise we get the components (C).

In case (B) we get a factor of $5!$ for the choice of labeling of the marked points, a factor of $\binom{d-4}{2}$ for the choice of two transverse intersection points of Y with the bitangent, and a factor of 2 for the choice of bitangency point over which we take only one inverse image point to be marked. In case (C) the second ramification point is not fixed, so we get one-dimensional families of such curves. Every such family has a 2:1 map to the bitangent given by the image of the moving ramification point; the two stable maps in a fiber of this map differ by exchanging the marked points over one bitangency point. The map is simply ramified over the two stable maps where the moving ramification point is one of the bitangency points. Hence every such family is a $\mathbb{P}^1$. The number of such families is $d-4$ (for the choice of transverse intersection point of the bitangent with Y) times $\frac{1}{4} \cdot 5!$ (for the labeling

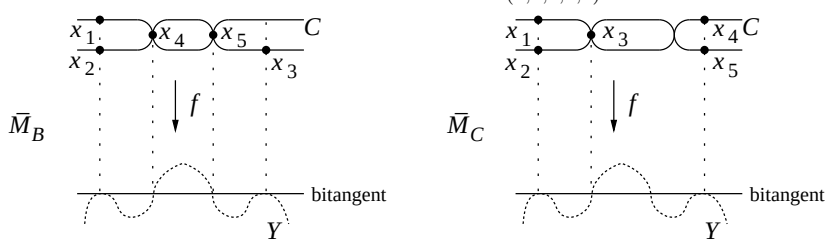
of the marked points, taking into account that exchanging the marked points over the bitangency points does not give us a new family). □

Of course, the virtual fundamental class of $\bar{M}_{(2,2,2,2,2)}$ splits naturally into a sum of virtual fundamental classes on each of the connected components that we have just identified. As it is well-known that the number of bitangents of Y is $\frac{1}{2}d(d+3)(d-2)(d-3)$ (see e.g. [H] exercise IV.2.3 f), we get the following corollary.

Corollary 2.5. *We have*

$$N_d = n_d + \frac{1}{2}d(d+3)(d-2)(d-3)(d-4)(d-5)b_d + \frac{1}{8}d(d+3)(d-2)(d-3)(d-4)c_d,$$

where b_d is the degree of the part of the virtual fundamental class of $\bar{M}_{(2,2,2,2,2)}$ supported on the point $\bar{M}_B \in \bar{M}_{(2,2,2,2,2)}$ below, and c_d is the corresponding degree supported on the smooth rational curve $\bar{M}_C \subset \bar{M}_{(2,2,2,2,2)}$ below.



3. Computation of the virtual fundamental classes

In this section we will do the necessary computations to determine the numbers b_d and c_d of corollary 2.5. Most of them are simple calculations in local coordinates, so we will only sketch these parts and leave the details to the reader.

The computation of b_d is quite simple, as the component $\bar{M}_B$ of $\bar{M}_{(2,2,2,2,2)}$ has the expected dimension.

Lemma 3.1. $b_d = 1$ *for all d.*

Proof. By remark 1.4 we just have to show that the 10 equations of vanishing of the 1-jets of$ev_i^* F$ for $i = 1, \dots, 5$ locally cut out the point $\bar{M}_B$ in $\bar{M}_{0,5}(\mathbb{P}^2, 2)$ scheme-theoretically with multiplicity 1. Let us start with the 1-jets at the points x_1, x_2, x_3, i.e. with the space $\bar{M}_{(2,2,2)}$. We can choose the coordinates on $\mathbb{P}^2$ such that the bitangent is $\{z_2 = 0\} \subset \mathbb{P}^2$ and the bitangency points are $(1:0:0)$ and $(0:1:0)$. This means that

$$a_{(d,0,0)} = a_{(d-1,1,0)} = 0, \quad a_{(d-1,0,1)}, a_{(d-2,2,0)} \neq 0 \qquad \text{(tangency at } (1:0:0)),$$
$$a_{(0,d,0)} = a_{(1,d-1,0)} = 0, \quad a_{(0,d-1,1)}, a_{(2,d-2,0)} \neq 0 \qquad \text{(tangency at } (0:1:0)).$$

Moreover, we can choose coordinates on the source $\mathbb{P}^1$ such that the stable map $\bar{M}_B$ is given by $(s:t) \mapsto (s^2 - t^2 : st : 0)$, and the marked points are $x_1 = (1:0)$, $x_2 = (0:1)$, $x_3 = (1:1)$. Local coordinates of $\bar{M}_{0,3}(\mathbb{P}^2, 2)$ around this point are then $\epsilon_1, \dots, \epsilon_8$, where the stable map is given by

$$(s:t) \mapsto (s^2 - t^2 + \epsilon_1 s^2 + \epsilon_2 st + \epsilon_3 t^2 : st + \epsilon_4 s^2 + \epsilon_5 t^2 : \epsilon_6 s^2 + \epsilon_7 st + \epsilon_8 t^2).$$

The three tangency equations are that $F|_{s=1,t=\xi}$, $F|_{s=\xi,t=1}$, and $F|_{s=1,t=1+\xi}$ have no constant and linear ξ terms. It is an easy computation to see that these 6

equations, linearized in the ϵ_i, give

$$\epsilon_1 + \epsilon_2 + \epsilon_3 = \epsilon_4 = \epsilon_5 = \epsilon_6 = \epsilon_7 = \epsilon_8 = 0,$$

so $\bar{M}_{(2,2,2)}$ is smooth of dimension 2 at the point $\bar{M}_B$ (with the points x_4 and x_5 forgotten).

Now let us consider the two other tangency conditions at the points x_4 and x_5. As the four coordinates of $\bar{M}_{(2,2,2,0,0)}$ around $\bar{M}_B$ we can choose the images of the ramification points and a point in the domain of the stable map in the neighborhood of each ramification point. Considering only one ramification point for now, the two corresponding local coordinates are $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$, where the stable map is given locally in affine coordinates as $t \mapsto t^2 + \tilde{\epsilon}_1$, and the marked point is $t = \tilde{\epsilon}_2$. Tangency means that the constant and linear ξ terms of $(\tilde{\epsilon}_2 + \xi)^2 + \tilde{\epsilon}_1$ vanish, so linearly in the $\tilde{\epsilon}_i$ we get $\tilde{\epsilon}_1 = \tilde{\epsilon}_2 = 0$. The same is true for the other ramification point, so we see that $\bar{M}_B$ is a smooth point of $\bar{M}_{(2,2,2,2,2)}$. □

To study the space $\bar{M}_C$, we need a lemma that tells us how the stable maps in $\bar{M}_C$ can be deformed if we relax some of the multiplicity conditions.

Lemma 3.2. *Let H be a line in $\mathbb{P}^2$, and let $P \in H$ be a point where Y is simply tangent to H. Let $\mathcal{C} = (C, x_1, x_2, f) \in \bar{M}_{(2,2)}$ be a (possibly reducible) double cover of H, such that $f^{-1}(P) = \{x_1, x_2\}$. Then every stable map in $\bar{M}_{(2,2)}$ in a neighborhood of $\mathcal{C}$ is also a double cover of a (maybe different) line.*

Proof. It is obvious that $\mathcal{C}$ cannot be deformed into a union of two distinct lines in $\bar{M}_{(2,2)}$. So we have to show that $\mathcal{C}$ cannot be deformed to an irreducible smooth conic in $\bar{M}_{(2,2)}$.

We use the classical space of complete conics (which is isomorphic to $\bar{M}_{0,0}(\mathbb{P}^2, 2)$). Recall that this space is the closure in $\mathbb{P}^5 \times (\mathbb{P}^5)^\vee$ of the set $(C, C^\vee)$, where C is an irreducible conic and $C^\vee$ its dual. For our given point $\mathcal{C}$ (with the two marked points forgotten for the moment), C is the double line H, and $C^\vee$ is the union of the two lines in $\mathbb{P}^\vee$ that correspond to the two ramification points of f in $\mathbb{P}^2$. Assume that we can deform $(C, C^\vee)$ in the space of complete conics to an irreducible conic that is still tangent to Y at two points in the neighborhood of P. In particular, we would then deform $C^\vee$ to an irreducible conic that is tangent to the dual of Y (or more precisely if H is a bitangent of Y: to the branch of the dual of Y that corresponds to the point P) at two points in this neighborhood. By the continuity of intersection products this means that both lines of $C^\vee$ must actually be the line corresponding to the point P. Hence both ramification points of f would have to be P. This means that $\mathcal{C}$ must have a contracted rational component over P, in contradiction to the assumption $f^{-1}(P) = \{x_1, x_2\}$. □

We want to reduce the computation of c_d to spaces that have the expected dimension. To do this, we use equation (2) from example 1.7. Note that by remark 1.12 this equation is true in the Chow group of the geometric zero locus of the section σ (that describes the tangency condition and whose zero locus has class $ev_5^* Y + \psi_5$), so it makes sense to restrict the equation to a connected component of this zero locus. Using lemma 3.2 it is easy to see that $\bar{M}_C$ is such a connected component, so we will restrict equation (2) to $\bar{M}_C$ and denote this restriction by $|_{\bar{M}_C}$.

Note first that on $\bar{M}_C$ the point x_5 can only come close to x_4, but never to the other three marked points. Hence equation (2) restricted to $\bar{M}_C$ reads

$$\left((ev_5^*Y + \psi_5) \cdot [\bar{M}_{(2,2,2,2,1)}]\right)|_{\bar{M}_C} = c_d + 3\,[\bar{M}_{(2,2,2,3)}]|_{\bar{M}_C}. \tag{3}$$

Let us first compute the virtual fundamental classes occurring in this equation.

Lemma 3.3.

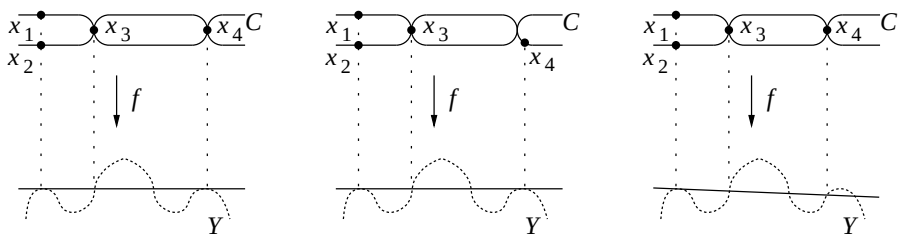
(i) *The (one-dimensional) virtual fundamental class of $\bar{M}_{(2,2,2,2,1)}$ on $\bar{M}_C$ is twice the usual one.*

(ii) *The degree of the (zero-dimensional) virtual fundamental class of $\bar{M}_{(2,2,2,3)}$ on $\bar{M}_C$ is 3.*

Proof. We will only sketch the computations.

(i): It is enough to do the computation at a general point of $\bar{M}_C$. We have seen in the proof of lemma 3.1 that $\bar{M}_{(2,2,2,2,0)}$ is smooth of dimension 2 at a general point of $\bar{M}_C$. But requiring multiplicity 1 at the point x_5 gives us a factor of 2 (i.e. ev_5^*Y cuts out $\bar{M}_C$ in $\bar{M}_{(2,2,2,2,0)}$ with multiplicity 2) because x_5 lies on a tangency point of the stable map with Y.

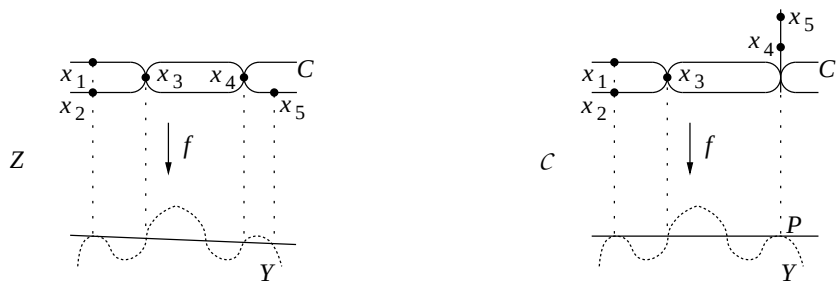
(ii): The only point of $\bar{M}_{(2,2,2,3)}$ in $\bar{M}_C$ is the stable map in the following picture on the left:



Its multiplicity can be computed using remark 1.4. Let us study the space $\bar{M}_{(2,2,2,2)}$ at this point first. By lemma 3.2 every stable map in $\bar{M}_{(2,2,2,2)}$ in a neighborhood of this point must also be a double cover of a line. It follows easily that, locally around this point, $\bar{M}_{(2,2,2,2)}$ is reducible, with two smooth one-dimensional components coming together: one of them is keeping the image line $f(C)$ to be the bitangent and moving the ramification point at x_4, while keeping x_4 on an inverse image point of the bitangency point (picture in the middle). A local coordinate for this component is ϵ, where the stable map is given locally in affine coordinates as $t \mapsto z = t(t+\epsilon)$, and the marked point x_4 has coordinate $t = 0$. Now the equation F of Y vanishes on the bitangent with multiplicity 2 in z, so this equation pulled back to the curve is locally $t^2(t+\epsilon)^2$. Its t^2 coefficient vanishes to order 2 in ϵ, so this component contributes 2 to the virtual fundamental class of $\bar{M}_{(2,2,2,3)}$. The other component is deforming the line in $\mathbb{P}^2$ away from the bitangent, with marked points as in the picture above on the right. Requiring multiplicity 3 at x_4 now restricts the line back to the bitangent with multiplicity 1. Hence the total degree of the virtual fundamental class of $\bar{M}_{(2,2,2,3)}$ on $\bar{M}_C$ is $2 + 1 = 3$. □

To evaluate the left hand side of equation (3) it is not enough to compute the integral of $ev_5^*Y + \psi_5$ on $\bar{M}_C$. There may also be contributions from components of $\bar{M}_{(2,2,2,2,1)}$ that just intersect $\bar{M}_C$, if the section σ above (whose zero locus has class $ev_5^* + \psi_5$) vanishes on them at a point of $\bar{M}_C$. Let us compute these contributions.

Lemma 3.4. *The only component Z of $\bar{M}_{(2,2,2,2,1)}$ that meets $\bar{M}_C$ but is not $\bar{M}_C$ itself corresponds to double covers of simple tangent lines of Y, with ramification and marked points as in the picture on the left:*



Its virtual fundamental class is equal to the usual one. It intersects $\bar{M}_C$ in the point $\mathcal{C}$ in the picture on the right. The section σ on Z vanishes at $\mathcal{C}$ with multiplicity 4.

Proof. By lemma 3.2 (applied to the two marked points x_1 and x_2) no stable map in $\bar{M}_C$ can be deformed into an element in $\bar{M}_{(2,2)}$ that is not itself a double cover of a line. So the only possible deformation is that we still have a double cover, however not of the bitangent but rather a nearby line. It is now easy to see that such a deformation is only possible for the stable map $\mathcal{C} \in \bar{M}_C$ in the picture above on the right, and that the deformation has to be the one in the picture on the left.

The computation of the virtual fundamental class on this component Z is completely analogous to similar calculations in previous lemmas and is therefore omitted. It remains to compute the order of vanishing of the section σ on Z at $\mathcal{C}$. Let us forget for a moment that the stable map splits off a contracted rational component at this point. Let $P = f(x_4) = f(x_5)$ be the bitangency point of $\mathcal{C}$. Choose local affine coordinates z_1, z_2 of $\mathbb{P}^2$ around P such that the local equation of Y is $z_2 = z_1^2 + O(z_1^3)$, and $z_2 = 0$ is the bitangent. As a local coordinate for Z around $\mathcal{C}$ we can choose ϵ, where the stable map is given locally around P by $t \mapsto (z_1 = t^2 - \frac{1}{2}\epsilon^2, z_2 = \frac{1}{4}\epsilon^4 + O(\epsilon^5) + t^2\, O(\epsilon^4))$, and the marked points x_4 and x_5 are $t = 0$ and $t = \epsilon$. (Note that this stable map is still a double cover of a line, $t = 0$ is a ramification point that maps to Y, and $t = \epsilon$ another point that maps to Y.)

Now actually the stable map splits off a rational contracted component at $\mathcal{C}$, which corresponds to blowing up the point $(\epsilon = 0, t = 0)$ in our family of stable maps. Hence the true local coordinates of this family of stable maps are the coordinates of this blow-up, i.e. t/ϵ and ϵ instead of t and ϵ. So the marked points x_4 and x_5 have coordinates $t = 0$ and $t/\epsilon = 1$; in particular they do not coincide any more at $\epsilon = 0$.

The vanishing of the section σ is the condition of tangency of f to Y at x_5. So to compute its order of vanishing at $\mathcal{C}$ we have to look at the linear ξ coefficient of the equation of Y evaluated at the point $t/\epsilon = 1 + \xi$, i.e. of

$$z_2 - z_1^2 + O(z_1^3) = \frac{1}{4}\epsilon^4 - (\frac{1}{2}\epsilon^2 + \xi\epsilon^2)^2 + O(\epsilon^5) = -\xi\epsilon^4 - \xi^2\epsilon^4 + O(\epsilon^5).$$

Hence the linear ξ coefficient vanishes with multiplicity 4 at $\epsilon = 0$, which proves the lemma. □

We can now assemble the results of our local calculations to compute the number c_d.

Lemma 3.5. *$c_d = -1$ for all d.*

Proof. We will evaluate equation (3)

$$\left((ev_5^*Y + \psi_5) \cdot [\bar{M}_{(2,2,2,2,1)}]\right)|_{\bar{M}_C} = c_d + 3\,[\bar{M}_{(2,2,2,3)}]|_{\bar{M}_C}.$$

The right hand side is c_d+9 by lemma 3.3 (ii). The left hand side gets a contribution from the components of $\bar{M}_{(2,2,2,2,1)}$ that intersect $\bar{M}_C$, and a contribution from $\bar{M}_C \subset \bar{M}_{(2,2,2,2,1)}$ itself. The former is 4 by lemma 3.4. The latter is twice the degree of $ev_5^*Y + \psi_5$ on $\bar{M}_C$ by lemma 3.3 (i). Note that the degree of ev_5^*Y on $\bar{M}_C$ is zero, as the image point of x_5 is fixed in $\bar{M}_C$.

To compute the integral of ψ_5 on $\bar{M}_C$, we will give a section of the cotangent line bundle L_5 and compute its zero locus. Let z be a local coordinate around the bitangency point $f(x_4) = f(x_5)$. Then f^*dz defines a section of L_5. This section vanishes only at the point where the moving ramification point comes to x_4 and x_5, i.e. at the point $\mathcal{C}$ in the picture of lemma 3.4 on the right. The computation of the order of vanishing is very similar to the calculation in lemma 3.4. Ignoring the fact that $\mathcal{C}$ splits off a rational contracted component for x_4 and x_5, a local coordinate for $\bar{M}_C$ around $\mathcal{C}$ is ϵ, where the stable map is given locally by $t \mapsto z = t(t-\epsilon)$, the points x_4 and x_5 are $t = 0$ and $t = \epsilon$, and the moving ramification point is at $t = \frac{1}{2}\epsilon$. Now, as in the proof of the previous lemma, taking into account the contracted rational component of $\mathcal{C}$ means that we have to blow up the point $(t = 0, \epsilon = 0)$, and the coordinates are actually t/ϵ and ϵ. Now we see that

$$f^*dz = \frac{\partial}{\partial \frac{t}{\epsilon}}\left(\epsilon^2 \cdot \frac{t}{\epsilon}\left(\frac{t}{\epsilon} - 1\right)\right) d\frac{t}{\epsilon} = \epsilon^2\left(2\frac{t}{\epsilon} - 1\right) d\frac{t}{\epsilon},$$

which vanishes with multiplicity 2 in ϵ around 0 at the point x_5. Hence the integral of ψ_5 over $\bar{M}_C$ is 2.

Putting everything together, we get $4 + 2 \cdot 2 = c_d + 9$, and therefore $c_d = -1$. □

We can now insert the values for N_d, b_d and c_d from proposition 1.10, lemma 3.1, and lemma 3.5, respectively, into the equation from lemma 2.5, and get the following final result.

Corollary 3.6. *For $d \geq 5$, the enumerative number of conics 5-fold tangent to Y is*

$$n_d = \frac{1}{5!}\, d\,(d-3)\,(d-4)\,(d^7 + 12d^6 - 18d^5 - 540d^4 + 251d^3 + 5712d^2 - 1458d - 14580).$$

Remark 3.7. The first few values of n_d are given in the following table.

d	5	6	7	8	9	10
n_d	2015	70956	684222	3878736	16137873	54575640

4. Application to rational curves on K3 surfaces

Let X be a K3 surface, and let $\beta \subset H_2(X, \mathbb{Z})$ be the class of a holomorphic curve in X. The moduli space of stable maps to X of class β has virtual dimension -1, so there is no corresponding Gromov-Witten invariant of X. However, we have chosen X such that it contains rational curves of class β, and we would like to count them. The reason for the mismatch in the virtual dimension is that, in the space of all K3 surfaces, only a 1-codimensional subset of K3 surfaces contains curves in the class β at all. So, if $\mathcal{X}$ is a general 1-dimensional family of K3 surfaces with X as central fiber, and $\tilde{\beta} \in H_2(\mathcal{X}, \mathbb{Z})$ is the class induced by β via the inclusion $X \subset \mathcal{X}$, then

the only rational curves in $\mathcal{X}$ of class $\tilde{\beta}$ are in fact curves of class β in X. But now the virtual dimension of the space of stable maps to $\mathcal{X}$ of class β is 0, hence there is a corresponding Gromov-Witten invariant. This invariant counts curves in $\mathcal{X}$ of class $\tilde{\beta}$ and therefore curves in X of class β; so we would like to call this number n_β "the number of rational curves in X of class β". A rigorous definition of the invariant n_β of X along these lines has been given in [BL]. The number n_β does not depend on a family $\mathcal{X}$ chosen to define it.

The numbers n_β have been computed in various papers ([YZ], [B], [G], [BL]) under the assumption that the class β is primitive, i.e. not a non-trivial multiple of a smaller integral homology class. The result is that n_β is equal to the q^d coefficient in the series

$$\begin{aligned} G(q) = \prod_{i>0} \frac{1}{(1-q^i)^{24}} &= 1 + 24q + 324q^2 + 3200q^3 + 25650q^4 + 176256q^5 + \cdots \\ &=: \sum_{d\geq 0} G_d q^d. \end{aligned}$$

where $d = \frac{1}{2}\beta^2 + 1$. It is not known yet what the numbers are if β is not primitive.

The results of this paper allow us to compute the number n_β explicitly in a case where β is not primitive. Let $Y \subset \mathbb{P}^2$ be a general sextic curve, and let $\pi : X \to \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ branched along Y. It is well-known that X is a K3 surface. Let us start by considering curves on X in the (primitive) class $\beta = \pi^*\ell$, where ℓ is the class of a line. The pull-back of a general line in $\mathbb{P}^2$ will be a 2:1 cover of $\mathbb{P}^1$, branched along the 6 intersection points of Y with the line, hence it is a curve of genus 2. We get a rational curve (with 2 nodes) on Y as a pull-back of a line if and only if the line is a bitangent of Y: the pull-back is then again a 2:1 cover, but with 2 nodes (the bitangency points), and only 2 ramification points (the remaining 2 intersection points of Y with the line). So we see that $n_{\pi^*\ell}$ has to be the number of bitangents of Y. In fact, this number is 324 (see e.g. [H] exercise IV.2.3 f), which is equal to G_2 (note that $\frac{1}{2}(\pi^*\ell)^2 + 1 = 2$).

Now let us consider rational curves in X of class $2\pi^*\ell$, i.e. pull-backs of conics — this class is not primitive any more. The pull-back of a general conic will be a 2:1 cover of the conic ramified at 12 points, so it is a curve of genus 5. We can get rational curves in the following ways:

(i) Pull-backs of (smooth) conics that are 5-fold tangent to Y. These will be 2:1 covers of the conic, with 5 nodes over the tangency points, and only 2 ramification points (the remaining 2 intersection points of Y with the conic). By corollary 3.6, there are 70956 such curves.
(ii) Pull-backs of unions of two distinct lines: these give a rational curve only if the pull-backs of both lines are rational, i.e. they are both bitangents. The pull-back is then a union of two (2-nodal) rational curves on X that intersect in 2 points. To make this into a rational stable map we can glue these two components at either intersection point. Hence there are $2 \cdot \binom{324}{2} = 104652$ such stable maps.
(iii) Double covers of pull-backs of a line, necessarily again of a bitangent. The pull-back of such a bitangent is a 2-nodal rational curve C. There are two possible ways of double covers of such a curve:

(a) Double covers that factor through the normalization of C. The space of these curves is the same as that of double covers of a smooth rational curve; it has dimension 2.
(b) Double covers that do not factor through the normalization. They have two components that are both mapped to C with degree 1, and glued over one of the nodes of C in such a way that, locally around this node, the morphism of the stable map is an isomorphism onto C. There are $2 \cdot 324 = 648$ such curves.

Adding up just the numbers from (i), (ii), and (iii b), we get

$$70956 + 104652 + 648 = 176256,$$

which is exactly G_5 (and $\frac{1}{2}(2\pi^*\ell)+1 = 5$). So we see that, for our non-primitive class β, the corresponding invariant from the series $G(q)$ does give the correct number, *except* for a correction (iii a) for double covers of curves of class $\frac{1}{2}\beta$ that factor through the normalization of these curves. Let us compute what this correction term is.

Lemma 4.1. *With notations as above, the double covers of type (iii a) of the pull-back of a bitangent contribute* $\frac{1}{8}$ *to the invariant* $n_{2\pi^*\ell}$.

Proof. Let $D \cong \mathbb{P}^1$ be the normalization of the nodal rational curve in X. The moduli space of the double covers that factor through the normalization is then just $\bar{M}_{0,0}(D, 2)$, which has dimension 2. As the normal bundle of the (local) immersion $D \to X$ is $\mathcal{O}(-2)$, the rank-3 obstruction bundle for the corresponding Gromov-Witten invariant would be $R^1\pi_* f^*\mathcal{O}(-2)$, where $\pi : \bar{M}_{0,1}(\mathbb{P}^1, 2) \to \bar{M}_{0,0}(\mathbb{P}^1, 2)$ is the forgetful map and $f : \bar{M}_{0,1}(\mathbb{P}^1, 2) \to \mathbb{P}^1$ the evaluation.

As explained above, the K3-invariants of X are defined as the ordinary Gromov-Witten invariants of a 1-dimensional family $\mathcal{X}$ of K3 surfaces in which X is the only surface that contains rational curves in the given homology class. This means that the obstruction bundle for the K3 invariants is obtained from the usual Gromov-Witten obstruction bundle by taking the quotient with $\pi_* f^* N_{X/\mathcal{X}} = \pi_* f^*\mathcal{O} = \mathcal{O}$. So the integral that we want to compute is

$$c_{top}(R^1\pi_* f^*\mathcal{O}(-2)/\mathcal{O}) \cdot [\bar{M}_{0,0}(\mathbb{P}^1, 2)].$$

This is easily done: from the two exact sequences on $\bar{M}_{0,1}(\mathbb{P}^2, 2)$

$$0 \to f^*\mathcal{O}(-1) \to f^*\mathcal{O} \to f^*\mathcal{O}_P \to 0$$
$$0 \to f^*\mathcal{O}(-2) \to f^*\mathcal{O}(-1) \to f^*\mathcal{O}_P \to 0$$

(where $P \in \mathbb{P}^1$ is a point) we get the exact sequences of vector bundles on $\bar{M}_{0,0}(\mathbb{P}^2, 2)$

$$0 \to \mathcal{O} \to \pi_* f^*\mathcal{O}_P \to R^1\pi_* f^*\mathcal{O}(-1) \to 0$$
$$0 \to \pi_* f^*\mathcal{O}_P \to R^1\pi_* f^*\mathcal{O}(-2) \to R^1\pi_* f^*\mathcal{O}(-1) \to 0$$

from which it follows that

$$c_2(R^1\pi_* f^*\mathcal{O}(-2)/\mathcal{O}) = c_2(R^1\pi_* f^*(\mathcal{O}(-1) \oplus \mathcal{O}(-1))),$$

i.e. the contribution of the double covers under consideration is the same as the double cover contribution for rational curves on Calabi-Yau threefolds with balanced normal bundle. This contribution is well-known to be $\frac{1}{8}$, see e.g. [Ga2] example 6.3. □

So we see that

$$n_{2\pi^*\ell} = G_5 + \frac{1}{8} \cdot G_2.$$

We conjecture that this pattern continues, i.e. that the numbers n_β receive multiple cover corrections similarly to the case of Gromov-Witten invariants of Calabi-Yau threefolds:

$$n_\beta = \sum_k \frac{1}{k^3} \cdot G_{\frac{1}{2}(\frac{\beta}{k})^2+1},$$

where the sum is taken over all $k > 0$ such that $\frac{\beta}{k}$ is an integral homology class.

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