

INTERSECTING PSI-CLASSES ON TROPICAL $\mathcal{M}_{0,n}$

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ABSTRACT. We apply the tropical intersection theory as suggested by G. Mikhalkin and developed in detail by L. Allermann and J. Rau to compute intersection products of tropical Psi-classes on the moduli space of rational tropical curves. We show that in the case of zero-dimensional (stable) intersections, the resulting numbers agree with the intersection numbers of Psi-classes on the moduli space of n -marked rational curves computed in algebraic geometry.

1. INTRODUCTION

A rational n -marked tropical curve is a metric tree with n labeled leaves and without 2-valent vertices. Those curves are parametrized by the combinatorial structure of the underlying (non-metric) tree and the length of each interior edge. The tropical moduli space $\mathcal{M}_{0,n}$ (the space which parametrizes these curves) has the structure of a polyhedral complex, obtained by gluing several copies of the positive orthant $\mathbb{R}_{\geq 0}^{n-3}$ — one copy for each 3-valent combinatorial graph with n leaves (see section 2).

Recently, G. Mikhalkin (see [M1]) introduced tropical Psi-classes on the moduli space of rational tropical curves: for $k \in [n]$, the tropical Psi-class Ψ_k is the subcomplex of cones of $\mathcal{M}_{0,n}$ corresponding to tropical curves which have the property that the leaf labeled with the number k is adjacent to a vertex of valence at least 4 (see definition 3.1).

The aim of this article is to apply the concepts of tropical intersection theory suggested by G. Mikhalkin and developed in detail by L. Allermann and J. Rau ([M2], [AR]) to compute the intersection products of an arbitrary number of these Psi-classes.

In order to do this, we first recall the embedding of the moduli space $\mathcal{M}_{0,n}$ of n -marked rational tropical curves into some real vector space \mathcal{Q}_n (and other preliminaries) in section 2. On this space \mathcal{Q}_n , we construct in section 3 a tropical rational function f_k for all $k \in [n]$ with the property that the Cartier divisor of the restriction of f_k to (the embedding of) $\mathcal{M}_{0,n}$ is (a multiple of) the k -th Psi-class Ψ_k . We use this description in section 4 to compute the weights on the maximal cones of the tropical fan obtained by intersecting an arbitrary number of tropical Psi-classes. As a special case, we compute the weights of (0-dimensional) intersections of $n - 3$ tropical Psi-classes — they agree with the 0-dimensional intersection product of $n - 3$ Psi-classes on the moduli space of rational n -marked curves computed in algebraic geometry.

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2. PRELIMINARIES

In the sequel, n will always denote an integer greater than 2.

An n -marked (rational) abstract tropical curve is a metric tree Γ (that is, a tree together with a length function assigning to each non-leaf edge a positive real number) without 2-valent vertices and with n leaves, labeled by numbers $\{1, \dots, n\}$ (see [GaMa], definition 2.2). The space $\mathcal{M}_{0,n}$ of all n -marked tropical curves has the structure of a polyhedral fan of dimension $n - 3$ obtained from gluing copies of the space $\mathbb{R}_{>0}^k$ for $0 \leq k \leq n - 3$ — one copy for each combinatorial type of a tree with n leaves and exactly k bounded edges. Its face lattice is given by $\tau \prec \sigma$ if and only if the tree corresponding to τ is obtained from the tree corresponding to σ by contracting bounded edges. For more details, see [BHV], section 2, or [GaMa], section 2.

In order to recall how $\mathcal{M}_{0,n}$ can be embedded from [GKM], we need the following notations:

Let $\mathcal{T} := \{S \subset [n] : |S| = 2\}$ denote the set of two-element subsets of $[n] := \{1, \dots, n\}$. Consider the space $\mathbb{R}^{\binom{n}{2}}$ indexed by the elements of \mathcal{T} and define a map to this space via

$$\begin{aligned} \Phi_n : \mathbb{R}^n &\longrightarrow \mathbb{R}^{\binom{n}{2}} \\ a &\longmapsto (a_i + a_j)_{\{i,j\} \in \mathcal{T}}. \end{aligned}$$

Let \mathcal{Q}_n denote the quotient vector space $\mathbb{R}^{\binom{n}{2}} / \text{im}(\Phi_n)$, which has dimension $\binom{n}{2} - n$.

Furthermore, we define a map

$$\begin{aligned} \varphi_n : \mathcal{M}_{0,n} &\longrightarrow \mathbb{R}^{\binom{n}{2}} \\ C &\longmapsto \text{dist}(\{i, j\})_{\{i,j\} \in \mathcal{T}} \end{aligned}$$

where $\text{dist}(\{i, j\})$ denotes the sum of the lengths of all bounded edges on the (unique) path between the leaf marked i and the leaf marked j .

Theorem 2.1. *Using the map φ_n , $\mathcal{M}_{0,n}$ can be embedded as a tropical fan into \mathcal{Q}_n .*

For a proof, see theorem 3.4 of [GKM] or theorem 3.4 of [SS].

Note that $\mathcal{M}_{0,n}$ is a marked fan (see definition 2.12 of [GKM]): Let τ be a cone and C be the corresponding tropical curve where all lengths of bounded edges are chosen to be one. Then τ is generated by the rays $\varphi_n(C_i)$, where C_i denotes a curve obtained from C by shrinking all but one bounded edge to length 0. In particular, $\mathcal{M}_{0,n}$ is a simplicial fan and the $\varphi_n(C_i)$ form a basis for the span of the cone. They even form a unimodular basis which follows from proposition 5.4 of [GiMa] (note that there, a different lattice and a different embedding of $\mathcal{M}_{0,n}$ is used).

Notation 2.2. For each subset $I \subset [n]$ of cardinality $1 < |I| < n - 1$, define a vector $v_I \in \mathbb{R}^{\binom{n}{2}}$ via

$$(v_I)_T := \begin{cases} 1, & \text{if } |I \cap T| = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Note that v_I is the image under Φ_n of a tree with one bounded edge of length one, the marked ends with labels in I on one side of the bounded edge and the marked ends with labels in $[n] \setminus I$ on the other, hence $v_I = v_{[n] \setminus I}$.

For $k \in [n]$, we define $V_k := \{v_I : k \notin I \text{ and } |I| = 2\}$.

Lemma 2.3. *For any $k \in [n]$, the linear span of the set V_k equals $\mathcal{Q}_n = \mathbb{R}^{\binom{n}{2}} / \text{im}(\Phi_n)$.*

Proof. We prove that for $S = \{s_1, s_2\} \in \mathcal{T}$, the S -th standard unit vector $e_S \in \mathbb{R}^{\binom{n}{2}}$ is the sum of a linear combination of elements in V_k and an element of $\text{im}(\Phi_n)$.

First assume that $k \notin S = \{s_1, s_2\}$. Then it follows immediately from the definitions of v_S and Φ_n that $e_S = (-v_S + \Phi_n(e_{s_1} + e_{s_2}))/2$, where e_{s_i} denotes the s_i -th unit vector in \mathbb{R}^n .

Now assume that $S = \{s_1, k\}$. We claim that

$$e_S = \frac{1}{2} \left(\sum_{I \in \mathcal{T} : I \cap S = \{s_1\}} v_I - \Phi_n(a) \right), \quad (1)$$

where $a \in \mathbb{R}^n$ is the vector with entries

$$a_i := \begin{cases} n - 4, & \text{if } i = s_1 \\ 0, & \text{if } i = k \\ 1 & \text{otherwise} \end{cases}$$

Check this equality in each component $T = \{t_1, t_2\}$.

The entry there is equal to one if and only if $|I \cap T| = 1$ — note that $s_1 \in I$.

If $S \cap T = \emptyset$, then $\Phi_n(a)_T = 2$ and we have $|I \cap T| = 1$ iff I contains one element of T .

If $S \cap T = \{k\}$, then $\Phi_n(a)_T = 1$ and we have $|I \cap T| = 1$ iff $I = \{s_1\} \cup T \setminus \{k\}$.

If $S \cap T = \{s_1\}$, then $\Phi_n(a)_T = n - 3$ and we have $|I \cap T| = 1$ iff $I \neq T$.

If $S \cap T = \{s_1, k\}$, then $\Phi_n(a)_T = n - 4$ and we have $|I \cap T| = 1$ for all $(n - 2)$ choices of I .

It follows that for $T \neq S$, we have

$$\left(\sum_{I \in \mathcal{T} : I \cap S = \{s_1\}} v_I - \Phi_n(a) \right)_T = 0$$

and for $T = S$, we have

$$\left(\sum_{I \in \mathcal{T} : I \cap S = \{s_1\}} v_I - \Phi_n(a) \right)_T = n - 2 - (n - 4) = 2,$$

hence equation (1) holds. \square

Lemma 2.4. *The sum over all elements $v_S \in V_k$ (from notation 2.2) is an element in $\text{im}(\Phi_n)$, hence*

$$\sum_{v_S \in V_k} v_S = 0 \in Q_n.$$

Proof. If $k \notin T$, then there are $\binom{n-3}{1} \binom{2}{1}$ two-element subsets $S \in [n] \setminus \{k\}$ with the property that $S \cap T = 1$, hence $(\sum v_S)_T = 2(n-3)$ in this case. If $k \in T$, then the number of subsets $S \in [n] \setminus \{k\}$ satisfying $|S \cap T| = 1$ equals $n-2$, hence $(\sum v_S)_T = n-2$ in this case. It follows that

$$\sum_{v_S \in V_k} v_S = \Phi_n(n-3, \dots, n-3, 1, n-3, \dots, n-3)$$

with entry 1 at position k . □

Lemma and Definition 2.5. *Every element $v \in Q_n$ has a unique representation*

$$v = \sum_{v_S \in V_k} \lambda_S v_S$$

with $\lambda_S \geq 0$ for all S and $\lambda_S = 0$ for at least one $S \in \mathcal{T}$. We will call such a representation in the future a positive representation of v with respect to V_k .

Proof. As $|V_k| = \binom{n-1}{2} = \dim(Q_n) + 1$ (see notation 2.2), the vectors of V_k subdivide Q_n into a fan whose $\dim(Q_n) + 1$ top-dimensional cones are spanned by a choice of $\dim(Q_n)$ vectors of V_k . Each v lies in a unique cone and its positive representation is given by the linear combination of the spanning vectors of the cone. Given a representation

$$v = \sum_{v_S \in V_k} \lambda_S v_S$$

with $\lambda_S > 0$ for all S , the unique positive representation with respect to V_k can be found by subtracting $\sum_{v_S \in V_k} (\min_S \lambda_S) v_S$. □

Remark 2.6. It follows that a map from V_k to $\mathbb{R}_{\geq 0}$ gives rise to a well-defined convex piecewise-linear function on the space Q_n via $f(\sum \lambda_S v_S) := \sum \lambda_S f(v_S)$.

Lemma 2.7. *Let $I \subset [n]$ with $1 < |I| < n-1$ and assume without restriction that $k \notin I$. Then a positive representation of $v_I \in Q_n$ with respect to V_k (as in definition 2.5 and notation 2.2) is given by*

$$v_I = \sum_{S \subset I, v_S \in V_k} v_S.$$

Proof. Let $|I| = m, I = \{i_1, \dots, i_m\}$. We claim that

$$v_I = \left(\sum_{S \subset I, v_S \in V_k} v_S \right) - (m-2) \cdot \Phi_n(e_{i_1} + \dots + e_{i_m}).$$

Check this equality in each component $T = \{t_1, t_2\}$. If $T \subset I$, then $(v_I)_T = 0$. There are $m-2$ choices for v_S such that S contains t_1 and not t_2 , and the same number of choices such that S contains t_2 and not t_1 . Hence the first sum of the right hand side contributes $2(m-2)$. As

$$\Phi_n(e_{i_1} + \dots + e_{i_m})_T = 2$$

we get 0 altogether. If $|T \cap I| = 1$, then $(v_I)_T = 1$. On the right hand side, there are $m - 1$ choices of S such that S contains $T \cap I$, and

$$\Phi_n(e_{i_1} + \dots + e_{i_m})_T = 1.$$

If $T \cap I = \emptyset$, both sides are equal to 0. \square

3. PSI-CLASSES AS DIVISORS OF RATIONAL FUNCTIONS

Let us start by reviewing some of the tropical intersection theory from [AR]. A *cycle* X is a balanced, weighted, pure-dimensional, rational and polyhedral fan in \mathbb{R}^n . The integer weights assigned to each top-dimensional cone σ are denoted by $\omega(\sigma)$. By $|X|$, we denote the union of all cones of X in \mathbb{R}^n . *Balanced* means that the weighted sum of the primitive vectors of the facets σ_i around a cone $\tau \in X$ of codimension 1

$$\sum_i \omega(\sigma_i) u_{\sigma_i/\tau}$$

lies in the linear vector space spanned by τ , denoted by V_τ . Here, a *primitive vector* $u_{\sigma_i/\tau}$ of σ_i modulo τ is a integer vector in \mathbb{Z}^n that points from τ towards σ and fulfills the primitive condition: The lattice $\mathbb{Z}u_{\sigma_i/\tau} + (V_\tau \cap \mathbb{Z}^n)$ must be equal to the lattice $V_{\sigma_i} \cap \mathbb{Z}^n$. Slightly differently, in [AR] the class of $u_{\sigma_i/\tau}$ modulo V_τ is called primitive vector and $u_{\sigma_i/\tau}$ is just a representative of it.

Cycles are only considered up to refinements, i.e. we will consider two cycles equivalent if they have a common refinement.

A (*non-zero*) *rational function on X* is a continuous piece-wise linear function $\varphi : |X| \rightarrow \mathbb{R}$ that is linear with rational slope on each cone. The *Weil-divisor of φ on X* , denoted by $\text{div}(\varphi)$, is the balanced subcomplex (resp. subfan) of X defined in construction 3.3. of [AR], namely the codimension one skeleton of X together with weights $\omega(\tau)$ for each cone $\tau \in X$ of codimension 1. These weights are given by the formula

$$\omega(\tau) = \sum_i \varphi(\omega(\sigma_i) u_{\sigma_i/\tau}) - \varphi\left(\sum_i \omega(\sigma_i) u_{\sigma_i/\tau}\right),$$

where the sum goes again over all top-dimensional neighbours of τ .

Now let us repeat the definition of tropical Psi-class.

Definition 3.1 (see [M1], definition 3.1.). For $k \in [n]$, the tropical Psi-class $\Psi_k \subset \mathcal{M}_{0,n}$ is defined to be the weighted fan consisting of those closed $(n - 4)$ -dimensional cones that correspond to tropical curves with the property that the leaf marked with the number k is adjacent to a vertex with valence 4. The weight of each cone is defined to be equal to one.

A motivation for this definition is given in [M1]. Another motivation is that if one evaluates the classical ψ_i on 1-strata of $\overline{\mathcal{M}}_{0,n}$, i.e. on rational curves whose dual graphs are 3-valent except for one 4-valent vertex; we get 0 if the 4-valent vertex is not adjacent to the leaf i . To the author's knowledge, E. Katz is about to prepare a preprint that explains more about the connection of tropical and classical Psi-classes.

In a recent article, G. Mikhalkin defines an embedding of the space $\mathcal{M}_{0,n}$ as a tropical fan into $\mathbb{R}^{\binom{n}{2} \binom{n-2}{2}}$, with the property that the tropical Psi-class Ψ_k has the structure of a tropical subfan, i.e. satisfies the balancing condition (see [M1], theorem 3.1 and proposition 3.2).

Here, we prefer to work with the embedding of $\mathcal{M}_{0,n}$ in \mathcal{Q}_n as described in section 2. We will see later in this section that $\varphi_n(\Psi_k) \subset \mathcal{Q}_n$ is a tropical fan — this will follow directly from the proof of proposition 3.5, which states that (a multiple of) $\varphi_n(\Psi_k)$ is the Weil-divisor associated to a rational function.

By abuse of notation we will in the following not distinguish between Ψ_k and $\varphi_n(\Psi_k) \subset \mathcal{Q}_n$.

Notation 3.2. For any $k \in [n]$, let f_k be the extension of the map $V_k \ni v_S \mapsto 1$ to \mathcal{Q}_n (see notation 2.2 and remark 2.6).

Lemma 3.3. *The map f_k is linear on each cone of $\mathcal{M}_{0,n}$.*

Proof. Let τ be a cone and C the tropical curve of the combinatorial type corresponding to τ with all lengths equal to one. The cone τ is generated by vectors v_i corresponding to curves where all but one bounded edge of C are shrunk to length 0. Assume τ is generated by v_{I_1}, \dots, v_{I_r} , and let $k \notin I_i$ for all i . Then each point p in τ is given by a linear combination $p = \sum_{i=1}^r \mu_i v_{I_i}$, where the μ_i are non-negative. We can find a positive representation

$$v_{I_i} = \sum_{v_S \in V_k} \lambda_{i,S} v_S$$

for each v_{I_i} using lemma 2.7 (where each $\lambda_{i,S}$ is either 1 or 0, depending on whether $S \subset I_i$ or not). We claim that

$$p = \sum_{v_S \in V_k} \left(\sum_{i=1}^r \mu_i \lambda_{i,S} \right) v_S \quad (2)$$

is a positive representation of p with respect to V_k . It is obvious that the $\sum_{i=1}^r \mu_i \lambda_{i,S}$ are non-negative. It remains to show that there is at least one S such that $\sum_{i=1}^r \mu_i \lambda_{i,S} = 0$. Let $a, b \in [n]$ be leaves in different connected components of $C \setminus k$ (where $C \setminus k$ denotes the graph produced from C by removing the closure of the unbounded edge labeled k , i.e. including the end vertex of k .) There are at least two such connected components, due to the fact that k is adjacent to an at least 3-valent vertex. Then $T := \{a, b\}$ is not contained in any of the sets I_i . Hence $\lambda_{i,T} = 0$ for all i . In particular $\sum_{i=1}^r \mu_i \lambda_{i,T} = 0$ and the equation 2 is a positive representation. Therefore

$$f_k(p) = f_k \left(\sum_{v_S \in V_k} \left(\sum_{i=1}^r \mu_i \lambda_{i,S} \right) v_S \right) = \sum_{v_S \in V_k} \left(\sum_{i=1}^r \mu_i \lambda_{i,S} \right) f_k(v_S)$$

and f_k is linear on τ . □

Remark 3.4. Let us explain in more detail how to compute $\text{div}(f_k)$ for a d -cycle Z of $\mathcal{M}_{0,n}$. We require that Z is supported on the cones of $\mathcal{M}_{0,n}$ corresponding to combinatorial types. As f_k is linear on each cone by lemma 3.2, the locus of non-differentiability of f_k is contained in cones of codimension 1 of Z . Given a cone τ of codimension 1 in Z , we need to compute the weight $\omega(\tau)$ of the cone τ

$$\omega(\tau) = \sum_i f_k(\omega(\sigma_i) u_{\sigma_i/\tau}) - f_k \left(\sum_i \omega(\sigma_i) u_{\sigma_i/\tau} \right)$$

where σ_i denote the top-dimensional neighboring cones of τ , $\omega(\sigma_i)$ their weight and $u_{\sigma_i/\tau}$ their primitive vectors. In our case the primitive vectors are given by the structure of a marked fan: each primitive vector corresponds to a tropical curve with only one bounded edge of length 1. Each $u_{\sigma_i/\tau}$ is equal to v_{I_i} for a subset $I_i \subset [n]$ (assume $k \notin I_i$ for all i). (The marked leaves in I_i are on one side of the bounded edge.) As f_k is defined on the

set V_k , we need to find a positive representation of the $u_{\sigma_i/\tau}$ with respect to V_k (see 2.5 and 2.6). This can be done using lemma 2.7. Also, we need to find a positive representation of $\sum_i \omega(\sigma_i) u_{\sigma_i/\tau}$. Given a representation $u_{\sigma_i/\tau} = \sum_{T \in V_k} \lambda_{i,T} v_T$ (note that all $\lambda_{i,T}$ are either 0 or 1 by the above), the following equality holds

$$\sum_i \omega(\sigma_i) u_{\sigma_i/\tau} = \sum_i \omega(\sigma_i) \left(\sum_{T \in V_k} \lambda_{i,T} v_T \right) = \sum_{T \in V_k} \left(\sum_i \omega(\sigma_i) \lambda_{i,T} \right) v_T.$$

Of course $\sum_i \omega(\sigma_i) \lambda_{i,T} \geq 0$ for all T , but the equation above is not necessarily a positive representation, since it is possible that none of the coefficients is zero. To make it a positive representation, we have to subtract $\sum_{T \in V_k} \min_{T \in V_k} (\sum_i \omega(\sigma_i) \lambda_{i,T}) v_T$. Then

$$\begin{aligned} & \sum_i f_k(\omega(\sigma_i) u_{\sigma_i/\tau}) - f_k \left(\sum_i \omega(\sigma_i) u_{\sigma_i/\tau} \right) \\ &= \sum_i \sum_{T \in V_k} \omega(\sigma_i) \lambda_{i,T} - \left(\sum_{T \in V_k} \left(\sum_i \omega(\sigma_i) \lambda_{i,T} \right) - \sum_{T \in V_k} \min_{T \in V_k} \left(\sum_i \omega(\sigma_i) \lambda_{i,T} \right) \right) \\ &= \min_{T \in V_k} \left(\sum_i \omega(\sigma_i) \lambda_{i,T} \right) \cdot \binom{n-1}{2}, \end{aligned}$$

because $|V_k| = \binom{n-1}{2}$. That is, in order to determine the weight of the cone τ in $\text{div}(f_k)$, we only have to determine the value $\min_{T \in V_k} (\sum_i \omega(\sigma_i) \lambda_{i,T})$. Recall that $\lambda_{i,T} = 1$ if $T \subset I_i$ and 0 else. Hence

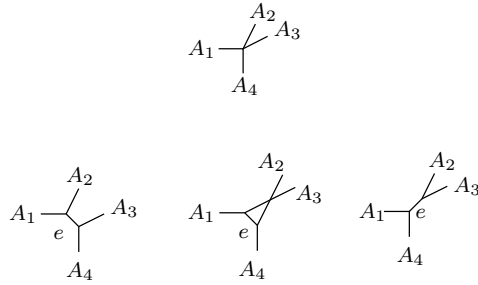
$$\min_{T \in V_k} \left(\sum_i \omega(\sigma_i) \lambda_{i,T} \right) = \min_{T \in V_k} \left(\sum_{i: T \subset I_i} \omega(\sigma_i) \right).$$

Proposition 3.5. *Let f_k be as in notation 3.2. Then the divisor of f_k in $\mathcal{M}_{0,n}$ is*

$$\text{div}(f_k) = \binom{n-1}{2} \Psi_k,$$

where $\text{div}(f_k)$ is defined in 3.4 of [AR] and Ψ_k is defined in definition 3.1.

Proof. We may assume without restriction that $k = 1$. By lemma 3.3, the locus of non-differentiability of f_1 is contained in the cones of codimension 1 of $\mathcal{M}_{0,n}$. A cone τ of codimension 1 in $\mathcal{M}_{0,n}$ corresponds to the combinatorial type of a tropical curve C with one 4-valent vertex. Let A_1 denote the subset of $[n]$ consisting of the ends which can be reached from this 4-valent vertex via the adjacent edge e_1 , A_2 the subset which can be reached via e_2 and so on. Without restriction $1 \in A_1$ and $2 \in A_2$. The three neighboring cones σ_1 , σ_2 and σ_3 of τ are given by the three possible resolutions of the 4-valent vertex in two 3-valent vertices (i.e. the three possible ways to add a new edge which produces two 3-valent vertices instead of the one 4-valent).



The primitive vector $u_{\sigma_i/\tau}$ is given by the tropical curve where all edges except the new edge e (which is introduced by resolving) are shrunk to length 0. Hence the three primitive vectors are

$$v_{A_1 \cup A_2} = v_{A_3 \cup A_4}, \quad v_{A_1 \cup A_3} = v_{A_2 \cup A_4} \quad \text{and} \quad v_{A_1 \cup A_4} = v_{A_2 \cup A_3}.$$

By lemma 2.7, the positive representations of these three primitive vectors with respect to V_1 (see 2.5) are

$$\sum_{T \subset A_3 \cup A_4} v_T, \quad \sum_{T \subset A_2 \cup A_4} v_T, \quad \text{respectively} \quad \sum_{T \subset A_2 \cup A_3} v_T,$$

(where in each case the sum goes over all $T \in V_k$).

By remark 3.4, the weight of $\text{div}(f_1)$ along τ is equal to $\min_{T \in V_k} (|\{i : T \subset I_i\}|) \cdot \binom{n-1}{2}$, where $I_1 = A_3 \cup A_4$, $I_2 = A_2 \cup A_4$ and $I_3 = A_2 \cup A_3$ (since all weights of the neighbouring cones are 1). There are two cases to distinguish. Assume first that $\{1\} \neq A_1$, that is, there exists a number $1 \neq a_1 \in A_1$. Then for $T = \{a_1, 2\}$ the number $|\{i : T \subset I_i\}|$ is zero. Hence the weight of $\text{div}(f_1)$ at τ is zero, and τ is not part of $\text{div}(f_1)$. Assume next that $A_1 = \{1\}$. Then all vectors $v_T \in V_1$ appear in the sum of the three primitive vectors. Let $a_2 \in A_2$ and $a_3 \in A_3$, then $T = \{a_2, a_3\}$ appears only once, that is, the number $|\{i : T \subset I_i\}|$ is one. Hence the weight of $\text{div}(f_1)$ along τ is $\binom{n-1}{2}$. In particular, $\text{div}(f_1)$ consists of the cones given by a 4-valent tropical curves such that leaf 1 is adjacent to the 4-valent vertex, and each such cone has weight $\binom{n-1}{2}$. It follows that $\text{div}(f_1) = \binom{n-1}{2} \cdot \Psi_1$. \square

4. INTERSECTING TROPICAL PSI-CLASSES

In the sequel of this section, given integers $k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}$ and a subset $I \subset [n]$, we denote $K(I) = \sum_{i \in I} k_i$. Using this notation, the main theorem of this section reads as follows:

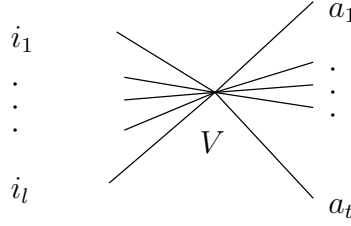
Theorem 4.1. *The intersection $\Psi_1^{k_1} \cdot \dots \cdot \Psi_n^{k_n}$ is the subfan of $\mathcal{M}_{0,n}$ consisting of the closure of the cones of dimension $n - 3 - K([n])$ corresponding to abstract tropical curves C such that for each vertex V of C we have $\text{val}(V) = K(I_V) + 3$, where I_V denotes the set*

$$I_V = \{i \in [n] : \text{leaf } x_i \text{ is adjacent to } V \text{ and } k_i \geq 1\} \subset [n].$$

The weight of the facet $\sigma(C)$ containing the point $\phi_n(C)$ equals

$$\omega(\sigma(C)) = \frac{\prod_{V \in V(C)} K(I_V)!}{\prod_{i=1}^n k_i!}.$$

Proof. We prove by induction on $K([n])$, that is, we compute the weight $\omega(\tau)$ of a codimension one cell $\tau \subset \prod_{i=1}^n \Psi_i^{k_i}$ in the intersection product $\Psi_1 \prod_{i=1}^n \Psi_i^{k_i}$. Let τ be a ridge of $\prod_{i=1}^n \Psi_i^{k_i}$ and let C be a curve parameterized by τ . As τ is of codimension one, there is by construction exactly one vertex V of C of valence one higher than expected, i.e. $\text{val}(V) = K(I_V) + 4$ — apart from the leaves x_i with $i \in I_V = \{i_1, \dots, i_l\}$ there are exactly $K(I_V) + 4 - |I_V|$ edges a_1, \dots, a_r adjacent to V as indicated in the picture below.



Using remark 3.4, we compute

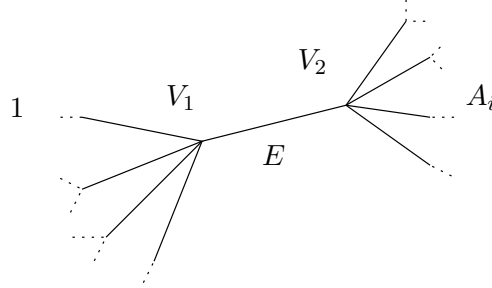
$$\omega(\tau) = \min_{\tau \in V_k} \left(\sum_{i: T \subset A_i} \omega(\sigma_i) \right)$$

where σ_i denote the facets containing τ and $v_{\sigma_i/\tau} = v_{A_i}$ denote their primitive vectors.

Facets σ containing the ridge τ parameterize curves C' such that C is obtained from C' by collapsing an edge E with vertices V_1 and V_2 to the vertex V . By assumption, we know that the weight of such a facet equals

$$\omega(\sigma_i) = \frac{\prod K(I_{V'})!}{\prod k_i!} \cdot K(I_{V_1})!K(I_{V_2})! =: W \cdot K(I_{V_1})!K(I_{V_2})!,$$

where the first product goes over all vertices V' of C' different from V_1 and V_2 . The normal vector of such a cone σ_i is by definition given by $v_{\sigma_i/\tau} = v_{A_i}$, where A_i is the set of labels on one of connecting component of $C' \setminus \{E\}$. Assume without loss of generality that x_1 is in the connected component of $C' \setminus \{E\}$ containing V_1 and let $A_i \subset [n]$ denote the subsets of labels of the other connected component.



Assume first that x_1 is not adjacent to V . Then there exist a leaf $x_s \neq x_1$ in the connected component of $C \setminus \{V\}$ containing x_1 , hence the label s is not contained in any of the sets A_i . For any T containing s we thus have $\sum_{i: T \subset A_i} \omega(\sigma_i) = 0$ and hence the minimum over all T is 0, too. Consequently, $\omega(\tau) = 0$.

Assume that x_1 is adjacent to V , let $s_1, s_2 \in [n] \setminus \{1\}$ be two labels. It suffices to prove that

$$\sum_{T \subset A_i} \omega(\sigma_i) = \frac{\prod K(I_{V'})!}{\prod k_i!} \cdot \frac{(K(I_V) + 1)!}{k_1 + 1} = W \cdot \frac{(K(I_V) + 1)!}{k_1 + 1}$$

where the first product goes over all vertices V' of C different from V .

Note that a cell σ satisfying the conditions above corresponds to a partition $J_{V_1} \cup J_{V_2} = I_V \setminus \{1, s_1, s_2\} =: M$ and a distribution of the labels a_i among the leaves not labelled by

some x_i , $i \in M \cup \{1, s_1, s_2\}$. The total number of labels a_i equals $K(I_V) + 4 - |M|$ and there are exactly $K(J_{V_1}) + k_1 + 1 - |J_{V_1}|$ non-labelled edges at vertex V_1 . Hence we get

$$\omega(\tau) = W \cdot \sum_{J_{V_1} \cup J_{V_2} = M} (K(J_{V_1}) + k_1)! (K(J_{V_2}) + k_{s_1} + k_{s_2})! \binom{K(I_V) + 1 - |M|}{K(J_{V_1}) + k_1 + 1 - |J_{V_1}|}$$

and we want to see that this is equal to

$$W \cdot \frac{(K(I_V) + 1)!}{k_1 + 1}.$$

The equality follows after proving the following identity

$$\sum_{I \subset M} (K(I) + k_1)! (K - (K(I) + k_1))! \binom{K + 1 - m}{K(I) + k_1 + 1 - |I|} = \frac{(K + 1)!}{k_1 + 1} \quad (3)$$

setting $I = J_{V_1}$, $m = |M|$ and $K = K(I_V)$.

To see that this identity holds, multiply the left hand side by $\frac{k_1 + 1}{(K + 1 - m)!}$ and simplify:

$$\begin{aligned} & (k_1 + 1) \cdot \left(\sum_{I \subset M} \frac{(K(I) + k_1)! \cdot (K - (K(I) + k_1))!}{(K(I) + k_1 + 1 - |I|)! \cdot (K - K(I) - k_1 - m + |I|)!} \right) \\ &= (k_1 + 1) \cdot \left(\frac{(k_1)! \cdot (K - k_1)!}{(k_1 + 1)! \cdot (K - k_1 - m)!} + \sum_{\emptyset \neq I \subset M} (K(I) + k_1)^{|I| - 1} (K - K(I) - k_1)^{m - |I|} \right) \\ &= (K - k_1)^{[m]} + \sum_{\emptyset \neq I \subset M} (k_1 + 1) (K(I) + k_1)^{|I| - 1} (K - K(I) - k_1)^{m - |I|} \end{aligned}$$

where we use the falling power notation

$$x^{[p]} = x \cdot (x - 1) \cdot \dots \cdot (x - p + 1).$$

If we multiply the right hand side of equation (3) by $\frac{k_1 + 1}{(K + 1 - m)!}$ we get $(K + 1)^{[m]}$. Thus the identity follows from lemma A.2. \square

Corollary 4.2. *If the intersection $\Psi_1^{k_1} \cdot \dots \cdot \Psi_n^{k_n}$ is 0-dimensional, i.e. $K([n]) = n - 3$, then the (stable) intersection $\Psi_1^{k_1} \cdot \dots \cdot \Psi_n^{k_n}$ is just the origin $\{0\}$, with weight*

$$\omega(\{0\}) = \frac{(n - 3)!}{k_1! \dots k_n!} = \binom{n - 3}{k_1, \dots, k_n}.$$

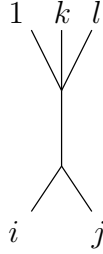
Remark 4.3. For $n \geq 3$, let $\overline{M}_{0,n}$ denote the space of n -pointed stable rational curves, that is, tuples (C, p_1, \dots, p_n) consisting of a connected algebraic curve C of arithmetic genus 0 with simple nodes as only singularities and a collection p_1, \dots, p_n of distinct smooth points on C such that the number of automorphisms of C with the property that the points p_i are fixed is finite.

Define the line bundle \mathcal{L}_i on $\overline{M}_{0,n}$ to be the unique line bundle whose fiber over each pointed stable curve (C, p_1, \dots, p_n) is the cotangent space of C at p_i and let $\Psi_i \in A^1(\overline{M}_{0,n})$ denote its first Chern class. For $1 \leq i \leq n$, let $k_i \in \mathbb{Z}_{\geq 0}$ such that $\sum_{i=1}^n k_i = \dim(\overline{M}_{0,n}) = n - 3$. Then the following equation holds (see e.g. [HM], section 2.D)

$$\int_{\overline{M}_{0,n}} \Psi_1^{k_1} \Psi_2^{k_2} \dots \Psi_n^{k_n} = \frac{(n - 3)!}{\prod_{i=1}^n k_i!}.$$

Hence the 0-dimensional intersection products of Psi-classes on the moduli space of n -marked rational algebraic curves coincides with its tropical counterpart.

Example 4.4 (Psi-classes on $\mathcal{M}_{0,5}$). If we intersect two Ψ -classes, the intersection is 0-dimensional. Hence there are (up to symmetry) only two different intersection products to compute: Ψ_1^2 and $\Psi_1 \cdot \Psi_2$. Let us compute both. Let us start with $\text{div}(f_1) \cdot \Psi_1$. By lemma 3.3 we know that we only have to check the cones of codimension 1 in Ψ_1 — that is, the cone $\{0\}$. The neighbors of $\{0\}$ in Ψ_1 — that is, in this case, the top-dimensional cones of Ψ_1 — correspond to tropical curves with 1 at a 4-valent vertex:

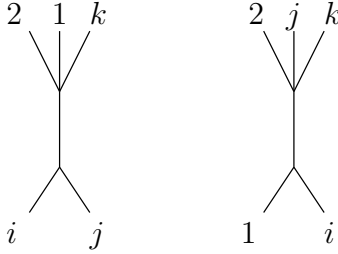


(Here we assume that $\{i, j, k, l\} = \{2, 3, 4, 5\}$.) There are $\binom{4}{2} = 6$ of these cones. Each such cone is generated by the primitive vector $v_{\{i,j\}}$. The sum over all primitive vectors is 0. Hence the weight of $\{0\}$ is given by

$$\sum_{i,j \in \{2,3,4,5\}, i \neq j} f_1(v_{\{i,j\}}) - f_1(0) = 6.$$

(By definition, all cones of Ψ_1 are of weight 1.) Thus the weight of $\{0\}$ in $\text{div}(f_1) \cdot \Psi_1$ is 6, and using proposition 3.5, the weight of $\{0\}$ in $\Psi_1 \cdot \Psi_1$ is 1.

Now let us compute the weight of $\{0\}$ in $\text{div}(f_1) \cdot \Psi_2$. Three of the neighbors of $\{0\}$ correspond to a curve as on the right, the other three to a curve as on the left:

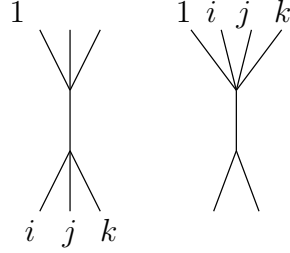


(We assume $\{i, j, k\} = \{3, 4, 5\}$.) The primitive vectors of the first type are $v_{i,j}$ and they are given as a positive combination with respect to V_1 . A positive combination for the primitive vectors of the second type is $v_{\{2,j\}} + v_{\{2,k\}} + v_{\{j,k\}}$. Their sum is of course again 0. Hence the weight of $\{0\}$ is

$$\begin{aligned} & f_1(v_{\{3,4\}}) + f_1(v_{\{3,5\}}) + f_1(v_{\{4,5\}}) + f_1(v_{\{2,3\}} + v_{\{2,4\}} + v_{\{3,4\}}) \\ & + f_1(v_{\{2,3\}} + v_{\{2,5\}} + v_{\{3,5\}}) + f_1(v_{\{2,4\}} + v_{\{2,5\}} + v_{\{4,5\}}) - f_1(0) \\ & = 1 + 1 + 1 + 3 + 3 + 3 = 12. \end{aligned}$$

Thus — using proposition 3.5 again — the weight of $\{0\}$ in $\Psi_1 \cdot \Psi_2$ is 2.

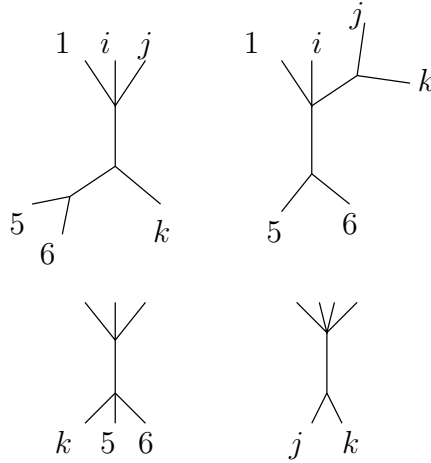
Example 4.5 (Psi-classes on $\mathcal{M}_{0,6}$). Let us compute Ψ_1^2 , respectively $\text{div}(f_1) \cdot \Psi_1$, on $\mathcal{M}_{0,6}$. To do so, we have to compute the weight of a cone of codimension 1 in Ψ_1 . Such a cone corresponds to a tropical curve with either another 4-valent vertex (as on the left) or with a 5-valent vertex, to which 1 is adjacent (as on the right):



A cone corresponding to the curve on the left has 3 neighbors corresponding to the 3 possible resolutions of the lower vertex. These three cones are generated by the primitive vectors $v_{\{i,j\}}$, $v_{\{i,k\}}$ and $v_{\{j,k\}}$. Thus the weight of such a cone is

$$f_1(v_{\{i,j\}}) + f_1(v_{\{i,k\}}) + f_1(v_{\{j,k\}}) - f_1(v_{\{i,j\}} + v_{\{i,k\}} + v_{\{j,k\}}) = 0,$$

and it does not belong to Ψ_1^2 . A cone corresponding to the curve on the right (where we assume now $\{i, j, k\} = \{2, 3, 4\}$ for simplicity) has 6 neighbors in Ψ_1 , 3 as on the right and 3 as on the left (below the curve, the corresponding normal vector for the cones are shown):



Hence, the weight of this cone is

$$\begin{aligned} & f_1(v_{\{2,5\}} + v_{\{2,6\}} + v_{\{5,6\}}) + f_1(v_{\{3,5\}} + v_{\{3,6\}} + v_{\{5,6\}}) + f_1(v_{\{4,5\}} + v_{\{4,6\}} + v_{\{5,6\}}) \\ & + f_1(v_{\{2,3\}}) + f_1(v_{\{2,4\}}) + f_1(v_{\{3,4\}}) \\ & - f_1(v_{\{2,5\}} + v_{\{2,6\}} + v_{\{5,6\}} + v_{\{3,5\}} + v_{\{3,6\}} + v_{\{5,6\}} + v_{\{4,5\}} + v_{\{4,6\}} \\ & \quad + v_{\{5,6\}} + v_{\{2,3\}} + v_{\{2,4\}} + v_{\{3,4\}}) \\ & = 3 + 3 + 3 + 1 + 1 + 1 - f_1(2v_{\{5,6\}}) = 12 - 2 = 10, \end{aligned}$$

because the sum over all primitive vectors is not automatically written as a positive combination with respect to V_1 and we have to subtract $\sum_{i,j \in [6] \setminus \{1\}, i \neq j} v_{\{i,j\}}$ to make a positive combination. By proposition 3.5, the weight of each such cone in Ψ_1^2 is one.

APPENDIX A. EQUATIONS NEEDED FOR THE PROOF THE MAIN THEOREM, 4.1

Let k_1, \dots, k_m and K be integers. For a subset $I \subset [m] := \{1, \dots, m\}$ we use the notation $K(I) = \sum_{i \in I} k_i$ and the falling power notation

$$x^{[p]} = x \cdot (x-1) \cdot \dots \cdot (x-p+1).$$

Lemma A.1. *The following equation is satisfied:*

$$\sum_{\emptyset \neq I \subset [m]} K(I)^{|I|-1} (K - K(I))^{m-|I|} = m \cdot K^{[m-1]}. \quad (4)$$

Proof. The proof is an induction. We assume that equation (4) holds for any $p < m$ and prove that it holds for m . The induction beginning holds trivially.

We use the well-known multinomial identity for falling powers:

$$(x_1 + \dots + x_n)^{[p]} = \sum_{a_1 + \dots + a_r = p} \frac{p!}{a_1! \cdot \dots \cdot a_r!} x_1^{[a_1]} \cdot \dots \cdot x_r^{[a_r]} \quad (5)$$

where the a_i are nonnegative integers. Set $x_i = k_i$ for all $i = 1, \dots, m$ and $x_0 = K - \sum_{i \in [m]} k_i$. Express each side of equation (4) as a linear combination of monomials $x_0^{[a_0]} \cdot \dots \cdot x_r^{[a_r]}$ with $a_0 + \dots + a_r = m-1$ and compare the coefficients. The coefficient of such a monomial on the right hand side of (4) equals $m \cdot \frac{(m-1)!}{a_0! \cdot \dots \cdot a_r!} = \frac{m!}{a_0! \cdot \dots \cdot a_r!}$. The coefficient on the left hand side equals

$$\sum_I \frac{(|I|-1)!(m-|I|)!}{a_0! \cdot \dots \cdot a_r!}$$

where the sum is over all non-empty subsets of $[m]$ satisfying $\sum_{i \in I} a_i = |I| - 1$. To prove (4) we thus have to show the following identity for any tuple (a_1, \dots, a_m) of nonnegative integers satisfying $a_1 + \dots + a_m < m$:

$$\sum_I (|I|-1)!(m-|I|)! = m! \quad (6)$$

where again the sum goes over all $I \subset [m]$ satisfying $\sum_{i \in I} a_i = |I| - 1$. We use induction on $\sum_{i \in [m]} a_i$ to prove equation (6). If all $a_i = 0$ then the only subsets $I \subset [m]$ satisfying $\sum_{i \in I} a_i = |I| - 1$ are one-element subsets. There are m of those and they all contribute a summand of $(m-1)!$ to the left hand side, so we have $m!$ altogether which equals the right hand side. Now assume that a_1, \dots, a_p are positive and $a_{p+1} = \dots = a_m = 0$ for some p . Then $p < m$ since $a_1 + \dots + a_m < m$. We can write any subset $I \subset [m]$ as a disjoint union $I = J \cup I'$ where $J \subset [p]$ and $I' \subset \{p+1, \dots, m\}$. The sum $\sum_{i \in I} a_i$ depends only on J , i.e. it is equal to $\sum_{i \in J} a_i$. As before we denote $\sum_{i \in J} a_i$ by $K(J)$. If we fix a set J we can produce several possible subsets I satisfying $K(J) = \sum_{i \in J} a_i = \sum_{i \in I} a_i = |I| - 1$ by just adding $K(J) - |J| + 1$ elements of $\{p+1, \dots, m\}$. Therefore we can write the left hand side

of equation (6) as

$$\begin{aligned}
& \sum_{J \subset [p]} \binom{m-p}{K(J) - |J| + 1} K(J)! (m-1 - K(J))! \\
&= (m-p)(m-1)! + \sum_{\emptyset \neq J \subset [p]} \frac{(m-p)! K(J)! (m-1 - K(J))!}{(K(J) - |J| + 1)! (m-p - K(J) + |J| - 1)!} \\
&= (m-p)(m-1)! + (m-p)! \sum_{\emptyset \neq J \subset [p]} K(J)^{|J|-1} (m-1 - K(J))^{m-|J|}.
\end{aligned}$$

Subtract the contribution of the empty set from the right hand side of equation (6) and divide by $(m-p)!$. Then we get

$$\frac{m! - (m-p)(m-1)!}{(m-p)!} = \frac{(m-1)!}{(m-p)!} (m-m+p) = p \cdot (m-1)^{[p-1]}.$$

Hence (6) follows if

$$\sum_{\emptyset \neq J \subset [p]} K(J)^{|J|-1} (m-1 - K(J))^{m-|J|} = p \cdot (m-1)^{[p-1]}$$

which holds by the induction assumption on (4). \square

Lemma A.2. *Let $M = \{2, \dots, m+1\}$ and let $K = \sum_{i \in [m+1]} k_i$. Then the following equation holds:*

$$(K - k_1)^{[m]} + \sum_{\emptyset \neq I \subset M} (k_1 + 1) \cdot (K(I) + k_1)^{|I|-1} (K - K(I) - k_1)^{m-|I|} = (K+1)^{[m]}. \quad (7)$$

Proof. The proof is a double induction on k_1 and m . We show that the equation is true for $k_1 = 0$ and for all m . Next, we assume that it is true for all $k_1 - 1$ and any m and for any k_i and $m - 1$ and show that it is true for k_1 and m . For $k_1 = 0$, the equation reads

$$K^{[m]} + \sum_{\emptyset \neq I \subset M} (K(I))^{|I|-1} (K - K(I))^{m-|I|} = (K+1)^{[m]}.$$

If we subtract $K^{[m]}$ from the right hand side, we get

$$\begin{aligned}
& (K+1)^{[m]} - K^{[m]} \\
&= ((K+1) - (K-m+1))(K \cdot \dots \cdot (K-m+2)) \\
&= m \cdot K^{[m-1]}.
\end{aligned}$$

Thus the equation for $k_1 = 0$ follows from lemma A.1 after relabeling the index set M . Now we assume that the equation is true for $k_1 - 1$. Remember that K is defined as $K = k_1 + \sum_{i \in M} k_i$, so if we replace k_1 by $k_1 - 1$ then we also have to replace K by $K - 1$. Then the equation reads

$$(K - k_1)^{[m]} + \sum_{\emptyset \neq I \subset M} k_1 \cdot (K(I) + k_1 - 1)^{|I|-1} (K - K(I) - k_1)^{m-|I|} = K^{[m]}. \quad (8)$$

We subtract the left hand side of equation (8) from the left hand side of equation (7) and get

$$\begin{aligned}
& \sum_{\emptyset \neq I \subset M} (k_1 + 1) \cdot (K(I) + k_1)^{|I|-1} (K - K(I) - k_1)^{m-|I|} \\
& - \sum_{\emptyset \neq I \subset M} k_1 \cdot (K(I) + k_1 - 1)^{|I|-1} (K - K(I) - k_1)^{m-|I|} \\
& = \sum_{I=\{j\}, j \in M} (K - k_j - k_1)^{m-1} \\
& + \sum_{|I| \geq 2} ((k_1 + 1)(K(I) + k_1) - k_1(K(I) + k_1 - |I| + 1)) \\
& \quad \cdot (K(I) + k_1 - 1)^{|I|-2} (K - K(I) - k_1)^{m-|I|}
\end{aligned}$$

which can be simplified to

$$\begin{aligned}
& \sum_{I=\{j\}, j \in M} (K - k_j - k_1)^{m-1} \\
& + \sum_{|I| \geq 2} (K(I) + k_1 \cdot |I|)(K(I) + k_1 - 1)^{|I|-2} (K - K(I) - k_1)^{m-|I|}. \quad (9)
\end{aligned}$$

Now we subtract the right hand side of equation (8) from the right hand side of equation (7) and get

$$(K + 1)^{[m]} - K^{[m]} = m \cdot K^{[m-1]} = \sum_{j \in M} K^{[m-1]}. \quad (10)$$

We want to apply equation (7) for $m - 1$ and any k_i differently for each of the m summands above. To do so, we need to interpret $K - 1$ as a sum of m numbers. We choose the first summand (i.e. the analogue of k_1) to be $k_1 + k_j - 1$ and the other summands to be the k_i (except k_j). If we replace k_1 by $k_1 + k_j - 1$, K by $K - 1$ and m by $m - 1$ in equation (7) it reads

$$\begin{aligned}
& (K - k_1 - k_j)^{[m-1]} \\
& + \sum_{\emptyset \neq J \subset M \setminus \{j\}} (k_1 + k_j) \cdot (K(J) + k_1 + k_j - 1)^{|J|-1} (K - K(J) - k_1 - k_j)^{m-|J|-1} = K^{[m-1]}.
\end{aligned}$$

Thus equation (10) equals

$$\begin{aligned}
& \sum_{j \in M} (K - k_1 - k_j)^{[m-1]} \\
& + \sum_{\emptyset \neq J \subset M \setminus \{j\}} (k_1 + k_j) \cdot (K(J) + k_1 + k_j - 1)^{|J|-1} (K - K(J) - k_1 - k_j)^{m-|J|-1}. \quad (11)
\end{aligned}$$

It remains to show that the expression (9) equals expression (11). To see this, note that every $I \subset M$ yields a possible J for every $j \in I$ by just deleting j . Thus $|I| - 1 = |J|$. Any I contributes a factor of $(k_1 + k_j)$ in the summand for j in (11). Thus it contributes $K(I) + k_1 \cdot |I|$ in total, which equals the contribution in (9). \square

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