TROPICAL DESCENDANT GROMOV-WITTEN INVARIANTS

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ABSTRACT. We define tropical Psi-classes on $\mathcal{M}_{0,n}(\mathbb{R}^2, d)$ and consider intersection products of Psi-classes and pull-backs of evaluations on this space. We show a certain WDVV equation which is sufficient to prove that tropical numbers of curves satisfying certain Psi- and evaluation conditions are equal to the corresponding classical numbers. We present an algorithm that generalizes Mikhalkin's lattice path algorithm and counts rational plane tropical curves satisfying certain Psi- and evaluation conditions.

1. INTRODUCTION

Psi-classes ψ_i are certain divisor classes on spaces of stable curves or stable maps, $\overline{M}_{g,n}$ or $\overline{M}_{g,n}(\mathbb{P}^r, d)$, which arise as the first Chern class of the line bundle whose fiber over a point $(C, x_1, \ldots, x_n)$ (or $(C, x_1, \ldots, x_n, f)$) is the cotangent space of C at x_i. They are for example useful to count curves with tangency conditions. To count curves that satisfy incidence conditions (e.g. pass through given points), one defines evaluation maps on the space of stable maps, $\mathrm{ev}_i : \overline{M}_{g,n}(\mathbb{P}^r, d) \to \mathbb{P}^r$ that send a stable map $(C, x_1, \ldots, x_n, f)$ to the image $f(x_i)$ of the marked point i. Then we can pullback the conditions via the evaluation map. Finally, we can intersect pullbacks of evaluation maps and Psi-classes on $\overline{M}_{g,n}(\mathbb{P}^r, d)$. The degrees of such top-dimensional intersection products are called descendant Gromov-Witten invariants and have been studied in Gromov-Witten theory.

The aim of this paper is to define tropical analogues of rational descendant Gromov-Witten invariants in the plane and to show that, under certain assumptions on the distribution of the Psi- and evaluation conditions, they coincide with their conventional counterparts.

To do so, we use the constructions of moduli spaces of abstract and parameterized rational tropical curves as tropical varieties and the definition of evaluation maps on the latter ones ([SS04], [GKM07], [Mi07]). Moreover, [Mi07] already defines Psi-classes on the space of abstract tropical curves $\mathcal{M}_{0,n}$ and [KM07] deals with their intersections. In this paper, we define Psi-classes on the space of parameterized tropical curves. Together with the intersection theory of [Mi06] and [AR07], we have all tools to define descendant Gromov-Witten invariants. We show that these invariants are independent of the position and "type" of the evaluation conditions and that they fulfill the string and divisor equation. Then we show that the invariants also fulfill a certain WDVV equation which can be used to determine the numbers recursively. As the classical numbers fulfill the same equations, it remains to compare the initial numbers appearing in the recursion to show that the classical and tropical invariants coincide.

These results can only be achieved for invariants such that the Psi-conditions come together with point conditions, and not alone or with line conditions. Note that one should expect such restrictions as we work with a non-compact moduli space that does *not* parametrize curves with components in the "boundary" of $\mathbb{R}^n$. Hence the tropical descendant Gromov-Witten invariants are different from the classical ones in some cases, namely whenever tropical curves are "missing" in the corresponding tropical count. However, we show that this does not happen when Psi-conditions always come together with point conditions.

To prove the WDVV equation we show that the weight of a curve in an intersection product can be computed locally as the determinant of a map (which basically collects all evaluation maps) and then proceed similarly to [GM05]. Finally, we present a tropical algorithm similar to Mikhalkin's lattice path

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count ([Mi03]) to determine the numbers of rational plane tropical curves passing through points and satisfying Psi-conditions.

Our definition of tropical descendant invariants partly agrees with Mark Gross' definition which was found independently in his study of mirror symmetry ([Gr09]).

The paper is organized as follows. In section 2, we recall some facts about tropical moduli spaces and tropical intersection theory that we need. Then we define Psi-classes on the space of parametrized tropical curves and tropical descendant invariants. In section 3 we define what it means for incidence conditions to be general and what consequences arise for our tropical descendant invariants if we choose the conditions to be general. In section 4, we show that every tropical curve in an intersection product of Psi-classes, point and line evaluations *and* the pullback of a point with a large coordinate in $\mathcal{M}_{0,4}$ under the forgetful map contains a contracted bounded edge. Thus the tropical curve can be interpreted as a reducible curve by cutting it along this contracted bounded edge. In section 5 we show that the weight of a tropical curve in a zero-dimensional intersection product can be computed using a determinant of a linear matrix. We use this in section 6 to show that the weight of tropical curves with a contracted bounded edge can be (almost) split into two factors corresponding to the irreducible components. In section 7, we show the string equation and the divisor equation for our tropical descendant invariants. In section 8 finally, we collect our results to prove that our tropical descendant invariants satisfy a certain WDVV equation, and we conclude that the tropical invariants are equal to the corresponding classical invariants that satisfy the same recursion. In section 9, we describe an algorithm similar to the lattice path count that determines tropical descendant invariants.

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2. Defining the invariants

First of all, let us briefly recall the constructions from [AR07] that we need here:

A *cycle* X is a balanced (weighted, pure-dimensional, rational and polyhedral) complex (resp. fan) in $\mathbb{R}^n$. The top-dimensional polyhedra (resp. cones) in X are called *facets*, the codimension one polyhedra (resp. cones) are called *ridges*. The integer weights assigned to each facet σ are denoted by $\omega(\sigma)$. *Balanced* means that the weighted sum of the primitive vectors of the facets σ around a ridge $\tau \in X$

$$\sum_{\substack{\sigma \in X^{(\dim(X))} \\ \tau < \sigma}} \omega(\sigma) v_{\sigma/\tau}$$

vanishes "modulo τ", or, precisely, lies in the linear vector space spanned by τ, denoted by V_τ. Here, a *primitive vector* $v_{\sigma/\tau}$ *of* σ *modulo* τ is a integer vector in $\mathbb{Z}^n$ that points from τ towards σ and fulfills the primitive condition: The lattice $\mathbb{Z}v_{\sigma/\tau} + (V_\tau \cap \mathbb{Z}^n)$ must be equal to the lattice $V_\sigma \cap \mathbb{Z}^n$. Slightly differently, in [AR07] the class of $v_{\sigma/\tau}$ modulo V_τ is called primitive vector and $v_{\sigma/\tau}$ is just a representative of it.
For us, a polyhedron σ is always understood to be closed. The *(relative) interior* $\mathrm{Int}(\sigma)$ is the topological interior of σ in its affine span (e.g. $\mathrm{Int}(\{P\}) = \{P\}$). The *support of* X, denoted by $|X|$, is the union of all facets in X with non-zero weight.

A *(non-zero) rational function on* X is a function $\varphi : |X| \to \mathbb{R}$ that is affine (resp. linear) with rational slope on each polyhedron (resp. cone). The *divisor of* φ, denoted by $\mathrm{div}(\varphi) = \varphi \cdot X$, is the balanced subcomplex (resp. subfan) of X constructed in [AR07, 3.3], namely the codimension one skeleton $X \setminus X^{(\dim X)}$ together with the weights $\omega_\varphi(\tau)$ for each ridge $\tau \in X$. These weights are given by the formula

$$\omega_\varphi(\tau) = \sum_{\substack{\sigma \in X^{(\dim(X))} \\ \tau < \sigma}} \omega(\sigma)\varphi_\sigma(v_{\sigma/\tau}) - \varphi_\tau\Big(\sum_{\substack{\sigma \in X^{(\dim(X))} \\ \tau < \sigma}} \omega(\sigma) v_{\sigma/\tau}\Big),$$

where $\varphi_\sigma : V_\sigma \to \mathbb{R}$ denotes the linear part of the affine function $\varphi|_\sigma$. Note that the balancing condition of X around τ ensures that the argument of φ_τ is an element of V_τ. If φ is globally affine (resp. linear),

all weights are zero, which we denote by $\varphi \cdot X = 0$. Let the *support of* φ, denoted by $|\varphi|$, be the subcomplex of X containing the points where φ is not locally affine. Then we have $|\varphi \cdot X| \subseteq |\varphi|$. Furthermore, the intersection product is bilinear (see [AR07, 3.6]). As the restriction of a rational function to a subcycle is again a rational function, we can also form multiple intersection products $\varphi_1 \cdot \ldots \cdot \varphi_l \cdot X$. In this case we will sometimes omit "$\cdot X$" to keep formulas shorter. Note that multiple intersection products are commutative (see [AR07, 3.7]).

By abuse of notation, a *cycle* is also a class of balanced fans with common refinement and agreeing weights. A *rational function* φ on such a class is just a rational function on a fan X contained in the class. We can generalize our intersection product to such classes of fans $[X]$ by defining $\varphi \cdot [X] := [\varphi \cdot X]$. In the following, we try to avoid these technical aspects whenever possible. We will also omit the brackets distinguishing between fans and their classes, hoping that no confusion arises.

A *morphism of cycles* $X \subseteq \mathbb{R}^n$ *and* $Y \subseteq \mathbb{R}^m$ is a map $f : |X| \to |Y|$ that comes from a linear map from $\mathbb{Z}^n$ to $\mathbb{Z}^m$ and that maps each polyhedron (resp. cone) of X into one of Y. Such a morphism *pulls back rational functions* φ on Y to rational functions $f^*(\varphi) = \varphi \circ f$ on X. Note that the second condition of a morphism, which is not required in [AR07], makes sure that we do not have to refine X further. $f^*(\varphi)$ is already affine (resp. linear) on each cone. Furthermore, we can *push forward subcycles* Z of X to subcycles $f_*(Z)$ of Y. This is due to [GKM07, 2.24 and 2.25] in the case of fans and can easily be generalized to complexes. We can omit further refinements here if we assume that $f(\sigma) \in Y$ for all $\sigma \in X$. The *projection formula* (see [AR07, 4.8]) connects all the above constructions via

$$f_*(f^*(\varphi) \cdot X) = \varphi \cdot f_*(X).$$

Moreover, let us recall the basic facts of rational equivalence introduced in [AR07, section 8]. The *degree* of a zero-dimensional cycle Z is just the sum of all weights. Hence the push-forward of a zero-dimensional cycle preserves degree. If X is a one-dimensional cycle, and φ is a *bounded* rational function, then $\deg(\varphi \cdot X) = 0$ (see [AR07, 8.3]). The pull-back of a bounded rational function is again bounded. Two functions are called *rationally equivalent* if they differ by the sum of a bounded and a globally linear function. Hence (and by linearity of the pull-back) rational equivalence is preserved when pulled back.

An example for functions that are rationally equivalent is given by translations of functions on $\mathbb{R}^n$.

Lemma 2.1 (Translations are rationally equivalent)**.** *Let h be a rational function on $\mathbb{R}^n$, choose $v \in \mathbb{R}^n$ and consider h' with $h'(x) := h(x + v)$. Then h and h' are rationally equivalent.*

Proof. Let X be a subdivision of $\mathbb{R}^n$ on which h is a rational function. For each cone $\sigma \in X$, let h_σ be the linear part of the affine function $h|_\sigma$. Take the maximum of the finitely many $h_\sigma(v), \sigma \in X$ and call it c. Now, X subdivides the line segment $x + \lambda v, \lambda \in [0, 1]$ into q line segments of length λ_i contained in some polyhedron σ_i. This means $h(x + v)$ can be expressed as $h(x) + h_{\sigma_1}(\lambda_1 v) + \ldots + h_{\sigma_q}(\lambda_r v)$, where $\sum_i \lambda_i = 1$. This implies

$$h(x + v) - h(x) \leq c,$$

which proves that $h' - h$ is bounded. □

In the following, we will apply these constructions and results to the case of Psi- and evaluation classes on the space of rational plane curves.

The tropical analogue $\mathcal{M}_{0,n}$ of the space of stable n-marked curves is the space of trees, or (a quotient of) the tropical Grassmanian ([SS04], [GKM07], [Mi07]). Thus an abstract tropical curve is just a tree with n marked ends and whose bounded edges e are equipped with a length $l(e) \in \mathbb{R}_{>0}$. The fan $\mathcal{M}_{0,n}$ is stratified by cones corresponding to combinatorial types of trees. The facets correspond to 3-valent trees.

The tropical analogue $\mathcal{M}_{0,n}(\mathbb{R}^2, d)$ of the space of stable maps has been studied in [GKM07]. An element of $\mathcal{M}_{0,n}(\mathbb{R}^2, d)$ is an abstract tropical curve Γ (i.e. a tree) together with a map $h : \Gamma \to \mathbb{R}^2$ such that the image satisfies the balancing condition and marked ends are contracted to a point. An important

feature of this definition is that it also allows to contract bounded edges, as it will happen in section 4 and 6. If we furthermore also label the *non-contracted* ends, we obtain the space $\mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^2,d)$. The advantage of this space is that, after choosing the vertex of one marked end as root vertex, we can identify $\mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^2,d)$ with $\mathcal{M}_{n+3d}\times\mathbb{R}^2$, where the second factor describes the position of the root vertex in $\mathbb{R}^2$ (cf. [GKM07]). In particular, in this sense $\mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^2,d)$ is a tropical variety. For enumerative purposes, its difference to $\mathcal{M}_{0,n}(\mathbb{R}^2,d)$ cumulates in nothing but a factor $(d!)^3$ by which each invariant in $\mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^2,d)$ must be divided to get the corresponding one in $\mathcal{M}_{0,n}(\mathbb{R}^2,d)$. Note that, independent of the choice of a root vertex, there exists a forgetful map $\text{ft}':\mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^2,d)\to\mathcal{M}_{n+3d}$ forgetting just the position of the image of a curve in $\mathbb{R}^2$. This forgetful map $\text{ft}':\mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^2,d)\to\mathcal{M}_{n+3d}$ is a morphism of tropical varieties, as after choosing a root vertex and identifying $\mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^2,d)$ with $\mathcal{M}_{n+3d}\times\mathbb{R}^2$, ft' is just the projection onto the first factor.

Analogues of Psi-classes on tropical $\mathcal{M}_{0,n}$ have been defined recently ([Mi07]). ψ_i with $i=1,\dots,n$ is the codimension one subcycle that consists of cones corresponding to trees where the marked end i is at a 4- or higher-valent vertex. How such Psi-classes intersect is discussed in [KM07]. To do so, Psi-classes $\psi_i, i=1,\dots,n$ are defined as divisors of rational functions f_i on $\mathcal{M}_{0,n}$ cf. [KM07, proposition 3.5]. As $\mathcal{M}_{0,n}$ is simplicial, the function f_i can be defined by specifying its values on the primitive vectors of the rays contained in $\mathcal{M}_{0,n}$. These rays are given by curves with only one bounded edge splitting up the marked ends into two sets $I\cup J=[n]$. Let $v_{I|J}$ be the corresponding primitive vector and assume w.l.o.g. $i\in I$, then f_i is defined by

$$f_i(v_{I|J})=\frac{|J|\,(|J|-1)}{(n-1)(n-2)}.$$

Note that we denote by f_i a multiple of what is called f_i in [KM07], such that we obtain $\text{div}(f_i)=\psi_i$. We use these functions to pull back Psi-classes to $\mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^2,d)$.

Definition 2.2 (Psi-classes for parameterized curves)**.** For $i=1,\dots,n$ we define *the i-th Psi-class on* $\mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^2,d)$ to be $\psi_i:=\text{div}(\text{ft}'^*(f_i))$.

Remark 2.3. It can be shown that two rational functions on $\mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^2,d)$ (or $\mathcal{M}_{0,n}$) defining the same divisor cycle only differ by the restriction of a globally linear function. Hence, the choice of the functions defining our Psi-classes is not really important for intersection-theoretic purposes. This justifies that throughout our paper we use the specific function $\text{ft}^*(f_i)$ to describe ψ_i and in particular define

$$\psi_i\cdot Y:=\text{ft}'^*(f_i)\cdot Y,$$

where Y is an arbitrary subcycle of $\mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^2,d)$. Note also that for our purposes we do not really need that the function describing ψ_i is (nearly) unique. The only thing we need to know is contained in the following lemma.

Lemma 2.4 (Products of Psi-classes)**.** *Let $r_1,\dots,r_n$ be positive integers and let*

$$X=\prod_{k=1}^n\psi_k^{r_k}\cdot\mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^2,d)$$

be a product of Psi-classes. Then X is the codimension-$\sum_k r_k$-subfan of $\mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^2,d)$ consisting of cones σ corresponding to trees such that for each vertex V we have $\text{val}(V)=K(I_V)+3$, where I_V denotes the set

$$I_V=\{k\in[n]\ :\ \textit{end } x_k \textit{ is adjacent to } V\}\subset[n]$$

and $K(I)$ is a short notation for $K(I)=\sum_{k\in I}r_k$. The weight of σ equals

$$\omega(\sigma)=\frac{\prod_V K(I_V)!}{\prod_{k=1}^n r_k!}.$$

Proof. Choose a root vertex and identify $\mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^2,d)$ with $\mathcal{M}_{0,n+3d}\times\mathbb{R}^2$. Then ft' is just the projection on the first factor and we can apply [AR07, 9.6], i.e. instead of intersecting the pull-backs of the f_k on the product, we can just intersect the f_k on the first factor and then multiply with $\mathbb{R}^2$. Thus,

$$X=\left(\prod_{k=1}^{n}\psi_k^{r_k}\cdot\mathcal{M}_{0,n+3d}\right)\times\mathbb{R}^2,$$

where here ψ_k denotes a Psi-class in $\mathcal{M}_{0,n+3d}$. But in the case of non-parameterized curves, it is proved in [KM07, 4.1] that the valence of the vertices and the weights of the facets satisfy the formulas of the statement. Multiplying with $\mathbb{R}^2$ does not disturb this, as the weight of $\mathbb{R}^2$ is one and as the combinatorics of a curve remain unchanged under ft'. □

Remark 2.5. In particular the preceding lemma says that ψ_i consists of those curves whose marked end i is adjacent to an at least 4-valent vertex (where bounded edges as well as marked ends and non-contracted ends count towards the valence).

Later on, we will also use the *forgetful map*

$$\text{ft}:\mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^2,d)\to\mathcal{M}_{0,4},$$

which forgets the map of a given curve C to $\mathbb{R}^2$ and all its ends but the first four marked ends (it also "stabilizes", which means that, after forgetting one marked end, it replaces all two-valent vertices by straight edges while adding up lengths).

Lemma 2.6. *The forgetful map* $\text{ft}:\mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^2,d)\to\mathcal{M}_{0,4}$ *is a morphism of cycles.*

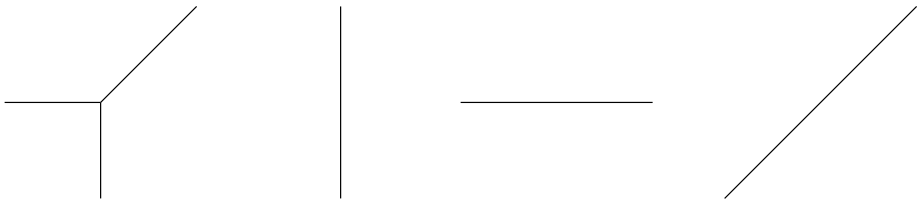
Proof. Let $\text{ft}_n:\mathcal{M}_{0,n}\to\mathcal{M}_{0,n-1}$ be the forgetful map that just forgets the i-th end of an i-marked non-parameterized curve. It is shown in [GKM07, 3.9] that ft_n is a morphism for all integers $n\geq 4$. As mentioned above, the map ft' is a morphism, too. Thus, the statement follows from the formula $\text{ft}=\text{ft}_5\circ\ldots\circ\text{ft}_n\circ\text{ft}'$. □

Moreover, we use the *evaluation maps*

$$\text{ev}_i:\mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^2,d)\to\mathbb{R}^2$$

assigning to a curve C the position of its i-th marked end. It is shown in [GKM07, 4.8] that these maps are also morphisms of cycles. Along these morphisms we will pull back lines and points.

Definition 2.7 (Lines)**.** A *line* G is a one-dimensional cycle in $\mathbb{R}^2$ that is the divisor of a tropical polynomial of degree one. In other words, lines are divisors of translations of the functions $\max\{x,y,0\}$, $\max\{x,0\}$, $\max\{y,0\}$ or $\max\{x,y\}$.



$\max\{x,y,0\}$ $\max\{x,0\}$ $\max\{y,0\}$ $\max\{x,y\}$

Lines of type $\max\{x,y,0\}$ are also called *non-degenerated*.

We would like to pull back lines and points along an evaluation map ev_i. However, up to now, pull backs are only defined for functions, not for cycles. Of course, we can choose rational functions cutting out the line resp. point in question and pull them back instead. In the following lemma we will show that, for our purposes, the choice of describing functions plays no role.

Notation 2.8. We use the following notation: We have a total number of $l+m+n$ marked ends, which are subdivided into the three sets $L \cup M \cup N = \{1, \ldots, l+m+n\}$, such that $|L| = l$, $|M| = m$ and $|N| = n$. In the following, the ends $i \in L$ are unrestricted, the ends $j \in M$ are restricted by lines G_j (see 2.7) and the ends $k \in N$ have to meet points P_k. Furthermore we fix numbers $r_k, k \in N$ describing how many Psi-classes we require at $k \in N$.

Lemma 2.9. *Consider the intersection product*

$$Z := \prod_{j \in M} \mathrm{ev}_j^*(G_j) \prod_{k \in N} \mathrm{ev}_k^*(P_k)\psi_k^{r_k} \cdot \mathcal{M}_{0,l+m+n}^{lab}(\mathbb{R}^2, d),$$

where $\mathrm{ev}_j^*(G_j)$ *stands for* $\mathrm{ev}_j^*(h)$ *with a function* h *cutting out* G_j *and* $\mathrm{ev}_k^*(P_k)$ *stands for* $\mathrm{ev}_k^*(h_1) \cdot \mathrm{ev}_k^*(h_2)$ *with function* h_1, h_2 *cutting out* P_k*. Then* Z *is well-defined, i.e. it does not depend on the chosen rational functions.*

Proof. Let $\mathrm{ev} := \mathrm{ev}_i$ be an evaluation map and G be a line. First we check that the intersection product $\mathrm{ev}^*(G) \cdot \mathcal{M}_{0,l+m+n}^{\mathrm{lab}}(\mathbb{R}^2, d)$ does not depend on the rational function describing G: Choose the vertex of the end i as root vertex and identify $\mathcal{M}_{0,l+m+n}^{\mathrm{lab}}(\mathbb{R}^2, d)$ with $\mathcal{M}_{0,l+m+n+3d} \times \mathbb{R}^2$. Then ev is just the projection onto the second factor. By [AR07, 9.6] we deduce

$$\mathrm{ev}^*(G) \cdot (\mathcal{M}_{0,l+m+n+3d} \times \mathbb{R}^2) = \mathcal{M}_{0,l+m+n+3d} \times G,$$

which shows independence of the describing function.
Now let $X = \varphi_1 \cdot \ldots \cdot \varphi_r \cdot \mathcal{M}_{0,l+m+n}^{\mathrm{lab}}(\mathbb{R}^2, d)$ be a cycle given by arbitrary rational functions $\varphi_1, \ldots, \varphi_r$. Then, by commutativity of the intersection product, the cycle

$$\mathrm{ev}^*(G) \cdot X = \varphi_1 \cdot \ldots \cdot \varphi_r \cdot \mathrm{ev}^*(G) \cdot \mathcal{M}_{0,l+m+n}^{\mathrm{lab}}(\mathbb{R}^2, d)$$

is also well-defined. The same arguments work if we consider a point P instead of G. But this suffices to conclude inductively that the big intersection product Z is also well-defined. Moreover note that the same argument also shows that our choice of the function f_i describing ψ_i does not matter in this intersection product. □

We are now ready to define our tropical descendant Gromov-Witten invariants.

Proposition and Definition 2.10. *Let* d, l, m, n *and* $r_k, k \in N$ *be positive integers such that*

$$l + m + n + 3d - 3 + 2 = m + 2n + \sum_{k \in N} r_k. \tag{1}$$

Then we define the tropical descendant Gromov-Witten invariant $\langle \tau_0(0)^l \tau_0(1)^m \prod_{k \in N} \tau_{r_k}(2) \rangle_d$ *to be the number*

$$\langle \tau_0(0)^l \tau_0(1)^m \prod_{k \in N} \tau_{r_k}(2) \rangle_d := \frac{1}{(d!)^3} \deg \left(\prod_{j \in M} \mathrm{ev}_j^*(G_j) \prod_{k \in N} \mathrm{ev}_k^*(P_k)\psi_k^{r_k} \cdot \mathcal{M}_{0,l+m+n}^{lab}(\mathbb{R}^2, d) \right).$$

As indicated by the notation, this number only depends on $d, l, m, n, r_k, k \in N$, but not *on the lines* G_j *and the points* P_k.

Proof. Lemma 2.1 says that we can move around our points and lines arbitrarily, namely by translating the describing functions, without changing the degree. It remains to show that the type of the lines does not matter, for example the type of G_1. We will show that $\mathrm{ev}_1^*(G_1) \cdot F$ does not depend on the choice of the line G_1 for a one-dimensional cycle F, where

$$F = \prod_{j \in M \setminus \{1\}} \mathrm{ev}_j^*(G_j) \prod_{k \in N} \mathrm{ev}_k^*(P_k)\psi_k^{r_k} \cdot \mathcal{M}_{0,l+m+n}^{\mathrm{lab}}(\mathbb{R}^2, d)$$

To see this, we have to use lemma 3.7 which requires general conditions and therefore is stated and proven in the next section of this article. It states that $\mathrm{ev}_{1*}(F)$ has only standard outer directions $-e_1, -e_2$ and $e_1 + e_2$. Knowing this, we push forward $\mathrm{ev}_1^*(G_1) \cdot F$ via ev_1, which does not change the

degree and use the projection formula ([AR07]). It tells us that $\mathrm{ev}_{1*}(\mathrm{ev}_1^*(G_1)\cdot F) = G_1 \cdot \mathrm{ev}_{1*}(F)$. Now, as $\mathrm{ev}_{1*}(F)$ has only standard outer directions (at least for general conditions, which we can assume), any line intersects $\mathrm{ev}_{1*}(F)$ in the same number of points, not depending on the type. Note that lemma 3.7 does not care about the types of the lines appearing in the product of F. Thus we can apply the above argument inductively and see that the types of all lines G_j can be changed arbitrarily without changing the degree of the intersection product. □

Remark 2.11. The dimension of the space $\mathcal{M}^{\mathrm{lab}}_{0,l+m+n}(\mathbb{R}^2,d) = \mathcal{M}_{0,m+n+l+3d}\times\mathbb{R}^2$ is $l+m+n+3d-3+2$ since a 3-valent tree with $m+l+n+3d$ ends has $l+m+n+3d-3$ bounded edges. The codimension of the intersection of Psi-classes is $\sum_{k\in N} r_k$. The pullback of a line has codimension 1 and the pullback of a point codimension 2. Hence the requirement (1) is equivalent to a 0-dimensional expected dimension of the intersection.

Notation 2.12. We will use the τ-notation in a more general meaning: A product

$$\big(\prod_{i\in I}\tau_{r_i}(c_i)\big)_d$$

(with round brackets) stands for *a cycle in* $\mathcal{M}^{lab}_{0,|I|}(\mathbb{R}^2,d)$, obtained as the intersection product where we replace the i-th factor $\tau_{r_i}(c_i)$ by $\psi_i^{r_i}\,\mathrm{ev}^*(C_i)$. Here, C_i is some point P_i if $c_i=2$, some line G_i (of some type) if $c_i=1$ and the whole space $\mathbb{R}^2$ (which means you can omit this pull-back) if $c_i=0$; thus c_i describes the codimension of C_i. If $\big(\prod_{i\in I}\tau_{r_i}(c_i)\big)_d$ is zero-dimensional, we denote, as before, by

$$\langle\prod_{i\in I}\tau_{r_i}(c_i)\rangle_d = \frac{1}{(d!)^3}\deg\big(\prod_{i\in I}\tau_{r_i}(c_i)\big)_d$$

the degree of the product above divided by $(d!)^3$. Note that a factor $\tau_0(0)$ can not be dropped in this notation as it stands for a marked end that does not have to meet any condition at all.

Remark 2.13. Later on, we will also allow the factor $\mathrm{ft}^*(\lambda)$ in this notation, where λ is an element in $\mathcal{M}_{0,4}$ and $\mathrm{ft}^*(\lambda)$ stands for the pull-back of a rational function on $\mathcal{M}_{0,4}$ describing λ. Two such functions differ by an affine one, and so do the pull-backs. Hence, the intersection product containing $\mathrm{ft}^*(\lambda)$ as factor is still well-defined.

3. General incidence conditions

The invariants defined in 2.10 are well-defined also for "special" incidence conditions, e.g. if we choose all points P_i to coincide. In this case the *set* of curves fulfilling the conditions is of too big dimension, but our intersection theory ensures that the corresponding intersection product still has the correct dimension and degree. However, many of the following arguments still require a notion of "general incidence conditions" that ensures that our intersection product equals the set-theoretical count of curves fulfilling the incidence conditions (up to weights).

Let us start with the case of pulling back a single line in $\mathbb{R}^2$. Let X be a subcomplex of $\mathcal{M}^{\mathrm{lab}}_{0,n}(\mathbb{R}^2,d)$, let $f: X\to\mathbb{R}^2$ be a map that is the restriction of a linear map (e.g. morphisms like $f=\mathrm{ev}_i$) and let G be a line in $\mathbb{R}^2$. Let $f^{-1}(G)$ be the subcomplex of X containing all polyhedra $\sigma\cap f^{-1}(\delta)$ for all $\sigma\in X$ and $\delta\in G$ (where δ denotes a cone in the polyhedral complex G). Recall that the interior of a polyhedron $\mathrm{Int}(\sigma)$ denotes its topological interior in its affine span.

Lemma 3.1. *There exists a open dense subset $U\subset\mathbb{R}^2$ such that for $v\in U$ and a translation $G':=G+v$ of G, it holds:*

(a) *The subcomplex $f^{-1}(G')$ is either empty or of pure codimension 1 in X.*
(b) *The interior of a facet of $f^{-1}(G')$ is contained in the interior of a facet of X.*
(c) *For an element C in the interior of a facet of $f^{-1}(G')$, the image $f(C)$ lies in the interior of a facet of G.*

Proof. Let σ be a facet of X and δ a one-dimensional polyhedron of G. Consider the map $q \circ f_\sigma$, where $f_\sigma : V_\sigma \to \mathbb{R}^2$ is the extension of $f|_\sigma$ to V_σ and $q : \mathbb{R}^2 \to \mathbb{R}^2/V_\delta$ is the quotient map. This composition has either rank 1 (in which case $\ker(q \circ f_\sigma)$ has codimension one in V_σ; hence, for a general translation δ' of δ, the polyhedron $\sigma \cap f^{-1}(\delta')$ is either empty or of codimension 1 and intersecting the interior of σ) or has rank 0 (then $\sigma \cap f^{-1}(\delta')$ is empty for a general translation of δ' of δ). As there are only finitely many pairs σ, δ, the set of vectors $v \in \mathbb{R}^2$ such that these statements are true simultaneously is still open and dense. But note that all facets of $f^{-1}(G')$ can be obtained in this way for some pair σ, δ. This shows part (a) and (b).
Furthermore, let $V \in \mathbb{R}^2$ be the vertex of G (if G is of type $\max\{x, y, 0\}$). Applying the same argument to V shows that for a general translation $V' := V + v$, the preimage $f^{-1}(V')$ has at least codimension 2, which proves part (c). □

Definition 3.2. Let Z be an intersection product of the form $(\prod_{i\in I} \tau_{r_i}(c_i))_d$ with incidence conditions C_i. Define $X := \prod_{i\in I} \psi_i^{r_i} \cdot \mathcal{M}^{\text{lab}}_{0,|I|}(\mathbb{R}^2, d)$. We call the conditions *general* if the following holds:

(a) The subcomplex S of X containing all points $C \in X$ fulfilling $\mathrm{ev}_i(C) \in C_i$ has dimension $\dim(S) = \dim(Z)$.
(b) The interior of a facet of S is contained in the interior of a facet of X.
(c) The interior of a facet σ of S maps to the interior of a facet of C_i under ev_i.
(d) Any intersection $C_i \cap C_j$, $i, j \in I$ has expected codimension $c_i + c_j$.

Remark 3.3. Let S be the subcomplex of X containing all the curves $C \in X$ fulfilling $\mathrm{ev}_i(C) \in C_i$. Note that Z is a subcomplex of S. This follows from the facts that the support of an intersection product is contained in the support of the intersecting rational function and that the support of a pull-back is contained in the preimage of the support of the pulled-back function. Note that in general we have $S = Z$ (as sets) if $\dim(S) = \dim(Z)$ is satisfied, the only thing that can happen in principle is that there are facets of Z which get 0 as a weight in the intersection product, although they are facets of S. For the intersection products we work with, this cannot happen though, since we only have a weight of 0 if the set S is of higher dimension (see section 5). Hence for us the incidence conditions being general implies that $|Z|$ equals the set of curves satisfying the incidence conditions, and $\deg(Z)$ equals the number of curves satisfying the conditions, counted with weight.

Lemma 3.4. *The set of general conditions in the space of all conditions (which can be identified with some big $\mathbb{R}^N$ collecting all the translation vectors) is open and dense.*

Proof. The set of conditions fulfilling 3.2 (d) is obviously open and dense. The remaining follows from recursively applying 3.1 to X and ev_1, then $X \cap \mathrm{ev}_1^{-1}(C_1)$ and ev_2, and so on. More precisely, if $c_i = 0$ we have nothing to do in this step, if C_i is a line, we apply 3.1, and if C_i is a point, we apply 3.1 twice for two lines intersecting set-theoretically in the single point C_i. □

Remark 3.5. We also consider the following case: Let X be a 1-dimensional subcycle of $\mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^2, d)$ and consider the forgetful map $\mathrm{ft} : \mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^2, d) \to \mathcal{M}_{0,4}$. We call $\lambda \in \mathcal{M}_{0,4}$ *general*, if $\lambda \notin \mathrm{ft}(X^{(0)}) \cup \mathcal{M}^{(0)}_{0,4}$, where $X^{(0)}$ denotes the vertices of X and $\mathcal{M}^{(0)}_{0,4}$ denotes the single vertex of $\mathcal{M}_{0,4}$. This ensures that all points in $\mathrm{ft}|_X^{-1}(\lambda)$ lie in the interior of a one-dimensional polyhedron of X.

The following lemma describes the combinatorial type of the curves which satisfy general incidence conditions.

Lemma 3.6. *Let Z be an intersection product of the form $(\tau_0(0)^l \tau_0(1)^m \prod_{k\in N} \tau_{r_k}(2))_d$ with general conditions. Then*

(b') *For a curve C in the interior of a facet the following holds: All ends $k \in M \cup N$ lie at different vertices and the valence of a vertex is $r_k + 3$ if $k \in N$ is adjacent to it and 3 otherwise.*

Proof. Because of remark 3.3 we know that $Z \subset S$. In addition, condition 3.2 (a) says that Z and S have the same dimension and therefore (b) and (c) also hold for curves in the interior of a facet of Z.
Let C be in the interior of a facet of Z. Condition 3.2 (d) implies that $\mathrm{ev}_i(C) \neq \mathrm{ev}_j(C)$ for all $i \in M \cup N$, $j \in N$, as in this case $C_i \cap P_j$ is empty. If $i, j \in M$ would lie at the same vertex this would induce either a contracted bounded edge (which contradicts 3.2 (a)) or valence greater than 3 of this vertex (which contradicts 3.2 (b)). Hence all ends in $M \cup N$ must lie at different vertices. The statement about the valence of the vertices follows from 3.2 (b) and the description of X in 2.4. □

As a first application of our notion of general conditions we can now prove the lemma which we promised and needed in the independence statement 2.10.

Lemma 3.7. *Let F be a one-dimensional cycle of the form $(\tau_0(0)\tau_0(0)^l\tau_0(1)^m \prod_{k\in N} \tau_{r_k}(2))_d$ with general conditions. Let x denote the marked end corresponding to the first factor $\tau_0(0)$. Then all of the unbounded rays of the push-forward $\mathrm{ev}_{x*}(F)$ have standard directions $-e_1, -e_2$ and $e_1 + e_2$.*

Proof. Let σ be a facet of F. For a curve in the interior of σ two possibilities can occur: Either x is adjacent to a higher-valent vertex. Then by 3.6 also an end $k \in N$ interpolating the point P_k lies at this vertex. Therefore, $\mathrm{ev}_x(\sigma) = \mathrm{ev}_k(\sigma) = \{P_k\}$.
Secondly, x might be adjacent to a 3-valent vertex. Since x itself is contracted, the two other edges which are adjacent are mapped to lines with opposite direction (because of the balancing condition). That means locally the image looks like a straight line with the marked point $h(x)$ on it. We can deform a curve in σ in a one-dimensional family (thus covering σ) by changing the length of the two adjacent edges and thus making the point $h(x)$ move on the line. This movement is unbounded if and only if one of these two edges is an end. But then $\mathrm{ev}_x(\sigma)$ points to the same direction as this end, which is by definition one of the standard directions or 0. □

4. Contracted edges

Let F be a one-dimensional cycle of the form $(\tau_0(0)^l\tau_0(1)^m \prod_{k\in N} \tau_{r_k}(2))_d$ with general conditions. Remember that this implies that $|F|$ equals the set of curves satisfying the conditions.

Notation 4.1. We fix the type of the first four ends in the sense that we assume from now on $1 \in L$, $2 \in M$ and $3, 4 \in N$.

As before we denote by ft the forgetful map $\mathrm{ft} : \mathcal{M}^{\mathrm{lab}}_{0,l+m+n}(\mathbb{R}^2, d) \to \mathcal{M}_{0,4}$, which forgets the embedding and all ends but the first four marked ends. It is the aim of this section to show that for a very large $\mathcal{M}_{0,4}$-coordinate λ, the curves in $\mathrm{ft}^{-1}(\lambda) \cap F$ (i.e. curves with such a large $\mathcal{M}_{0,4}$-coordinate) must contain a contracted bounded edge. We will use the contracted bounded edge in section 6 to split such curves into two components.

Definition 4.2. Let C be a curve in $\mathcal{M}^{\mathrm{lab}}_{0,l+m+n}(\mathbb{R}^2, d)$. For two different marked ends i_1, i_2, we denote by $S(i_1, i_2)$ the smallest connected subgraph of C containing i_1 and i_2 and call it *the string of i_1 and i_2*. Such a string $S(i_1, i_2)$ is called *movable* if $i_1, i_2 \in L \cup E$, where E denotes the set of non-contracted ends, and if $S(i_1, i_2)$ does not intersect (the closure) of any k for $k \in N$.

Lemma 4.3. *Let C be a curve in the interior of a facet of F. Then C contains a movable string S.*

Proof. We know $\dim(F) = 1$, $\mathrm{codim}(F) = m + 2n + \sum_{k\in N} r_k$ and $\dim(\mathcal{M}^{\mathrm{lab}}_{0,l+m+n}(\mathbb{R}^2, d)) = l + m + n + 3d - 3 + 2$. Plugging in all this in $\dim(F) + \mathrm{codim}(F) = \dim(\mathcal{M}^{\mathrm{lab}}_{0,l+m+n}(\mathbb{R}^2, d))$ leads to

$$l + 3d = n + \sum_{k\in N} r_k + 2.$$

On the other hand we can compute the number of connected components of $\Gamma \setminus \bigcup_{k\in N} \bar{k}$: Removing k increases the number of connected components by $r_k + 1$ as the valence of the adjacent vertex is $r_k + 3$

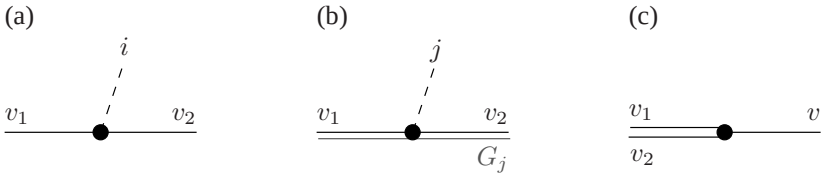
by 3.6. So, after removing all n ends, we arrive at $1 + n + \sum_{k \in N} r_k$ connected components. The above equation tells us that there is one more end in $L \cup E$ then there are connected components and therefore at least two ends $i_1, i_2 \in L \cup E$ lie in the same component. Hence $S(i_1, i_2)$ is a movable string. □

By construction all vertices of a movable string are 3-valent.

Lemma 4.4. *Let σ be a facet of F such that the corresponding interior curves do* not *contain a contracted bounded edge. Then the image of σ under* ft *is bounded.*

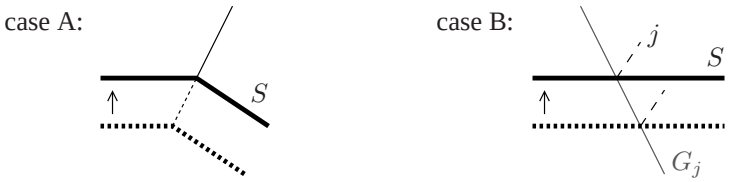
Proof. Let C be a curve in the interior of σ. We will deform C in a one-dimensional family inside σ. Since σ is one-dimensional itself, this family covers σ. By lemma 4.3 there exists a movable string S in C. In the following, we show that either σ is bounded (i.e. the deformation of C is bounded) or ft is constant on σ (i.e. the deformation of C does not affect ft).
Let V be a vertex in S. We call V *degenerated* if we can deform C one-dimensionally locally around V, i.e. if

(a) either one of the adjacent edges is a marked end $i \in L$,
(b) or one of the adjacent edges is a marked end $j \in M$ and the linear spans of the corresponding line G_j at $\mathrm{ev}_j(C)$ and of the other two edges adjacent to V coincide (i.e. if the curve C and the line G do not intersect transversally at $\mathrm{ev}_j(C)$),
(c) or all edges adjacent to V are non-contracted, but their span near V is still only one-dimensional; w.l.o.g. we denote the edge alone on one side of V by v and the two edges on the other side by v_1, v_2.



If such a degenerated vertex exists, the 1-dimensional deformation of the curves inside σ is given by moving this vertex and changing the lengths of the adjacent edges accordingly. We show that this movement is either bounded or, if not, the changed lengths do not influence ft.
Consider the cases (a) and (b) and let v_1, v_2 be the two other edges adjacent to V. At least one of the two edges, say v_1, is bounded. Then the movement is unbounded only if v_2 is unbounded. But v_2 cannot be contracted, as then v_1 would also be contracted (*and* bounded). But this means that ft forgets v_2 and therefore also the length of v_1.
Now consider the case (c). The balancing condition says $v = v_1 + v_2$ (by abuse of notation we denote the direction vectors by the same letters as the edges), which in particular implies that v is not primitive and hence the edge v has to be bounded. Now again, if we require the movement of V to be unbounded, v_1 and v_2 must be unbounded. But they are also non-contracted which means that ft forgets them and the length of v.
So we are left with the case that all vertices of S are non-degenerated. We can still describe the deformation of the curves inside σ using the movement of the string: Take one of the ends of the string (which is necessarily non-contracted) and move it slightly in a non-zero direction modulo its linear span. Consider the next vertex V and let v be the adjacent edge not contained in the string. Then two things can happen:
If v is non-contracted (case A), our moved end will meet the affine span of v at some point P (as V is non-degenerated). So we change the length of v such that it ends at P (while keeping the position of its second vertex fixed). Then we also move the second edge of the string to P and go on to the next vertex.
If v is contracted (case B), our assumptions ensure that it is a marked end $j \in M$ and that the corresponding line G_j intersects our curve transversally at V. Thus our moved edge will again meet G_j at some point and by changing the lengths of the adjacent edges appropriately, the obtained curve will still

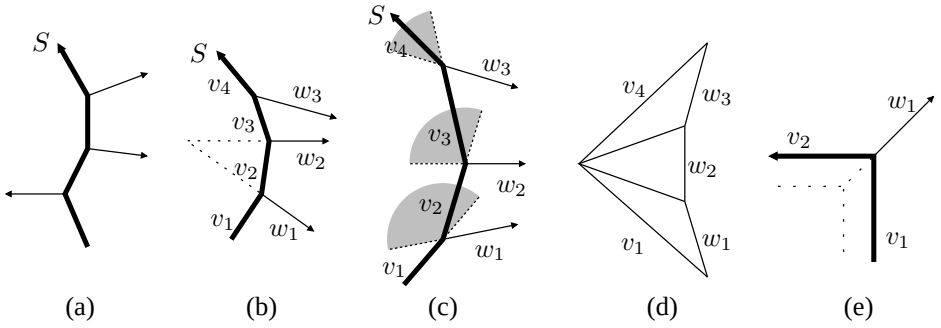
meet G_j.



In this way we can make our way through the string and finally obtain a deformation of the whole curve. Note that the non-degeneracy of all the vertices ensures that all edges of the string *must* change their positions modulo their linear span and, hence, that all edges adjacent to, but not contained in the string *must* change their length. In particular this means we cannot have more non-contracted ends adjacent to our string: Then we would have two different strings providing two independent deformations of the curves inside σ, which is a contradiction as σ is one-dimensional.

Let us summarize: Our string S is generated by two unique non-contracted ends i_1, i_2, all of its vertices are 3-valent and the adjacent edges not contained in the string are either bounded or marked ends in M, where the corresponding line G_j intersects transversally. The deformation only moves the string S; the adjacent edges are shortened or elongated and the other parts of the curve remain fixed. We want to show that, even if the movement is unbounded, the considered $\mathcal{M}_{0,4}$-coordinate is bounded.

If there are bounded edges adjacent to S to both sides of S as in picture (a) below then the movement of the string is bounded. (This is true because if we move the string to either side, we can only move until the length of one of the adjacent bounded edges shrinks to 0.) So we only have to consider the case when all adjacent bounded edges of S are on the same side of S, say on the right side as in picture (b) below. Label the edges of S (respectively, their direction vectors) by $v_1, \dots, v_k$ and the adjacent bounded edges of the curve by $w_1, \dots, w_{k-1}$ as in the picture. As above the movement of the string to the right is bounded. If one of the directions w_{i+1} is obtained from w_i by a left turn (as it is the case for $i = 1$ in the picture) then the edges w_i and w_{i+1} meet on the left of S. This restricts the movement of the string to the left, too, since the corresponding edge v_{i+1} then shrinks to length 0.



So we can assume that for all i the direction w_{i+1} is either the same as w_i or obtained from w_i by a right turn as in picture (c). The balancing condition then shows that for all i both the directions v_{i+1} and $-w_{i+1}$ lie in the angle between v_i and $-w_i$ (shaded in the picture above). Therefore, all directions v_i and $-w_i$ lie within the angle between v_1 and $-w_1$. In particular, the image of the string S cannot have any self-intersections in $\mathbb{R}^2$. We can therefore pass to the (local) dual picture (d) where the edges dual to w_i correspond to a concave side of the polygon whose other two edges are the ones dual to v_1 and v_k.

But note that there are no such concave polygons *with integer vertices* if the two outer edges (dual to v_1 and v_k) are two of the vectors $\pm(1,0)$, $\pm(0,1)$, $\pm(1,-1)$ that can occur as dual edges of an end of a plane tropical curve of degree d. Therefore the string can consist at most of the two ends i_1 and i_2

that are connected to the rest of the curve by exactly one bounded edge w_1. This situation is shown in picture (e).

In this case the movement of the string is indeed not bounded to the left. Note that then w_1 is the only internal edge whose length is not bounded. But by our assumptions 1, 3 and 4 cannot lie on S but must lie on the other side of w_1; hence its length does not influence ft. This finishes the proof.

□

5. Computing weights

In this section we prove that the weight of a curve in a zero-dimensional intersection product can be computed as the (absolute value of a) determinant of the linear map that basically collects all the evaluation morphisms (and forgetful morphism if present). We will use this to express the weight of a reducible curve (in the sense that it contains a contracted bounded edge) in terms of the weights of the two components.

The following statement connects intersection products with determinants.

Let $V = \mathbb{R} \otimes \Lambda$ be a real vector space of dimension n with underlying lattice Λ and let $h_1, \ldots, h_n \in \Lambda^\vee$ be linear functions on Λ resp. V. By $H : V \to \mathbb{R}^n$ we denote the linear map given by $x \mapsto (h_1(x), \ldots, h_n(x))$. Choose lattice bases of Λ and $\mathbb{Z}^n$ and consider the matrix representation of H with respect to these bases. Obviously, the absolute value of the determinant of this matrix is independent of the choice of bases; hence we denote it by $|\det(H)|$.
On the other hand, we can consider the rational functions $\varphi_i = \max\{h_i, 0\}$ on V. To do so, we give V the fan structure consisting of all cones on which each h_i is either positive or zero or negative. These rational functions form a zero-dimensional intersection product, which obviously consists of only $\{0\}$ with a certain weight.

Lemma 5.1. *The weight of $\{0\}$ appearing in $\varphi_n \cdot \ldots \cdot \varphi_1 \cdot V$ is equal to $|\det(H)|$.*

Proof. Let us first assume that $h_1, \ldots, h_n$ form a lattice basis of $\Lambda^\vee$ with dual basis $\tilde{h}_1, \ldots, \tilde{h}_n$.
We compute $\varphi_1 \cdot V$: For each ridge τ of V (with the fan structure described above) there exists a unique j such that $\tau \subseteq h_j^\perp$. Then there are two facets containing τ (where h_j is positive resp. negative) and the corresponding (representatives of the) primitive vectors are $\tilde{h}_j$ resp. $-\tilde{h}_j$. Therefore the weight of τ in the intersection product $\varphi_1 \cdot V$ can be computed as

$$\omega_{\varphi_1}(\tau) = \max\{h_1(\tilde{h}_j), 0\} + \max\{h_1(-\tilde{h}_j), 0\} = h_1(\tilde{h}_j) + 0 = h_1(\tilde{h}_j).$$

Hence, when omitting cones with weight 0, $\varphi_1 \cdot V$ consists of all cones contained in $h_1^\perp$ and all weights are 1. Now we can apply induction on $h_2|_{h_1^\perp}, \ldots, h_n|_{h_1^\perp}$ and conclude that $\varphi_n \cdot \ldots \cdot \varphi_1 \cdot V$ produces $\{0\}$ with weight 1.
On the other hand, the matrix representation of H with respect to the basis $\tilde{h}_1, \ldots, \tilde{h}_n$ for Λ and the standard basis for $\mathbb{Z}^n$ is just the unit matrix. Hence, $|\det(H)| = 1$, which proves the statement in the special case.
General case: For general $h_1, \ldots, h_n$ we can choose a lattice basis $l_1, \ldots, l_n$ of $\Lambda^\vee$ such that

$$\begin{aligned} h_1 &= a_{1,1} l_1, \\ h_2 &= a_{2,1} l_1 + a_{2,2} l_2, \\ &\vdots \\ h_n &= a_{n,1} l_1 + \ldots + a_{n,n} l_n, \end{aligned}$$

where the $a_{i,j}$ are integers. Then we get $|\det(H)| = |\det((a_{i,j}))| = |a_{1,1} \cdot \ldots \cdot a_{n,n}|$.
On the other hand, let us compute that $\varphi_n \cdot \ldots \cdot \varphi_1 \cdot V$ produces $\{0\}$ with weight $|a_{1,1} \cdot \ldots \cdot a_{n,n}|$. We saw in the special case that $\max\{l_1, 0\} \cdot V$ is $l_1^\perp$ with weight 1. The above equations tell us that $\max\{h_1, 0\} = |a_{1,1}| \cdot \max\{l_1, 0\}$. Using the linearity of the intersection product, we deduce that $\varphi_1 \cdot V$ is $l_1^\perp$ with weight $|a_{1,1}|$. Now we apply induction on n again: After restricting all functions to $l_1^\perp$, we

can omit all terms $a_{i,1}l_1$ in the above equations (in particular we can omit the first equation). Hence we can apply our induction hypothesis and conclude that $\varphi_n \cdot \ldots \cdot \varphi_2 \cdot l_1^\perp$ produces $\{0\}$ with weight $|a_{2,2} \cdot \ldots \cdot a_{n,n}|$. Hence, $\varphi_n \cdot \ldots \cdot \varphi_2 \cdot (\varphi_1 \cdot V) = |a_{1,1}| \cdot \varphi_n \cdot \ldots \cdot \varphi_2 \cdot l_1^\perp = \{0\}$ with weight $|a_{1,1} \cdot \ldots \cdot a_{n,n}|$. □

With this tool we can express all weights occurring in a zero-dimensional intersection product in terms of absolute values of determinants.

Notation 5.2. Let Z be a zero-dimensional intersection product of the form $(\tau_0(0)^l\tau_0(1)^m \prod_{k\in N} \tau_{r_k}(2))_d$ or of the form $(\mathrm{ft}^*(\lambda) \cdot \tau_0(0)^l\tau_0(1)^m \prod_{k\in N} \tau_{r_k}(2))_d$ with general condition G_j and P_k (and λ, resp.). The set of curves S that fulfill the incidence conditions set-theoretically is finite by 3.2 (a). For simplicity, let us furthermore assume that all points $P_k = (p_1, p_2)$ are described by the rational functions $\max\{x, p_1\}$ and $\max\{y, p_2\}$ on $\mathbb{R}^2$ and that all lines G_j are vertical, i.e. of type $\max\{x, 0\}$ (i.e. are given by a rational function $\max\{x, c_j\}$). λ can be described by $\max\{x, \lambda\}$, where x is the coordinate of the ray in whose interior λ lies (see 3.5).

Denote $X := \prod_{k\in N} \psi_k^{r_k} \cdot \mathcal{M}_{0,n}^{\mathrm{lab}}(\mathbb{R}^2, d)$. We then consider the morphisms

$$\begin{aligned} \mathrm{ev} : X &\to \mathbb{R}^m \times (\mathbb{R}^2)^n, \\ C &\mapsto \Big(\big(\mathrm{ev}_j(C)_x\big)_{j\in M}, \big(\mathrm{ev}_k(C)\big)_{k\in N} \Big), \end{aligned}$$

respectively

$$\begin{aligned} \mathrm{ft} \times \mathrm{ev} : X &\to \mathcal{M}_{0,4} \times \mathbb{R}^m \times (\mathbb{R}^2)^n, \\ C &\mapsto \Big((\mathrm{ft}(C), \big(\mathrm{ev}_j(C)_x\big)_{j\in M}, \big(\mathrm{ev}_k(C)\big)_{k\in N} \Big), \end{aligned}$$

where $\mathrm{ev}_j(C)_x$ denotes the first coordinate of the point $\mathrm{ev}_j C \in \mathbb{R}^2$. Thus, these morphisms evaluate at each end $i \in M \cup N$ and keep all coordinates if $i \in N$ and only the first coordinate if $i \in M$.
Let C be a curve in the interior of a facet σ of X (and with $\mathrm{ft}(C)$ not being the vertex of $\mathcal{M}_{0,4}$). Then ev (resp. $\mathrm{ft} \times \mathrm{ev}$) is affine in a neighborhood of C and we define $|\mathrm{det}_C(\mathrm{ev})|$ (resp. $|\mathrm{det}_C(\mathrm{ft} \times \mathrm{ev})|$) to be $|\det(H)|$, where H is the linear part of ev (resp. $\mathrm{ft} \times \mathrm{ev}$) at C.

Theorem 5.3. *The zero-dimensional intersection product Z (as in notation 5.2) can be computed as*

$$Z = \sum_{C\in S} |\mathrm{det}_C(\mathrm{ev})| \cdot C,$$

resp.

$$Z = \sum_{C\in S} |\mathrm{det}_C(\mathrm{ft} \times \mathrm{ev})| \cdot C,$$

i.e. the weight of a curve $C \in Z$ is just $|\mathrm{det}_C(\mathrm{ev})|$ (resp. $|\mathrm{det}_C(\mathrm{ft} \times \mathrm{ev})|$).

Proof. Each $C \in Z$ is contained in the interior of a facet σ of X (see 3.2 (a)). In 2.4 the weight of σ in X was computed to be

$$\omega(\sigma) = \frac{\prod_V K(I_V)!}{\prod_{k=1}^n r_k!}.$$

But we know from 3.6 that no two marked ends lie at a common vertex and hence

$$K(I_V) = \begin{cases} r_k & \text{if } k \text{ is adjacent to } V, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we can cancel the fraction defining $\omega(\sigma)$ down to 1.
As the computation of the weight of C is local, we can replace X by $V := \mathbb{R} \cdot \sigma$. On the other hand, locally around C, all the pull backs along ev and ft are of the form $\max\{a, c\}$, where a is an affine function on V and c is a constant. To be more precise, a is exactly one of the coordinate functions of ev (resp. $\mathrm{ft} \times \mathrm{ev}$). Now, up to translations and subtracting constant terms, we are in the situation of 5.1: The weight of C equals the absolute value of the determinant of linear map H whose coordinate functions are the linear parts of the affine functions a. Thus H is the linear part of ev (resp. $\mathrm{ft} \times \mathrm{ev}$)

on V, and we can conclude that the weight of C in Z is precisely $|\det(H)| = |\det_C(\mathrm{ev})|$ (resp. $= |\det_C(\mathrm{ft} \times \mathrm{ev})|$). □

Remark 5.4. If we dropped the requirement that Psi-conditions are only allowed at marked ends that are also restricted by a point condition, we could still prove a formula similar to the above one: For a zero-dimensional intersection product of arbitrary Psi- and evaluation classes, the weights can still be computed as the absolute value of an appropriate determinant times the weight of the corresponding facet in X. In particular, this shows that such weights are always positive, as well as the degree of the product. Hence, whenever classical descendant Gromov-Witten invariants are negative (e.g. $\langle \tau_1(1)\tau_0(2)\rangle_1^{\mathrm{alg}} = -1$), we have an example of classical invariants that do *not* coincide with their tropical counterparts as we define them.

6. Splitting curves

Now we want to think of curves with a contracted bounded edge as reducible curves. We do that basically by cutting the contracted bounded edge. We have to show that the weight of the curve C is (almost) the product of the two weights of the two curves that arise after cutting. We use the description of weight in terms of determinants from section 5.

Notation 6.1. As in section 4 we assume $1 \in L$, $2 \in M$ and $3, 4 \in N$. Additionally we require from now on $L = \{1\}$ (i.e. the marked end 1 is the only "free" end) and $r_3 \geq 1$ (i.e. the marked end is restricted by at least one Psi-class).

Let Z be a zero-dimensional cycle of the form $(\mathrm{ft}^*(\lambda) \cdot \tau_0(0)\tau_0(1)^m \prod_{k\in N} \tau_{r_k}(2))_d$ with general condition G_j and P_k and λ, as in 5.2. Let $C \in Z$.

Construction 6.2. Assume C satisfying $|\det_C(\mathrm{ev} \times \mathrm{ft})| \neq 0$ has a contracted bounded edge e. Cut the bounded edge e, thus producing two marked ends. In this way we get two curves C_1 and C_2 that both have a new marked end in the place of e. Let L_i, M_i and N_i be the subsets of L, M and N of marked ends in C_i. Let l_i, m_i and n_i be the sizes of these subsets. Let d_i be the degree of C_i. Denote by

$$\mathrm{ev}_{M_i\cup N_i} : \prod_{j\in N_i} \psi_j^{r_j} \cdot [\mathcal{M}^{\mathrm{lab}}_{0,l_i+n_i+m_i+1}(\mathbb{R}^2, d_i)] \to \mathbb{R}^{m_i+2n_i}$$

the map that evaluates the first coordinate for the points in M_i and both coordinates for the ends in N_i (as in 5.2). Denote by

$$\mathrm{ev}_e : \prod_{j\in N_i} \psi_j^{r_j} \cdot [\mathcal{M}^{\mathrm{lab}}_{0,l_i+n_i+m_i+1}(\mathbb{R}^2, d_i)] \to \mathbb{R}^2$$

the evaluation at e at both coordinates, and by $(\mathrm{ev}_e)_x$ the evaluation at the first coordinate. Denote by $\tilde{C}_i$ the curve C_i where we remove the marked end e and straighten the 2-valent vertex which appears.

Let

$$Z_i := \prod_{j\in M_i} \mathrm{ev}_j^*(G_j) \cdot \prod_{k\in N_i} \mathrm{ev}_k^*(P_k) \cdot \psi_k^{r_k} \cdot [\mathcal{M}^{\mathrm{lab}}_{0,l_i+n_i+m_i+1}(\mathbb{R}^2, d_i)]$$

denote the corresponding intersection products.

Notation 6.3. We pullback a general point $\lambda \in \mathcal{M}_{0,4}$ (i.e. not the vertex) via the forgetful map $\mathrm{ft} : \mathcal{M}^{\mathrm{lab}}_{0,1+m+n}(\mathbb{R}^2, d) \to \mathcal{M}_{0,4}$. There are 3 types of such general points, corresponding to the 3 types of abstract tropical curves with 4 marked ends. The ends 1 and 2 can be together at a vertex, or the ends 1 and 3, or the ends 1 and 4. We use the following short notation: if $\mathrm{ft}(C)$ is in the ray corresponding to the type where 1 and 2 are together at a vertex, we say $\mathrm{ft}(C) = 12/34$ (and analogously in the other cases).

Lemma 6.4. *Let C be as in construction 6.2 and stick to the notations from there. If $\mathrm{ft}(C) = 12/34$ (then $\{1\} = L_1$, so $l_1 = 1$ and $l_2 = 0$), then either $d_1 = 0$ and $L_1 \cup M_1 \cup N_1 = \{1, 2\}$, or $d_1, d_2 > 0$ and there are 3 cases to distinguish (of which the first and last are symmetric):*

(a) $\dim(Z_1) = 0$ *and* $\dim(Z_2) = 2$,
(b) $\dim(Z_1) = 1$ *and* $\dim(Z_2) = 1$, *or*
(c) $\dim(Z_1) = 2$ *and* $\dim(Z_2) = 0$.

If $\mathrm{ft}(C) = 13/24$ *then* $d_1, d_2 > 0$ *and the analogous* 3 *cases are to distinguish.*

Proof. If there were two contracted edges, then all evaluations (i.e. $2n + m$ coordinates) would depend only on $1 + m + n + 3d - 3 - \sum r_i = m + 2n - 1$ coordinates, so we get $|\det_C(\mathrm{ev} \times \mathrm{ft})| = 0$. So we can assume now there is only one contracted bounded edge e. Since e has to count towards the $\mathcal{M}_{0,4}$-coordinate to satisfy $|\det_C(\mathrm{ev} \times \mathrm{ft})| \neq 0$, $1, j \subset L_1 \cup M_1 \cup N_1$ and $k, l \subset L_2 \cup M_2 \cup N_2$ if $\mathrm{ft}(C) = 1j/kl$.
Let us first consider the case where one of the d_i's is zero. This implies that all edges of the corresponding curve C_i are contracted. As we cannot have more contracted bounded edges, C_i is a star-shaped curve containing only a single vertex V. But 3.6 states that the ends $2, 3, 4 \in M \cup N$ all lie at different vertices. Thus the cases $d_1 = 0, \mathrm{ft}(C) = 13/24$ and $d_2 = 0$ cannot occur, whereas in the remaining case $d_1 = 0, \mathrm{ft}(C) = 12/34$ the single vertex V must be 3-valent which is the same as $L_1 \cup M_1 \cup N_1 = \{1, 2\}$.
Let us now assume $d_1, d_2 > 0$. It remains to show that $\dim(Z_1) + \dim(Z_2) = 2$ which follows since

$$\begin{aligned}&\dim(Z_1) + \dim(Z_2)\\ =&3d_1 - \sum_{k\in N_1} r_k - m_1 - 2n_1 + 3d_2 - \sum_{k\in N_2} r_k - m_2 - 2n_2\\ =&\, 3d - \sum_{k\in N} r_k - m - 2n = 2\end{aligned}$$

where the last equality follows since Z is zero-dimensional and thus $3d - \sum_{k\in N} r_k - 1 = m + 2n + 1$. □

Remark 6.5. In the following, we will choose bases in order to write down an explicit matrix representation for the map $\mathrm{ev} \times \mathrm{ft}$ or ev locally on a cone. For a cone σ of $\mathcal{M}^{\mathrm{lab}}_{0,n}(\mathbb{R}^2, d)$ corresponding to a combinatorial type (i.e. an abstract graph Γ (without length) together with all direction vectors) we pick a root vertex V of Γ and choose the coordinates of the point $h(V) \in \mathbb{R}^2$ to which this vertex is mapped as two coordinates. The remaining coordinates of σ are given by the lengths of the bounded edges. For the spaces $\mathbb{R}^2$ or $\mathbb{R}$ that describe our incidence conditions locally, we choose the standard basis vectors. It follows from remark 3.2 of [GM05] that the absolute value of the determinant does not depend on any of the choices we make.

Lemma 6.6. *Let* C *be as in construction 6.2 and stick to the notations from there. If* $\mathrm{ft}(C) = 12/34$ *and* $d_1 = 0$ *we want to show* $|\det_C(\mathrm{ev} \times \mathrm{ft})| = (G_2 \cdot \tilde{C_2})_{h(e)} \cdot |\det_{\tilde{C_2}}(\mathrm{ev}_{M_2 \cup N_2})|$. *For the other three cases from lemma 6.4 we want to show:*

(a) $|\det_C(\mathrm{ev} \times \mathrm{ft})| = |\det_{C_1}(\mathrm{ev}_{M_1 \cup N_1})| \cdot |\det_{C_2}(\mathrm{ev}_{M_2 \cup N_2} \times \mathrm{ev}_e)|$,
(b) $|\det_C(\mathrm{ev} \times \mathrm{ft})| = (\tilde{C_1} \cdot \tilde{C_2})_{h(e)} \cdot |\det_{\tilde{C_1}}(\mathrm{ev}_{M_1 \cup N_1})| \cdot |\det_{\tilde{C_2}}(\mathrm{ev}_{M_2 \cup N_2})|$, *or*
(c) $|\det_C(\mathrm{ev} \times \mathrm{ft})| = |\det_{C_1}(\mathrm{ev}_{M_1 \cup N_1} \times \mathrm{ev}_e)| \cdot |\det_{C_2}(\mathrm{ev}_{M_2 \cup N_2})|$.

Proof. For all cases, note first that the matrix of $|\det_C(\mathrm{ev} \times \mathrm{ft})|$ has a column with only zeros except one 1. This is the column corresponding to e. Since e is contracted, it is not needed for any evaluation. But it is needed for the $\mathcal{M}_{0,4}$-coordinate, so it has zeros except a 1 in the ft-row. We can delete this row and column without changing the absolute value of the determinant. Call the matrix with the deleted row and column A. Then $|\det(A)| = |\det_C(\mathrm{ev} \times \mathrm{ft})|$.

Now let $\mathrm{ft}(C) = 12/34$ and $d_1 = 0$, it follows $L_1 \cup M_1 \cup N_1 = \{1, 2\}$. We want to show that the boundary vertex V of e in C_2 is 3-valent, too. Assume it is not, then there has to be a marked end with a Psi-condition adjacent to V. But this marked end is in N and thus required to meet a point. This is a contradiction, since the point is not on the line that 2 is required to meet (cf. 3.2 (d)). So let e_1 and e_2 be the two other edges adjacent to V and assume first that both of them are bounded. Denote their

common direction vector (up to sign) by $v = (v_1, v_2)$ and their lengths by $l(e_1), l(e_2)$. Assume that the root vertex is on the e_1-side of e. Then the entries of the matrix A corresponding to $l(e_1)$ and $l(e_2)$ are

$\downarrow$ evaluation at...	$l(e_1)$	$l(e_2)$
2 (1 row)	v_1	0
points reached via e_1 from 2 (1 or 2 rows)	0	0
points reached via e_2 from 2 (1 or 2 rows)	v	v

We see that after subtracting the $l(e_2)$-column from the $l(e_1)$-column we again get one column with only one non-zero entry v_1. So for the determinant we get v_1 as a factor, dropping the corresponding row and column (which means removing e and straightening the 2-valent vertex), so we get $|\det(A)| = v_1 \cdot |\det_{\tilde{C}_2}(\mathrm{ev}_{M_2\cup N_2})| = (\tilde{C}_2 \cdot G_2)_{h(e)} \cdot |\det_{\tilde{C}_2}(\mathrm{ev}_{M_2\cup N_2})|$. Essentially the same argument holds if one of the adjacent edges — say e_2 – is unbounded: in this case there is only an $l(e_1)$-column which has zeroes everywhere except in the one 2-row where the entry is v_1.

Next, let $\dim(Z_1) = 0$ and $\dim(Z_2) = 2$. Denote by a_i the dimension of

$$\prod_{k\in N_i} \psi_k^{r_k} \cdot \mathcal{M}^{\mathrm{lab}}_{0+l_i+m_i+n_i+1}(\mathbb{R}^2, d_i),$$

that is,

$$a_i = 3d_i + l_i + m_i + n_i + 1 - \sum_{k\in N_i} r_k - 1.$$

Since $\dim(Z_1) = 0$ we have $m_1 + 2n_1 = a_1$ and since $\dim(Z_2) = 2$ we have $m_2 + 2n_2 = a_2 - 2$. Let the boundary vertex V of e in C_1 be the root vertex for C. Choose the following order of coordinates: start with the root vertex, then bounded edges in C_1, next bounded edges in C_2. Start with the marked ends in C_1 and then add the marked ends in C_2. Then the matrix A is in block form: because the points on C_1 need only the root vertex and the bounded edges of C_1, they need the first $a_1 = m_1 + 2n_1$ coordinates, and have 0 after that. So there is a 0 block on the top right, and the top left is just the matrix of $\mathrm{ev}_{M_1\cup N_1}$ at C_1. So $|\det(A)| = |\det_{C_1}(\mathrm{ev}_{M_1\cup N_1})| \cdot |\det(B)|$ where B denotes the lower right box. Consider the matrix of $\mathrm{ev}_{M_2\cup N_2} \times \mathrm{ev}_e$ at C_2, and let the root vertex be the boundary vertex of e in C_2. Then this matrix has two more rows and columns than B, namely the root vertex columns and the rows corresponding to ev_e. But since these two rows start with a 2×2 unit matrix block and have zeros after that, we can see that $|\det(B)| = |\det_{C_2}(\mathrm{ev}_{M_2\cup N_2} \times \mathrm{ev}_e)|$.

The third case is symmetric. Finally, assume $\dim(Z_1) = 1$ and $\dim(Z_2) = 1$, i.e. $m_1 + 2n_1 = a_1 - 1$ and $m_2 + 2n_2 = a_2 - 1$. First we want to show that the two vertices of e are 3-valent. Assume the vertex in C_1, V, is not 3-valent, then there must be a marked end i with a Psi-class adjacent to V. But this end is in N then, so it is required to meet a point $P_i \in \mathbb{R}^2$. Since $\dim(Z_1) = 1$ we can move C_1 locally in a 1-dimensional family such that all incidence conditions are still defined. Let C_1' be an element of this family. Since C_1' has to meet P_i as well, we can glue C_1' to C_2 thus producing a curve C in Z. This is a contradiction since the dimension of Z is 0.

Since the argument is symmetric it follows that both vertices of e are 3-valent. Denote the two edges adjacent to e in C_1 by e_1 and e_2 and the two edges in C_2 adjacent to e by e_3 and e_4. Assume first that all of those edges are bounded. Let the boundary vertex V of e in C_1 be the root vertex for C. Then the matrix A reads:

			lengths in C_1					lengths in C_2
		root	($a_1 - 4$ cols)	$l(e_1)$	$l(e_2)$	$l(e_3)$	$l(e_4)$	($a_2 - 4$ cols)
($2n_1 + m_1$	ends behind e_1	I_2	$*$	v	0	0	0	0
rows)	ends behind e_2	I_2	$*$	0	$-v$	0	0	0
($2n_2 + m_2$	ends behind e_3	I_2	0	0	0	w	0	$*$
rows)	ends behind e_4	I_2	0	0	0	0	$-w$	$*$

where I_2 is the 2×2 unit matrix, and $*$ denotes arbitrary entries. Now add v times the root columns to the $l(e_2)$-column, subtract the $l(e_1)$-column from the $l(e_2)$-column and the $l(e_4)$-column from the

$l(e_3)$-column to obtain the following matrix with the same determinant:

		root	lengths in C_1 (a_1-4 cols)	$l(e_1)$	$l(e_2)$	$l(e_3)$	$l(e_4)$	lengths in C_2 (a_2-4 cols)
$(2n_1+m_1$ rows)	ends behind E_1	I_2	$*$	v	0	0	0	0
	ends behind E_2	I_2	$*$	0	0	0	0	0
$(2n_2+m_2$ rows)	ends behind E_3	I_2	0	0	v	w	0	$*$
	ends behind E_4	I_2	0	0	v	w	$-w$	$*$

Note that this matrix has a block form with a zero block at the top right. Denote the top left block (of size $2n_1+m_1 = 2+a_1-4+1$) by A_1 and the bottom right (of size $2n_2+m_2 = 3+a_2-4$) by A_2, then $|\det(A)| = |\det A_1 \cdot \det A_2|$.

The matrix A_1 is precisely the matrix for the evaluation map $\mathrm{ev}_{M_1\cup N_1}$ of $\tilde{C}_1$ (which arises from C_1 after forgetting the marked end corresponding to e) if we choose the other vertex of e_2 as the root vertex. Hence $|\det A_1| = |\det_{\tilde{C}_1}(\mathrm{ev}_{M_1\cup N_1})|$. In the same way the matrix for the evaluation map $\mathrm{ev}_{M_2\cup N_2}$ of $\tilde{C}_2$, if we again forget the marked end corresponding to e and now choose the other vertex of e_3 as the root vertex, is the matrix A_2' obtained from A_2 by replacing v and w in the first two columns by the first and second unit vector, respectively. But A_2 is simply obtained from A_2' by right multiplication with the matrix

$$\begin{pmatrix} v & w & 0 \\ 0 & 0 & I_{2n_2+2} \end{pmatrix}$$

which has determinant $\det(v,w)$. So we conclude that

$$|\det A_2| = |\det(v,w)| \cdot |\det A_2'| = (C_1\cdot C_2)_{h(e)} \cdot |\det_{\tilde{C}_2}(\mathrm{ev}_{M_2\cup N_2})|.$$

□

Remark 6.7. The following "converse" of lemma 6.4 and lemma 6.6 is also true: For each choice of $\tilde{C}_2$ satisfying all conditions but 2 and each choice of an intersection point of $\tilde{C}_2$ with G_2 we can add a contracted bounded edge and the two marked ends $1,2$ on the other side to built exactly one possible C. The curve C then contributes $(G_2\cdot\tilde{C}_2)_{h(e)}|\det_{\tilde{C}_2}(\mathrm{ev}_{M_2\cup N_2})|$ to the count. By Bézout's theorem ([RST03]), each choice of $\tilde{C}_2$ contributes $d_2\cdot|\det_{\tilde{C}_2}(\mathrm{ev}_{M_2\cup N_2})| = d\cdot|\det_{\tilde{C}_2}(\mathrm{ev}_{M_2\cup N_2})|$.

For each choice of C_1 satisfying the conditions in $L_1\cup M_1\cup N_1$ and each choice of C_2 satisfying the conditions in $L_2\cup M_2\cup N_2$ plus in addition the condition $h(e)=p$ we get exactly one possible C by gluing the two curves along e. This curve C contributes to the count with weight $|\det_{C_1}(\mathrm{ev}_{M_1\cup N_1})|\cdot|\det_{C_2}(\mathrm{ev}_{M_2\cup N_2}\times\mathrm{ev}_e)|$ (and the other way round).

For each choice of $\tilde{C}_1$ and $\tilde{C}_2$ satisfying the conditions in $L_1\cup M_1\cup N_1$ and $L_2\cup M_2\cup N_2$ and for each choice of points $P\in\tilde{C}_1$ and $Q\in\tilde{C}_2$ that map to the same image point in $\mathbb{R}^2$ we can glue P and Q along a contracted bounded edge and thus built exactly one possible C. The curve C contributes to the count with weight $(C_1\cdot C_2)_{h(e)}\cdot|\det_{\tilde{C}_1}(\mathrm{ev}_{M_1\cup N_1})|\cdot|\det_{\tilde{C}_2}(\mathrm{ev}_{M_2\cup N_2})|$. By Bézout's theorem, each choice of $\tilde{C}_1$ and $\tilde{C}_2$ thus contributes $d_1\cdot d_2\cdot|\det_{\tilde{C}_1}(\mathrm{ev}_{M_1\cup N_1})|\cdot|\det_{\tilde{C}_2}(\mathrm{ev}_{M_2\cup N_2})|$.

7. String and divisor equation

In this section we prove two lemmas which deal with the case of an extra end in a top-dimensional intersection product that is restricted either by no condition at all (string equation) or by only a line condition (divisor equation).

Lemma 7.1 (String equation)**.** *For tropical descendant Gromov-Witten invariants the following equality holds:*

$$\begin{aligned}&\langle \tau_0(0)\cdot\tau_0(0)^l\cdot\tau_0(1)^m\cdot\prod_{k\in N}\tau_{r_k}(2)\rangle_d\\ =&\sum_{\substack{k\in N\\ r_k>0}}\langle\tau_0(0)^l\cdot\tau_0(1)^m\cdot\tau_{r_k-1}(2)\cdot\prod_{k\neq k'\in N}\tau_{r_{k'}}(2)\rangle_d\end{aligned}$$

Proof. Choose incidence conditions G_j, P_k such that they are general for all the derived intersection products

$$\begin{aligned}Z&:=(\tau_0(0)\cdot\tau_0(0)^l\cdot\tau_0(1)^m\cdot\prod_{k\in N}\tau_{r_k}(2))_d,\\ Z_k&:=(\tau_0(0)^l\cdot\tau_0(1)^m\cdot\tau_{r_k-1}(2)\cdot\prod_{k\neq k'\in N}\tau_{r_{k'}}(2))_d\end{aligned}$$

(note that Z lives in $\mathcal{M}^{\text{lab}}_{0,1+l+m+n}(\mathbb{R}^2,d)$, whereas the Z_k lives in $\mathcal{M}^{\text{lab}}_{0,l+m+n}(\mathbb{R}^2,d)$). Then 3.2 (a) tells us that the products just consist of the set of curves fulfilling the incidence conditions and having required valences, with the additional data of a weight for each curve.
Let C' be a curve in Z_k. Then we obtain a curve $C\in Z$ by attaching the additional end, say x, to the vertex V_k at which the end k lies. Let us check that the weight of C' in Z_k and C in Z coincide. As our conditions are general, C' lies in a facet σ' of $\psi_k^{r_k-1}\cdot\prod_{k\neq k'\in N}\psi_{k'}^{r_{k'}}\cdot\mathcal{M}^{\text{lab}}_{0,l+m+n}(\mathbb{R}^2,d)$ and C lies in a facet σ of $\prod_{k\in N}\psi_k^{r_k}\cdot\mathcal{M}^{\text{lab}}_{0,1+l+m+n}(\mathbb{R}^2,d))$. Moreover, the map

$$\text{ft}_x:\mathcal{M}^{\text{lab}}_{0,1+l+m+n}(\mathbb{R}^2,d)\to\mathcal{M}^{\text{lab}}_{0,l+m+n}(\mathbb{R}^2,d)$$

forgetting the additional end x maps σ $\mathbb{Z}$-isomorphically to σ' (the inverse is given by adding 1 to V_k as above). The evaluation maps ev_k on σ are just obtained as pull-backs $\text{ft}_{x*}(\text{ev}'_k)$, where ev'_k denotes the corresponding evaluation map on σ'. Hence, the weights of C and C' coincide.
It remains to check that each $C\in Z$ is obtained in the above way from $C'\in Z_k$ for unique $k\in N$. Uniqueness is clear, as by 3.6 (b') all ends $k\in N$ lie at pairwise different vertices and hence x can not be adjacent to more than one end $k\in N$. On the other hand, to show that it is adjacent to a $k\in N$ with $r_k>0$, it suffices to show that x cannot be adjacent to a 3-valent vertex. If it were, at least one of the other two adjacent edges, say E would be bounded (otherwise the abstract graph were not connected). But then, we could change the length of E (and accordingly the length of the other edge if necessary) without changing the coordinates of the marked ends, which contradicts the fact that the set of curves fulfilling our given conditions is finite by 3.2 (a).

□

Lemma 7.2 (Divisor equation)**.** *For tropical descendant Gromov-Witten invariants the following equality holds:*

$$\langle\tau_0(1)\cdot\tau_0(0)^l\cdot\tau_0(1)^m\cdot\prod_{k\in N}\tau_{r_k}(2)\rangle_d=d\cdot\langle\tau_0(0)^l\cdot\tau_0(1)^m\cdot\prod_{k\in N}\tau_{r_k}(2)\rangle_d$$

Proof. First we choose general incidence conditions. Because of 2.10 we can assume that the line conditions are all vertical lines, i.e. of type $\max\{x,0\}$. Then for all curves C in $(\tau_0(1)\cdot\tau_0(0)^l\cdot\tau_0(1)^m\cdot\prod_{k\in N}\tau_{r_k}(2))_d$ we know that their weight is equal to $|\det_C(\text{ev})|$, where ev denotes the product of all evaluation maps (evaluation of the x-coordinate for all lines, both coordinates for all points) (theorem 5.3). Assume x is the additional marked end with line condition G (but without Psi-condition). x has to be adjacent to a 3-valent vertex (see 3.6). Exactly as in lemma 6.6 we can see that $|\det_C(\text{ev})|=(G\cdot\tilde{C})_{h(x)}\cdot|\det_{\tilde{C}}(\text{ev}_x)|$ where $\tilde{C}$ is the curve we get when forgetting x (i.e. removing it from C and straightening the 2-valent vertex) and ev_x is the product of all other evaluations. Thus any curve in $(\tau_0(1)\cdot\tau_0(0)^l\cdot\tau_0(1)^m\cdot\prod_{k\in N}\tau_{r_k}(2))_d$ gives us a curve in $(\tau_0(0)^l\cdot\tau_0(1)^m\cdot\prod_{k\in N}\tau_{r_k}(2))_d$ by removing

the marked end x. Conversely, given a curve $\tilde{C}$ in $(\tau_0(0)^l \cdot \tau_0(1)^m \cdot \prod_{k\in N} \tau_{r_k}(2))_d$ we can pick a point $p \in (G \cdot \tilde{C})$ and attach a marked end to get a curve $C \in (\tau_0(1) \cdot \tau_0(0)^l \cdot \tau_0(1)^m \cdot \prod_{k\in N} \tau_{r_k}(2))_d$. Since $(G \cdot \tilde{C}) = \sum_p (G \cdot \tilde{C})_p = d$ by tropical Bézout's theorem ([RST03]), the statement follows. □

8. Recursion

Now we sum up the results of the preceding sections to a certain WDVV equation. We also show in this section that this WDVV equation together with the string and the divisor equation are sufficient to show that the tropical invariants coincide with the classical ones. To distinguish our tropical invariants that we denote by $\langle \tau_0(0)^l \tau_0(1)^m \prod_{k\in N} \tau_{r_k}(2)\rangle_d$ from the classical ones, we use the notation $\langle \tau_0(0)^l \tau_0(1)^m \prod_{k\in N} \tau_{r_k}(2)\rangle_d^{\text{alg}}$ for the classical invariants.

Theorem 8.1. *The tropical descendant invariants as defined in 2.10 satisfy the following WDVV equation if $r_3 > 0$:*

$$\begin{aligned}
&\langle \tau_0(1)^m \prod_{k\in N} \tau_{r_k}(2)\rangle_d + \\
&\sum D \cdot \langle \tau_0(0)\tau_0(1)\tau_0(1)^{m_1} \prod_{k\in N_1} \tau_{r_k}(2)\tau_0(e)\rangle_{d_1} \cdot \langle \tau_{r_3}(2)\tau_{r_4}(2)\tau_0(1)^{m_2} \prod_{k\in N_2} \tau_{r_k}(2)\tau_0(f)\rangle_{d_2} \\
=&\sum D \cdot \langle \tau_0(0)\tau_{r_3}(2)\tau_0(1)^{m_1} \prod_{k\in N_1} \tau_{r_k}(2)\tau_0(e)\rangle_{d_1} \cdot \langle \tau_0(1)\tau_{r_4}(2)\tau_0(1)^{m_2} \prod_{k\in N_2} \tau_{r_k}(2)\tau_0(f)\rangle_{d_2}
\end{aligned}$$

where

$$D = \frac{(d_1!)^3 \cdot (d_2!)^3}{d!^3}$$

and the sums range over all

$$\begin{aligned}
&e + f = 2,\ e, f \geq 0, \\
&M_1 \dot\cup M_2 = M \setminus \{2\}, \\
&N_1 \dot\cup N_2 = N \setminus \{3, 4\} \text{ and} \\
&E_1 \dot\cup E_2 = E, E_1, E_2 \neq \emptyset.
\end{aligned}$$

Here, E denotes the set of non-contracted ends, and E_1 is subset of non-contracted ends such that each of the standard directions $-e_1, -e_2, e_1 + e_2$ appears d_1 times.

The equation can be rewritten as

$$\begin{aligned}
&\langle \tau_0(1)^m \prod_{k\in N} \tau_{r_k}(2)\rangle_d + \\
&\sum \langle \tau_0(0)\tau_0(1)\tau_0(1)^{m_1} \prod_{k\in N_1} \tau_{r_k}(2)\tau_0(e)\rangle_{d_1} \cdot \langle \tau_{r_3}(2)\tau_{r_4}(2)\tau_0(1)^{m_2} \prod_{k\in N_2} \tau_{r_k}(2)\tau_0(f)\rangle_{d_2} \\
=&\sum \langle \tau_0(0)\tau_{r_3}(2)\tau_0(1)^{m_1} \prod_{k\in N_1} \tau_{r_k}(2)\tau_0(e)\rangle_{d_1} \cdot \langle \tau_0(1)\tau_{r_4}(2)\tau_0(1)^{m_2} \prod_{k\in N_2} \tau_{r_k}(2)\tau_0(f)\rangle_{d_2}
\end{aligned}$$

where now the sums range over all

$$\begin{aligned}
&e + f = 2, \\
&M_1 \dot\cup M_2 = M \setminus \{2\}, \\
&N_1 \dot\cup N_2 = N \setminus \{3, 4\} \text{ and} \\
&d_1 + d_2 = d, d_1, d_2 > 0.
\end{aligned}$$

Proof. It follows from rational equivalence that

$$\begin{aligned}&\langle\tau_0(0)\tau_0(1)^m\prod_{k\in N}\tau_{r_k}(2)\,\mathrm{ft}^*(\lambda)\rangle_d\\&=\frac{1}{(d!)^3}\deg\left(\prod_{j\in M}\mathrm{ev}_j^*(G_j)\prod_{k\in N}\mathrm{ev}_k^*(P_k)\psi_k^{r_k}\cdot\mathrm{ft}^*(\lambda)\cdot\mathcal{M}^{\mathrm{lab}}_{0,1+m+n}(\mathbb{R}^2,d)\right)\end{aligned}$$

does not depend on the choice of $\lambda\in\mathcal{M}_{0,4}$. Thus we can pick a very large λ_1 on the ray $12/34$ of $\mathcal{M}_{0,4}$ and a very large λ_2 on the ray $13/24$ and set the degree equal for those two values. Denote by

$$Z:=\deg\left(\prod_{j\in M}\mathrm{ev}_j^*(G_j)\prod_{k\in N}\mathrm{ev}_k^*(P_k)\psi_k^{r_k}\cdot\mathrm{ft}^*(\lambda_1)\cdot\mathcal{M}^{\mathrm{lab}}_{0,1+m+n}(\mathbb{R}^2,d)\right).$$

We show that the left hand side of the above sum equals $\frac{1}{d!^3}$ times the degree of Z. Analogously one can show that the right hand side equals $\frac{1}{d!^3}$ times the degree of the analogous intersection product with λ_2, which finishes the proof. By theorem 5.3 we know that

$$Z=\sum_{C\in S}|\mathrm{det}_C(\mathrm{ft}\times\mathrm{ev})|\cdot C,$$

where S is the set of curves in $\mathcal{M}^{\mathrm{lab}}_{0,1+m+n}(\mathbb{R}^2,d)$ satisfying the point and line conditions and mapping to λ_1 under ft. Let $F=(\tau_0(0)\tau_0(1)^m\prod_{k\in N}\tau_{r_k}(2))_d$, then F is a one-dimensional cycle. Let σ be a cone of F corresponding to curves without a contracted bounded edge. Then lemma 4.4 says that the image of σ under ft is bounded. Since we picked λ_1 to be very large, we therefore know that σ cannot contribute to the degree of Z. Hence all $C\in S$ contain a contracted bounded edge. Pick a curve $C\in S$, then we know by 6.4 that we can cut the contracted edge thus producing two curves C_1 and C_2 with an extra marked end e.

If the degree of C_1, d_1, equals 0 then we know by 6.6 that

$$|\mathrm{det}_C(\mathrm{ft}\times\mathrm{ev})|=(G_2\cdot\tilde{C}_2)_{h(e)}\cdot|\mathrm{det}_{\tilde{C}_2}(\mathrm{ev}_{M_2\cup N_2})|,$$

where G_2 denotes the line condition for the marked end 2 and $\tilde{C}_2$ denotes the curve that we get from C_2 by forgetting the additional marked end e. By 6.7 we know that each choice of $\tilde{C}_2$ satisfying all conditions in $L_2\cup M_2\cup N_2=L\cup M\cup N\setminus\{1,2\}$ contributes $d\cdot|\mathrm{det}_{\tilde{C}_2}(\mathrm{ev}_{M_2\cup N_2})|$ possible curves C (counted with weight). Thus the contribution to Z from curves C such that $d_1=0$ equals

$$d\cdot\prod_{j\in M\setminus\{2\}}\mathrm{ev}_j^*(G_j)\prod_{k\in N}\mathrm{ev}_k^*(P_k)\psi_k^{r_k}\cdot\mathcal{M}^{\mathrm{lab}}_{0,m-1+n}(\mathbb{R}^2,d)$$

which by the divisor equation (7.2) equals

$$\prod_{j\in M}\mathrm{ev}_j^*(G_j)\prod_{k\in N}\mathrm{ev}_k^*(P_k)\psi_k^{r_k}\cdot\mathcal{M}^{\mathrm{lab}}_{0,m+n}(\mathbb{R}^2,d).$$

Multiplying by the factor $\frac{1}{d!^3}$, we can see that those curves contribute

$$\langle\tau_0(1)^m\prod_{k\in N}\tau_{r_k}(2)\rangle_d$$

to $\frac{1}{d!^3}\deg Z$.

Now assume that $d_1>0$ and denote as in 6.2

$$Z_i:=\prod_{j\in M_i}\mathrm{ev}_j^*(G_j)\cdot\prod_{j\in N_i}\mathrm{ev}_j^*(p_j)\cdot\prod_{j\in N_i}\psi_j^{r_j}\cdot[\mathcal{M}^{\mathrm{lab}}_{0,l_i+n_i+m_i+1}(\mathbb{R}^2,d_i)].$$

Then we know by 6.4 that one of the following three cases hold:

(a) $\dim(Z_1)=0$ and $\dim(Z_2)=2$ or
(b) $\dim(Z_1)=1$ and $\dim(Z_2)=1$ or

(c) $\dim(Z_1) = 2$ and $\dim(Z_2) = 0$.

We know by 6.6 that in the first case,

$$|\det_C(\mathrm{ev} \times \mathrm{ft})| = |\det_{C_1}(\mathrm{ev}_{M_1 \cup N_1})| \cdot |\det_{C_2}(\mathrm{ev}_{M_2 \cup N_2} \times \mathrm{ev}_e)|,$$

where ev_e now denotes the evaluation on both coordinates of the new marked end e. By 6.7 we know that for each choice of C_1 and C_2 satisfying the conditions we get exactly one possible C. But by 5.3 we know that

$$Z_1 = \sum_{C_1} |\det_{C_1}(\mathrm{ev}_{M_1 \cup N_1})| \cdot C_1,$$

and analogously

$$\mathrm{ev}_e^*(P) \cdot Z_2 = \sum_{C_2} |\det_{C_2}(\mathrm{ev}_{L_2 \cup M_2 \cup N_2} \times \mathrm{ev}_e)| \cdot C_2,$$

Thus we get a contribution of $\deg(Z_1) \cdot \deg(\mathrm{ev}_e^*(P) \cdot Z_2)$ to $\deg(Z)$, respectively

$$\frac{(d_1!)^3 \cdot (d_2!)^3}{d!^3} \langle \tau_0(0)\tau_0(1)\tau_0(1)^{m_1} \prod_{k \in N_1} \tau_{r_k}(2) \rangle_{d_1} \cdot \langle \tau_{r_3}(2)\tau_{r_4}(2)\tau_0(1)^{m_2} \prod_{k \in N_2} \tau_{r_k}(2)\tau_0(2) \rangle_{d_2}$$

to $\frac{1}{d!^3} \deg Z$.

Analogously, we get a contribution of

$$\frac{(d_1!)^3 \cdot (d_2!)^3}{d!^3} \langle \tau_0(0)\tau_0(1)\tau_0(1)^{m_1} \prod_{k \in N_1} \tau_{r_k}(2)\tau_0(2) \rangle_{d_1} \cdot \langle \tau_{r_3}(2)\tau_{r_4}(2)\tau_0(1)^{m_2} \prod_{k \in N_2} \tau_{r_k}(2) \rangle_{d_2}$$

in the last case.

In the second case, we know by 6.6 that

$$|\det_C(\mathrm{ev} \times \mathrm{ft})| = (\tilde{C}_1 \cdot \tilde{C}_2)_{h(e)} \cdot |\det_{\tilde{C}_1}(\mathrm{ev}_{M_1 \cup N_1})| \cdot |\det_{\tilde{C}_2}(\mathrm{ev}_{M_2 \cup N_2})|$$

and by 6.7 we know that each choice of $\tilde{C}_1$ and $\tilde{C}_2$ satisfying the conditions gives us

$$d_1 \cdot d_2 \cdot |\det_{\tilde{C}_1}(\mathrm{ev}_{M_1 \cup N_1})| \cdot |\det_{\tilde{C}_2}(\mathrm{ev}_{M_2 \cup N_2})|.$$

Since

$$(\mathrm{ft}_e)_*(Z_i) = \sum_{\tilde{C}_i} |\det_{\tilde{C}_i}(\mathrm{ev}_{M_i \cup N_i})| \cdot \tilde{C}_i$$

(where ft_e denotes the map which forgets the marked point e) and since

$$d_1 \cdot \langle \tau_0(0)\tau_0(1)\tau_0(1)^{m_1} \prod_{k \in N_1} \tau_{r_k}(2) \rangle_{d_1} = \langle \tau_0(0)\tau_0(1)\tau_0(1)^{m_1} \prod_{k \in N_1} \tau_{r_k}(2)\tau_0(1) \rangle_{d_1}$$

and

$$d_2 \cdot \langle \tau_{r_3}(2)\tau_{r_4}(2)\tau_0(1)^{m_2} \prod_{k \in N_2} \tau_{r_k}(2) \rangle_{d_2} = \langle \tau_{r_3}(2)\tau_{r_4}(2)\tau_0(1)^{m_2} \prod_{k \in N_2} \tau_{r_k}(2)\tau_0(1) \rangle_{d_2}$$

by the divisor equation we get a contribution of

$$\frac{(d_1!)^3 \cdot (d_2!)^3}{d!^3} \langle \tau_0(0)\tau_0(1)\tau_0(1)^{m_1} \prod_{k \in N_1} \tau_{r_k}(2)\tau_0(1) \rangle_{d_1} \cdot \langle \tau_{r_3}(2)\tau_{r_4}(2)\tau_0(1)^{m_2} \prod_{k \in N_2} \tau_{r_k}(2)\tau_0(1) \rangle_{d_2}.$$

Finally, there are $\binom{d}{d_1}^3$ choices of the sets E_1 and E_2 if we fix d_1 and d_2.

□

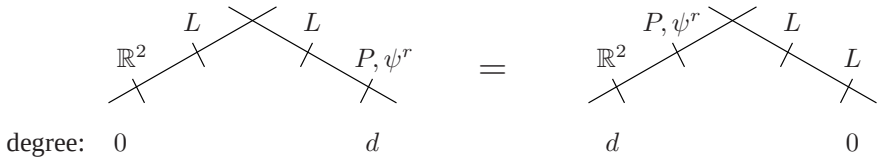
Lemma 8.2. *Choose strictly positive integers r, d such that $1 + 3d - 3 + 2 = 2 + r$. Then the classical one-marked-point invariant $\langle \tau_r(2) \rangle_d^{alg}$ equals*

$$\langle \tau_r(2) \rangle_d^{alg} = \frac{1}{(d!)^3}.$$

Proof. We use two (classical) WDVV equations ([FP95] or, more detailed but unpublished, [Ko]) with four marked points. If we compute $\langle \tau_0(0)\tau_0(1)^2\tau_r(2)\,\mathrm{ft}^*(\lambda)\rangle_d^{\text{alg}}$ for the two special points $\lambda = 12/34$ and $\lambda = 13/24$ on $M_{0,4}$, then we get

$$\langle \tau_0(1)^2\tau_r(2)\rangle_d^{\text{alg}} = \langle \tau_0(0)\tau_r(2)\tau_0(2)\rangle_d^{\text{alg}}$$

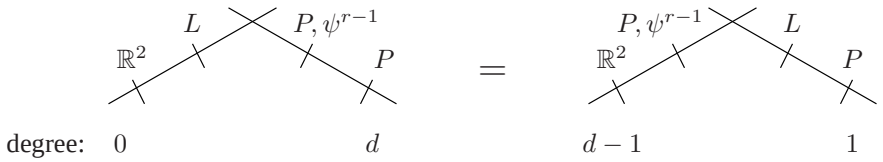
as illustrated by the following picture:



For $\langle \tau_0(0)\tau_0(1)\tau_0(2)\tau_{r-1}(2)\,\mathrm{ft}^*(\lambda)\rangle_d^{\text{alg}}$ we get

$$\langle \tau_0(1)\tau_{r-1}(2)\tau_0(2)\rangle_d^{\text{alg}} = \langle \tau_0(0)^2\tau_{r-1}(2)\rangle_{d-1}^{\text{alg}}$$

as illustrated by



Now, applying string and divisor equation where possible and plugging in the left hand side of the first equation in the right hand side of the second equation produces

$$d^3 \cdot \langle \tau_r(2)\rangle_d^{\text{alg}} = \langle \tau_{r-3}(2)\rangle_{d-1}^{\text{alg}}.$$

Together with the initial invariant $\langle \tau_0(0)^2\tau_0(2)\rangle_0^{\text{alg}} = 1$, this proves the lemma. □

Lemma 8.3. *Choose strictly positive integers r, d such that $1 + 3d - 3 + 2 = 2 + r$. Then the tropical one-marked-end invariant $\langle \tau_r(2) \rangle_d^{trop}$ equals*

$$\langle \tau_r(2) \rangle_d^{trop} = \frac{1}{(d!)^3}.$$

Proof. Choosing the single marked end e as root vertex, we get $\mathcal{M}_{0,1}^{\text{lab}}(\mathbb{R}^2, d) = \mathcal{M}_{0,1+3d} \times \mathbb{R}^2$ and the two projections are ft' and ev_e. Recall that Psi-classes for parameterized curves are just pull-backs of Psi-classes along ft. Using [AR07, 9.6], we get

$$\begin{aligned}
\langle \tau_r(2)\rangle_d^{\text{trop}} &= \frac{1}{(d!)^3} \deg\left((\mathrm{ft}'(\psi_e))^r \cdot \mathrm{ev}_e(P) \cdot (\mathcal{M}_{0,1+3d} \times \mathbb{R}^2)\right) \\
&= \frac{1}{(d!)^3} \deg\left((\psi_e^r \cdot \mathcal{M}_{0,1+3d}) \times (P \cdot \mathbb{R}^2)\right) \\
&= \frac{1}{(d!)^3},
\end{aligned}$$

where in the last step we use $\deg(\psi_e^r \cdot \mathcal{M}_{0,1+3d}) = 1$ (cf. [KM07, 4.2]). □

Theorem 8.4. *Let d, l, m, n and $r_k, k \in N$ be positive integers with $d > 0$ such that*

$$l + m + n + 3d - 3 + 2 = m + 2n + \sum_{k \in N} r_k.$$

Then the corresponding tropical and classical descendant invariants coincide, i.e.

$$\langle \tau_0(0)^l \tau_0(1)^m \prod_{k \in N} \tau_{r_k}(2) \rangle_d^{trop} = \langle \tau_0(0)^l \tau_0(1)^m \prod_{k \in N} \tau_{r_k}(2) \rangle_d^{alg}.$$

Proof. The tropical string, divisor and WDVV equations proved in the preceding sections are also fulfilled by the corresponding classical invariants. Hence, we can use these equations to reduce our invariants to such ones for which we know or can prove that they coincide.
1. case: $r_k = 0$ for all $k \in N$ (i.e. no Psi-classes)
After applying string and divisor equation, we can assume that $l = 0 = m$. Using 5.3, we see that the numbers $I^{\text{trop}}(d; 0, 0, n; 0)$ are equal to the numbers N_d defined in [GM05, 3.4, 3.9]. It is well-known that these numbers coincide with the classical ones (see [Mi03] and [GM05, 5.6]).
2. case: exists $k \in N$ with $r_k > 0$ (i.e. at least one Psi-class)
subcase I: $n = 1$
After applying string and divisor equation, we can assume that $l = 0$ and $m = 0$. The two last preceding lemmas show that in this case the classical and tropical invariants coincide.
subcase II: $n \geq 2$
After applying string and divisor equation, we can assume that $l = 0$ and $m = 1$. In particular, if $m = 0$, we can use the divisor equation to add a line condition, which introduces a factor $\frac{1}{d}$ and therefore leads to rational numbers. Then we can use both the tropical (see theorem 8.1) and the classical WDVV equation ([FP95] or [Ko]) and express $\langle \tau_0(1) \prod_{k \in N} \tau_{r_k}(2) \rangle_d$ in terms of invariants $\langle \tau_0(0)^{l'} \tau_0(1)^{m'} \prod_{k \in N'} \tau_{r'_k}(2) \rangle_{d'}$ with $n' + \sum_{k \in N'} r'_k < n + \sum_{k \in N} r_k$. Repeating this procedure, we eventually end up with $n' = 1$ (subcase I) or $r'_k = 0$ for all $k \in N'$, which is the 1. case. □

9. Lattice paths

In this section, we present a lattice-paths algorithm to determine the numbers

$$\langle \prod_{k \in N} \tau_{r_k}(2) \rangle_d = \frac{1}{(d!)^3} \deg \left(\prod_{k \in N} \text{ev}_k^*(P_k) \psi_k^{r_k} \cdot \mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^2, d) \right)$$

i.e. numbers of curves with Psi- and point conditions (and no line conditions; all other numbers can be easily computed using string and divisor equation). Note that in this case we need

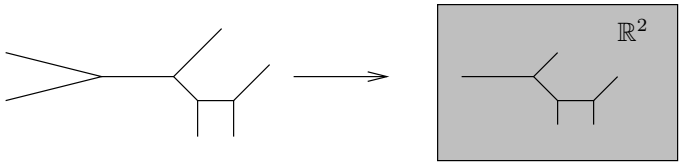
$$3d - 1 = n + \sum_{k \in N} r_k$$

to get a zero-dimensional cycle.

We use the fact that if we choose general point conditions, the intersection product $\prod_{k \in N} \text{ev}_k^*(P_k) \psi_k^{r_k} \cdot \mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^2, d)$ equals set-theoretically the set of all points corresponding to curves satisfying the Psi- and the point conditions (see 3.3). Each such curve C has to be counted with weight, and it is counted with the weight $\frac{1}{d!^3} |\det_C(\text{ev})|$ (see theorem 5.3), where ev denotes the product of all evaluation maps (at both coordinates). Note that no such curve can have a string since this would provide a deformation of the curve described in the proof of 4.4, which contradicts 3.2 (a).

We pick a certain configuration of points and count dual Newton subdivisions of curves passing through the points and satisfying the Psi-conditions. The dual Newton subdivisions are in fact dual to the image $h(\Gamma) \subset \mathbb{R}^2$ of the graph in the plane. In particular, the labels of the non-contracted ends are lost. That means we have to count tropical curves without labels on the non-contracted ends, and then multiply with the number of possibilities to set labels.

There is a map $c : \mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^2, d) \to \mathcal{M}_{0,n}(\mathbb{R}^2, d)$ which forgets the labels of the non-marked ends. This map is a cover, the number of preimages is the number of ways to set labels. The biggest number of preimages is $d!^3$. However, not every point in a facet has this number of preimages: the curve in $\mathcal{M}_{0,0}(\mathbb{R}^2, 2)$ pictured below has only 4 preimages, not 8, since the two ends in direction $(-1, 0)$ are not distinguishable.



Let $C' \in \prod_{k\in N} \psi_k^{r_k} \cdot \mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^2, d)$ and let C be the curve after forgetting the labels of the non-contracted ends. Assume that the facet σ in $M_{0,n}(\mathbb{R}^2, d)$ in which C lies has s preimages under the cover above. Thus C has to be counted with $\frac{s}{d!^3} |\det_{C'}(\text{ev})|$. Assume that C has t vertices $V_1, \ldots, V_t$ such that b_{ij} non-contracted ends of the same direction $-e_j$ are adjacent to V_i (where j goes from 0 to 2 and $e_0 := -e_1 - e_2$). Then $\nu_C := \prod_{i=1}^{t} \prod_{j=0}^{2} \frac{1}{b_{ij}!} = \frac{s}{d!^3}$ and we have to count C with $\nu_C |\det_{C'}(\text{ev})|$.

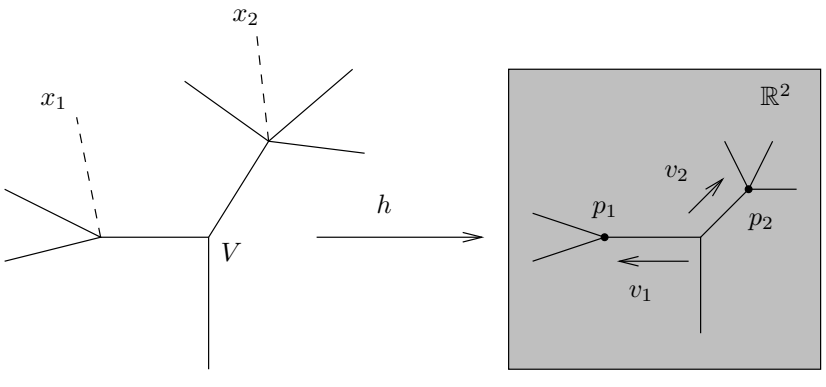
First, we want to understand this weight locally in terms of vertex multiplicities. We define another weight that we denote by $\text{mult}(C)$ and we show that it is equal to $\nu_C |\det_{C'}(\text{ev})|$.

Definition 9.1. Let $C' \in \prod_{k\in N} \psi_k^{r_k} \cdot \mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^2, d)$ and let C be the curve after forgetting the labels of the non-contracted ends. Define the weight $\text{mult}(C)$ as ν_C times the product of the multiplicities of those (necessarily 3-valent) vertices without any marked ends on them (see [Mi03], definition 2.16).

Example 9.2. Let C be the curve as in the picture below. (For this example, we chose some other degree, not d, to keep the picture nice.) As in remark 6.5 we choose coordinates to write down an explicit matrix for ev. Choose V to be the root vertex. Then the matrix of ev is

$$\begin{pmatrix} 1 & 0 & v_{1,1} & 0 \\ 0 & 1 & v_{1,2} & 0 \\ 1 & 0 & 0 & v_{2,1} \\ 0 & 1 & 0 & v_{2,2} \end{pmatrix}.$$

The absolute value of the determinant is equal to $|\det(v_1, v_2)|$, which is the multiplicity of the 3-valent vertex V.



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Lemma 9.3. *Let $C' \in \prod_{k\in N} \psi_k^{r_k} \cdot \mathcal{M}_{0,n}^{lab}(\mathbb{R}^2, d)$ and let C be the curve after forgetting the labels of the non-contracted ends. Then $\nu_C |\det_{C'}(\text{ev})| = \text{mult}(C)$ if C has no string.*

Proof. We have to show that $|\det_{C'}(\text{ev})|$ equals the product of multiplicities of all 3-valent vertices without marked end.

This is an induction on the number of bounded edges. Curves with no bounded edge satisfy $|\det_{C'}(\mathrm{ev})| = \prod_V \mathrm{mult}(V) = 1$ (the product is empty). Curves with 2 bounded edges (as the one in the example 9.2) need to have the two marked ends at the two "outer" vertices, because otherwise there is a string. So as in the example, there is one "interior" 3-valent vertex without a marked end, and $|\det_{C'}(\mathrm{ev})|$ is the multiplicity of the vertex.

Now we can assume we have 4 or more bounded edges. (The number of bounded edges is even, $2n-2$.) We choose one such that there are still bounded edges on both sides of it. (If such an edge does not exist, it means we have a "star-shaped" tropical curve with one vertex in the middle and all bounded edges around. If one of those bounded edges was not adjacent to a marked end, C has a string, so we can assume that all bounded edges are adjacent to a marked end. If there is no marked end in the middle, we then have $b+2 = 2b$ where b is the number of bounded edges, since $b+2$ is the number of coordinates of this cone, and $2b$ is the number of coordinates of the b marked ends. So $b = 2$ and we are in the situation of the example. If there is a marked end in the middle, we have $b+2 = 2b+2$ so $b = 0$, so this curve has no bounded edge and counts one. Now we can assume we do not have a star-shaped curve, and there is in fact a bounded edge with bounded edges on both sides.) Then we cut this edge to get two curves C_1 and C_2. Let I_i denote the subset of marked ends on C_i and let e_i be the number of non-contracted ends of C_i. We make the cut edge a new non-contracted end of C_i, so C_i has in fact $e_i + 1$ non-contracted ends, one of them the special new end. Assume

$$\#I_1 \leq e_1 - 2 - \sum_{k\in I_1} r_k,$$

then if we remove all the closures of marked ends (as in lemma 4.3) we get

$$\sum_{k\in I_1} r_k + \#I_1 + 1$$

connected components, which is less than or equal to

$$\sum_{k\in I_1} r_k + e_1 - 2 - \sum_{k\in I_1} r_k + 1 = e_1 - 1.$$

So there must be a connected component which has two non-contracted ends of C (not the new end of C_1). Hence C has a string, which contradicts the assumption. We have

$$\#I_1 + \#I_2 = n = 3d - 1 - \sum r_k = e_1 + e_2 - 1 - \sum r_k.$$

Therefore

$$\#I_1 = e_1 - \sum_{k\in I_1} r_k \text{ and } \#I_2 = e_2 - 1 - \sum_{k\in I_2} r_k$$

without restriction. As in remark 6.5, we pick coordinates to write down an explicit matrix for ev. C_1 has

$$e_1 + \#I_1 - 2 - \sum_{k\in I_1} r_k = 2\#I_1 - 2$$

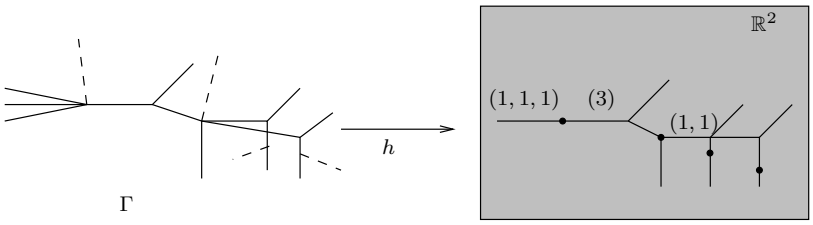
bounded edges. We pick the root vertex to be the boundary vertex of the cut edge in C_1. We order the basis elements such that the root vertex comes first, then the bounded edges in C_1, then the cut edge, then the bounded edges in C_2. We order the basis of $\mathbb{R}^{2n}$ such that the marked ends in C_1 come first and then the marked ends in C_2. Then the matrix of ev for C is a block matrix. The block on the top left is just the matrix of ev for C_1 — so by induction, the product of multiplicities of 3-valent unmarked vertices of C_1. The top right block is 0, because no marked end on C_1 needs a bounded edge of C_2. The bottom right block has the same determinant as the matrix ev for C_2, when we add a marked end on the cut edge and make its end vertex the root vertex. So the determinant of this block is again by induction the product of multiplicities of 3-valent unmarked vertices in C_2. This proves the claim. □

Now we know that $\frac{1}{(d!)^3} \deg\left(\prod_{k\in N} \mathrm{ev}_k^*(P_k)\psi_k^{r_k} \cdot \mathcal{M}_{0,n}^{\mathrm{lab}}(\mathbb{R}^2, d)\right)$ equals the number of all curves $C \in \mathcal{M}_{0,n}(\mathbb{R}^2, d)$ satisfying the Psi- and the point conditions, each counted with weight $\mathrm{mult}(C)$. We want

to simplify this count even further: we do not want to count parametrized tropical curves $C = (\Gamma, x_i, h)$, but we want to count their images in $\mathbb{R}^2$.

Definition 9.4. Let $C = (\Gamma, x_i, h) \subset \prod_{k\in N} \psi_k^{r_k} \cdot \mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^2, d)$. In the image $h(\Gamma)$, some edges may lie on top of each other. Mark each edge in the image $h(\Gamma)$ by a partition reflecting the weights of all edges which map onto this image edge. The image $h(\Gamma)$ together with those partitions is called the *labelled image* of C.

Example 9.5. The following picture shows a tropical curve and its labelled image.



Given a labelled image, there can be different possible parametrizations. Ambiguity may for example arise if the labelled image comes from a parametrization that maps vertices on top of each other. We could then also parametrize this labelled image with a graph where the two vertices are replaced by only one. To avoid this ambiguity, we need a slightly more special notion of general conditions, which we call restricted general conditions. This definition is cooked up in such a way that we exactly avoid all ambiguity and make parametrizations unique.

Definition 9.6. A curve $C = (\Gamma, x_i, h)$ is called *simple*, if it satisfies:

(a) the map h is injective on vertices,
(b) if $h(V) \in h(e)$ for a vertex V and an edge e then V is adjacent to an edge e' which is mapped on top of e,
(c) if two edges e and e' are mapped on top of each other, then they share a vertex,
(d) assume $p \in \mathbb{R}^2$ is a point through which more than two edges pass. Divide the edges into equivalence classes depending on the slope of the line to which they are mapped. Then we have at most 2 equivalence classes.

Definition 9.7. The subset of $\mathbb{R}^{2n}$ of *restricted general conditions* is defined to be the subset of the set of general conditions such that only simple curves $C \in \prod_{k\in N} \psi_k^{r_k} \cdot \mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^2, d)$ pass through the points (i.e. satisfy $\text{ev}(C) = (P_1, \ldots, P_n)$).

Remark 9.8. It is easy to see that the subset of restricted general conditions is still open and dense. Points which are not restricted general admit a non-simple curve. Being not simple sums up to codimension 1 conditions, hence only the image under ev of certain lower-dimensional subsets of $\prod_{k\in N} \psi_k^{r_k} \cdot \mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^2, d)$ is not restricted general.

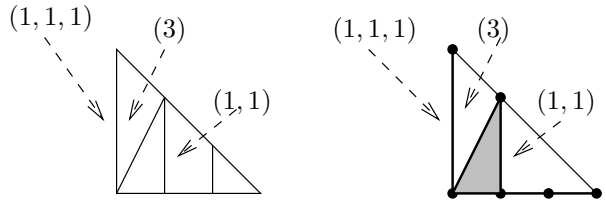
Lemma 9.9. *Given a labelled image of a tropical curve through restricted general conditions, there is exactly one abstract tropical curve* (Γ, x_i) *and one map* h *parametrizing this labelled image and sending the marked ends to the* P_i.

Proof. Clearly there is a parametrization $C = (\Gamma, x_i, h)$ of the labelled image, we just need to show that it is unique. Since the P_i are general, C cannot have a contracted bounded edge. If it had a contracted bounded edge, we could vary the length of this edge without changing the image, in contradiction to 3.2(a). Hence all edges can be seen in the image $h(\Gamma)$. Due to the conditions for being simple, we can also distinguish the images of vertices and the images of all edges in the labelled image. Because of the labels we know whether edges lie on top of each other. If there are edges lying on top of each other, then we know that they have to share a vertex. If there is a vertex V with two edges of the same direction, it has to be more than 3-valent. If it was 3-valent, then by the balancing condition the 3 edges would

be mapped to a line. At least one of the 3 edges is bounded, and so we could change the length of this edge (and accordingly the lengths of the other two edges, if necessary) without changing the image of the curve. That contradicts 3.2(a). Since V is more than 3-valent, there must be a marked end adjacent to it. It is not possible that 2 (or more) of the points P_i lie on a line with a direction that can be the direction of an edge. Thus if we have two edges in the labelled image on top of each other, there must be exactly one adjacent vertex which passes through a point P_i. Thus we know that the edges have to be connected at that vertex when we built the parametrization. □

Definition 9.10. Let C be a curve in $\prod_{k\in N}\psi_k^{r_k}\cdot\mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^2,d)$ passing through restricted general conditions. Draw a dual Newton subdivision to the image $h(\Gamma)$ and label the dual edges also with the partitions belonging to the edges of the labelled image $h(\Gamma)$. This is called a *labelled dual Newton subdivision*. Mark the polygons dual to vertices which are adjacent to a marked end x_i. Those marked polygons in Δ_d together with the partitions belonging to their boundary edges is called the *set of dual marked polygons of* C.

Example 9.11. The following picture shows the labelled dual Newton subdivision to the labelled image from example 9.5. Next to it, we can see the set of dual marked polygons of C.



Our aim is to count dual marked polygons to curves in $\prod_{k\in N}\psi_k^{r_k}\cdot\mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^2,d)$. To do that, we have to choose a special point configuration $\mathcal{P}$ as our condition. This configuration is chosen is such a way that the set of dual marked polygons can be described as something like a generalized lattice path that we call a *rag rug*. We will now first introduce labelled lattice paths and rag rugs, and then show that the count of rag rugs equals the count of labelled images of curves in $\prod_{k\in N}\psi_k^{r_k}\cdot\mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^2,d)$ passing through $\mathcal{P}$ (with weight $\text{mult}(C)$).

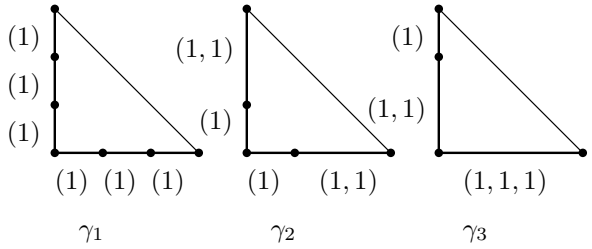
Let Δ_d be the triangle with endpoints $(0,0)$, $(d,0)$ and $(0,d)$. Fix λ to be a linear map of the form

$$\lambda:\mathbb{R}^2\to\mathbb{R}:\ (x,y)\mapsto x-\varepsilon y,$$

where ε is a small irrational number. Recall that a path $\gamma:[0,n]\to\mathbb{R}^2$ is called a lattice path if $\gamma|_{[j-1,j]}, j=1,\dots,n$ is an affine-linear map and $\gamma(j)\in\mathbb{Z}^2$ for all $j=0\dots,n$. For $n=1,\dots,n$, we call $\gamma|_{[j-1,j]}([j-1,j])$ a *step* (the j-th step) of the lattice path γ. A lattice path is called λ-increasing if $\lambda\circ\gamma$ is strictly increasing. Let $p:=(0,d)$ and $q:=(d,0)$ be the points in $\Delta:=\Delta_d$ where $\lambda|_\Delta$ reaches its minimum (resp. maximum). Let G be a line in $\mathbb{R}^2$ orthogonal to $\ker(\lambda)$. Then G divides the plane into two halfplanes. We will denote the upper one by H^+ and the lower one by H^-.

Definition 9.12. A *labelled λ-increasing lattice path* in Δ is a λ-increasing lattice path from p to q such that the k-th step is labelled by a partition $\alpha_k=((\alpha_k)_1,\dots,(\alpha_k)_{r_k})$ of the integer length of this step, that is $(\alpha_k)_1+\ldots+(\alpha_k)_{r_k}=\#(\mathbb{Z}^2\cap\gamma([k-1,k]))-1$.

Remark 9.13. Let δ be a labelled λ-increasing lattice path from p to q whose image is contained in the boundary $\partial\Delta$ and whose steps are labelled with partitions consisting of only ones. All those paths will be possible end paths for the recursion defining multiplicity. The following picture shows 3 examples for Δ_3.



Definition 9.14. We define the *positive multiplicity* μ_+ (resp. *negative multiplicity* μ_-) of a labelled λ-increasing lattice path recursively as follows:

(a) For a possible end path δ as in remark 9.13 going *clockwise* from p to q (resp. *counterclockwise*) with n steps we define $\mu_\pm(\delta) := \prod_{k=1}^{n} 1/(|\alpha_k|!)$, where $|\alpha_k|$ denotes the size of the partition of the k-th step (recall it has to be a partition with only ones as entries).

(b) For a labelled λ-increasing lattice path γ which is not a possible end path, assume that the k-th and the $k+1$-th step form the first *left* (resp. *right*) corner of the path γ. (If no such turn exists, we define $\mu_\pm(\gamma) := 0$.)

Define a finite set of lattice paths as follows:
- pick an integer r with $0 \leq r \leq \min\{|\alpha_k|, |\alpha_{k+1}|\}$,
- pick a set S of r pairs
$$S = \big\{[(\alpha_k)_{i_1}, (\alpha_{k+1})_{j_1}], \ldots, [(\alpha_k)_{i_r}, (\alpha_{k+1})_{j_r}]\big\}$$
such that the multiset $\{(\alpha_k)_{i_1}, \ldots, (\alpha_k)_{i_r}\}$ is a subset of the multiset $\{(\alpha_k)_1, \ldots, (\alpha_k)_{r_k}\}$ and the multiset $\{(\alpha_{k+1})_{j_1}, \ldots, (\alpha_{k+j})_{j_r}\}$ is a subset of the multiset $\{(\alpha_{k+1})_1, \ldots, (\alpha_{k+1})_{r_{k+1}}\}$.

For each $l = 1, \ldots, r$, build a triangle $T_{r,S,l}$ with one edge of integer length $(\alpha_k)_{i_l}$ and one edge of integer length $(\alpha_{k+1})_{j_l}$ (in the direction of the k-th resp. $k+1$-th step). Let $M_{r,S}$ be the Minkowski sum of all triangles $T_{r,S,l}$ for $l = 1, \ldots, r$, and edges e_s in direction of the k-th step of integer length $(\alpha_k)_s$ for all s which are not one of the i_l and edges f_t in direction of the $k+1$-th step of integer length $(\alpha_{k+1})_t$ for all t which are not one of the j_l. Label each edge E of $M_{r,S}$ with a partition reflecting the integer lengths of edges e_s, f_t, and edges of triangles $T_{r,S,l}$ that contribute to E. Think of the polygon $M_{r,S}$ as sitting in the corner built by step k and $k+1$ of γ, and define a new labelled λ-increasing lattice path $\gamma_{r,S}$ by going the other way around $M_{r,S}$. If $M_{r,S}$ does not fit inside the polygon Δ, we define $\mu_\pm(\gamma_{r,S}) = 0$. The positive multiplicity of this new labelled λ-increasing lattice path is known recursively, because it includes a smaller area with the possible end paths. We define

$$\mu_\pm(\gamma) = \sum_r \sum_S \mathrm{Area}(T_{r,S,1}) \cdot \ldots \cdot \mathrm{Area}(T_{r,S,r}) \cdot \mu_\pm(\gamma_{r,S}),$$

(where $\mathrm{Area}(T)$ is the normalized lattice area, i.e. the area of the simplex with vertices $(0,0)$, $(1,0)$ and $(0,1)$ is defined to be 1).

Example 9.15. For the 3 possible end paths from remark 9.13, we have multiplicity $\mu_-(\gamma_1) = 1$, $\mu_-(\gamma_2) = \frac{1}{4}$ and $\mu_-(\gamma_3) = \frac{1}{12}$.

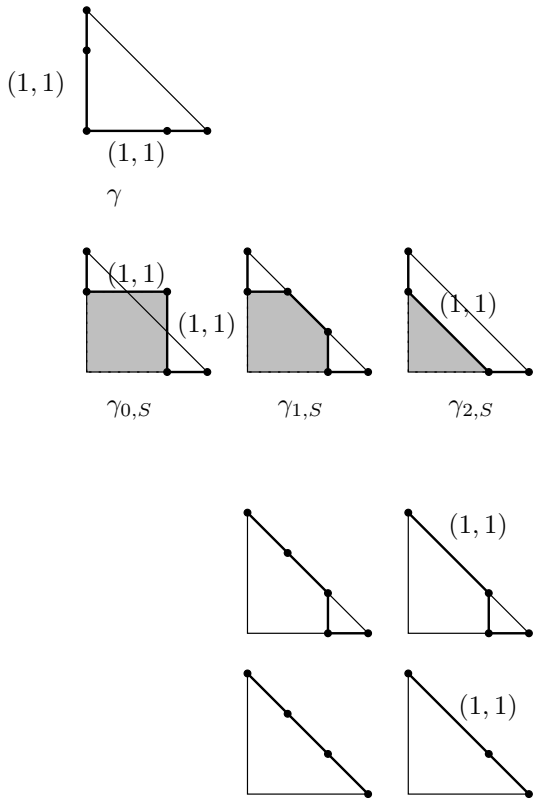
Example 9.16. The following picture shows an example of the recursion from definition 9.14 to compute the positive multiplicity of a labelled path γ in Δ_3. The first left turn is from step 2 to step 3. We have 3 choices for r: $r = 0$, $r = 1$ or $r = 2$, since both the partition of step 2 as the partition of step 3 contain 2 elements. No matter what we choose for r, there is just one choice for the set S (r pairs consisting of all ones), since both partitions contain only ones.

For $r = 0$ and $S = \emptyset$, $M_{0,\emptyset}$ is a square of size 2 which does not fit inside Δ_3. Therefore the multiplicity of $\gamma_{0,\emptyset} = 0$.

For $r = 1$ and $S = \{(1, 1)\}$, $M_{1,S}$ is a pentagon. The integer length of each new side is one. The new side of direction $(0, 1)$ is labelled by the partition (1), because it comes from the edge e_s in direction of the 2-nd step of integer length 1 which is not one of the i_l in the set of pairs S. The new side of $M_{1,S}$ of direction $(1, -1)$ comes with label (1), because it comes from a side of the triangle $T_{1,S,1}$ of integer length 1. The side of direction $(1, 0)$ comes from an edge of the 3-rd step which is not part of S, and gets label (1) as well. The area of $T_{1,S,1}$ is one.

For $r = 2$, we have $S = \{(1, 1), (1, 1)\}$, and $M_{2,S}$ is a triangle of size 2 whose new side gets the label $(1, 1)$ because it comes from 2 sides of the two triangles $T_{2,S,1}$ and $T_{2,S,2}$. The area of both triangles $T_{2,S,1}$ and $T_{2,S,2}$ is one.

The picture shows how the recursion goes on after the first step. The choices where $r = 0$ for $\gamma_{1,S}$ or where $r = 1$ for $\gamma_{2,S}$ are left out because they yield to a path of multiplicity 0. We end up with one path of multiplicity 1 and one of multiplicity $\frac{1}{2}$, so $\mu_+(\gamma) = \frac{3}{2}$.



Definition 9.17. Let F be a set of n convex polytopes $Q_1, \ldots, Q_n$ inside Δ whose endpoints are lattice points of Δ and whose boundary edges e are labelled by partitions. It is possible that a polygon Q_i is 1-dimensional, i.e. just an edge itself, then it has two partitions as labels, one for each outward pointing normal vector. We call F a *rag rug* of the form $(r_1, \ldots, r_n)$ if the following conditions are satisfied:

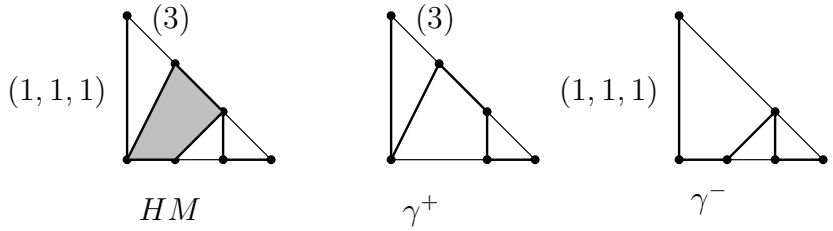
(a) the (outside) label α_e of an edge e in the boundary of Δ_d is $\alpha_e = (1, 1, \ldots, 1)$,
(b) two polygons Q_i and Q_j intersect in at most one point,
(c) boundary edges whose outward normal vector points into H^+ (starting at G) (with their corresponding labels) form a labelled λ-increasing lattice path from p to q that we will denote by γ^+,
(d) boundary edges whose outward normal vector points into H^- (with their corresponding labels) form a labelled λ-increasing lattice path from p to q that we will denote by γ^-,

(e) the order of the polytopes $Q_1, \dots, Q_n$ agrees with the obvious order given by the paths γ^+ resp. γ^-,
(f) the sum of the sizes of the partitions of the boundary edges of Q_i is equal to $r_i + 2$,

$$\sum_{e|e \text{ edge of } Q_i} |\alpha_e| = r_i + 2.$$

We define the multiplicity $\mu(F)$ to be $\mu_+(\gamma_+) \cdot \mu_-(\gamma_-)$.

Example 9.18. The following picture shows a rag rug F of the form $(2,2,0,0)$ in Δ_3, and the two labelled λ-increasing lattice paths γ_+ and γ_-. For all edges of integer length one, the corresponding partitions are just (1) and we did not mark this in the picture. We have $\mu_+(\gamma_+) = 3$ and $\mu_-(\gamma_-) = \frac{1}{6}$, so $\mu(F) = \frac{1}{2}$.



Definition 9.19. Given d, n and numbers $(r_1, \dots, r_n)$ we define $N_{\mathrm{rr}}(d, n, (r_1, \dots, r_n))$ to be the number of rag rugs of form $(r_1, \dots, r_n)$, counted with multiplicity as defined in 9.17.

Remark 9.20. Note that this definition generalizes Mikhalkin's lattice path count (see [Mi03]). A λ-increasing lattice path γ from p to q is a rag rug of form $(0, \dots, 0)$. We have to attach labels (1) to each edge. The two paths γ_+ and γ_- agree with γ. In the recursion for the lattice path count, we define $\mathrm{mult}_\pm(\gamma)$ depending on the multiplicity of two other paths γ' and γ''. γ' is the path that cuts the corner, and γ'' is the path that completes the corner to a parallelogram. In our definition, we can choose $r = 0$ or $r = 1$. For $r = 0$, we have $S = \emptyset$ as only choice. The polygon $M_{0,\emptyset}$ is the parallelogram which is equal to the Minkowski sum of the two steps of the corner. For $r = 1$, we have $S = \{(1,1)\}$ as only choice, and $M_{1,S}$ is the triangle formed by the two steps of the corner. Since all partitions are just (1), also the end paths have only those partitions, so that there is in fact only one end path, the path $\delta_\pm$. It has multiplicity 1. Therefore our definition gives the same multiplicity in this case.

It is not true that $N_{\mathrm{rr}}(d, n, (r_1, \dots, r_n)) = \frac{1}{(d!)^3} \deg\left(\prod_k \mathrm{ev}_k^*(P_k)\psi_k^{r_k} \cdot \mathcal{M}_{0,n}^{\mathrm{lab}}(\mathbb{R}^2, d)\right)$, since we count also reducible curves with the rag rugs.

We therefore have to modify the count and throw away the dual subdivisions corresponding to reducible tropical curves.

Definition 9.21. Given a rag rug γ and the two corresponding lattice paths γ_+ and γ_-, perform the recursion to compute their multiplicity and keep track of the polygons $M_{r,S}$ that the new paths $\gamma_{r,S}$ in the recursion enclose with $\gamma_\pm$. This way we end up with a set of labelled Newton subdivisions. We call this the set of *possible labelled Newton subdivisions for* γ. The recursion allows us to assign a multiplicity to a possible labelled Newton subdivision, so that the multiplicity of γ is equal to the sum of the multiplicities of the possible labelled Newton subdivisions for γ.

Definition 9.22. Given a labelled Newton subdivision, draw a dual labelled image and then the unique tropical curve mapping to this image. This is well-defined up its position in $\mathbb{R}^2$ and the lengths of its bounded edges. We say that the Newton subdivision is *reducible* if the tropical curve mapping to a dual labelled image is *reducible* (again, this does not depend on the choice of dual labelled image). Otherwise, we say it is irreducible.

Remark 9.23. It is possible to express the reducibility condition in terms of the Newton subdivision itself and not in terms of the dual tropical curve. A labelled marked Newton subdivision is reducible

if and only if it admits a mixed subdivision where the marked polygons Q_i remain unmixed (i.e. come from sums of the form $Q_i + v_1 + \ldots + v_t$, where the v_j are vertices of the subdivision of the j-th summand). For details, see [M08].

Definition 9.24. For a rag rug γ as defined in 9.17 define its irreducible multiplicity $\mathrm{mult}'(\gamma)$ to be the multiplicity $\mathrm{mult}(\gamma)$ minus the number of possible reducible Newton subdivision (counted with multiplicity). Given d, n and numbers $(r_1, \ldots, r_n)$ we say $N'_{\mathrm{rr}}(d, n, (r_1, \ldots, r_n))$ is the number of rag rugs counted with their irreducible multiplicity.

Definition 9.25. Let $\mathcal{P}_\lambda = (P_1, \ldots, P_n)$ denote n restricted general point conditions on the line G orthogonal to $\ker(\lambda)$ such that the distance between P_i and P_{i+1} is much bigger than the distance between P_{i-1} and P_i.

Lemma 9.26. *Let $C \in \prod_k \psi_k^{r_k} \cdot \mathcal{M}^{lab}_{0,n}(\mathbb{R}^2, d)$ with $\mathrm{ev}(C) = \mathcal{P}_\lambda$. Then the set of dual marked polygons of C is a rag rug of the form $(r_1, \ldots, r_n)$.*

Proof. The polygon Q_i dual to P_i is convex and has to satisfy

$$\sum_{e|e \text{ edge of } Q_i} |\alpha_e| = r_i + 2,$$

where α_e denotes the partition belonging to e. This is true since the marked end $x_i \subset \Gamma$ is adjacent to a vertex of valence $r_i + 3$ and we can see all edges (except the contracted end x_i) in the labelled image (and thus in their labelled dual Newton subdivision, too). (Outside) labels of edges in the boundary of Δ_d have only ones as entries, since the ends of C are all of weight 1. That the boundaries of those polygons form labelled λ-increasing lattice paths follows analogously to [Mi03], 8.27 (or [Ma06], 5.48 for more details). □

Theorem 9.27. *The number $N'_{\mathrm{rr}}(d, n, (r_1, \ldots, r_n))$ from definition 9.24 equals the intersection product $\frac{1}{(d!)^3} \deg\left(\prod_k \mathrm{ev}_k^*(P_k)\psi_k^{r_k} \cdot \mathcal{M}^{lab}_{0,n}(\mathbb{R}^2, d)\right) = \langle \prod_k \tau_{r_k}(2) \rangle$.*

Proof. To determine $\frac{1}{(d!)^3} \deg\left(\prod_k \mathrm{ev}_k^*(P_k)\psi_k^{r_k} \cdot \mathcal{M}^{\mathrm{lab}}_{0,n}(\mathbb{R}^2, d)\right)$, we can draw all labelled images of tropical curves that pass through $\mathcal{P}_\lambda$ and count them each with their weight $\mathrm{mult}(C)$ which is ν_C times the product of the multiplicities of non-marked vertices. We show that this count is equivalent to counting irreducible possible labelled Newton subdivision for all rag rugs of the form $(r_1, \ldots, r_n)$.

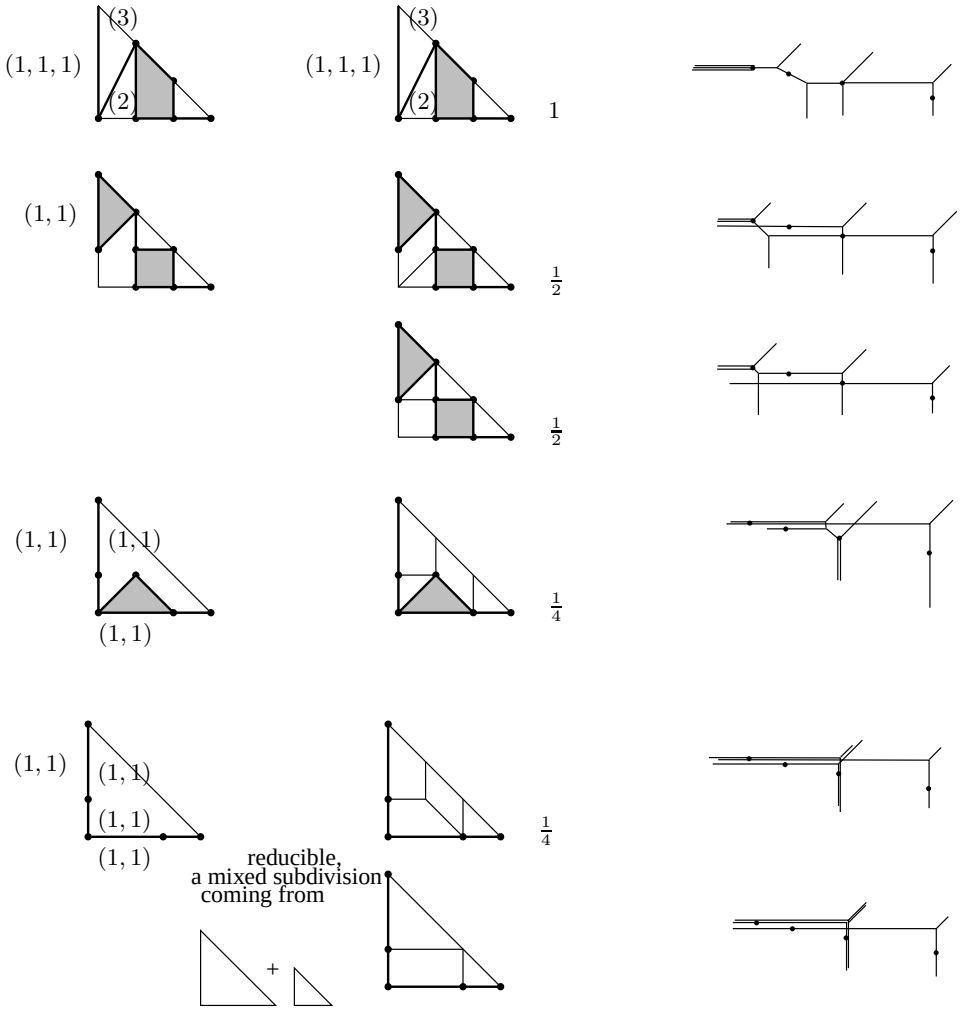
The proof is a generalization of the proof of theorem 2 of [Mi03]. We know that each C leads to a rag rug γ_C as in lemma 9.26. Each rag rug yields a set of possible labelled Newton subdivision, with multiplicity. We will show that for each such possible labelled Newton subdivision, there is a dual tropical curve C through $\mathcal{P}_\lambda$ of the same weight. At the same time, we show that for each curve C through $\mathcal{P}_\lambda$, the dual labelled Newton subdivision is possible for the rag rug γ_C.

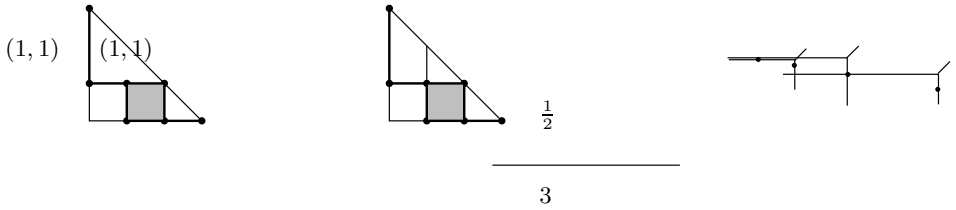
Let γ be a rag rug. The recursion for γ_+ yields possible subdivision of Δ_d above the polygons Q_i. They correspond to the part of a tropical curve above G. Analogously, possible subdivisions for γ_- correspond to the parts of tropical curves below G. From lemma 9.3 it follows that weight of a tropical curve can be computed locally, so the weight of C is equal to the weight of the part above G times the weight of the part below G. The same is true for the multiplicity of the dual Newton subdivision. Therefore it is enough to show that for each subdivision above the Q_i, there is a dual part of a tropical curve above G of the same weight, and that each part of a tropical curve above G is dual to a possible subdivision above the Q_i. The corresponding statement for subdivisions below the Q_i and parts of tropical curves below G follows analogously, and thus the complete statement follows.

For each point $P_i \in \mathcal{P}_\lambda$, draw edges emanating from P_i of directions dual to the boundary edges of Q_i and with the same partitions as labels. Draw a line G' in H_+ parallel to G, such that the strip between G and G' encloses one intersection of the edges we have drawn through the P_i. This intersection of edges corresponds to the first left turn of the path γ_+, since the distances between the P_i are increasing. Let us determine the possibilities how the tropical curve can go on at this point. We should think about both

edges as a set of edges of weights given by the partition. Edges can either meet in a 3-valent vertex, or intersect. First, we pick r less than the smaller number of edges in a set to determine how many edges should meet in a 3-valent vertex. Then we pick a set of r pairs of weights to determine which edges should meet in a 3-valent vertex. The other edges intersect. The weight locally in the strip between G and G' is equal to the product of areas of triangles dual to the 3-valent vertices because of lemma 9.3. The dual polygon is the Minkowski sum of those triangles and the remaining edges which intersect. Therefore the recursion for the multiplicity of γ_+ corresponds to the possibilities for a labelled image of a tropical curve in the strip between G and G' and keeps track of the weight. The end paths which do not have zero multiplicity are exactly those dual to ends of direction $(1,1)$ and weight one. The multiplicity of such an end path corresponds to the correction factor ν_C with which we have to divide the weight of a tropical curve if more than one non-contracted end is adjacent to the same vertex. □

Example 9.28. The following picture shows how to count $\langle \tau_2(2)^2\tau_0(2)^2\rangle_3$ using rag rugs. The left column shows all rag rugs of the form $(2,0,2,0)$ in the triangle Δ_3. The middle column shows the possible Newton subdivisions for the rag rugs and their multiplicity. The third column shows sketches of the dual tropical curves.





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