Algorithmic aspects of tropical intersection theory

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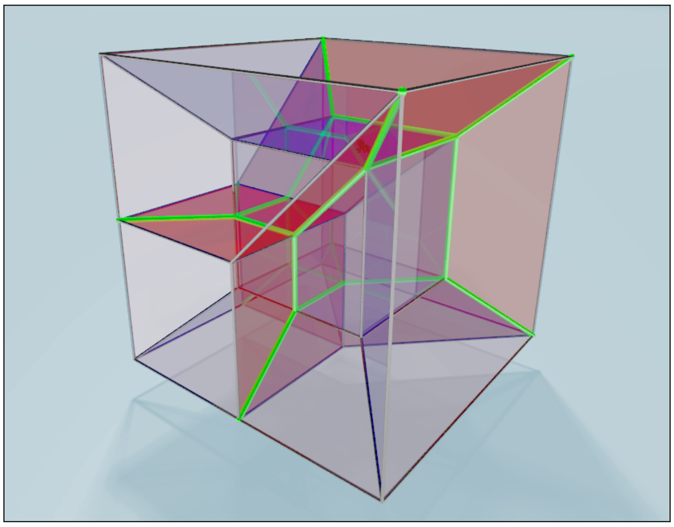

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1. Introduction

1.1. Motivation

Tropical intersection theory

Tropical intersection theory has proven to be a very powerful tool, especially in enumerative geometry. Using the famous correspondence theorem [M1], many well-known enumerative results have been reproved in a combinatorial, i.e. elementary fashion (see for example [GM] and [KM1]). New results, for example in real algebraic geometry, have also been found (e.g. [IKS],[GS]). However, without tropical intersection theory these results always necessitate some ad-hoc proof of invariance of counts under a change of parameters (usually points through which curves are supposed to pass). One big advantage of intersection theory in algebraic geometry is that this invariance is automatic due to intersection products being defined on equivalence classes of cycles. The discovery of tropical moduli spaces of lines and rational curves ([SS], [GKM]) made it even more desirable to have a similar concept in tropical geometry.

The basic ideas for a tropical intersection theory were laid out in [M3], using the *stable intersection* defined in [RGST]. The notion turned out to be closely related to the *fan displacement rule* found in [FS2], used to describe toric intersection theory. However, this approach only provided a good theory for intersection products in $\mathbb{R}^n$, as the computation required one to shift two cycles such that they intersect transversely. In general ambient tropical varieties this is not always possible. Allermann and Rau then developed a more general approach based on Mikhalkin's ideas in [AR2]: By finding rational functions cutting out the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$, they were able to define an intersection product for cycles in arbitrary position. Also, the same concept would obviously work for any ambient variety which provided rational functions to cut out the diagonal. This was later used in [FR] to define an intersection product on Bergman fans. A different approach was taken in [S2] to find a recursive definition of an intersection product on Bergman fans using projections and modifications. Recently, Jensen and Yu have provided another equivalent definition of an intersection product in $\mathbb{R}^n$ [JY], closely related to the fan displacement rule, which turns out to be much more suitable for computations then previous definitions.

Having an intersection product on Bergman fans automatically produced an intersection theory on $\mathcal{M}_{0,n}^{\text{trop}}$, the moduli space of tropical rational curves. For tropical enumerative geometers it is very desirable to be able to compute such intersection products on moduli spaces - most prominently tropical descendant Gromov-Witten invariants, i.e. products of Psi-classes and pull-backs of points via evaluation maps. Such examples can be very hard to compute by hand, so the development of a computational tool became necessary.

Computations in tropical geometry

Due to its combinatorial nature, tropical geometry has always been a popular candidate for computational approaches. One of the most prominent tools in tropical geometry is probably `gfan` [J1] which can (among other things) compute tropicalizations of algebraic varieties. Felipe Rincón has developed `TropLi` to compute Bergman fans of linear matroids [R2]. However, the objects computed in this fashion tend to be rather large and complex and difficult to analyze in detail. So far, there has been no tool for dealing with general tropical varieties and especially for doing intersection theory on or with them. It is the aim of `a-tint` (= `a`lgorithmic `t`ropical `int`ersection theory) to provide such a tool. `a-tint` is a software package developed by the author as an extension for `polymake`. The latter was originally created for the analysis of polytopes [GJ], but has become much more versatile and powerful since its birth. It also provides tools for dealing with polyhedral complexes, groups, graphs, matroids, simplicial complexes and *tropical convexity* - a topic not directly related to intersection theory, but still a focus of much interest ([DS], [J3], [BY],...). This versatility, together with its extension system and its strong focus on polyhedral computations make `polymake` a very good starting point for writing algorithms in tropical intersection theory. `a-tint` now provides a large set of functions for tropical computations with a strong emphasis on moduli spaces of rational curves (Chapter 8 in the Appendix contains a list of features).

Tropical Hurwitz cycles

Roughly speaking, Hurwitz numbers count covers of $\mathbb{P}^1$ by complex curves C - but with a given degree and some special ramification profile over a certain number of points. For example, *simple* Hurwitz numbers require the cover $C \to \mathbb{P}^1$ to have a specific ramification profile over some special point (usually ∞) and only simple ramification elsewhere. These numbers have played a significant role in the study of the intersection theory of the moduli spaces $\bar{\mathcal{M}}_{g,n}$ of curves. The *ELSV formula* relates Hurwitz numbers to certain intersection products of tautological classes on $\bar{\mathcal{M}}_{g,n}$. This was then used by Okounkov and Pandharipande to prove Witten's conjecture [OP].

To obtain *double* Hurwitz numbers, we fix the ramification over two points in $\mathbb{P}^1$, usually 0 and ∞. These numbers not only occur in algebraic geometry, but also in representation theory and combinatorics - thus providing a strong connection between a wide variety of disciplines. An overview over the different definitions of double Hurwitz numbers can for example be found in [J2]. An ELSV-type formula has been conjectured by Goulden, Jackson and Vakil in [GJV], where it is also shown that these numbers are piecewise polynomial in terms of the ramification profile. By convention, one writes the profile as $x \in \mathbb{Z}^n$ with $\sum x_i = 0$. The interpretation of this is that the positive part x^+ gives the ramification profile over 0 and the negative part x^- gives the ramification profile over ∞. A special feature of double Hurwitz numbers is the fact that the number of simple ramification points only depends on the length of the ramification profile, not on the multiplicities. The number of additional simple ramification points is then $n - 2 + 2g$. This fact will be very helpful in defining higher-dimensional cycles.

The generalization to Hurwitz cycles is achieved by letting one or more of the images of simple ramification points "move around" in $\mathbb{P}^1$. In the general case, these loci were defined and studied by Graber and Vakil in [GV]. In the genus 0 case, Bertram, Cavalieri and Markwig proved that these cycles are linear coefficients of cycles with coefficients that are piecewise polynomial in the entries of the ramification profile [BCM]. They also considered a tropical version $\mathbb{H}_k^{\text{trop}}(x,p), \tilde{\mathbb{H}}_k^{\text{trop}}(x,p)$ of double Hurwitz loci and showed that their combinatorics relate very nicely to the combinatorics of the different strata of the algebraic loci via dualizing of graphs. Here $\tilde{\mathbb{H}}_k^{\text{trop}}(x,p)$ differs from $\mathbb{H}_k^{\text{trop}}(x,p)$ in that the preimages of the "simple ramification points" p_i are also marked.

The tropical Hurwitz cycles are obtained by "tropicalizing" a Gromov-Witten type formula for the algebraic version. While the definition is rather simple and involves only the well-known tropical moduli space of rational curves, the properties of the tropical Hurwitz cycles themselves are a mystery at first. There are several natural questions that one might ask: Are they connected in codimension one? Are they irreducible? Can they be realized as divisors of rational functions? All of these questions can now be answered very easily for specific cycles using `a-tint`, which was of significant use in finding the theoretical results in the second part of this thesis.

1.2. Summary

Outline and main results

We will conclude this chapter with some preliminary definitions and notations concerning polyhedral and tropical geometry. The thesis itself is then split into three parts:

In the first part of this thesis we discuss the algorithmic challenges presented by tropical intersection theory and tropical geometry in general. We prove several theoretical results that help us compute tropical varieties and their properties. A strong focus is put on moduli spaces of tropical rational curves, whose high combinatorial complexity requires special methods for dealing with them.

In the second part we will show how the machinery developed in the first part can be used to obtain theoretical results in tropical enumerative geometry. We consider tropical double Hurwitz cycles, which are tropical versions of algebraic Hurwitz cycles. The tropical cycles have a rather complex combinatorial nature, so it is very difficult to study them purely "by hand". Being able to compute examples has been very helpful in coming up with the subsequent theoretical results. The main results of this part can be summarized as follows:

Theorem. *For any* $k \geq 1, x \in \mathbb{Z}^n$ *with* $\sum x_i = 0$ *and* $p = (p_0, \dots, p_{n-3-k}) \in \mathbb{R}^{n-2-k}$ *the marked and unmarked Hurwitz cycles* $\tilde{\mathbb{H}}_k^{\text{trop}}(x,p)$ *and* $\mathbb{H}_k^{\text{trop}}(x,p)$ *are connected in codimension one. If the* p_i *are pairwise different, then* $\tilde{\mathbb{H}}_k^{\text{trop}}(x,p)$ *is an integer multiple of an irreducible cycle.*

The third part contains the Appendix with information on the software package `a-tint` and some benchmarks.

Summary of Part I

In chapter 2 we discuss how to compute some basic properties and operations on tropical cycles. The *lattice normal vector* of a polyhedral cell with respect to a codimension one face plays a central role in most tropical computations. We discuss how it can be computed in 2.1. Most intersection-theoretic formulas and constructs are given in terms of *rational functions* and their divisors, which we introduce in 2.2. In 2.3 we define what it means for a tropical variety to be *irreducible* and we show that this can be computed regardless of a particular polyhedral subdivision by solving certain linear equations induced by the balancing condition at each codimension one cell. In addition we will see that all irreducible subvarieties can be found by computing the rays of the cone of possible nonnegative weight vectors. Two tropical varieties are considered the same if they have a common refinement. Hence it is a natural question whether there exists a canonical or *coarsest* polyhedral structure for any tropical variety. This is true (and has been well-known) for hypersurfaces, but not at all clear in the general case. In 2.4 we make some efforts to answer this question at least partially. We conjecture that any tropical variety which is connected in codimension one has a coarsest polyhedral structure. The approach is constructive: One can define an equivalence relation on the maximal cells of a polyhedral subdivision. This relation is very naturally the inverse operation to refining a polyhedral complex. We prove that if one can show that the support of the equivalence classes is a polyhedron for varieties of codimension two, then this statement is true in any codimension. In 2.5 we discuss the different existing definitions of *intersection products* of tropical varieties in $\mathbb{R}^n$ and how they fare when we try to compute them. Finally, in 2.6, we show how `a-tint` handles *local* computations. In many situations, we are only interested in the local result of some operation. Instead of computing globally, then restricting to the part of interest to us, it is of course much more efficient to do all computations locally from the beginning. This section shows how this can be done using an object called a *local weighted complex* and how different operations can be performed on this object.

In chapter 3 we introduce *Bergman fans*. In tropical geometry, they play the role of local building blocks for "smooth" varieties. Hence they obviously constitute a very important class of objects. We give three equivalent definitions. They induce different ways of computing Bergman fans, which we discuss in 3.2. After briefly describing the current state of the art concerning intersection products in Bergman fans in 2.5, we go on to prove that *every Bergman fan has a coarsest polyhedral structure* in 3.4.

In chapter 4 we put our focus on $\mathcal{M}_{0,n}^{\text{trop}}$, the moduli space of *rational tropical curves*. We define rational tropical curves and $\mathcal{M}_{0,n}^{\text{trop}}$ in 4.1. The next two sections are then dedicated to computing this space. While in theory $\mathcal{M}_{0,n}^{\text{trop}}$ can be computed as the Bergman fan of the complete graph K_{n-1} on $n-1$ vertices, this proves to be very inefficient. Instead we use the fact that trees on a fixed number of vertices are in bijection to so-called *Prüfer sequences*. They are introduced in 4.2 and it is shown that a special subset of these sequences is in bijection to combinatorial types of trivalent trees, i.e. to maximal cones of $\mathcal{M}_{0,n}^{\text{trop}}$. This provides a very efficient way of computing the moduli space and also gives us an easy proof of the well-known fact that the number of maximal cones of $\mathcal{M}_{0,n}^{\text{trop}}$ is the Schröder number $(2n-5)!!$. Since the combinatorics of Prüfer sequences very naturally correspond to the combinatorics of rational curves, we can also use them to compute certain subsets of $\mathcal{M}_{0,n}^{\text{trop}}$ (as long as they are defined by "combinatorial conditions"). We make use of this in 4.3 to compute *products of Psi-classes*. $\mathcal{M}_{0,n}^{\text{trop}}$ can appear in several different "coordinate systems": As a Bergman fan, as the space of leaf distances (i.e. as a space of tree metrics) or in terms of combinatorics, by specifying the partitions on $\{1,\dots,n\}$ induced by the bounded edges of a curve. Each of these representations has its merits, so one would like to be able to convert each description into one of the others. The isomorphism between the Bergman fan and the space of metrics has been discussed in [FR], so we focus on *obtaining the combinatorial description* from the metric in 4.4. Finally, we discuss what the *local picture* of $\mathcal{M}_{0,n}^{\text{trop}}$ looks like in 4.5. We prove that locally the space is again a Cartesian product of $\mathcal{M}_{0,m_i}^{\text{trop}}$'s for some $m_i \le n$.

In chapter 5 we discuss an alternative concept of tropical cycles that could for example be useful for computing push-forwards. Normally, one has to refine a tropical cycle before being able to compute a push-forward, such that its image is again a polyhedral complex. So far we do not know an efficient way to do this. The concept of *layerings* circumvents this problem: In 5.1 we define this to be a collection of polyhedra with the additional data of how these polyhedra "intersect": For two cells σ_i, σ_j we fix a common face τ_{ij} - which need not be the set-theoretic intersection! I.e. one can consider a layering to be a topological quotient space obtained by gluing the σ_i along their faces τ_{ij}. In 5.2 we show that one can still do meaningful tropical geometry on these objects. We define rational functions and their divisors and show that taking divisors commutes with "flattening" a layering to an actual tropical variety. Finally, we define morphisms of layerings and show how push-forwards can be computed very easily.

Summary of Part II

Chapter 6 mainly consists of definitions: We give a (very short) definition of algebraic Hurwitz cycles in 6.1. The notion of tropical stable maps is introduced in 6.2, which is essential for the tropical definition of *marked* and *unmarked Hurwitz cycles* (6.3). We give some examples to illustrate the definition, then go on to discuss how these cycles can be computed in 6.4.

Chapter 7 contains the results of this second part. It is concerned with studying several different properties of Hurwitz cycles. 7.1 is concerned with *irreducibility.* We first show that all Hurwitz cycles, marked or unmarked are connected in codimension one. This implies that for a generic choice of simple ramification points, all marked cycles are irreducible. However, we computationally find examples that show that in any other case, Hurwitz cycles are generally not irreducible. In fact, the experimental data motivates our conjecture that unmarked Hurwitz cycles are *never* irreducible - though we still lack a candidate for a canonical decomposition into irreducible parts. In 7.2 we show how to write the codimension one Hurwitz cycle as the *divisor of a rational function* if all ramification points are equal. This function can be considered in two ways: It has a very intuitive geometric interpretation, adding up the pairwise distances of vertex images in a cover. Alternatively, we can write it as the *push-forward* of the rational function cutting out the corresponding marked Hurwitz cycle. In fact, we use this notion to prove the statement. However, we can again use the computer to see that this statement cannot be easily generalized to higher codimension: For almost all examples, there is no rational function at all cutting out the codimension two cycle from the codimension one cycle. In 7.3 we consider an *alternative definition* of Hurwitz cycles. This definition comes from a naive tropicalization of an algebraic representation of Hurwitz cycles as linear combinations of boundary divisors. We show that the resulting polyhedral fan is equivalent to the tropical Hurwitz cycle - where the equivalence relation is induced by the Chow ring of the toric variety $X(\mathcal{M}_{0,n}^{\text{trop}})$.

Summary of Part III

In chapter 8 we describe the software `a-tint` in more detail and discuss how polyhedral complexes are represented in `polymake`. Chapter 9 contains some benchmarks. We compare two different methods to compute lattice normal vectors, show how divisor computation reacts to certain parameters and compare successive divisor computation to computation of intersection products. Finally, we compare the different methods of computing Bergman fans.

Note that we do not include a formal discussion of complexity issues. This is due to the fact that many of the algorithms underlying our computations - e.g. convex-hull algorithms or computation of Hermite normal forms - have a complexity that is difficult to predict. They can be exponential in worst-case scenarios but are often better. This makes any results on time complexity we could state either completely uninteresting or too involved for the scope of this thesis.

Published work and software

The software package `a-tint` can be obtained under `https://bitbucket.org/hampe/atint`

(see Chapter 8 for more details).

Most of Part I can be found in

> S. Hampe, *a-tint: A polymake extension for algorithmic tropical intersection theory*, European J. of Combinatorics, **36C** (2014), pp. 579-607

The results of Part II will be published on the `arXiv` in spring 2014.

1.3. Preliminaries

1.3.1. Polyhedra and polyhedral complexes

Let $\Lambda \cong \mathbb{Z}^n$ be a lattice and denote by $V_\Lambda = \Lambda \otimes \mathbb{R} \cong \mathbb{R}^n$ its associated vector space.

A *rational polyhedron* or *polyhedral cell* in V_Λ is a set of the form

$$\sigma = \{x : g_i(x) \geq \alpha_i; i = 1, \dots, s\}$$

where $g_i \in \Lambda^\vee$ and $\alpha_i \in \mathbb{R}$ for $i = 1, \dots, s$, i.e. it is an intersection of finitely many halfspaces. We call σ a *cone* if we can choose a representation such that $\alpha_i = 0$ for all i.

Equivalently, any polyhedron σ can be described as

$$\sigma = \operatorname{conv}\{p_1, \dots, p_k\} + \mathbb{R}_{\geq 0} r_1 + \dots + \mathbb{R}_{\geq 0} r_l + L$$

where $p_1, \dots, p_k, r_1, \dots, r_l \in \mathbb{R}^n$, L is a linear subspace of $\mathbb{R}^n$ and $+$ denotes the Minkowski sum of sets:

$$A + B = \{a + b; a \in A, b \in B\}.$$

The first description is often called an $\mathcal{H}$-description of σ and the second is a $\mathcal{V}$-description of σ.

It is a well known algorithmic problem in convex geometry to compute one of these descriptions from the other. In fact, both directions are computationally equivalent and there are several well-known *convex-hull-algorithms*. Most notable are the double-description method [MRTT], the reverse search method [AF] and the beneath-and-beyond algorithm (e.g. [G],[E]). Generally speaking, each of these algorithms behaves very well in terms of complexity for a certain class of polyhedra, but very badly for some other types (see [ABS] for a more in-depth discussion of this). Since in tropical geometry all kinds of polyhedra can occur, it is very difficult to pick an optimal algorithm. It is also an open problem, whether there exists a convex hull algorithm with polynomial complexity in both input and output. All of the above mentioned algorithms are implemented in `polymake` or in libraries used by `polymake`. At the moment, all algorithms in `a-tint` use the implementation of the double-description algorithm by Fukuda [F].

For any polyhedron σ we denote by V_σ the vector space associated to the affine space spanned by σ, i.e.

$$V_\sigma := \langle a-b; a,b \in \sigma\rangle$$

We denote by $\Lambda_\sigma := V_\sigma \cap \Lambda$ its associated lattice. The *dimension* of σ is the dimension of V_σ.

A *face* of σ is any subset τ that can be written as $\sigma \cap H$, where $H = \{x : g(x) = \lambda\}, g \in \Lambda^\vee, \lambda \in \mathbb{R}$ is an affine hyperplane such that σ is contained in one of the halfspaces $\{x : g(x) \geq \lambda\}$ or $\{x : g(x) \leq \lambda\}$ (i.e. we change one or more of the inequalities defining σ to an equality). If τ is a face of σ, we write this $\tau \leq \sigma$ (or $\tau < \sigma$ if the inclusion is proper). By convention we will also say that σ is a face of itself.

Finally, the *relative interior* of a polyhedron is the set

$$\operatorname{relint}(\sigma) := \sigma \smallsetminus \bigcup_{\tau<\sigma} \tau$$

A *polyhedral complex* is a set Σ of polyhedra that fulfills the following properties:

- For each $\sigma \in \Sigma$ and each face $\tau \leq \sigma$ we have $\tau \in \Sigma$
- For each two $\sigma, \sigma' \in \Sigma$, the intersection is a face of both.

If all of the polyhedra in Σ are cones, we call Σ a *fan*.

We will denote by $\Sigma^{(k)}$ the set of all k-dimensional polyhedra in Σ and set the dimension of Σ to be the largest dimension of any polyhedron in Σ. The set-theoretic union of all cells in Σ is denoted by $|\Sigma|$, the *support* of Σ. We call Σ *pure-dimensional* or *pure* if all inclusion-maximal cells are of the same dimension. The *lineality space* of Σ is the intersection of all its cells. We call Σ *rational* if all polyhedral cells are defined by inequalities $Ax \geq b$ with rational coefficient matrix A. If not explicitly stated otherwise, all complexes and fans in this paper will be pure and rational.

Note that a polyhedral complex is uniquely defined by giving all its top-dimensional cells. Hence we will often identify a polyhedral complex with its set of maximal cells.

A polyhedral complex Σ' is a *refinement* of a complex Σ, if $|\Sigma'| = |\Sigma|$ and each cell of Σ' is contained in a cell of Σ.

We will sometimes want to look at a complex Σ locally: Let $\tau \in \Sigma$ be any cell and $\Pi : V \to V/V_\tau$ the residue morphism. We define

$$\operatorname{Star}_\Sigma(\tau) := \{\mathbb{R}_{\geq 0} \cdot \Pi(\sigma - \tau); \tau < \sigma \in \Sigma\} \cup \{0\}$$

which is a fan in V/V_τ.

The *Cartesian product* of two polyhedral complexes Σ and Σ' is the polyhedral complex

$$\Sigma \times \Sigma' := \{\sigma \times \sigma'; \sigma \in \Sigma, \sigma' \in \Sigma'\}.$$

The last definition we need is the *normal fan* of a *polytope*, i.e. a bounded polyhedron: Let σ be a polytope, τ any face of σ. The *normal cone* of τ in σ is

$$N_{\tau,\sigma} := \overline{\{w \in \Lambda^\vee \otimes \mathbb{R} : w(t) = \max\{w(x); x \in \sigma\} \text{ for all } t \in \tau\}}$$

i.e. the closure of the set of all linear forms which take their maximum on τ. These sets are in fact cones and it is well known that they form a fan, the *normal fan* N_σ of σ.

1.3.2. Tropical geometry

Tropical cycles

Let X be a pure d-dimensional rational polyhedral complex in V_Λ. Let $\sigma \in X^{(d)}$ and assume $\tau \leq \sigma$ is a face of dimension $d-1$. The *primitive normal vector* of τ with respect to σ is defined as follows: By definition there is a linear form $g \in \Lambda^\vee$ such that its minimal locus on σ is τ. Then there is a unique generator of $\Lambda_\sigma / \Lambda_\tau \cong \mathbb{Z}$, denoted by $u_{\sigma/\tau}$, such that $g(u_{\sigma/\tau}) > 0$.

A *tropical cycle* (X, ω) is a pure rational d-dimensional complex X together with a *weight function* $\omega : X^{(d)} \to \mathbb{Z}$ such that for all codimension one faces $\tau \in X^{(d-1)}$ it fulfills the *balancing condition*:

$$\sum_{\sigma > \tau} \omega(\sigma) u_{\sigma/\tau} = 0 \in V/V_\tau$$

We call X a *tropical variety* if furthermore all weights are positive.

Two tropical cycles are considered *equivalent* if their polyhedral structures have a common refinement that respects both weight functions. Usually, tropical cycles are only considered modulo equivalence, but we will sometimes distinguish between a tropical cycle X and a specific polyhedral structure $\mathcal{X}$.

We also want to define the local picture of $\mathcal{X}$ around a given cell: $\mathrm{Star}_\mathcal{X}(\tau)$ is a fan in V/V_τ (on the lattice Λ/Λ_τ). If we equip it with the weight function $\omega_{\mathrm{Star}}(\mathbb{R}_{\geq 0} \cdot \Pi(\sigma - \tau)) = \omega_X(\sigma)$ for all maximal σ, then $(\mathrm{Star}_\mathcal{X}(\tau), \omega_{\mathrm{Star}})$ is a tropical fan cycle. It is now easy to see that a weighted complex $\mathcal{X}$ is balanced around a codimension cell τ, if and only if the one-dimensional fan $\mathrm{Star}_\mathcal{X}(\tau)$ is balanced. An example for this construction can be found in Figure 1.1.

Convention. Throughout this thesis, if not stated otherwise, we will always assume that Λ is the standard lattice $\mathbb{Z}^n$ and $V_\Lambda = \mathbb{R}^n$ in its usual coordinates.

Tropical morphisms

A *morphism* of tropical cycles $f : X \to Y$ is a map from $|X|$ to $|Y|$ which is locally a linear map and respects the lattice, i.e. maps Λ_X to Λ_Y.

The *push-forward* of X is defined as follows: By [GKM, Construction 2.24] there exists a refinement $\mathcal{X}$ of the polyhedral structure on X such that $\{f(\sigma); \sigma \in \mathcal{X}\}$ is a polyhedral complex. We then set

$$f_*(X) = \{f(\sigma); \sigma \in \mathcal{X}^{(\dim X)}; f \text{ injective on } \sigma\}$$

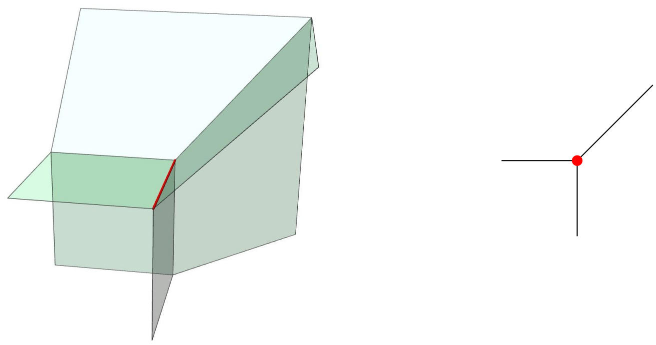

Figure 1.1.: A tropical plane L and its local picture $\mathrm{Star}_L(\tau)$, where τ is the codimension one face marked in red.

(recall that defining the maximal cells is sufficient to define a complex) with weights

$$\omega_{f_*(X)}(f(\sigma)) = \sum_{\sigma' : f(\sigma') = f(\sigma)} \left|\Lambda_{f(\sigma)}/f(\Lambda_{\sigma'})\right| \omega_X(\sigma').$$

It is shown in [GKM, Proposition 2.25] that this gives a tropical cycles and does not depend on the choice of $\mathcal{X}$.

Recession fans and rational equivalence

The *recession cone* of a polyhedron $\sigma \subseteq \mathbb{R}^n$ is the set

$$\mathrm{rec}(\sigma) := \{v \in \mathbb{R}^n ; \exists x \in \sigma \text{ such that } x + \mathbb{R}_{\geq 0} v \subseteq \sigma\}.$$

If X is a tropical cycle, then by there exists a refinement $\mathcal{X}$ of its polyhedral structure such that $\delta(X) := \{\mathrm{rec}(\sigma); \sigma \in \mathcal{X}\}$ is a polyhedral fan (One can use a construction similar to the one used for defining push-forwards). If we define a weight function

$$\omega_\delta(\mathrm{rec}(\sigma)) := \sum_{\sigma' : \mathrm{rec}(\sigma') = \mathrm{rec}(\sigma)} \omega_X(\sigma'),$$

then $(\delta(X), \omega_\delta)$ is a tropical cycle by [R1, Theorem 1.4.12].

We call two tropical cycles *rationally equivalent* if $\delta(X) = \delta(Y)$ (up to refinement, of course).

Part I.

Algorithms

2. Basic computations in tropical geometry

2.1. Primitive normal vectors and lattice bases

The primitive normal vector $u_{\sigma/\tau}$ defined in the previous section is an essential part of most formulas and calculations in tropical geometry. Hence we will need an algorithm to compute it. An important tool in this computation is the *Hermite normal form* of an integer matrix:

Definition 2.1.1. Let $M \in \mathbb{Z}^{m\times n}$ be a matrix with $n \geq m$ and assume M has full rank m. We say that M is in *Hermite normal form* (HNF) if it is of the form

$$M = (0_{m\times(n-m)}, T)$$

where $T = (t_{i,j})$ is an upper triangular matrix with $t_{i,i} > 0$ and for $j > i$ we have $t_{i,i} > t_{i,j} \geq 0$.

Remark 2.1.2. We are actually only interested in the fact that T is an upper triangular invertible matrix. Furthermore, it is known that for any $A \in \mathbb{Z}^{m\times n}$ of full rank there exists a $U \in \mathrm{GL}_n(\mathbb{Z})$ such that $B = AU$ is in HNF (see for example [C, 2.4]).

Proposition 2.1.3. *Let σ be a d-dimensional polyhedron in $\mathbb{R}^n$ and τ a codimension one face of σ. Let $A \in \mathbb{Z}^{(n-d+1)\times n}$, such that $V_\tau = \ker A$ and $V_\sigma = \ker \tilde{A}$, where $\tilde{A}$ denotes A without its first row. Let $U \in \mathrm{GL}_n(\mathbb{Z})$ such that*

$$AU = (0_{(n-d+1)\times(d-1)}, T)$$

is in HNF. Denote by U_{*i} the i-th column of U. Then:

1. *$U_{*1}, \ldots, U_{*d-1}$ is a lattice basis for Λ_τ.*
2. *$U_{*1}, \ldots, U_{*d}$ is a lattice basis for Λ_σ.*

In particular $U_{*d} = \pm u_{\sigma/\tau}$ mod V_τ.

Proof. It is clear that $U_{*1}, \ldots, U_{*d-1}$ form an $\mathbb{R}$-basis for $\ker A$ and the fact that $\det U = \pm 1$ ensures that it is a lattice basis. Removing the first row of A corresponds to removing the first row of AU so we obtain an additional column of zeros. Hence $U_{*1}, \ldots, U_{*d}$ is a basis of $\ker \tilde{A}$ and U_{*d} is a generator of $\Lambda_\sigma/\Lambda_\tau$. □

Remark 2.1.4. In [C, 2.4.3], Cohen suggests an algorithm for computing the HNF of a matrix using integer Gaussian elimination. However, he already states that this algorithm is useless for practical applications, since the coefficients in intermediate steps of the computation explode too quickly. A more practical solution is an LLL-based normal form algorithm that reduces the coefficients in between elimination steps. `a-tint` uses an implementation based on the algorithm designed by Havas, Majevski and Matthews in [HMM].

Note that, knowing the primitive normal vector up to sign, it is easy to determine its final form, since we know that the linear form defined by $u_{\sigma/\tau}$ must be positive on σ. So we only have to compute the scalar product of U_{*d} with any ray in σ that is not in τ and check if it is positive.

We now present an algorithm to compute the primitive normal vector with respect to a cone. The algorithm can easily be adapted to the general case of polyhedra, though this involves some technicalities we wish to avoid here.

Algorithm 1 PrimitiveNormalVector

Input: A d-dimensional cone σ and a $(d-1)$-dimensional face $\tau < \sigma$, given in terms of their rays and lineality space:

$$\sigma = \mathbb{R}_{\geq 0} r_1 + \cdots + \mathbb{R}_{\geq 0} r_s + L$$
$$\tau = \mathbb{R}_{\geq 0} r_1 + \cdots + \mathbb{R}_{\geq 0} r_t + L, \text{ for some } t < s$$

Output: The primitive normal vector $u_{\sigma/\tau}$.
Compute matrices $M_\sigma = \ker \langle r_1, \ldots, r_s, L\rangle, M_\tau = \ker \langle r_1, \ldots, r_t, L\rangle$.
(i.e. each matrix contains a basis of the respective kernel as row vectors).
Find a row z of M_τ, such that $z \notin \ker \langle r_1, \ldots, r_s, L\rangle$.
Let A be the matrix obtained by attaching z to the top of M_σ.
Compute a transformation matrix $U \in \mathrm{GL}_n(\mathbb{Z})$, such that AU is in Hermite normal form.
Let $u := U_{*d}$.
Let r be any ray of σ that is not in τ.
Let $\delta = \mathrm{sign}(\langle u, r\rangle)$.
return $\delta \cdot u$.

The above results now also allow us to compute the lattice spanned by a cone:

2.1.1. Lattice normal computation via projections

There is an additional trick that can make lattice normal computations speed up by a large factor:

Assume we want to compute normal vectors of a d-dimensional tropical variety in $\mathbb{R}^n$.

Algorithm 2 LATTICEBASIS

1: **Input:** A d-dimensional cone σ, given in terms of its rays and lineality space: $\sigma = \mathbb{R}_{\geq 0} r_1 + \dots + \mathbb{R}_{\geq 0} r_s + L$
2: **Output:** A $\mathbb{Z}$-basis for Λ_σ.
3: Let $M_\sigma = \ker \langle r_1, \dots, r_s, L \rangle$.
4: Compute a transformation matrix $U \in \mathrm{GL}_n(\mathbb{Z})$, such that $M_\sigma U$ is in Hermite normal form.
5: **return** $U_{*1}, \dots, U_{*d}$.

Let N be the number of maximal cones and for the sake of simplicity assume that each cone has the same number T of codimension one faces. In this case we would have to compute the HNF of $N \cdot T$ matrices with $d+1$ rows and n columns each. An alternative approach would be the following: First compute a lattice basis B_σ for each maximal cone σ. For each lattice normal $u_{\sigma/\tau}$ we now use B_σ to define the following projection:

Assume $B_\sigma = \{b_1, \dots, b_d\}$ and denote the corresponding matrix by

$$M_\sigma = (b_1, \dots, b_d).$$

We assume without restriction that

$$M_\sigma^d := \begin{pmatrix} b_1^{(1)} & \dots & b_d^{(1)} \\ & \dots & \\ b_1^{(d)} & \dots & b_d^{(d)} \end{pmatrix}$$

has full rank d (where $b_i^{(j)}$ is the j-th coordinate of b_i). Let $R_\sigma = (M_\sigma^d)^{-1}$. The projection matrix is now

$$P := \begin{pmatrix} R_\sigma & 0 & \dots & 0 \end{pmatrix} \in \mathbb{Z}^{d \times n}.$$

Since $P(b_i) = e_i$, the standard unit vector, P is a lattice isomorphism on Λ_σ and we obtain the following result:

Lemma 2.1.5. *Using the notation above, the lattice normal can be computed as*

$$u_{\sigma/\tau} = P^{-1}(u_{P(\sigma)/P(\tau)}) = B_\sigma \cdot u_{P(\sigma)/P(\tau)}.$$

Since $P(\tau)$ is defined by just one equation, computing $u_{P(\sigma)/P(\tau)}$ boils down to computing the HNF of a $(1 \times n)$-matrix, which is just an extended Euclidean algorithm. Summarizing, we see that the alternative approach requires us to compute:

- N HNFs of $(d \times n)$-matrices
- N inverses of $(d \times d)$-matrices
- $N \cdot T$ extended Euclidean algorithms for d integers.

Although the complexity of the HNF algorithm is difficult to predict (only the maximal complexity of the final entries has been studied in [VDK]), it is easy to see that this should be much faster if $(n - d)$ is large. A table comparing the performance of the two methods can be found in the Appendix in Section 9.

polymake example: Computing a lattice normal.
This creates two cones in $\mathbb{R}^2$, a two-dimensional cone σ and one if its codimension one faces. It then computes and prints the corresponding lattice normal.

```
atint >  $sigma = new polytope::Cone(RAYS=>[[1,2],[1,0]]);
atint >  $tau = new polytope::Cone(RAYS=>[[1,2]]);
atint >  print latticeNormalByCone($tau,$sigma);
0 -1
```

2.2. Divisors of rational functions

The most basic operation in tropical intersection theory is the computation of the divisor of a rational function. Let us first discuss how we define a rational function and its divisor. Our definition is the same as in [AR2]:

Definition 2.2.1. Let X be a tropical variety. A *rational function* on X is a continuous function $\varphi : X \to \mathbb{R}$ that is affine linear with integer slope on each cell of some arbitrary polyhedral structure $\mathcal{X}$ of X.

The *divisor* of φ on X, denoted by $\varphi \cdot X$, is defined as follows: Choose a polyhedral structure $\mathcal{X}$ of X such that φ is affine linear on each cell. Let $\mathcal{X}' = \mathcal{X}^{(\dim X - 1)}$ be the codimension one skeleton. For each $\tau \in \mathcal{X}'$, we define its weight via

$$\omega_{\varphi \cdot X}(\tau) = \left(\sum_{\sigma > \tau} \omega(\sigma) \varphi_\sigma(u_{\sigma/\tau}) \right) - \varphi_\tau \left(\sum_{\sigma > \tau} \omega(\sigma) u_{\sigma/\tau} \right)$$

where φ_σ and φ_τ denote the linear part of the restriction of φ to the respective cell. Then

$$\varphi \cdot X := (\mathcal{X}', \omega_{\varphi \cdot X})$$

Remark 2.2.2. While the computation of the weights on the divisor is relatively easy to implement, the main problem is computing the appropriate polyhedral structure. The most general form of a rational function φ on some cycle X would be given by its domain, a polyhedral complex Y with $|X| \subseteq |Y|$ together with the values and slopes of φ on the vertices and rays of Y. To make sure that φ is affine linear on each cell of X, we then have to compute the intersection of the complexes, which boils down to computing the pairwise intersection of all maximal cones of X and Y. Here lies the main problem of computing divisors: One usually computes the intersection of two cones by converting them to an $\mathcal{H}$-description and converting the joint description back to a $\mathcal{V}$-description via some convex hull algorithm. But as we discussed earlier, so far no convex hull algorithm is known that has polynomial runtime for all polyhedra. Also, [T] shows that computing the intersection of two $\mathcal{V}$-polyhedra is NP-complete.

Hence we already see a crucial factor for computing divisors (besides the obvious ones: dimension and ambient dimension): The number of maximal cones of the tropical cycle and the domain of the rational function. Table 9.2 in the appendix shows how divisor computation is affected by these parameters.

Example 2.2.3. The easiest example of a rational function is a *tropical polynomial*

$$\varphi(x) = \max\{\langle v_i, x\rangle + \alpha_i; i = 1, \dots, r\}$$

with $v_i \in \mathbb{Z}^n, \alpha_i \in \mathbb{R}$. To this function, we can associate its *Newton polytope*

$$P_\varphi = \text{conv}\{(\alpha_i, v_i); i = 1, \dots, r\} \subseteq \mathbb{R}^{n+1}$$

Denote by N_φ its normal fan and let $N_\varphi^1 := N_\varphi \cap \{x : x_0 = 1\}$ be the associated complete polyhedral complex in $\mathbb{R}^n$. Then it is easy to see that φ is affine linear on each cell of this complex. In fact, each cone in the normal fan consists of those vectors maximizing a certain subset of the linear functions $\langle v_i, (x_1, \dots, x_n)\rangle + \alpha_i \cdot x_0$ at the same time.

So for any tropical polynomial φ and any tropical variety X we can compute an appropriate polyhedral structure on X by intersecting it with N_φ^1. An example is given in Figure 2.1.

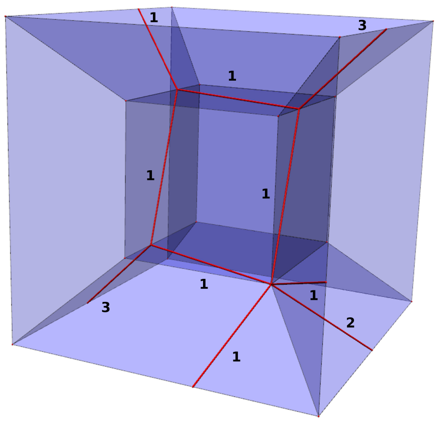

Figure 2.1.: The surface is $X = \max\{1, x, y, z, -x, -y, -z\} \cdot \mathbb{R}^3$ with weights all equal to 1. The curve is $\max\{3x + 4, x - y - z, y + z + 3\} \cdot X$, the weights are indicated by the labels.

polymake example: Computing a divisor.
This computes the divisors displayed in figure 2.1.

```
atint >  $f = new MinMaxFunction(
         INPUT_STRING=>"max(1,x,y,z,-x,-y,-z)");
atint >  $x = divisor(linear_nspace(3),$f);
atint >  $g = new MinMaxFunction(
         INPUT_STRING=>"max(3x+4,x-y-z,y+z+3)");
atint >  $c = divisor($x,$g);
```

Remark 2.2.4. `a-tint` can of course also handle more general rational functions. Given a tropical cycle X, one can define a rational function φ on X by giving:

- A polyhedral complex Y such that $|X| \subseteq |Y|$, the *domain* of φ.
- A list of the values φ takes on the rays and vertices of Y. Note that φ is supposed to be affine linear on each cell of Y, though `a-tint` does not check this explicitly.

2.2.1. The inverse divisor problem

When given a tropical variety X and a codimension one subcycle Y of X, a natural question to ask is whether there exists a rational function φ on X, such that $Y = \varphi \cdot X$.

It is rather obvious that this cannot always be the case. E.g., consider a tropical curve X of topological genus $g > 1$ and let Y be any point lying on an interior edge of a cycle of this curve. Any rational function on X has to break at least twice on this cycle, so its divisor can never be only Y.

However, when X and Y are given in such a way that the polyhedral structure $\mathcal{Y}$ of Y is also a subcomplex of the polyhedral structure $\mathcal{X}$ of X, it is very easy to answer this question computationally:

Let τ be a codimension one face of a maximal cone σ of X. Assume σ has vertices $p_0^\sigma, \dots, p_{n_\sigma}^\sigma$ and rays $r_1^\sigma, \dots, r_{m_\sigma}^\sigma$. We can express (a representative of) the lattice normal vector $u_{\sigma/\tau}$ in the form

$$u_{\sigma/\tau} = \sum_{i=1}^{n_\sigma} \lambda_i^\sigma (p_i^\sigma - p_0^\sigma) + \sum_{j=1}^{m_\sigma} \mu_j^\sigma r_j^\sigma; \qquad \lambda_i^\sigma, \mu_j^\sigma \in \mathbb{R}$$

by a simple computation in linear algebra. Similarly, if p_i^τ, r_j^τ are the vertices and rays of τ, we can find a similar expression for

$$\sum_{\sigma > \tau} \omega_X(\sigma) u_{\sigma/\tau} = \sum \lambda_i^\tau (p_i^\tau - p_0^\tau) + \sum \mu_j^\tau r_j^\tau$$

Any rational function on X whose divisor is Y is now given by values $\varphi(p_i^\sigma), \varphi(r_j^\sigma)$

fulfilling

$$\omega_Y(\tau) = \left(\sum_{\sigma>\tau} \omega_X(\sigma) \left(\sum_{i=1}^{n_\sigma} \lambda_i^\sigma (\varphi(p_i^\sigma) - \varphi(p_0^\sigma)) + \sum_{j=1}^{m_\sigma} \mu_j^\sigma \varphi(r_j^\sigma)\right)\right)$$
$$- \left(\sum_{i=1}^{n_\tau} \lambda_i^\tau (\varphi(p_i^\tau - p_0^\tau)) + \sum_{j=1}^{m_\tau} \mu_j^\tau \varphi(r_j^\tau)\right)$$

for all τ. This is an affine linear equation in the values of φ, thus defining an affine linear hyperplane in $\mathbb{R}^N$, where N is the number of rays and vertices of $\mathcal{X}$. By intersecting all these hyperplanes, we obtain the space of all rational functions defined on $\mathcal{X}$ whose divisor is Y.

2.3. Weight spaces and irreducibility

A property of classical varieties that one is often interested in is irreducibility and a decomposition into irreducible components. While one can easily define a concept of irreducible tropical cycles, there is in general no unique decomposition (see Figure 2.2). We can, however, still ask whether a cycle is irreducible and what the possible decompositions are.

Definition 2.3.1. We call a d-dimensional tropical cycle X *irreducible* if any other d-dimensional cycle Y with $|Y| \subseteq |X|$ is an integer multiple of X.

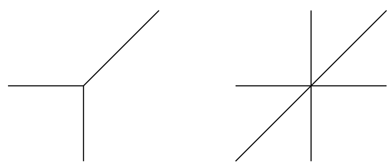

Figure 2.2.: The curve on the left is irreducible. The curve on the right is reducible and there are several different ways to decompose it.

To compute whether a cycle is irreducible, we have to introduce a few notations:

Definition 2.3.2. Let $\mathcal{X}$ be a pure rational polyhedral complex. Let N be the number of maximal cells $\sigma_1, \dots, \sigma_N$ of $\mathcal{X}$. We identify an integer vector $\omega \in \mathbb{Z}^N$ with the weight function $\sigma_i \mapsto \omega_i$. We define

- $\Omega_{\mathcal{X}} := \{\omega \in \mathbb{Z}^N : (\mathcal{X}, \omega) \text{ is balanced}\}$ (which is a lattice).
- $W_{\mathcal{X}} := \Omega_{\mathcal{X}} \otimes \mathbb{R}$

Now fix a codimension one cell τ in $\mathcal{X}$. Let $\mathcal{T}$ be the induced polyhedral structure of $\mathrm{Star}_{\mathcal{X}}(\tau)$. For an integer vector $\omega \in \mathbb{Z}^N$, we denote by $\omega_{\mathcal{T}}$ the induced weight function on $\mathcal{T}$.

We then define

- $\Omega^\tau_{\mathcal{X}} := \{\omega \in \mathbb{Z}^N : (\mathcal{T}, \omega_{\mathcal{T}}) \text{ is balanced}\}$
- $W^\tau_{\mathcal{X}} := \Omega^\tau_{\mathcal{X}} \otimes \mathbb{R}$

Remark 2.3.3. We obviously have $\Omega_{\mathcal{X}} = \bigcap_{\tau \in \mathcal{X}^{(\dim X-1)}} \Omega^\tau_{\mathcal{X}}$ and similarly for $W_{\mathcal{X}}$. Now let X be a tropical cycle with polyhedral structure $\mathcal{X}$. Clearly, if X is irreducible, then $\dim W_{\mathcal{X}}$ should be 1 and vice versa (assuming that $\gcd(\omega_1, \dots, \omega_N) = 1$, where the ω_i are the weights on $\mathcal{X}$). However, so far this definition is tied to the explicit choice of the polyhedral structure. We would like to get rid of this restriction, which we can do using Lemma 2.3.6. Hence we will also write W_X and Ω_X. We call W_X the *weight space* and Ω_X the *weight lattice* of X.

Definition 2.3.4. Let $\mathcal{X}$ be a d-dimensional pure rational polyhedral complex. We define an equivalence relation on the maximal cells of $\mathcal{X}$ in the following way: Two maximal cells σ, σ' are equivalent if and only if there exists a sequence of maximal cells $\sigma = \sigma_0, \dots, \sigma_r = \sigma'$, $\sigma_i \in \mathcal{X}^{(d)}$ such that for all $i = 0, \dots, r-1$, the intersection $\sigma_i \cap \sigma_{i+1}$ is a codimension one cell of $\mathcal{X}$, whose only adjacent maximal cells are σ_i and σ_{i+1}.

We denote the set of equivalence classes of this relation, which we call *subdivision classes*, by $\mathcal{S}(\mathcal{X})$. Furthermore, we write $\mathcal{S}(\sigma)$ for the subdivision class containing a given maximal cell σ.

Lemma 2.3.5. *Let (X, ω) be a tropical cycle with polyhedral structure $\mathcal{X}$ and assume σ, σ' are equivalent maximal cells of $\mathcal{X}$. Then:*

1. *$\omega(\sigma) = \omega(\sigma')$.*
2. *If $\omega(\sigma) \neq 0$, then $V_\sigma = V_{\sigma'}$.*

Proof.

We can assume that $\sigma \cap \sigma' =: \tau \in \mathcal{X}^{(\dim X - 1)}$.

1. X is balanced at τ if and only if $\operatorname{Star}_{\mathcal{X}}(\tau)$ is balanced, which is a one-dimensional fan with exactly two rays. Such a fan can only be balanced if the weights of the two rays are equal.
2. Choose any representatives $v_{\sigma/\tau}, v_{\sigma'/\tau}$ of the lattice normal vectors. Then

$$\omega(\sigma) v_{\sigma/\tau} + \omega(\sigma') v_{\sigma'/\tau} \in V_\tau$$

Let $g_1, \dots, g_r \in \Lambda^\vee$ such that

$$V_\sigma = \ker \begin{pmatrix} g_1 \\ \vdots \\ g_r \end{pmatrix}$$

Since $V_\tau \subseteq V_\sigma$, we have for all i:

$$\begin{aligned} 0 &= g_i(\omega(\sigma) v_{\sigma/\tau} + \omega(\sigma') v_{\sigma'/\tau'}) \\ &= \omega(\sigma') g_i(v_{\sigma'/\tau'}) \end{aligned}$$

Now $\omega(\sigma') = \omega(\sigma) \neq 0$ implies $v_{\sigma'/\tau} \in V_\sigma$ and since $V_{\sigma'} = V_\tau \times \langle v_{\sigma'/\tau} \rangle$, we have $V_{\sigma'} \subseteq V_\sigma$. The other inclusion follows analogously.

□

Lemma 2.3.6. *Let $\mathcal{X}$ and $\mathcal{X}'$ be two pure and rational polyhedral complexes, which have a common refinement. Then $\Omega_{\mathcal{X}} \cong \Omega_{\mathcal{X}'}$.*

Proof. We can assume without loss of generality that $\mathcal{X}'$ is a refinement of $\mathcal{X}$. Denote by $\{\sigma_1, \dots, \sigma_N\}$ and $\{\sigma'_1, \dots, \sigma'_{N'}\}$ the maximal cells of $\mathcal{X}$ and $\mathcal{X}'$, respectively. First of all, assume two maximal cones of $\mathcal{X}'$ are contained in the same maximal cone of $\mathcal{X}$. Since subdividing a polyhedral cell produces equivalent cells in terms of definition 2.3.4, they must have the same weight in any $\omega' \in \Omega_{\mathcal{X}'}$ by Lemma 2.3.5. Thus the following map is well-defined: We partition $\{1, \dots, N'\}$ into sets $S_1, \dots, S_N$ such that $j \in S_i \iff \sigma'_j \subseteq \sigma_i$ (where σ'_j and σ_i are maximal cells of $\mathcal{X}'$ and $\mathcal{X}$, respectively). Pick representatives $\{j_1, \dots, j_N\}$ from each partitioning set S_i and let $p : \Omega_{\mathcal{X}'} \to \mathbb{Z}^N$ be the projection on these coordinates j_k. By the previous considerations, the map does not depend on the choice of representatives. We claim that $\mathrm{Im}(p) \subseteq \Omega_{\mathcal{X}}$: Let τ be a codimension one cell of $\mathcal{X}$ and τ' any codimension one cell of $\mathcal{X}'$ contained in τ. Then $\mathrm{Star}_{\mathcal{X}}(\tau) = \mathrm{Star}_{\mathcal{X}'}(\tau')$, so if $\omega \in \mathbb{Z}^{N'}$ makes $\mathcal{X}'$ balanced around τ', then $p(\omega)$ makes $\mathcal{X}$ balanced around τ. Bijectivity of p is obvious, so $\Omega_{\mathcal{X}} \cong \Omega_{\mathcal{X}'}$.

□

The following is now obvious:

Theorem 2.3.7. *Let (X, ω) be a d-dimensional tropical cycle. Then X is irreducible if and only if $g := \gcd(\omega(\sigma), \sigma \in X^{(d)}) = 1$ and $\dim W_X = 1$.*

After having laid out these basics, we want to see how we can actually compute this weight space:

Proposition 2.3.8. *Let τ be a codimension one cell of a d-dimensional pure complex $\mathcal{X}$ in $\mathbb{R}^n$. Let $u_1, \dots, u_k \in \mathbb{Z}^n$ be representatives of the normal vectors $u_{\sigma/\tau}$ for all $\sigma > \tau$. Also, choose a lattice basis $l_1, \dots, l_{d-1}$ of Λ_τ. We define the following matrix:*

$$M_\tau := (u_1 \ \dots \ u_k \ l_1 \ \dots \ l_{d-1}) \in \mathbb{Z}^{n \times (k+d-1)}$$

Then $W_{\mathcal{X}}^\tau \cong \pi(\ker(M_\tau)) \times \mathbb{R}^{(N-k)}$, where π is the projection onto the first k coordinates and N is again the number of maximal cells in $\mathcal{X}$.

Proof. Let $\{\sigma_1, \dots, \sigma_N\}$ be the maximal cells of $\mathcal{X}$ and define

$$J := \{j \in [N] : \tau \text{ is not a face of } \sigma_j\}.$$

Then clearly $\mathbb{R}^{(N-k)} \cong \langle e_j; j \in J \rangle_{\mathbb{R}} \subseteq W_{\mathcal{X}}^\tau$ and it is easy to see that $W_{\mathcal{X}}^\tau$ must be isomorphic to $\mathbb{R}^{(N-k)} \times W_{\mathrm{Star}_{\mathcal{X}}(\tau)}$. Hence it suffices to show that $W_{\mathrm{Star}_{\mathcal{X}}(\tau)}$ is isomorphic to $\pi(\ker(M_\tau)$.

Let $(a_1,\ldots,a_k,b_1,\ldots,b_l) \in \ker(M_\tau) \cap \mathbb{Z}^{k+d-1}$, Then $\sum a_i u_i = \sum(-b_i) l_i \in \Lambda_\tau$, so $\text{Star}_{\mathcal{X}}(\tau)$ is balanced if we assign weights a_i. In particular $(a_1,\ldots,a_k) \in \Omega_{\text{Star}_{\mathcal{X}}(\tau)}$. Since $l_1,\ldots,l_{d-1}$ are a lattice basis, any choice of the a_i such that $\text{Star}_{\mathcal{X}}(\tau)$ is balanced fixes the b_i uniquely, so π is injective on $\ker(M_\tau)$ and surjective onto $W_{\text{Star}_{\mathcal{X}}(\tau)}$. □

A naive algorithm to compute $W_{\mathcal{X}}$ would now be to compute all $W^\tau_{\mathcal{X}}$ using Proposition 2.3.8 and take their intersection. However, we already saw in the proof of Lemma 2.3.6 that this would produce redundant information: Any codimension one face that is contained in exactly two maximal cones σ, σ' only provides the information that the weights of these two cones must be equal (and possibly that they must be zero if $V_\sigma \neq V_{\sigma'}$). Let us see how we can make use of this:

Definition 2.3.9. Let $\mathcal{X}$ be a pure polyhedral complex and let $\mathcal{S} := \mathcal{S}(\mathcal{X})$ be its set of subdivision classes. We set

$$\mathcal{S}_1 := \{S \in \mathcal{S}; V_\sigma = V_{\sigma'} \text{ for all } \sigma, \sigma' \in S\},$$
$$\mathcal{S}_0 := \mathcal{S} \smallsetminus \mathcal{S}_1.$$

Then we define the *subdivision signature* of a codimension one cell τ as

$$\text{sig}(\tau) := \{S \in \mathcal{S}_1 : \exists! \sigma > \tau \text{ with } \sigma \in S\}$$

and its *signature neighbors* as

$$\text{nsig}(\tau) := \{\sigma > \tau; \mathcal{S}(\sigma) \in \text{sig}(\tau)\}.$$

The *signature fan* of τ is the one-dimensional fan $\text{sigfan}(\tau)$ in $\mathbb{R}^n/V_\tau$ with rays $u_{\sigma/\tau}, \sigma \in \text{nsig}(\tau)$ and we call its weight space $W_{\text{sigfan}(\tau)}$ the *signature weight space* of τ.

Example 2.3.10. Assume τ is a codimension one cell lying in exactly two maximal cells σ, σ', In particular, both cells lie in the same subdivision class and the signature fan of τ is empty. This agrees with our notion that τ does not give any relevant information concerning balancing - except that σ and σ' must have equal weight or weight 0, if $V_\sigma \neq V_{\sigma'}$.

Note that for any τ and any subdivision class $S \in \mathcal{S}_1$, there can be at most two cells $\sigma, \sigma' \in S$, such that τ is a face of both. In that case $u_{\sigma/\tau} = -u_{\sigma/\tau}$, so the terms coming from σ and σ' vanish in the balancing equation at τ.

Remark 2.3.11. Note that $W_{\text{sigfan}(\tau)}$ can be computed in a similar fashion as in Proposition 2.3.8: Simply remove all columns u_i from M_τ that do not belong to signature neighbors. If we call the resulting matrix M^{sig}_τ, we obviously have

$$W_{\text{sigfan}(\tau)} = \pi(\ker(M^{\text{sig}}_\tau)),$$

where π is again the projection onto the coordinates corresponding to the lattice normals.

The idea is now that, since cells in the same subdivision class have the same weight, we might as well compute $W_{\mathcal{X}}$ as a subspace of $\mathbb{R}^{|\mathcal{S}|}$ instead of $\mathbb{R}^N$. Besides reducing the dimension of the space, we also remove all the redundant information about weights being equal (which we already know from the combinatorics of the complex). We define

$$W_{\mathcal{X}}^{\mathrm{sig}(\tau)} := W_{\mathrm{sigfan}(\tau)} \times \mathbb{R}^{(|\mathcal{S}|-|\mathrm{sig}(\tau)|)}$$

and consider it as a subspace of $\mathbb{R}^{\mathcal{S}}$ in the obvious manner. Then the precise statement is the following:

Proposition 2.3.12. *Let $\mathcal{X}$ be a pure d-dimensional polyhedral complex. Then*

$$W_{\mathcal{X}} \cong \left(\bigcap_{\tau \in \mathcal{X}^{(d-1)}} W_{\mathcal{X}}^{\mathrm{sig}(\tau)} \right) \cap \bigcap_{S \in \mathcal{S}_0} \{x_S = 0\} =: W_{\mathcal{S}} \subseteq \mathbb{R}^{\mathcal{S}}.$$

The actual isomorphism is

$$s_{\mathcal{X}} : (\omega_\sigma)_{\sigma \in \mathcal{X}^{(d)}} \mapsto (\omega_S := \omega_\sigma \text{ for some } \sigma \in S)_{S \in \mathcal{S}}$$

with inverse map

$$t_{\mathcal{X}} : (\omega_S)_{S \in \mathcal{S}} \mapsto (\omega_\sigma := \omega_{\mathcal{S}(\sigma)})_{\sigma \in \mathcal{X}^{(d)}}.$$

Proof. It is obvious that $s_{\mathcal{X}}$ and $t_{\mathcal{X}}$ are inverse to each other, so we only need to show that they are well-defined.

Let $\omega \in W_{\mathcal{X}}$. If $S \in \mathcal{S}_0$, then $\omega_\sigma = 0$ for any $\sigma \in S$ by Lemma 2.3.5. Hence we only need to show that for each codimension one cell τ the weight function induced by $s_{\mathcal{X}}(\omega)$ makes $\mathrm{sigfan}(\tau)$ balanced. To see this, note the following: Assume $\sigma > \tau$, but $\sigma \notin \mathrm{nsig}(\tau)$. If $\mathcal{S}(\sigma) \in \mathcal{S}_0$, we must have $\omega_\sigma = 0$. If $\mathcal{S}(\sigma) \in \mathcal{S}_1$, there must be exactly one other cone $\sigma' \in \mathcal{S}(\sigma)$ such that $\sigma' > \tau$. In this case $u_{\sigma'/\tau} = -u_{\sigma/\tau}$. So we see that the balancing equation at τ splits up as

$$\begin{aligned} V_\tau \ni & \sum_{\sigma > \tau} \omega_\sigma u_{\sigma/\tau} \\ = & \sum_{\substack{\sigma > \tau \\ \sigma \in \mathrm{nsig}(\tau)}} \omega_\sigma u_{\sigma/\tau} + \sum_{\substack{\sigma > \tau \\ \sigma \notin \mathrm{nsig}(\tau) \\ \mathcal{S}(\sigma) \in \mathcal{S}_0}} 0 + \sum_{\substack{\sigma, \sigma' > \tau \\ \mathcal{S}(\sigma) = \mathcal{S}(\sigma') \in \mathcal{S}_1}} \omega_\sigma \cdot (u_{\sigma/\tau} + u_{\sigma'/\tau}) \\ = & \sum_{\substack{\sigma > \tau \\ \sigma \in \mathrm{nsig}(\tau)}} \omega_\sigma u_{\sigma/\tau}, \end{aligned}$$

which is just the balancing equation of $\mathrm{sigfan}(\tau)$.

The same argument shows that $t_{\mathcal{X}}(V_{\mathcal{S}}) \subseteq W_{\mathcal{X}}$. □

This finally allows us to give an algorithm that computes $W_{\mathcal{X}}$ (we omit the precise description of the computation of $\mathcal{S}$ and $\mathcal{S}_0$, which is obvious but tedious):

Algorithm 3 WEIGHTSPACE($\mathcal{X}$)

Input: : A pure-dimensional polyhedral complex $\mathcal{X}$
Output: : Its weight space $W_{\mathcal{X}}$

Compute $\mathcal{S}$ and $\mathcal{S}_0$.
$W = \{\omega \in \mathbb{R}^{\mathcal{S}};\ \omega_S = 0 \text{ for all } S \in \mathcal{S}_0\}$
for τ a codimension one cell of $\mathcal{X}$ **do**
    Compute $W_{\mathcal{X}}^{\mathrm{sig}(\tau)}$ as described in Proposition 2.3.8/Remark 2.3.11.
    $W = W \cap W_{\mathcal{X}}^{\mathrm{sig}(\tau)}$
end for
return $t_{\mathcal{X}}(W)$

Remark 2.3.13. One is often interested in the *positive* weights one can assign to a complex X to make it balanced. This is now very easy using `polymake`: Simply intersect V_X with the positive orthant $(\mathbb{R}_{\geq 0})^N$ and you will obtain the *weight cone* of X, which we will denote by C_X. We will now see that this weight cone can be used to find all irreducible subvarieties (i.e. irreducible subcycles with nonnegative weight) of a cycle X.

Definition 2.3.14. Let X be a pure polyhedral complex and $w \in W_X$. We define

$$\mathrm{supp}(w) := \{\sigma \in X^{(\max)}; w(\sigma) \neq 0\}$$

the *support* of X. If $w \in \Omega_X$, then the *associated cycle* is the polyhedral complex generated by $\mathrm{supp}(w)$ with weight function w. We denote this cycle by $Z_X(w)$.

Lemma 2.3.15. *Let X be a pure polyhedral complex and $0 \neq w \in \Omega_X$. Then $Z_X(w)$ is irreducible if and only if the following hold:*

1. $\gcd(w(\sigma), \sigma \in \mathrm{supp}(w)) = 1$.
2. *w has minimal support, i.e. any $0 \neq w' \in \Omega_X$ with $\mathrm{supp}(w') \subseteq \mathrm{supp}(w)$ is a multiple of w.*

Proof. This is obvious from the definition of irreducibility. □

Theorem 2.3.16. *Let X be a pure polyhedral complex and $w \in \Omega_X \cap C_X$. Then $Z_X(w)$ is an irreducible tropical variety if and only if w is a primitive integral generator of a ray of C_X.*

Proof. First of all assume w is not a ray. Then there exist primitive linearly independent rays $w_1, \dots, w_k$ of C_X (with $k \geq 2$) and $\alpha_1, \dots, \alpha_k > 0$ such that

$$w = \sum_{i=1}^{k} \alpha_i w_i.$$

In particular, we must have $\mathrm{supp}(w_i) \subseteq \mathrm{supp}(w)$ for all i, so $Z_X(w)$ is not irreducible.

polymake example: Checking irreducibility.
This creates the six-valent curve from Figure 2.2 and computes its weight space (as the space generated by the row vectors of the matrix displayed). We then also compute the rays of the cone of nonnegative weight vectors, which correspond to all irreducible subvarieties.

```
atint >   $w = new WeightedComplex(
          RAYS=>[[1,0],[1,1],[0,1],[-1,0],[-1,-1],[0,-1]],
          MAXIMAL_CONES=>[[0],[1],[2],[3],[4],[5]],
          TROPICAL_WEIGHTS=>[1,1,1,1,1,1]);
atint >   print $w->IS_IRREDUCIBLE;
0
atint >   print $w->WEIGHT_SPACE;
1 -1 1 0 0 0
0 0 1 0 0 1
1 0 0 1 0 0
0 1 0 0 1 0
atint >   print $w->WEIGHT_CONE->RAYS;
1 0 0 1 0 0
1 0 1 0 1 0
0 1 0 1 0 1
0 1 0 0 1 0
0 0 1 0 0 1
```

Now assume w does not have minimal support, i.e. there exists $0 \neq w'$ with $\operatorname{supp}(w') \subseteq \operatorname{supp}(w)$ and w, w' are linearly independent. We want to see that we can assume that $w' \in C_X$:

Consider the line segment

$$L = [w', w] := \{\underbrace{(1-\lambda)w' + \lambda w}_{=:p_\lambda}; \lambda \in [0,1]\}.$$

Note that $\operatorname{supp}(p_\lambda) \subseteq \operatorname{supp}(w)$ and $p_\lambda \in W_X$ by definition. For any $\sigma \in X^{(\max)}$ with $w'(\sigma) < 0$, L intersects the hyperplane $H_\sigma := \{v \in W_X, v(\sigma) = 0\}$ in a rational point p_{λ_σ}. If we pick σ such that λ_σ is maximal, then $p_{\lambda_\sigma} \in C_X$. If we replace w' by an appropriate multiple of p_{λ_σ}, we obtain an element of $\Omega_X \cap C_X$ whose support is contained in the support of w. Hence we can assume without loss of generality that $w' \in C_X$.

In particular, $v := w - \epsilon w' \in C_X$ for small ϵ. This means that we can write $w = v + \epsilon w'$, so w is not a ray of C_X. □

Remark 2.3.17. Note that - in theory - we can apply this method to find *all* irreducible subcycles of X. Simply intersect W_X with the *signed orthant*

$$\{v; v(\sigma) \geq 0 \text{ for } \sigma \in S \text{ and } v(\sigma) \leq 0 \text{ otherwise}\}$$

for some $S \subseteq X^{(\max)}$. The rays of the resulting cone then correspond to irreducible subcycles whose weight vector has the appropriate signs.

All irreducible subcycles could thus be found by iterating through all 2^N orthants, where N is the number of maximal cells of X.

2.4. The coarse subdivision of a tropical variety

As we saw before, many computations with polyhedral complexes react very sensitively to the number of polyhedra involved. However, in tropical geometry, we don't really care about the specific polyhedral structure. Hence it is natural to ask for the existence and computability of a minimal or *coarsest* polyhedral structure on a given variety X:

Definition 2.4.1. Let X be a tropical variety. We call a polyhedral structure $\mathcal{X}$ on X *coarsest*, if every other polyhedral structure on X is a refinement of $\mathcal{X}'$.

A useful tool for studying the existence of such a structure are *subdivision classes* (see Definition 2.3.4). It was already discussed that refining a polyhedral structure does not change the support of the classes. Thus they provide, in a sense, an inverse operation to the process of refining a polyhedral structure. However, it is immensely difficult to see whether they produce an actual polyhedral complex in general and we will only be able to give some partial results in that respect. We will prove the existence of a coarsest subdivision in the case of Bergman fans in 3.4.

Definition 2.4.2. Let $\mathcal{X}$ be a polyhedral complex. For a subdivision class $S \in \mathcal{S}(\mathcal{X})$ we define its *support* as $|S| := \bigcup_{\sigma \in S} \sigma$ and we call $|\mathcal{S}|(\mathcal{X}) := \{|S|, S \in \mathcal{S}(\mathcal{X})\}$ the *subdivision support* of $\mathcal{X}$.

Lemma 2.4.3. *If two polyhedral complexes $\mathcal{X}$ and $\mathcal{X}'$ are equivalent, then*

$$|\mathcal{S}|(\mathcal{X}) = |\mathcal{S}|(\mathcal{X}').$$

Proof. We can assume that $\mathcal{X}'$ is a refinement of $\mathcal{X}$. Since subdividing cones does not change subdivision classes, the claim follows. □

Corollary 2.4.4. *Let X be a tropical variety with a polyhedral structure $\mathcal{X}$. If all codimension one cells τ in $\mathcal{X}$ have at least three adjacent maximal cells, then $\mathcal{X}$ is the coarsest polyhedral structure on X.*

Proof. Let $\mathcal{X}'$ be another polyhedral structure on X. Then $|\mathcal{S}|(\mathcal{X}') = |\mathcal{S}|(\mathcal{X})$ by the previous lemma. But by assumption we have $|\mathcal{S}|(\mathcal{X}) = \mathcal{X}^{(\dim X)}$, so any cell of $\mathcal{X}'$ must be contained in a cell of $\mathcal{X}$. □

Proposition 2.4.5. *Let X be a tropical variety, whose support is the k-skeleton of the normal fan of a polytope P in $\mathbb{R}^n$ for some $k = 1, \dots, n-1$. Then X has a coarsest polyhedral structure, equal to the polyhedral structure induced by the normal fan.*

Proof. By Corollary 2.4.4, we only need to show that the k-skeleton of the normal fan F of P has no two-valent $k-1$-dimensional cones. To see this, note that a $k-1$-dimensional cone τ of F corresponds to an $(n-k+1)$-dimensional face P_τ of P (we assume without restriction that P is fulldimensional). Since $k < n$, the dimension of P_τ is at least two. Hence it has at least three facets, which means that there are at least three k-dimensional cones containing τ. □

Corollary 2.4.6. *Let X be a tropical hypersurface in $\mathbb{R}^n$. Then X has a coarsest polyhedral structure.*

Proof. By [M1, Thm. 3.15], every tropical hypersurface is realizable, i.e. equal to the variety of a tropical polynomial. Hence its support is equal to the codimension one skeleton of the normal fan of the corresponding Newton polytope. □

Remark 2.4.7. Assuming that a variety has a coarsest structure, it is easy to find: We simply have to compute $|S|$ for each subdivision class S, which amounts to finding an irredundant description for the polyhedral cell generated by all vertices and rays of the cells in S. But this is a standard operation provided by convex hull algorithms such as the double description algorithm.

2.4.1. The general case

Conjecture 2.4.8. *Let X be a tropical variety in $\mathbb{R}^n$. Then all subdivision classes of X have convex support.*

This conjecture is motivated by the fact that it is true for hypersurfaces and Bergman fans (Proposition 3.4.1) and that we can almost prove it inductively: The key idea of Proposition 2.4.10 is that we can find a "nice" projection vector w for any tropical variety of codimension at least three. Here, "nice" means that by projecting onto the orthogonal complement of w a certain fixed subdivision class remains a subdivision class under push-forward. Regretfully, the argument does not work in codimension two: The linear spaces we exclude for choosing w can have full dimension, so they would cover all of the ambient space.

But first, we want to see that it suffices to prove the conjecture for fans:

Proposition 2.4.9. *Let X be a tropical variety. If Conjecture 2.4.8 holds for all $\mathrm{Star}_X(p), p \in |X|$, then it also holds for X.*

Proof. Let X be an arbitrary tropical variety and assume S is a subdivision class of X, whose support is not convex. To prove the claim, we need to find a point x in X, such that $\mathrm{Star}_X(x)$ already has a nonconvex subdivision class.

We write $\mathrm{relint}(S)$ and ∂S for the relative interior and relative boundary of $|S|$. For two points $p, q \in \mathbb{R}^n$ we write $[p,q] := \{(1-\lambda)p + \lambda q;\ \lambda \in [0,1]\}$ for their convex hull and $[p,q]_\lambda := (1-\lambda)p + \lambda q$. We will also write $(p,q], [p,q)$ for the corresponding half-open intervals.

Now, for two maximal cells σ, σ' we define their *ridge distance* in S to be the minimal integer k, such that there exists a ridge path $\sigma = \sigma_0, \dots, \sigma_k = \sigma'$ of cells $\sigma_i \in S$, such that two subsequent cells intersect in a codimension one face of which they are the only neighbors. We set

$$S^{\mathrm{nc}} := \{(\sigma, \sigma') \in S \times S;\ \text{there exist } p \in \mathrm{relint}(\sigma), p' \in \mathrm{relint}(\sigma') \text{ s.t. } [p, p'] \notin |S|\}.$$

Now pick $(\sigma, \sigma') \in S^{\mathrm{nc}}$ with minimal ridge distance k and corresponding ridge path $\sigma = \sigma_0, \dots, \sigma_k = \sigma'$. We now pick a point $q \in |S|$: If $k = 1$, both cells intersect in a codimension one face, so we choose q from $\mathrm{relint}(\sigma \cap \sigma')$. If $k > 1$, we choose q from $\mathrm{relint}(\sigma_1)$. In any case, we must have $[q, p], [q, p'] \subseteq \mathrm{relint}(S)$. We define

$$\lambda_{\min} := \min\{\lambda \in [0,1] : [[q,p]_\lambda, [q,p']_\lambda] \cap \partial S \neq \varnothing\}.$$

This exists, as S is closed and $[p, p'] \notin |S|$. We set $r := [q,p]_{\lambda_{\min}}, r' := [q,p']_{\lambda_{\min}}$ and $L := [r, r']$. Then $L \cap \partial S$ is a closed subset of L, i.e. a disjoint union of closed intervals with boundary points $[r,r']_{\lambda_1}, \dots, [r,r']_{\lambda_s}$, where $0 < \lambda_1 < \dots < \lambda_s < 1$ (recall that $r, r' \in \mathrm{relint}(S)$). We claim that $x := [r,r']_{\lambda_1}$ fulfills our requirements:

To see this, pick any open neighborhood U of x. We can assume without loss of generality that $q, r, r' \in U$ (otherwise replace them by points on the line segments $[q,x], [r,x], [r',x]$). By our construction of r and r', the relative interior of the triangle $\mathrm{conv}\{q, r, r'\}$ is contained in $\mathrm{relint}(S)$. In particular, r and r' still lie in the support of the same subdivision class of $\mathrm{Star}_X(x)$ (which is a subset of $|S| \cap U$). Pick an open d-dimensional ball $B := B_\epsilon(r) \subseteq \mathrm{relint}(S)$. Any point in B must now also lie in the support of the same subdivision class as r'. However, the convex hull of $B \cup \{r'\}$ contains an open ball around x, which must contain a point not contained in $|S|$, as $x \in \partial S$. This concludes the proof. ☐

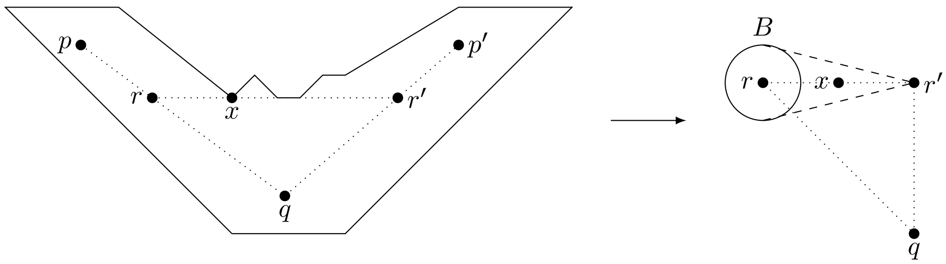

Figure 2.3.: Finding a point for local analysis of convexity

Proposition 2.4.10. *If the statement of Conjecture 2.4.8 is true in codimension two, it is true for all tropical varieties.*

Proof. Let X be a tropical variety. By Proposition 2.4.9 we can assume that X is a fan. Let $d := \dim X$. We prove this statement by induction on the codimension of X, so assume $n - d \geq 3$. Let S be any subdivision class of X (Note that, while S depends on the particular polyhedral structure of X, T does not). We now pick a vector $w \in \mathbb{R}^n$ with the following properties (we write $V_S := \langle S \rangle_{\mathbb{R}}$ for a subdivision class S):

- $w \notin V_S$
- For any subdivision class S' with $\dim(V_{S'} \cap V_S) \in \{d-1, d-2\}$, we have that $w \notin V_S + V_{S'}$. Note that $\dim(V_S + V_{S'}) = 2d - \dim(V_{S'} \cap V_S) \leq n-1$.

Since we only exclude a union of linear spaces of dimension at most $n-1$, such a w exists. We now consider the projection $\pi_w : \mathbb{R}^n \to \mathbb{R}^n / \langle w \rangle \cong \mathbb{R}^{n-1}$. The push-forward $\pi_{w*}(X)$ is still a variety of dimension d, so the codimension has decreased by one. Pick a suitable polyhedral structure on X compatible with the map π_w (again, this does not change the support of the subdivision classes). By our choice of w, $\pi_{w|S}$ is injective and $\pi_w(S)$ is still a subdivision class. Hence it must be convex by induction. □

Example 2.4.11. The natural question to ask next is of course when the subdivision classes form a polyhedral complex. A necessary condition is *connectedness in codimension one*: Consider two linear planes in $\mathbb{R}^4$ intersecting in a point. If we equip both planes with arbitrary weights, we can refine them such that we obtain a tropical variety X. However, the subdivision support of X obviously consists of the two planes, which do not intersect in a common face.

Conjecture 2.4.12. *Let X be a tropical variety and assume X is locally connected in codimension one. Then the subdivision supports of X form a polyhedral complex, which is the coarsest polyhedral structure on X.*

Remark 2.4.13. Note that we have stated these results and conjectures only for tropical *varieties*, i.e. assuming that all weights are positive. Of course the proof of Proposition 2.4.10 works equally well for arbitrary cycles. However, for general weights it is already unclear in codimension one how the subdivision classes behave. Hence we prefer to restrict ourselves to varieties for now.

2.5. Intersection products in $\mathbb{R}^n$

There are two main equivalent definitions for a tropical intersection product in $\mathbb{R}^n$, the *fan displacement rule* [RGST] and via rational functions [AR2]. At first sight, the computationally most feasible one seems to be the latter, since we can already compute it with the means available to us so far:

Let X, Y be tropical cycles in $\mathbb{R}^n$ and $\psi_i = \max\{x_i, y_i\} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Denote by $\pi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ the projection onto the first n coordinates. Then we define

$$X \cdot Y := \pi_*(\psi_1 \cdot \dots \cdot \psi_n \cdot (X \times Y))$$

(Here, applying π_* just means forgetting the last n coordinates) However, computing this directly turns out to be rather inefficient. The main reason is that, since we compute on the product $X \times Y$, we multiply the number of their maximal cones by each other and double the ambient dimension. As we have discussed earlier, both are factors to which the computation of divisors reacts very sensitively.

A different definition of the intersection product is given by Jensen and Yu:

Definition 2.5.1 ([JY, Definition 2.4])**.** Let X, Y be tropical cycles in $\mathbb{R}^n$ of dimension k and l respectively. Assume we have fixed some polyhedral structures $\mathcal{X}$ and $\mathcal{Y}$. Let σ be a $(k+l-n)$-dimensional cone in the complex $\mathcal{X} \cap \mathcal{Y} := \{\sigma \cap \sigma'; \sigma \in \mathcal{X}, \sigma' \in \mathcal{Y}\}$ and p any point in $\operatorname{relint}(\sigma)$. Then σ is a cell in $X \cdot Y$ if and only if the Minkowski sum

$$\operatorname{Star}_X(p) - \operatorname{Star}_Y(p)$$

is complete, i.e. its support is $\mathbb{R}^n$.

This definition is very close to the fan displacement rule and it is in fact not difficult to see that they are equivalent [JY, Proposition 2.7]. So, at first glance it would seem to be an unlikely candidate for an efficient intersection algorithm. In particular, for $n \geq 6$ it is in general algorithmically undecidable, whether a given fan is complete (see for example the appendix of [VKF]). However, one can also show that $\operatorname{Star}_X(p) - \operatorname{Star}_Y(p)$ can be made into a tropical fan (see [JY, Corollary 2.3] for more details). Since $\mathbb{R}^n$ is irreducible, a tropical fan is complete if and only if it is n-dimensional. In this case it is a multiple of $\mathbb{R}^n$.

The weight of the cone σ in the above definition is then computed in the following manner:

Definition 2.5.2 ([JY])**.** Let σ be a polyhedral cell in $X \cdot Y$. Let $p \in \operatorname{relint}(\sigma)$. Then

$$\omega_{X \cdot Y}(\sigma) = \sum_{\substack{\rho_1 \in \operatorname{Star}_X(p), \rho_2 \in \operatorname{Star}_Y(p) \\ \text{s.t. } p \in \operatorname{relint}(\rho_1 - \rho_2)}} \omega_X(\rho_1) \cdot \omega_Y(\rho_2) \cdot ((\Lambda_{\rho_1} + \Lambda_{\rho_2}) : \Lambda_{\rho_1 + \rho_2})$$

This now allows us to write down an algorithm based on these ideas (Algorithm 4).

---

polymake example: Computing an intersection product.
This computes the self-intersection of the standard tropical line in $\mathbb{R}^2$.

```
atint >  $l = tropical_lnk(2,1);
atint >  $i = intersect($l,$l);
atint >  print $i->TROPICAL_WEIGHTS;
1
```

2.6. Local computations

In many cases it is desirable to only compute a given divisor or intersection product *locally*, i.e. around a given point or cone. In these cases, one is usually interested in the

Algorithm 4 MINKOWSKIINTERSECTION

Input: Two tropical cycles X, Y in $\mathbb{R}^n$ of codimension k and l respectively, such that $k + l \leq n$
Output: Their intersection product $X \cdot Y$

Compute the $(n - (k + l))$-skeleton Z of $X \cap Y$
for σ a maximal cell in Z **do**
  Compute an interior point $p \in \text{relint}\sigma$
  Compute the local fans $\text{Star}_X(p), \text{Star}_Y(p)$
  if for any $\rho_1 \in \text{Star}_X(p), \rho_2 \in \text{Star}_Y(p)$ the cell $\rho_1 - \rho_2$ is n-dimensional **then**
    Compute weight $\omega_{X \cdot Y}$ of σ as described above
  else
    Remove σ
  end if
end for
return $(Z, \omega_{X \cdot Y})$

weight of only a certain set of cones or a certain property that can be checked locally. In any case, local computations often have the advantage of removing a large amount of cones, thus speeding up computations considerably. `a-tint` provides a mechanism for local computations based on the following data:

Definition 2.6.1. A *local weighted complex* is a tuple $(\mathcal{X}, \omega, L)$ consisting of a pure polyhedral complex $\mathcal{X}$ with a weight function $\omega : \mathcal{X}^{\dim(X)} \to \mathbb{Z}$ and a set of cells $L \subseteq \mathcal{X}$ (not necessarily of the same dimension) with the additional property that

- No cell of L is contained in another cell of L.
- Any maximal cell of X contains at least one cell of L.

We call L the *local restriction* of $\mathcal{X}$.

Remark 2.6.2. The semantics of the above definition is as follows: We want to do all tropical computations only in an infinitesimal open neighborhood of $\text{relint}(L)$, where

$$\text{relint}(L) := \bigcup_{\sigma \in L} \text{relint}(\sigma).$$

We will shortly see how this can be accomplished. First note that the two additional requirements for L are only necessary to avoid redundancy and useless information: If $\sigma \subseteq \sigma'$, we have of course that any neighborhood of $\text{relint}(\sigma')$ already contains $\text{relint}(\sigma)$ and the interior of any maximal cell not containing any cell in L will not play any role in our computations. Most algorithms in `a-tint` dealing with local computations expect these conditions to be fulfilled, however and there are special methods for creating local weighted complexes that ensure that they are.

The first thing we need to redefine for local complexes is their combinatorics:

Definition 2.6.3. Let $(\mathcal{X}, \omega, L)$ be a d-dimensional local weighted complex. For $k \leq d$ we define its *local k-skeleton* to be

$$\mathcal{X}_L^{(k)} := \{\sigma \in \mathcal{X}^{(k)},\ \text{there exists } \tau \in L \text{ such that } \tau \subseteq \sigma\}.$$

We now call a weighted local complex *balanced*, if it is balanced around each *local* codimension one face.

Example 2.6.4. Let $\mathcal{X}$ be the polyhedral complex in $\mathbb{R}$ consisting of the interval $\sigma := [-1, 1]$ and its two faces $\tau_1 := \{-1\}, \tau_2 := \{1\}$. We set $\omega(\sigma) = 1$ and $L = \{\sigma\}$. Now, while $\mathcal{X}$ has two codimension one faces, τ_1, τ_2, the *local complex* around L has none, since no codimension one face contains σ. In particular, $(\mathcal{X}, \omega, L)$ is trivially balanced.

2.6.1. Local divisors and refinements

Definition 2.6.5. Let $(\mathcal{X}, \omega, L)$ be a local weighted complex and assume $\varphi : |\mathcal{X}| \to \mathbb{R}$ is a rational function, i.e. affine linear on each cell of $\mathcal{X}$. The *local divisor* of φ on $\mathcal{X}$ with respect to L is defined as

$$(\mathcal{X}_L^{(\dim(\mathcal{X}-1))}, \omega_{\varphi \cdot \mathcal{X}}, L'),$$

where $\omega_{\varphi \cdot \mathcal{X}}$ is defined as in Section 2.2 and $L' := \{\tau \in L; \dim(\tau) < \dim(\mathcal{X})\}$ (i.e. we remove all maximal cones from the list of local restrictions).

Remark 2.6.6. While this definition seems fairly simple and straightforward, a different problem occurs when computing divisors of rational functions locally: Most rational functions will usually not be given on the existing polyhedral structure but will require a refinement. However, the list of local restrictions is tied directly to the choice of polyhedral structure, so it will have to be recomputed. It is not immediately obvious how this should be done such that the result should only reflect the divisor in a small neighborhood of the cells in L.

Example 2.6.7.

1. Let $\mathcal{X}$ be the interval complex from Example 2.6.4 and let $\varphi = \max\{0, x\} : |\mathcal{X}| \to \mathbb{R}$. Then there is an obvious refinement $\mathcal{X}'$ of $\mathcal{X}$ on which φ is piecewise affine linear: We insert an additional vertex τ_0 at 0, thus replacing the maximal cell σ by two maximal cells σ_1, σ_2. Following our semantics, the correct local restriction should *not* be $L := \{\sigma_1, \sigma_2\}$: Any neighborhood of $\text{relint}(\sigma_1) \cup \text{relint}(\sigma_2)$ is again this set, so the local divisor of φ would be empty. However, since $\text{relint}(\sigma)$ contains 0, we would like the point to be included in our result. If we include τ_0 in L, we then have to remove σ_1, σ_2 to avoid breaking the non-redundancy condition. Thus, $L = \{\tau_0\}$ is the correct choice.
2. Consider the two-dimensional local complex depicted in Figure 2.5. We define the local restriction to be the one-dimensional cell τ marked in red. The locus of non-differentiability of φ is marked by a dotted line. The local divisor of φ around τ should only be the part depicted on the right hand side. Furthermore, the divisor should again be a local complex, whose local restriction obviously has to be the new vertex τ'.

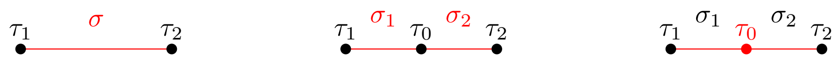

Figure 2.4.: Changing local restrictions during refinement: We have to pick the minimal interior cell(s) of the refinement of σ.

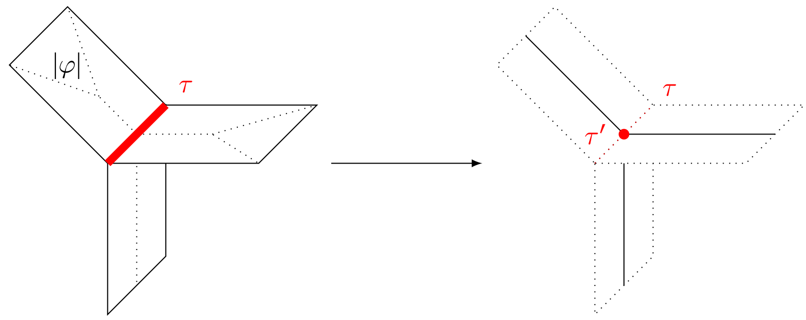

Figure 2.5.: We are only interested in that part of the divisor $\varphi \cdot X$ that intersects any open neighborhood of τ.

Definition 2.6.8. Let $\mathcal{S}$ be a subdivision of a polyhedron σ, i.e. a polyhedral complex whose support is σ. The *minimal interior cells* of $\mathcal{S}$ are the minimal nonempty elements (with respect to inclusion) of $\text{relint}(\mathcal{S}) := \{\tau \in \mathcal{S}; \text{relint}(\tau) \subseteq \text{relint}(\sigma)\}$. We denote this set by $\min \text{int}(\mathcal{S})$.

If L is a local restriction of a local complex $\mathcal{X}$ and $\mathcal{X}'$ is a refinement, then we define

$$L(\mathcal{X}') := \bigcup_{\sigma \in L} \min \text{int}(\mathcal{S}_\sigma),$$

where $\mathcal{S}_\sigma := \{\tau \in \mathcal{X}'; \tau \subseteq \sigma\}$.

Remark 2.6.9. We claim that this set is the "correct" choice for a new local restriction, if we want to compute divisors. In fact, let $(\mathcal{X}, L, \omega)$ be a d-dimensional weighted complex and $\mathcal{X}'$ a refinement of $\mathcal{X}$. Then obviously a cell $\tau \in \mathcal{X}'$ intersects all open neighborhoods of $\text{relint}(L)$, if and only if it contains a cell from $L(\mathcal{X}')$.

Now we can simply adapt our standard algorithm for computing divisors of rational functions: First, compute the appropriate refinement $\mathcal{X}'$ of $\mathcal{X}$ so that φ is affine linear on each cell. Choose $L(\mathcal{X}')$ as the new local restriction, remove all cells that contain no such cell and compute the divisor as in Definition 2.6.5.

It only remains to see how to compute the minimal interior cells of the subdivision of a cell σ:

Lemma 2.6.10. *Let $\mathcal{S}$ be a subdivision of a d-dimensional polyhedron σ containing at least two maximal cells. We define* $\operatorname{int}(\mathcal{S}) := \operatorname{relint}(\mathcal{S}) \cap \mathcal{S}^{(d-1)}$ *and* $\operatorname{ext}(\mathcal{S}) := \mathcal{S}^{(d-1)} \smallsetminus \operatorname{int}(\mathcal{S})$.

1. *Let τ be a codimension one cell of $\mathcal{S}$. Then $\tau \in \operatorname{int}(\mathcal{S})$, if and only if it is contained in at least two maximal cells.*
2. *Let ξ be a maximal cell of $\mathcal{S}$ and τ a face of ξ. Then τ is a minimal interior cell of $\mathcal{S}$, if and only if:*
   a) *There exists no cell $\tau' \in \operatorname{ext}(\mathcal{S})$ containing τ.*
   b) *There exist cells $\tau_1, \ldots, \tau_r \in \operatorname{int}(\mathcal{S})$, such that $\tau = \bigcap_{i=1}^{r} \tau_i$ and for any other cell $\tau' \in \operatorname{int}(\mathcal{S})$ we have $\tau \cap \tau' = \varnothing$.*

Proof. The first part is obvious and the second part follows from the fact that, in any polyhedron P, any nontrivial face is equal to the intersection of all codimension one faces containing it. □

Remark 2.6.11. By the previous lemma, we can now compute all minimal interior cells by iterating over the maximal cells of the subdivision of a local restriction cell σ (we can easily keep track of this when computing the refinement). We can use the first part to distinguish the interior from the exterior codimension one cells by purely combinatorial means. We then compute compute all minimal interior cells contained in such a maximal cell in the following manner: We find the maximal intersections (meaning: an intersection over as many cells as possible) of its interior codimension one faces such that the result does not lie in the boundary. We omit the actual algorithm here, which is straightforward.

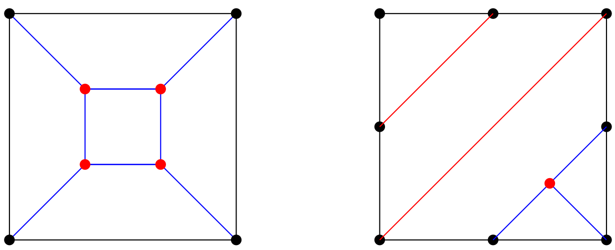

Figure 2.6.: Two different subdivisions of the square. The minimal codimension one cells of the subdivision are marked in blue, the minimal interior cells in red (In the second case, some interior codimension one cells are already minimal). Any red cell can be written as a maximal intersection of the interior codimension one faces of some maximal subdivision cell.

3. Bergman fans

3.1. Introduction

Matroid fans or *Bergman fans* are an important object of study in tropical geometry, since they are the basic building blocks of what we would consider as "smooth" varieties. There are several different but equivalent ways of associating a tropical fan to a matroid, see for example [AK, FR, FS1, S3, S]. We present some of them here (for basic matroid terminology we refer the reader to the standard reference [O]):

Definition 3.1.1. Let M be a matroid on n elements. Then $B(M)$, the *Bergman fan* of M, is the set of all elements $w \in \mathbb{R}^n$ such that for all circuits C of M, the maximum $\max\{w_i; i \in C\}$ is attained at least twice.

Proposition 3.1.2 ([FS1, Proposition 2.5])**.** *Let M be a matroid on n elements. For $w \in \mathbb{R}^n$ let M_w be the matroid whose bases are the bases σ of M of minimal w-cost $\sum_{i \in \sigma} w_i$. Then w lies in the* Bergman fan $B(M)$ *if and only if M_w has no loops, i.e. the union of its bases is the complete ground set.*

Remark 3.1.3. The convex hull of the incidence vectors of the bases of a matroid is a polytope in $\mathbb{R}^n$, the so-called *matroid polytope P_M*. So the vectors w minimizing a certain basis are exactly the vectors in the (inverted) normal cone of the vertex corresponding to that basis. Hence the Bergman fan has a canonical polyhedral structure as a subfan of the (inverted) normal fan of P_M. In addition, we know that it is of pure dimension $\operatorname{rank}(M)$ (this follows immediately from the other definitions of $B(M)$). If we set all weights to 1, we obtain a tropical variety.

Theorem 3.1.4 ([AK, Theorem 1])**.** *Let M be a (loopfree) matroid on ground set E. For each chain of flats*

$$\mathcal{F} = \varnothing \subsetneq F_1 \subsetneq \cdots \subsetneq F_r = E$$

we let $C_{\mathcal{F}}$ be the cone in $\mathbb{R}^{|E|}$ spanned by rays $v_{F_1}, \ldots, v_{F_{r-1}}$ and with lineality space v_{F_r}. Here $v_F = -\sum_{i \in F} e_i$, where e_i is the i-th standard basis vector. Then the $B(M)$ is the support of the fan consisting of all cones $C_{\mathcal{F}}$ for each chain of flats. Again, if we set the weight of each maximal cone to 1, we obtain a tropical variety.

Remark 3.1.5. A polyhedral structure on $B(M)$ can also be defined in the case of Definition 3.1.1, but it is a little more involved. However, all three definitions yield equivalent tropical varieties, i.e. the set $B(M)$ is always the same. The specific polyhedral structures, however, differ. Of course, in the context of tropical geometry, this does not really matter. Hence we will also use the notation $B(M)$ for the tropical variety obtained by setting all weights to 1.

Proposition 3.1.2 obviously is a likely candidate for the coarsest polyhedral structure as a subfan of a normal fan. We will call this the *coarse subdivision* (and prove at the end of the chapter, that it actually is the coarsest polyhedral structure). 3.1.1 gives the *cyclic subdivision*, a refinement of the coarse one. The structure defined by 3.1.4, the *fine subdivision* is again a refinement of the cyclic subdivision. A whole range of different subdivisions is given in [FS1, Theorem 4.1], which shows that each *building set* of the lattice of flats defines a refinement of the Bergman fan.

3.2. Computing Bergman fans

Considering our discussions in the previous sections, it might seem desirable to compute the Bergman fan in its coarsest possible structure to reduce the number of cones. However, for some applications it is sometimes necessary to have a Bergman fan given with a different polyhedral structure (e.g. if a function is defined on the fine subdivision). Also, we can always obtain the coarse subdivision by applying Remark 2.4.7. We compare the computation of all three subdivisions for some examples in Table 9.4 in the Appendix.

3.2.1. The coarse subdivision

An algorithm to compute the Bergman fan in its coarse subdivision is almost immediately obvious from the definition:

Algorithm 5 BERGMANFANCOARSE

Input: A matroid M on n elements, given in terms of its bases.
Output: The Bergman fan $B(M)$ in its coarse subdivision.

Compute the normal fan F of the matroid polytope P_M.
S = the rank(M)-skeleton of F.
for ξ a maximal cone in S **do**
    Let ρ be the corresponding face of P_M maximized by ξ
    Let $\sigma_1, \dots, \sigma_d$ be the bases corresponding to the vertices of ρ.
    if $\bigcup \sigma_i \subsetneq [n]$ **then**
        Remove ξ from S
    end if
end for
return $(S, \omega \equiv 1)$

While this algorithm is fairly simple to implement, it is highly inefficient for two reasons: Computing the skeleton of a fan from its maximal cones can be rather expensive, especially if we want to compute a low-dimensional skeleton. But mainly, the problem is that from the potentially many cones of S we often only retain a small fraction. Hence we compute a lot of superfluous information.

Note however, that we can always obtain the coarse subdivision from any other. As we will see later, it is the coarsest polyhedral structure, i.e. any other is a refinement of this one. Given a Bergman fan, we can thus just compute subdivision classes. The support of these classes then forms the coarse subdivision.

3.2.2. The cyclic subdivision

In [R1], the cyclic subdivision is used to compute the Bergman fans of linear matroids, i.e. matroids associated to matrices. The algorithm requires the computation of a *fundamental circuit* $C(e,I)$ for an independent set I and some element $e \notin I$ such that $I \cup \{e\}$ is dependent:

$$C(e,I) = \{e\} \cup \{i \in I | (I \setminus \{i\}) \cup \{e\} \text{ is independent}\}$$

It is an advantage of linear matroids that fundamental circuits can be computed very efficiently purely in terms of linear algebra. For general matroids it can still be computed using brute force. With this modified computation of fundamental circuits the algorithm of Rincón can be used to compute Bergman fans of general matroids. It turns out that this is still much faster than the normal fan algorithm above. `a-tint` by default uses its own implementation of this algorithm to compute Bergman fans.

3.2.3. The fine subdivision

This subdivision is obviously very bad in terms of time complexity, since it requires the computation of all flats and their chains. [KBE⁺] gives an *incremental* polynomial time algorithm, but [S1] states that already the number of hyperplanes (i.e. flats of rank $\text{rank}(M) - 1$) can be exponential in $|E|$. However, `a-tint` still contains an implementation to compute this subdivision, as it is feasible for small matroids and sometimes necessary for computations. In fact, we will see in the Appendix (Table 9.4) that it is still much faster than the normal fan algorithm.

3.3. Intersection products on matroid fans

Intersection products on matroid fans have been studied in [S2],[FR]. Both approaches however are not really suitable for computation. While the approach in [S2] is more theoretical (except for surfaces, where its approach might lead to a feasible algorithm) and will not be discussed here, the description in [FR] might seem applicable at first. The authors define rational functions which, applied to $B(M) \times B(M)$, cut out the diagonal. Hence they can define an intersection product similar to [AR2].

polymake example: Computing matroid fans.
This computes the Bergman fan of a matrix matroid and of the uniform matroid $U_{3,4}$ (the first in its cyclic subdivision, the second in its cyclic and its fine subdivision). The first function can only be applied to matrix matroids and makes use of linear algebra to compute fundamental circuits.

```
atint >  $m = new Matrix<Rational>([[1,-1,0,0],[0,0,1,-1]]);
atint >  $bm = bergman_fan_linear($m);
atint >  $u = matroid::uniform_matroid(3,4);
atint >  $bm2 = bergman_fan_matroid($u);
atint >  print $bm2->MAXIMAL_CONES->rows;
6
atint >  $bm3 = bergman_fan_flats($u);
atint >  print $bm3->MAXIMAL_CONES->rows;
12
```

However, these rational functions are defined on the fine subdivision of $B(M)$. We already saw that this subdivision is hard to compute. Also, recall that this approach to computing an intersection product already proved to be inefficient in $\mathbb{R}^n$. `a-tint` still contains an implementation of this, but it is only feasible in very small cases.

It remains to be seen whether there might be a more suitable criterion for computation of matroid intersection products, maybe similar to Definition 2.5.1.

3.4. The coarsest subdivision of Bergman fans

In this section we will prove that each Bergman fan has a coarsest polyhedral structure, which is equal to the coarse subdivision defined above. Note that we write $\mathcal{F}(M)$ for the set of flats of a matroid M.

Proposition 3.4.1. *Let M be a loopfree matroid of rank $r < n$ on $[n]$. Then $B(M)$ has a coarsest polyhedral structure as a subfan of the normal fan of the matroid polytope.*

This result needs some preparation:

Lemma 3.4.2. *Let M be a loopfree matroid of rank r on $[n]$. Then there exists a matroid N, such that the rank 1 flats of N are the 1-element sets, $B(M) \cong B(N)$ as tropical varieties and there is a poset isomorphism between the lattices of flats of M and N.*

Proof. Let $F_1, \dots, F_k$ be the rank one flats of M. For an arbitrary flat F of M, we now define $F' := \{i \in [k] : F_i \subseteq F\}$. Then $\mathcal{F}' := \{F'; F \in \mathcal{F}(M)\}$ obviously defines a lattice of sets that is isomorphic to the lattice $\mathcal{F}(M)$. Hence it is the lattice of flats of a matroid N on $[k]$. We now define a linear map $\varphi : \mathbb{R}^n \to \mathbb{R}^k$, which maps e_i to e_j, if $i \in F_j$. It is easy to see that this map induces an isomorphism $B(M) \cong B(N)$. □

Lemma 3.4.3. *Let M be a loopfree matroid of rank r on $[n]$ and assume $B(M)$ is a linear space, i.e. isomorphic to some $\mathbb{R}^k$. Then P_M is either a point or all facets of P_M are contained in the boundary of the simplex $\Delta_{n-1} = \operatorname{conv}\{e_i, i = 1, \dots, n\}$.*

Proof. First, let us assume that the rank one flats of M are $\{1\}, \dots, \{n\}$. In particular $-e_i \in B(M)$ for all $i = 1, \dots, n$. Since $B(M)$ is a linear space, this implies that $\sum_{i \in I} -e_i \in B(M)$ for all $I \subseteq [n]$. Hence all subsets $I \subseteq [n]$ are flats of M, so $M = U_{n,n}$, whose matroid polytope is the point $(1, \dots, 1)$.

Now assume M has rank one flats $F_1, \dots, F_l$ of arbitrary cardinality. We use the previous lemma to see that the lattice of flats of M is isomorphic to the lattice of a matroid N on $\{1, \dots, l\}$, whose rank one flats are all sets of cardinality one. Since $B(M) \cong B(N)$ is a linear space, we see again that $N = U_{l,l}$ (where l is the number of rank one flats of M). Note that the lattice of flats of $U_{l,l}$ is just the subset lattice of $[l]$. In particular, we see that $l = r$. The bases of M must now be all subsets of $[n]$, which contain exactly one element from each F_i.

By [FS1, Proposition 2.3], the matroid polytope P_M is defined by the equation $\sum_{i=1}^n x_i = r$ and the inequalities

1. $x_i \geq 0, i = 1, \dots, n$
2. $\sum_{i \in F} x_i \leq \operatorname{rank}(F), F$ a flat of M

Since $\mathcal{F}(M) \cong \mathcal{F}(N)$, any flat F of M is of the form

$$F_I := \bigcup_{i \in I} F_i$$

for some $I \subseteq [l]$. Since each basis of M contains exactly one element from each F_i, all the inequalities of the second type actually hold with equality. Thus a facet of P_M can only be obtained by requiring $x_i = 0$ for some i. But then the corresponding facet must already be contained in the boundary of the simplex. □

Proof. (of Proposition 3.4.1) Let $\mathcal{M}$ be the polyhedral structure on $B(M)$ induced by the normal fan of P_M. Recall that by [FS1, Proposition 2.5] $B(M)$ consists of the normal cones τ of those faces P_τ of P_M which intersect the interior of the simplex Δ_{n-1}. By Corollary 2.4.4 we have to show that there are at least three maximal cones adjacent to each codimension one cone τ.

Now assume there exists a codimension one cone τ in $\mathcal{M}$, which is only contained in two maximal cones. Thus P_τ is a face of P_M of dimension $n - r + 1$, which has exactly two facets that intersect the interior of the simplex. But P_τ is also a matroid polytope, whose Bergman fan is $\operatorname{Star}_{B(M)}(p)$, where p is any point in $\operatorname{relint}(\tau)$. This is a linear space by assumption, so Lemma 3.4.3 tells us, that *all* facets of P_τ must lie in the boundary of the simplex. This is a contradiction, so the claim follows. □

4. Moduli spaces of rational curves

4.1. Basic notions

We only present the basic notations and definitions related to tropical moduli spaces. For more detailed information, see for example [GKM].

Definition 4.1.1. An *n-marked rational tropical curve* is a metric tree with n unbounded edges, labeled with numbers $\{1,\dots,n\}$, such that all vertices of the graph are at least trivalent. We can associate to each such curve C its metric vector $(d(C)_{i,j})_{i<j} \in \mathbb{R}^{\binom{n}{2}}$, where $d(C)_{i,j}$ is the distance between the unbounded edges (called *leaves*) marked i and j determined by the metric on C.

Define $\Phi_n : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{2}}, a \mapsto (a_i + a_j)_{i<j}$. Then

$$\mathcal{M}_{0,n}^{\text{trop}} := \{d(C); C \text{ } n\text{-marked curve}\} \subseteq \mathbb{R}^{\binom{n}{2}}/\Phi_n(\mathbb{R}^n)$$

is the *moduli space* of n-marked rational tropical curves.

Remark 4.1.2. The space $\mathcal{M}_{0,n}^{\text{trop}}$ is also known as the *space of phylogenetic trees* [SS]. It is shown (e.g. in [GKM]) that $\mathcal{M}_{0,n}^{\text{trop}}$ is a pure $(n-3)$-dimensional fan and if we assign weight 1 to each maximal cone, it is balanced (though [GKM] does not use the standard lattice, as we will see below). Points in the interior of the same cone correspond to curves with the same *combinatorial type*: The combinatorial type of a curve is its equivalence class modulo homeomorphisms respecting the labelings of the leaves. I.e. morally we forget the metric on each graph. In particular, maximal cones correspond to curves where each vertex is exactly trivalent. We call this particular polyhedral structure on $\mathcal{M}_{0,n}^{\text{trop}}$ the *combinatorial subdivision.*

The lattice for $\mathcal{M}_{0,n}^{\text{trop}}$ under the embedding defined above is generated by the rays of the fan. These correspond to curves with exactly one bounded edge. Hence each such curve defines a partition or *split* $I|I^c$ on $\{1,\dots,n\}$ by dividing the set of leaves into those lying on the "same side" of e. We denote the resulting ray by v_I (note that $v_I = v_{I^c}$). Similarly, given any rational n-marked curve, each bounded edge E_i of length α_i induces some split $I_i|I_i^c, i = 1,\dots d$ on the leaves. In the moduli space, this curve is then contained in the cone spanned by the v_{I_i} and can be written as $\sum \alpha_i v_{I_i}$. In particular $\mathcal{M}_{0,n}^{\text{trop}}$ is a simplicial fan.

While this description of $\mathcal{M}_{0,n}^{\text{trop}}$ is very useful to understand the moduli space in terms of combinatorics, it is not very suitable for computational purposes. By dividing out $\text{Im}(\Phi_n)$, we have to make some choice of projection, which would force us to do a lot of tedious (and unnecessary) calculations. Also, the special choice of a lattice would make normal vector computations difficult. However, there is a different representation of $\mathcal{M}_{0,n}^{\text{trop}}$: It was proven in [AK] and [FR] that

$$\mathcal{M}_{0,n}^{\text{trop}} \cong B(K_{n-1})/\langle(1,\ldots,1)\rangle_{\mathbb{R}}$$

as a tropical variety, where K_{n-1} is the matroid of the complete graph on $n-1$ vertices. In particular, matroid fans are always defined with respect to the standard lattice. Dividing out the lineality space $\langle(1,\ldots,1)\rangle$ of a matroid fan can be done without much difficulty, so we will usually want to represent $\mathcal{M}_{0,n}^{\text{trop}}$ internally in matroid fan coordinates, while the user should still be able to access the combinatorial information hidden within.

While the description of $\mathcal{M}_{0,n}^{\text{trop}}$ as a matroid fan automatically gives us a way to compute it, it turns out that this is rather inefficient. Furthermore, as soon as we want to compute certain subsets of $\mathcal{M}_{0,n}^{\text{trop}}$, e.g. Psi-classes, the computations quickly become infeasible due to the sheer size of the moduli spaces. Hence we would like a method to compute $\mathcal{M}_{0,n}^{\text{trop}}$ (or parts thereof) in some combinatorial manner. The main instrument for this task is presented in the following subsection.

4.2. Prüfer sequences

Cayley's Theorem states that the number of spanning trees in the complete graph K_n on n vertices is n^{n-2}. One possible proof uses so-called *Prüfer sequences*: A Prüfer sequence of length $n-2$ is a sequence $(a_1,\ldots,a_{n-2})$ with $a_i \in \{1,\ldots,n\}$ (Repetitions allowed!). One can now give two very simple algorithms for converting a spanning tree in K_n into such a Prüfer sequence (Algorithm 6) and vice versa (Algorithm 7).

Algorithm 6 PRUEFERSEQUENCEFROMGRAPH(T)

Input: A spanning tree T in K_n
Output: A sequence $P = (a_1,\ldots,a_{n-2}), a_i \in [n]$
$P := ()$;
while number of nodes of $T > 2$ **do**
  Find the smallest node i that is a leaf and let v be the adjacent node
  Remove i from T
  $P = (P, v)$
end while

It is easy to see that this induces a bijection between Prüfer sequences and spanning trees (see for example [AZ, Chapter 30]). An example for this is given in Figure 4.1.

As one can see from the picture, tropical rational n-marked curves with d bounded edges can also be considered as graphs on $n+d+1$ vertices: We convert the unbounded leaves into terminal vertices, labeled $1,\ldots,n$ and arbitrarily attach labels

Algorithm 7 GRAPHFROMPRUEFERSEQUENCE(P)

Input: A sequence $P = (a_1, \ldots, a_{n-2}), a_i \in [n]$
Output: A spanning tree T in K_n
$T :=$ graph on n nodes with no edges
$V := [n]$
while $|V| > 2$ **do**
  Let $i := \min V \setminus P$
  Let j be the first element of P
  Connect nodes i and j in T
  Remove i from V and the first element from P
end while
Connect the two nodes left in V

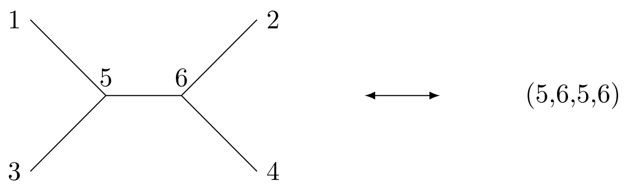

Figure 4.1.: An example for converting a spanning tree on K_6 into a Prüfer sequence and back. The tree can also be considered as a 4-marked rational curve with additional labels at the interior vertices.

$n+1, \ldots, n+d+1$ to the other vertices. This will allow us to establish a bijection between combinatorial types of rational curves and a certain kind of Prüfer sequence:

Definition 4.2.1. A *moduli* Prüfer sequence of order n and length d is a sequence $(a_1, \ldots, a_{n+d-1})$ for some $d \geq 0, n \geq 3$ with $a_i \in \{n+1, \ldots, n+d+1\}$ such that each entry occurs at least twice.

We call such a sequence *ordered* if after removing all occurrences of an entry but the first, the sequence is sorted ascendingly.

We denote the set of all sequences of order n and length d by $\mathcal{P}_{n,d}$ and the corresponding ordered sequences by $\mathcal{P}^<_{n,d}$.

Example 4.2.2. The sequences $(6,7,8,7,8,6)$ and $(6,7,6,7,8,8)$ are ordered moduli sequences of order 5 and length 2, but the sequence $(6,8,8,7,7,6)$ is not ordered.

Definition 4.2.3. For fixed n and d we call two sequences $p, q \in \mathcal{P}_{n,d}$ *equivalent* if there exists a permutation $\sigma \in \mathbb{S}(\{n+1, \ldots, n+d+1\})$ such that $q_i = \sigma(p_i)$ for all i.

Remark 4.2.4. It is easy to see that for fixed n and d the set $\mathcal{P}^<_{n,d}$ forms a system of representatives of $\mathcal{P}_{n,d}$ modulo equivalence, i.e. each sequence of order n and length d is equivalent to a unique ordered sequence.

We will need this equivalence relation to solve the following problem: As stated above, we want to associate Prüfer sequences to rational tropical curves by assigning vertex labels $n+1,\dots,n+d+1$ to all interior vertices. There is no canonical way to do this, so we can associate different sequences to the same curve. But two different choices of labellings will then yield two equivalent sequences.

Proposition 4.2.5. *The set of combinatorial types of n-marked rational tropical curves is in bijection to* $\bigcup_{d=0}^{n-3}\mathcal{P}_{n,d}^{<}$. *More precisely, the set of all combinatorial types of curves with d bounded edges is in bijection to* $\mathcal{P}_{n,d}^{<}$.

Proof. The bijection is constructed as follows: Given an n-marked rational curve C with d bounded edges, consider the unbounded leaves as vertices, labeled $\{1,\dots,n\}$. Assign vertex labels $\{n+1,\dots,n+d-1\}$ to the inner vertices. Then compute the Prüfer sequence $P(C)$ of this graph using Algorithm 6 and take the unique equivalent ordered sequence as image of C.

First of all, we want to see that $P(C)\in\mathcal{P}_{n,d}$. Since C has $n+d+1$ vertices if considered as above, the associated Prüfer sequence has indeed length $n+d-1$. Furthermore, the n smallest vertex numbers are assigned to the leaves, so they will never occur in the Prüfer sequence. Hence $P(C)$ has only entries in $\{n+1,\dots,n+d+1\}$. In addition, it is easy to see that each interior vertex v occurs exactly $\text{val}(v)-1$ times (since we remove $\text{val}(v)-1$ adjacent edges before the vertex becomes itself a leaf), i.e. at least twice.

Injectivity follows from the fact that if two curves induce the same ordered sequence, they can only differ by a relabeling of the interior vertices, so the combinatorial types are in fact the same. Surjectivity is also clear, since the graph constructed from any $P\in\mathcal{P}_{n,d}^{<}$ is obviously a labeling of a rational n-marked curve. □

We now prove that Algorithm 8, given a moduli sequence, computes the corresponding combinatorial type in terms of its edge splits (it is more or less just a slight modification of Algorithm 7):

Theorem 4.2.6. *In the notation of Algorithm 8 the procedure generates the set of splits* $I_1,\dots,I_d$ *of the combinatorial type corresponding to* P. *More precisely: If* $v(i)$ *is the element chosen from* V *in iteration* $i\in\{n+1,\dots,n+d-1\}$, *then* I_{i-n} *is the split on the leaves* $\{1,\dots,n\}$ *induced by the edge* $\{v(i),p_i\}$.

Proof. Let $v(i)$ be the element chosen in iteration i, corresponding to vertex w_i in the curve. In particular $i\notin P$. This means that i has already occurred $\text{val}(w_i)-1$ times in the sequence P as $P[0]$. Hence the node $v(i)$ is already $(\text{val}(w_i)-1)$-valent, i.e. connected to nodes $q_j; j=1,\dots,\text{val}(w_i)-1$. If q_j is a leaf (i.e. $q_j\le n$), then $q_j\in A_{v(i)}$. Otherwise, q_j must have been chosen as $v(k)$ in a previous iteration $k<i$. Hence it must already be $\text{val}(w_k)$-valent. Inductively we see that each node "behind" q_j is either a leaf or has already full valence. In particular, no further edges will be attached to any of these nodes.

By induction on i, the edge $\{v(i), q_j\}$ (assuming q_j is not a leaf) corresponds to the split A_{q_j}. In particular, A_{q_j} has been added to A_v. Hence $A_v = I_{i-n}$, the split induced by the edge $\{v(i), p_i\}$. □

Algorithm 8 COMBINATORIALTYPEFROMPRUEFERSEQUENCE(P,n)

Input: A moduli sequence $P = (p_1, \ldots, p_N) \in \mathcal{P}_{n,d}$
Output: The rational tropical n-marked curve associated to P in terms of the splits $I_1, \ldots, I_d$ induced by its bounded edges.

$d = N - n + 1$
$A_{n+1}, \ldots, A_{n+d+1} = \varnothing$
//First: Connect leaves
for $i = 1 \ldots n$ **do**
  $A_{p_i} = A_{p_i} \cup \{i\}$
end for
$V = \{n+1, \ldots, n+d+1\}$
$P = (p_{n+1}, \ldots, p_N)$
//Now create internal edges
for $i = n+1 \ldots n+d-1$ **do**
  $v = \min(V \smallsetminus P)$
  $I_{i-n} = A_v$
  if length$(P) > 0$ **then**
    //We denote by $P[0]$ the first element of the sequence P.
    $A_{P[0]} = A_{P[0]} \cup A_v$
    $V = V \smallsetminus \{i\}$
    $P = (p_{i+1}, \ldots, p_N)$
  end if
end for
//Create final edge
$I_d = A_{\min V}$

Example 4.2.7. Let us apply Algorithm 8 to the following sequence $P \in \mathcal{P}^{<}_{8,5}$ (see figure 4.2 for a picture of the corresponding curve):

$$P = (9, 9, 10, 10, 11, 11, 12, 12, 13, 13, 14, 14).$$

The algorithm begins by attaching the leaves $\{1, \ldots, 8\}$ to the appropriate vertices, i.e. after the first for-loop we have $A_9 = \{1,2\}, A_{10} = \{3,4\}, A_{11} = \{5,6\}, A_{12} = \{7,8\}$ and $P = (13, 13, 14, 14), V = \{9, \ldots, 14\}$. Now the minimal element of $V \smallsetminus P$ is $v = 9$. We set $I_1 = A_9 = \{1,2\}$ to be the split of the first edge. Then we connect the vertex 9 to the first vertex in P, which is 13. Hence $A_{13} = A_{13} \cup A_9 = \{1,2\}$. We remove 9 from V and set P to be $(13, 14, 14)$. Now $v = \min V \smallsetminus P = 10$. We obtain the second split $I_2 = A_{10} = \{3,4\}$. Then we connect vertex 10 to 13, so $A_{13} = A_{13} \cup A_{10} = \{1,2,3,4\}$. We set $V = \{11, 12, 13, 14\}$ and $P = (14, 14)$. In the next two iterations we obtain splits $I_3 = A_{11} = \{5,6\}, I_4 = A_{12} = \{7,8\}$ and we connect both 11 and 12 to 14, setting $A_{14} = \{5,6,7,8\}$. Now $P = ()$ and $V = \{13,14\}$, so we leave the for-loop and set the final split to be $I_5 = A_{13} = \{1,2,3,4\}$.

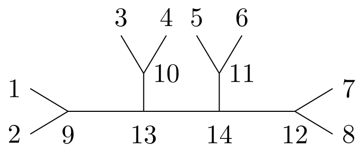

Figure 4.2.: The curve corresponding to the moduli sequence P, including labels for the interior vertices.

We now want to apply the results of the previous section to compute $\mathcal{M}_{0,n}^{\text{trop}}$. For this it is of course sufficient to compute all maximal cones. More precisely, we will only need to compute all combinatorial types corresponding to maximal cones, i.e. rational n-marked tropical curves whose vertices are all trivalent. Using Algorithm 8 we can then compute its rays $v_{I_1}, \dots, v_{I_{n-3}}$. These can easily be converted into matroid coordinates with the construction given in [FR, Example 7.2].

Proposition 4.2.5 directly implies the following:

Corollary 4.2.8. *The maximal cones of $\mathcal{M}_{0,n}^{\text{trop}}$ are in bijection to all ordered Prüfer sequences of order n and length $n-3$, i.e. sequences $(a_1, \dots, a_{2n-4})$ with a_i in $\{n+1, \dots, 2n-2\}$ such that each entry occurs exactly twice and removing the second occurrence of each entry yields an ordered sequence.*

This also gives us an easy way to compute the number of maximal cones of $\mathcal{M}_{0,n}^{\text{trop}}$, which is the *Schröder number*:

Lemma 4.2.9. *The number of maximal cones in the combinatorial subdivision of $\mathcal{M}_{0,n}^{\text{trop}}$ is*

$$(2n-5)!! = \prod_{i=0}^{n-4} (2(n-i)-5)$$

Proof. We prove this by constructing ordered Prüfer sequences of order n and length $n-3$. Each such sequence has $2n-4$ entries. Since it is ordered, the first entry must always be $n+1$. This entry must occur once more, so we have $2n-5$ possibilities to place it in the sequence. Assume we have placed all entries $n+1, \dots, n+k$, each of them twice. Then the first free entry must be $n+k+1$, since the sequence is ordered and we have $2(n-k)-5$ possibilities to place the remaining one. This implies the formula. □

As one can see, the complexity of this number is in $\mathcal{O}(n^{n-3})$, so there is no hope for a fast algorithm to compute all of $\mathcal{M}_{0,n}^{\text{trop}}$ for larger n (except using symmetries). As we will see later, however, we are sometimes only interested in certain subsets or local parts of $\mathcal{M}_{0,n}^{\text{trop}}$.

polymake example: Computing $\mathcal{M}_{0,n}^{\text{trop}}$.
This computes tropical $\mathcal{M}_{0,8}$ and displays the number of its maximal cones.

```
atint >  $m = tropical_m0n(8);
atint >  print $m->MAXIMAL_CONES->rows();
10395
```

4.3. Computing products of Psi-classes

In complex algebraic geometry, Psi-classes on the moduli spaces $\bar{\mathcal{M}}_{g,n}$ are the first Chern classes of the cotangent bundles of the sections of the universal family (see for example [K] for more details). They became especially interesting, when Witten discovered their relation to string theory and quantum gravity [W]. In enumerative geometry, they are useful to count curves satisfying certain tangency conditions. Combining these with pullbacks of evaluation maps to enforce incidence conditions one obtains the so-called descendant Gromov-Witten invariants.

For the genus 0 case, Mikhalkin suggested a tropical analogon [M2]: He defined the i-th Psi-class ψ_i as the (closure of the) locus of curves in $\mathcal{M}_{0,n}^{\text{trop}}$ with a unique four-valent vertex to which the i-th leaf is attached. A more detailed study of tropical Psi-classes on $\mathcal{M}_{0,n}^{\text{trop}}$ was then undertaken in [KM2]: The authors describe them as (multiples of) certain divisors of rational functions, but also in combinatorial terms. For nonnegative integers $k_1, \dots, k_n$ and $I \subseteq [n]$, they define $K(I) := \sum_{i \in I} k_i$. Then one of their main results is the following theorem:

Theorem 4.3.1 ([KM2, Theorem 4.1])**.** *The intersection product $\psi_1^{k_1} \cdot \dots \cdot \psi_n^{k_n} \cdot \mathcal{M}_{0,n}^{\text{trop}}$ is the subfan of $\mathcal{M}_{0,n}^{\text{trop}}$ consisting of the closure of the cones of dimension $n-3-K([n])$ corresponding to the abstract tropical curves C such that for each vertex V of C we have $\text{val}(V) = K(I_V) + 3$, where*

$$I_V = \{i \in [n] : \text{leaf } i \text{ is adjacent to } V\} \subseteq [n].$$

If we denote the set of vertices of C by $V(C)$, then the weight of the corresponding cone $\sigma(C)$ is

$$\omega(\sigma(C)) = \frac{\prod_{V \in V(C)} K(I_V)!}{\prod_{i=1}^n k_i!}.$$

In combination with Proposition 4.2.5, this allows us to compute these products in terms of Prüfer sequences:

Corollary 4.3.2. *The maximal cones in* $\psi_1^{k_1} \cdot \dots \cdot \psi_n^{k_n} \cdot \mathcal{M}_{0,n}^{\text{trop}}$ *are in bijection to the ordered moduli sequences* $P \in \mathcal{P}^{<}_{n,n-3-K([n])}$ *that fulfill the following condition:*

Let $d = n-3-K([n])$ *and* $k_i = 0$ *for* $i = n+1,\dots,n+d-1$. *For* $a \in \{n+1,\dots,n+d+1\}$ *let* $J_a := \{j \in \{1,\dots,n+d-1\}, P_j = a\}$ *be the indices of entries equal to* a *and* $l(a) := |J_a|$. *Then*

$$l(a) = 2 + \sum_{j \in J_a} k_j$$

Proof. Recall that any entry a corresponding to a vertex v_a in the curve $C(P)$ occurs exactly $\text{val}(v_a) - 1$ times. By the theorem above the valence of a vertex is dictated by the leaves adjacent to it. Furthermore, the leaves adjacent to a vertex v_a can be read off of the first n entries of the sequence: Leaf i is adjacent to v_a if and only if $P_i = a$.

So, given a curve in the Psi-class product, vertex v_a must have valence $3 + K(I_{v_a})$, so it occurs $2 + K(I_{v_a}) = 2 + \sum_{i:P_i=a} k_i$ times. Conversely, given a sequence fulfilling the above condition, we obviously obtain a curve with the required valences. □

We now want to give an algorithm that computes all of these Prüfer sequences. As it turns out, this is easier if we require the k_i to be in decreasing order, i.e. $k_1 \geq k_2 \geq \dots \geq k_n$. In the general case we will then have to apply a permutation to the k_i before computation and to the result afterwards. The general idea is that we recursively compute all possible placements of each vertex that fulfill the conditions imposed by the k_i (if we place vertex a at leaf i with $k_i > 0$, then it has to occur more often). Due to its length, the algorithm has been split into several parts: ITERATEPLACEMENTS goes through all possible entries of the Prüfer sequence recursively. It uses PLACEMENTS to compute all possible valid distributions of an entry, given a certain configuration of free spaces in the Prüfer sequence.

Algorithm 9 PSIPRODUCTSEQUENCESORDERED($k_1,\dots,k_n$)

1: **Input:** Nonnegative integers $k_1 \geq k_2 \geq \dots \geq k_n$
2: **Output:** All Prüfer sequences corresponding to maximal cones in $\psi_1^{k_1} \cdot \dots \cdot \psi_n^{k_n} \cdot \mathcal{M}_{0,n}^{\text{trop}}$
3: $K = \sum k_i$
4: current_vertex $= n+1$
5: current_sequence $= (0,\dots,0) \in \mathbb{Z}^{2n-4-K}$
6: exponents $= (k_1,\dots,k_n,0,\dots,0) \in \mathbb{Z}^{2n-4-K}$
7: ITERATEPLACEMENTS(current_vertex, current_sequence, exponents)

Proof. (of Algorithm 9) First of all we prove that PLACEMENTS computes indeed all possible subsets $J \in [m]$ such that $|J| = 2 + \sum_{j\in J} k_j$. So let $J = \{a_1,\dots,a_N\}$ be such a set with $a_1 \leq \dots \leq a_N$. It is easy to see that in each iteration of the while-loop we have $|J| = i-1$. Let $\delta = (2 + \sum_{j\in J} k_j) - |J|$.

ITERATEPLACEMENTS(current_vertex, current_sequence, exponents)

```
if current_vertex > 2n − 2 − K then
  if current_sequence contains no 0's then
    append current_sequence to result
  end if
else
  f = {i : current_sequence[i] = 0}
  for P ∈ PLACEMENTS(exponents[i], i ∈ f) do
    v = current_sequence
    Place current_vertex in v at positions indicated by P
    ITERATEPLACEMENTS(current_vertex+1,v,exponents)
  end for
end if
```

PLACEMENTS($k_1, \dots, k_m$)

```
Input: Nonnegative integers k_1 ≥ ··· ≥ k_m
Output: All subsets J ⊆ [m] such that |J| = 2 + ∑_{j∈J} k_j.
if ∑_{i=1}^m k_i > m − 2 then
  return empty list of solutions
end if
Let J = ∅, i = 1
used[j] = ∅ for j = 1, ..., m
while i > 0 do
  if |J| < 2 + ∑_{j∈J} k_j then
    Let l ∈ [m] ∖ J be minimal such that l > max J and l ∉ used[i].
    if There is no such l then
      STEPDOWN
    else
      used[i] = used[i] ∪ {l}
      i = i + 1
      J = J ∪ {l}
    end if
  else
    if |J| = 2 + ∑_{j∈J} k_j then
      Add J to list of solutions
    end if
    STEPDOWN
  end if
end while
return list of solutions
```

stepDown
1: used$[i] = \varnothing$
2: $J = J \smallsetminus \{\max(J)\}$
3: $i = i - 1$

One can see by induction on δ that, starting in any iteration of the while loop, the algorithm will eventually reach an iteration where i is one smaller. This proves termination of PLACEMENTS.

But we can only reach the iteration where $i = 0$ if in the previous iteration we have tried all indices $\{1, \dots, m\}$ as first element of J. In particular, there was a previous iteration, where we chose $l = a_1$ as first element of J. Now assume we are in the first iteration where $J = \{a_1, \dots, a_s\}, 1 \leq s < N$. Assuming $\delta > 0$, we can again only decrease i if we have tried all valid placements, including a_{s+1}. So assume $\delta = 0$. Then $\{a_1, \dots, a_s\}$ is a valid placement, i.e. $s = 2 + \sum_{i=1}^{s} k_{a_i}$. If we subtract this from the equation for J, we obtain

$$0 < N - s = \sum_{i=s+1}^{N} k_{a_i}$$

In particular, since the k_i are ordered, we must have $k_{a_{s+1}} \geq 1$ and hence also $k_{a_j} \geq 1$ for all $j \leq s$. This implies

$$s = 2 + \sum_{i=1}^{s} k_{a_i} \geq 2 + s$$

which is obviously a contradiction.

With this it is now easy to see that PSIPRODUCTSEQUENCESORDERED computes indeed all the required sequences. □

Example 4.3.3. We do indeed need that $k_1 \geq \dots \geq k_n$ to be able to compute all sequences. Assume $n = 7$ and $(k_1 \dots k_7) = (0,0,0,0,0,1,1)$. A valid sequence would be $(7,7,8,8,9,7,7,9)$, but this sequence would never occur in the algorithm: After having placed the first two 7's in PLACEMENTS we would already have $\delta = 0$, so the last two 7's are never tried out.

For completeness we also give the algorithm for the general case:

Algorithm 10 PSIPRODUCTSEQUENCES$(k_1, \dots, k_n)$
1: **Input:** A list of nonnegative integers $\underline{k} = k_1, \dots, k_n$
2: **Output:** All Prüfer sequences corresponding to maximal cones in $\psi_1^{k_1} \cdot \dots \cdot \psi_n^{k_n} \cdot \mathcal{M}_{0,n}^{\text{trop}}$
3: Let $\sigma \in \mathcal{S}_n$ such that $\sigma(\underline{k})$ is ordered descendingly
4: l = PSIPRODUCTSEQUENCESORDERED$(\sigma(\underline{k}))$
5: **return** $\sigma^{-1}(l)$ (applied elementwise to the first n entries of each sequence)

polymake example: Computing psi classes.
This computes $\psi_1^3 \cdot \psi_2^2 \cdot \psi_6 \cdot \mathcal{M}_{0,9}$ (which is a point) and displays its multiplicity.

```
atint >  $p = psi_product(9, new Vector<Int>(3,2,0,0,0,1,0,0,0));
atint >  print $p->TROPICAL_WEIGHTS;
60
```

4.4. Computing rational curves from a given metric

In previous sections we computed rational curves as elements of the moduli space given by their corresponding bounded edges, i.e. the v_I (as defined in Remark 4.1.2) that span the cone containing the curve. Usually, we will be given the curves either in the matroid coordinates of the moduli space or as a vector in $\mathbb{R}^{\binom{n}{2}}$, i.e. a metric on the leaves. It is relatively easy to convert the matroid coordinates to a metric (see [FR, Example 7.2]), but it is not trivial to convert the metric to a combinatorial description of the curve, i.e. a list of the splits induced by the bounded edges and their lengths.

The paper [B] describes an algorithm to obtain a tree from a metric d on $[n]$ that fulfills the *four-point-condition*, i.e. for all $x, y, z, t \in [n]$ we have

$$d(x,y) + d(z,t) \leq \max\{d(x,z) + d(y,t), d(x,t) + d(y,z)\}$$

and [BG, Theorem 2.1] shows that the metrics induced by semi-labeled trees (essentially: rational n-marked curves) are exactly those which fulfill this condition.

Recall that one possible embedding of $\mathcal{M}_{0,n}^{\text{trop}}$ was given by mapping each curve to its metric in $\mathbb{R}^{\binom{n}{2})}/\Phi_n(\mathbb{R}^n)$. In particular, a representative of a metric d might have negative coordinates. However, we can always assume $d(x,y) > 0$ for $x \neq y$ by adding an appropriate element from $\text{Im}(\Phi_n)$. More precisely, if we have an element $d \in \mathbb{R}^{\binom{n}{2}}$ that is equivalent to the metric of a curve modulo $\text{Im}(\Phi_n)$, there is a $k \in \mathbb{N}$ such that $d + k \cdot \Phi_n(\sum e_i)$ is a positive vector fulfilling the four-point-condition. In fact, if $m = d + \sum \alpha_i \Phi_n(e_i)$ is the equivalent metric, then $d + \sum(\alpha_i + |\alpha_i|)\Phi_n(e_i)$ still fulfills the four-point-condition, since adding positive multiples of $\Phi_n(e_i)$ preserves it.

Algorithm 11 gives a short sketch of the algorithm described in [B, Theorem 2]. As input, we provide a metric d. We then obtain a metric tree with leaves L labeled $\{1, \ldots, n\}$ such that the metric induced on L is equal to d. This tree corresponds to a rational n-marked curve: Just replace the bounded edges attached to the leaf vertices by unbounded edges. It is very easy to modify the algorithm such that it also computes the splits of all edges.

Algorithm 11 TREEFROMMETRIC(d) [B, Theorem 2]

Input: A metric d on the set $[n]$ fulfilling the four-point-condition.
Output: A metric tree T with leaf vertices L labeled $\{1,\dots,n\}$ such that the induced metric on L equals d.

Let $V = \{1,\dots,n\}$
while $|V| > 3$ **do**
  Find an ordered triple of distinct elements (p,q,r) from V, such that the value $d(p,r)+d(q,r)-d(p,q)$ is maximal
  Let t be a new vertex and define its distance to the other vertices by

$$d(t,p) = \frac{1}{2}\left(d(p,q) - d(p,r) - d(q,r)\right)$$
$$d(t,x) = d(x,p) - d(t,p) \text{ for } x \neq p$$

  If $d(t,x) = 0$ for any x, identify t and x, otherwise add t to V.
  Attach p and q to t. Then remove p and q from V
end while
Compute the tree on the remaining vertices using linear algebra.

4.5. Local bases of $\mathcal{M}_{0,n}^{\text{trop}}$

When computing divisors or intersection products on moduli spaces $\mathcal{M}_{0,n}^{\text{trop}}$, a major problem is the sheer size of the fans, in the number of cones and in the dimension of the ambient space. The number of cones can usually be reduced to an acceptable amount, since one often knows that only a handful of cells is actually relevant. However, the ambient dimension of $\mathcal{M}_{0,n}^{\text{trop}}$ is $\binom{n}{2}-n = \frac{n^2-3n}{2} \in \mathcal{O}(n^2)$. Convex hull computations and operations in linear algebra thus quickly become expensive. We will show, however, that locally at any point $0 \neq p \in \mathcal{M}_{0,n}^{\text{trop}}$, the span of $\text{Star}_{\mathcal{M}_{0,n}^{\text{trop}}}(p)$ has a much lower dimension. Hence we can do all our computations locally, where we embed parts of $\mathcal{M}_{0,n}^{\text{trop}}$ in a lower-dimensional space. Let us make this precise:

Definition 4.5.1. Let τ be a d-dimensional cone of $\mathcal{M}_{0,n}^{\text{trop}}$. We define

$$V(\tau) := \left\langle \{\sigma \geq \tau; \sigma \in \mathcal{M}_{0,n}^{\text{trop}}\} \right\rangle_{\mathbb{R}} = \langle U(\tau) \rangle_{\mathbb{R}},$$

where $U(\tau) = \bigcup_{\sigma \geq \tau} \text{relint}(\sigma)$. It is easy to see that for any $0 \neq p \in \mathcal{M}_{0,n}^{\text{trop}}$ and for τ the minimal cone containing p, the span of $\text{Star}_{\mathcal{M}_{0,n}^{\text{trop}}}(p)$ is exactly $V(\tau)$.

We are now interested in finding a basis for this space $V(\tau)$, preferably without having to do any computations in linear algebra. The idea for this is the following: Let C_τ be the combinatorial type of an abstract curve represented by an interior point of τ. We

polymake example: Converting curve descriptions.
This takes a ray from $\mathcal{M}_{0,6}$ (in its matroid coordinates) and displays it in different representations.

```
atint >  $m = tropical_m0n(6);
atint >  $r = $m->RAYS->row(0);
atint >  print $r;
-1 -1 -1 0 -1 -1 0 -1 0
atint >  $c = rational_curve_from_moduli($r);
atint >  print $c;
(1,2,3,4)       # This means that this ray represents v_{1,2,3,4}
atint >  print $c->metric_vector;
0 0 0 1 1 0 0 1 1 0 1 1 1 1 0 # Read as d(1,2),d(1,3),...,d(5,6)
```

want to find a set of rays v_I, all contained in some cones $\sigma \geq \tau$, that generate $V(\tau)$. Each such ray corresponds to separating edges and leaves at some higher-valent vertex p of C_τ along a new bounded edge (whose split is of course $I|I^c$). We will see that for a fixed vertex p with valence greater than 3, all the rays separating edges and leaves at that vertex span a space that has the same ambient dimension as $\mathcal{M}_{0,\text{val}(p)}^{\text{trop}}$. In fact, it is easy to see that they must be in bijection to the rays of that moduli space.

Hence the idea for constructing a basis is the following: In addition to the rays of τ, we choose a basis for the "$\mathcal{M}_{\text{val}(p)}$" at each higher-valent vertex p. This choice is similar to the one in [KM2, Lemma 2.3]. There the authors show that $V_k := \{v_S, |S| = 2, k \notin S\}$ is a generating set of the ambient space of $\mathcal{M}_{0,n}^{\text{trop}}$ for any $k \in [n]$ and it is easy to see that by removing any element it becomes a basis.

Now fix a vertex p of C_τ such that $s := \text{val}(p) > 3$. Denote by $I_1, \dots, I_s$ the splits on $[n]$ induced by the edges and leaves adjacent to p (in particular, some of the I_j might only contain one element). We now define

$$W_p := \{v_{I_i \cup I_j}; i, j \neq 1, i \neq j\}$$

(This corresponds to the set V_1 described above) and

$$B_p := W_p \smallsetminus \{v_{I_2 \cup I_3}\}.$$

Clearly all the following results also hold if we exclude some other I_k, $k > 1$ in the definition of W_p or remove a different element in the definition of B_p (in particular, because the numbering of the I_i is completely arbitrary). To make the proofs more concise, we will however stick to this particular choice. We introduce one final notation: For $|I_j| = 1$, we set $v_{I_j} := 0$.

Lemma 4.5.2 (see also [KM2, Lemmas 2.4 and 2.7]).

1. *Let p be a vertex of the curve C_τ and define $I_1, \ldots, I_s, W_p$ as above. Then*
$$\sum_{v \in W_p} v = (s-3)\left(\sum_{j>1} v_{I_j}\right) + v_{I_1} \equiv 0 \bmod V_\tau.$$

2. *Let v_I be a ray in some $\sigma \geq \tau$ and assume it separates some vertex p of C_τ. Define $I_1, \ldots, I_s, W_p$ as above. Assume without restriction that $I_1 \subseteq I^c$. Then*
$$v_I = \sum_{\substack{v_S \in W_p \\ S \subseteq I}} v_S - (m-2)\Big(\sum_{I_j \subseteq I} v_{I_j} + v_I\Big) \equiv \sum_{\substack{v_S \in W_p \\ S \subseteq I}} v_S \bmod V_\tau.$$

Proof.

1. We define $a = (a_i) \in \mathbb{R}^n$ via $a_i = 1$, if $i \in I_1$ and $a_i = (s-3)$ otherwise. Furthermore we define $b = (b_i) \in \mathbb{R}^n$ via
$$b_i = \begin{cases} 0, & \text{if } i \text{ is a leaf attached to } p \\ 1, & \text{if } i \text{ is not a leaf at } p \text{ and lies in } I_1 \\ (s-3), & \text{if } i \text{ is not a leaf at } p \text{ and does not lie in } I_1. \end{cases}$$

   We now prove the following equation (to be considered as an equation in $\mathbb{R}^{\binom{n}{2}}$, where each ray is represented by its metric vector):
$$\sum_{v \in W_p} v = (s-3)\left(\sum_{\substack{j>1 \\ |I_j|>1}} v_{I_j}\right) + v_{I_1} - \phi_n(b) + \phi_n(a). \qquad (*)$$

   We index $\mathbb{R}^{\binom{n}{2}}$ by all sets $\mathcal{T} = \{k_1, k_2\}, k_1 \neq k_2$. We have
$$\left(\sum_{v \in W_p} v\right)_{\mathcal{T}} = \begin{cases} 0, & \text{if } k_1, k_2 \in I_j, j = 1, \ldots, s \\ s-2, & \text{if } k_1 \in I_1, k_2 \in I_j, j > 1 \\ 2(s-3), & \text{if } k_1 \in I_i, k_2 \in I_j; i, j > 1; i \neq j. \end{cases}$$

   We now study the right hand side of (*) in four different cases:

   a) If $k_1, k_2 \in I_1$, then both are not leaves at p. Hence the right hand side yields $0 + 0 - 2 + 2 = 0$.

   b) If $k_1, k_2 \in I_j, j > 1$, again both are not leaves at p. The right hand side now yields $0 + 0 - 2(s-3) + 2(s-3) = 0$.

   c) Assume $k_1 \in I_i, k_2 \in I_j, i, j > 1$ and $i \neq j$. If both are not leaves at p, we get $2(s-3) + 0 - 2(s-3) + 2(s-3)$. If only one is a leaf, we get $(s-3) + 0 - (s-3) + 2(s-3)$. Finally, if both are leaves, we get $0 + 0 - 0 + 2(s-3)$. So in any of these cases the right hand side agrees with the left hand side.

d) Assume $k_1 \in I_1, k_2 \in I_j, j > 1$. If both are not leaves, we get $(s-3)+1-(s-3)-1+(s-2)$. The other cases are similar.

2. We know that I must be a union of some of the I_j and we assume without restriction that $I = \bigcup_{j \geq k} I_j$ for some $k > 1$. Furthermore we define

$$m := |\{i : I_i \subseteq I\}| = s - k + 1.$$

We now prove the following formula (again in $\mathbb{R}^{\binom{n}{2}}$. A similar formula for the representation of a ray v_I in V_k and a similar proof can be found in [KM2, Lemma 2.7]):

$$\begin{aligned} v_I = {} & \underbrace{\sum_{i,j \geq k} v_{I_i \cup I_j} - (m-2)\phi_n\left(\sum_{l \in I} e_l\right)}_{=:z} \\ & - (m-2)\underbrace{\left(\sum_{j \geq k} v_{I_j} + v_I - \phi_n\left(\sum_{i=1}^{n} e_i\right)\right)}_{=:w} \\ \equiv {} & \sum_{i,j \geq k} v_{I_i \cup I_j} \bmod V_\tau. \end{aligned} \tag{4.5.1}$$

To see that the equation holds, let us first compute w. We index $\mathbb{R}^{\binom{n}{2}}$ by all sets $\mathcal{T} := \{k_1, k_2\}, k_1 \neq k_2$. Then we have

$$\left(\sum_{j \geq k} v_{I_j}\right)_{\{k_1,k_2\}} = \begin{cases} 0, & \text{if } \{k_1,k_2\} \subseteq I_i \text{ for some } i \geq k \\ 1, & \text{if } k_1 \in I, k_2 \notin I \text{ or vice versa} \\ 2, & \text{if } k_1 \in I_i, k_2 \in I_j, i \neq j; i,j \geq k. \end{cases}$$

Hence

$$\begin{aligned} \left(\sum_{j \geq k} v_{I_j} + v_I\right)_{\{k_1,k_2\}} &= \begin{cases} 0, & \text{if } \{k_1,k_2\} \subseteq I_i \text{ for some } i \geq k \\ 2, & \text{if } k_1 \in I, k_2 \notin I \text{ or vice versa} \\ 2, & \text{if } k_1 \in I_i, k_2 \in I_j, i \neq j; i,j \geq k \end{cases} \\ &= \begin{cases} 0, & \text{if } \{k_1,k_2\} \subseteq I_i \text{ for some } i \geq k \\ 2, & \text{otherwise.} \end{cases} \end{aligned}$$

Finally we get

$$(w)_{\{k_1,k_2\}} = \begin{cases} -2, & \text{if } \{k_1,k_2\} \subseteq I_i \text{ for some } i \geq k \\ 0, & \text{otherwise.} \end{cases}$$

Thus it remains to prove that

$$(v_I - z)_{\{k_1,k_2\}} = \begin{cases} 2(m-2), & \text{if } \{k_1,k_2\} \subseteq I_i \text{ for some } i \geq k \\ 0, & \text{otherwise.} \end{cases}$$

For this, let $k_1 \neq k_2 \in [n]$. If $\mathcal{T} := \{k_1, k_2\} \subseteq I_i$ for some $i \geq k$, then $(v_I)_{\mathcal{T}} = (v_{I_i \cup I_j})_{\mathcal{T}} = 0$ for all $i, j \geq k$ and $(\phi_n(\sum_{l \in I} e_l))_{\mathcal{T}} = 2$. Thus the formula holds. Now if $k_1 \in I_i, k_2 \in I_j$ for $i \neq j$ and $i, j \geq k$, we still have $(v_I)_{\mathcal{T}} = 0$. Furthermore, there are $(m-2)$ choices for a ray $v_{I_i \cup I'_j}$ with $j' \neq j$ and $(m-2)$ choices for a ray $v_{I'_i \cup I_j}$ with $i' \neq i$. For these rays, the $\mathcal{T}$-th entry is 1, for all other rays $v_{I'_i \cup I'_j}$ it is 0. Hence

$$\left(\sum_{i,j \geq k} v_{I_i \cup I_j}\right)_{\mathcal{T}} = 2(m-2) = (m-2)\left(\phi_n(\sum_{l \in I} e_l)\right)_{\mathcal{T}}.$$

Finally, if $k_1 \in I$ (say $k_1 \in I_i$), $k_2 \notin I$, then $(v_I)_{\mathcal{T}} = 1$. There are $(m-1)$ choices for a ray v_{I_i, I_j} with $j \neq i$. Since $(\phi_n(\sum_{l \in I} e_l))_{\mathcal{T}} = 1$, we get

$$\left(\sum_{i,j \geq k} v_{I_i \cup I_j}\right)_{\mathcal{T}} = (m-1) = (m-2)\left(\phi_n(\sum_{l \in I} e_l)\right)_{\mathcal{T}} + 1.$$

Hence equation 4.5.1 holds.

□

Theorem 4.5.3. *Let $v_{E_1}, \dots, v_{E_t}$ be the rays of τ. Then the set*

$$B_\tau := \Big(\bigcup_{\substack{p \in C_\tau^{(0)} \\ \mathrm{val}(p) > 3}} B_p\Big) \cup \{v_{E_1}, \dots, v_{E_t}\}$$

is a basis for $V(\tau)$. In particular, the dimension of $V(\tau)$ can be calculated as

$$\dim V(\tau) = \dim \tau + \sum_{\substack{p \in C_\tau^{(0)} \\ \mathrm{val}(p) > 3}} \left(\binom{\mathrm{val}(p)}{2} - \mathrm{val}(p)\right).$$

Proof. By Lemma 4.5.2 these rays generate V_τ: We can write each v_I in some $\sigma \geq \tau$ in terms of W_p and the bounded edges at the vertex associated to it. The first part of the Lemma then yields that we can replace any occurrence of $v_{I_2 \cup I_3}$ to get a representation in B_p and the bounded edges.

To see that the set is linearly independent, we do an induction on n. For $n = 4$ the statement is trivial. For $n > 4$, assume τ is the vertex of $\mathcal{M}_{0,n}^{\mathrm{trop}}$. Then B_τ actually agrees with the set $V_k \setminus \{v_S\}$ for some S and we are done. So let p be a vertex of C_τ that has only one bounded edge attached and denote by i one of the leaves attached to it. It is easy to see that applying the forgetful map ft_i to B_τ, we get the set $B_{\mathrm{ft}_i(\tau)}$. If p is trivalent, then the ray corresponding to the bounded edge at p is mapped to 0 and all other elements of B_τ are mapped bijectively onto the elements of $B_{\mathrm{ft}_i(\tau)}$. Since the latter is independent by induction, so is B_τ.

If p is higher-valent, only rays from B_p might be mapped to 0 or to the same element. Hence, if we have a linear relation on the rays in B_τ, we can assume by induction that only the elements in B_p have non-trivial coefficients. But these are linearly independent as well: Let q be any other vertex with only one bounded edge and j any leaf at q. B_p is now preserved under the forgetful map ft_j and hence linearly independent by induction. □

---

polymake example: Local computations in $\mathcal{M}_{0,n}^{\text{trop}}$.
This computes a local version of $\mathcal{M}_{0,13}$ around a codimension 2 curve C with a single five-valent vertex, i.e. it computes all maximal cones containing the cone corresponding to C. `a-tint` keeps track of the local aspect of this complex, so it will actually consider it as balanced.

```
atint >  $c = new RationalCurve(N_LEAVES=>13,
         INPUT_STRING=>"(2,3) + (2,3,4) + (1,12) + (1,2,3,4,12) +
         (9,10) + (8,9,10) + (11,13) + (8,9,10,11,13)");
atint >  $m = local_m0n($c);
atint >  print $m->MAXIMAL_CONES->rows();
15
atint >  print $m->IS_BALANCED;
1
```

At the beginning of this section we introduced the notion that the rays resolving a certain vertex of a combinatorial type C_τ "look like $\mathcal{M}_{\text{val}(p)}$". The results above allow us to make this notion precise:

Corollary 4.5.4. *Let $\mathcal{M}$ be any polyhedral structure of $\mathcal{M}_{0,n}^{\text{trop}}$ (and hence a refinement of the combinatorial subdivision). Let $\tau \in \mathcal{M}$ be a d-dimensional cell. Let C_τ be the combinatorial type of a curve represented by a point in the relative interior of τ. Denote by $p_1,\dots,p_k$ its vertices and by l the number of bounded edges of the curve. Then there is an isomorphism of tropical varieties*

$$\text{Star}_{\mathcal{M}_{0,n}^{\text{trop}}}(\tau) \cong \mathbb{R}^{l-d} \times \mathcal{M}_{0,\text{val}(p_1)}^{\text{trop}} \times \dots \times \mathcal{M}_{0,\text{val}(p_k)}^{\text{trop}}$$

Proof. First assume $\mathcal{M}$ is the combinatorial subdivision of $\mathcal{M}_{0,n}^{\text{trop}}$. There is an obvious map

$$\psi_\tau : \text{Star}_{\mathcal{M}_{0,n}^{\text{trop}}}(\tau) \to \mathcal{M}_{\text{val}(p_1)} \times \dots \times \mathcal{M}_{\text{val}(p_k)},$$

defined in the following way: For each vertex p_i of C_τ fix a numbering of the adjacent edges and leaves, $I_1,\dots,I_{j_i}$. Now for each v_I in some $\sigma \geq \tau$, there is a unique $i \in \{1,\dots,k\}$ such that v_I separates edges/leaves at p_i. Let $S \subseteq \{1,\dots,j_i\}$ such that $I = \bigcup_{j\in S} I_j$. Again, this choice is unique. Now map v_I to v_S in $\mathcal{M}_{\text{val}(p_i)}$. It is easy to see that this map must be bijective.

First let us see that the map is linear. By Theorem 4.5.3 we only have to check that the map respects the relations given in Lemma 4.5.2. But this is clear, since analogous equations hold in $\mathcal{M}_{0,n}^{\text{trop}}$ (again, see [KM2, Lemmas 2.4 and 2.7] for details).

For any set of rays $v_{J_1}, \dots, v_{J_k}$ associated to the same vertex of C_τ it is easy to see that they span a cone in $\mathcal{M}_{0,n}^{\text{trop}}$ if and only if their images do. Now if $\sigma \geq \tau$ is any cone, we can partition its rays into subsets $S_j, j = 1, \dots, m$ that are associated to the same vertex p_j. Each of these sets of rays span a cone σ_j which is mapped to a cone in $\mathcal{M}_{\text{val}(p_j)}$. Since $\sigma = \sigma_1 \times \dots \times \sigma_m$, it is mapped to a cone in $\mathcal{M}_{\text{val}(p_1)} \times \dots \times \mathcal{M}_{\text{val}(p_m)}$. Hence ψ_τ is an isomorphism.

Finally, if $\mathcal{M}$ is any polyhedral structure, let τ' be the minimal cone of the combinatorial subdivision containing τ. Then $l = \dim \tau'$ and we have

$$\text{Star}_{\mathcal{M}_{0,n}^{\text{trop}}}(\tau) \cong \mathbb{R}^{l-d} \times \text{Star}_{\mathcal{M}_{0,n}^{\text{trop}}}(\tau')$$

□

5. Tropical layerings

Many operations in tropical geometry require us to refine a tropical variety in a certain way to make it compatible with these operations. For example, when defining the push-forward f_*X of a variety, we have to refine X, such that the images of its cells form a polyhedral complex. These refinements are usually fairly easy to define in theory but very hard to compute. For a push-forward, we have to intersect the variety with all halfspace complexes defined by all possible equations of the image cones (see [GKM, p.9] for a description of the construction). For relatively small and simple varieties this already means an immense amount of computations and one quickly finds this approach to be infeasible for all practical computational matters.

The idea is now to replace a tropical variety with a new data type that allows cells to overlap but still incorporates some polyhedral structure and locally looks like a tropical object. This data type can then be used to apply all tropical operations locally. As long as the final result of these computations is simple enough (usually one is interested in intersection numbers, which can easily be read off from a finite set of points with weights), this can still produce meaningful results in much less time.

5.1. Layerings

Definition 5.1.1. A d-dimensional *polyhedral layering* $(\Sigma, \tau(\cdot))$ consists of a finite multiset (i.e. some of the elements may appear more than once) of d-dimensional rational polyhedra $\Sigma = \{(1,\sigma_1),\dots,(r,\sigma_r)\}$ in $\mathbb{R}^n$ (The additional index is used to distinguish identical polyhedra in the multiset, but we will often write this $\{\sigma_1,\dots,\sigma_r\}$ for shortness), together with a collection of polyhedra

$$\tau(\sigma_i,\sigma_j) =: \tau_{ij} \text{ for all } i,j \in [r]$$

that fulfill the following criteria

1. $\tau_{ij} \leq \sigma_i, \tau_{ij} \leq \sigma_j$
2. $\tau_{ii} = \sigma_i$ and $\tau_{ij} = \tau_{ji}$
3. For all $i,j,k \in [r]$ we have $\tau_{ij} \cap \tau_{jk} \subseteq \tau_{ik}$ ("Intersection is associative")

We call Σ a *fan layering*, if all the σ_i are cones. A *weighted polyhedral layering* $(\Sigma, \tau(\cdot), \omega)$ is a polyhedral layering together with a weight function $\omega : \Sigma \to \mathbb{Z}$.

For any $i \in \{1, \ldots, r\}$ and any codimension one face $\tau \leq \sigma_i$, we denote by S^i_τ (or $S^{\sigma_i}_\tau$) the set of all maximal cells σ_j, such that $\tau \subseteq \tau_{ij}$. A weighted polyhedral layering is *balanced at* τ *in* σ_i, if

$$\sum_{\sigma \in S^i_\tau} \omega(\sigma) \cdot u_{\sigma/\tau} \in V_\tau$$

We call $(\Sigma, \tau(\cdot), \omega)$ a *tropical layering*, if it is balanced at every τ in every σ_i. By abuse of notation we will also denote a weighted layering by Σ.

Furthermore we define $\dim(\Sigma) = d$ and $|\Sigma| = \bigcup_{i=1}^r \sigma_i$.

Example 5.1.2. As an example, we will consider a double version of the tropical line in $\mathbb{R}^2$. More precisely, let $\Sigma = \{(1, \sigma), (2, \sigma'), (3, \sigma''), (4, \sigma), (5, \sigma'), (6, \sigma'')\}$, where

$$\begin{aligned} \sigma &:= \mathbb{R}_{\geq 0} \cdot (1, 1) \\ \sigma' &:= \mathbb{R}_{\geq 0} \cdot (-1, 0) \\ \sigma'' &:= \mathbb{R}_{\geq 0} \cdot (0, -1). \end{aligned}$$

For $\rho, \xi \in \{\sigma, \sigma', \sigma''\}$, we also define

$$\tau((i, \rho), (j, \xi)) := \begin{cases} \rho \cap \xi, & \text{if } \{i, j\} \subseteq \{1, 2, 3\} \text{ or } \{i, j\} \subseteq \{4, 5, 6\} \\ \varnothing, & \text{otherwise.} \end{cases}$$

One can easily check that this fulfills all axioms. In addition, if we set all weights to 1, we obtain a tropical layering. Topologically, this layering can be interpreted as the disjoint union of two tropical lines (see also Definition 5.1.4).

Remark 5.1.3. Before we start analyzing the properties of this new object, let us first discuss its definition: The idea is that, since the σ_i can intersect in any possible way (some of them may even coincide), we explicitly "define" their intersection via the τ_{ij}. Of course this can't be done in an arbitrary manner. The first property of the τ_{ij} ensures that cells only "intersect" in faces. The second property makes sure that the intersection of each cell with itself is again this cell and that intersection is commutative. Finally the third property ensures that the different intersections "fit together". For example, the following would not be a polyhedral layering: Let $\sigma_1 = \sigma_2 = \langle e_1, e_2 \rangle_{\geq 0}$ and $\sigma_3 = \langle e_2, e_3 \rangle_{\geq 0}$, where e_i is the i-th standard basis vector of $\mathbb{R}^3$. Now define $\tau_{12} = \tau_{23} = \langle e_2 \rangle_{\geq 0}$ and $\tau_{13} = \varnothing$. This would mean that σ_2 intersects the two other cones both in the same codimension one face, while the first and the third cone do not intersect at all, which is obviously not what we would want. In fact this contradicts the third property, as we have

$$\langle e_2 \rangle = \tau_{12} \cap \tau_{23} \nsubseteq \tau_{13} = \varnothing$$

Of course, any pure polyhedral complex gives rise to a polyhedral layering by defining $\tau(\sigma, \sigma') = \sigma \cap \sigma'$.

A good geometric picture of a polyhedral layering is given by the following topological space:

Definition 5.1.4. Let (Σ, τ) be a d-dimensional polyhedral layering. We define its *separated layer space* to be the topological space

$$\Sigma^{sep} := \left(\coprod_{i=1}^{r} \sigma_i \right) / \sim$$

where $\sigma_i \ni p \sim p' \in \sigma_j$ if and only if $p = p' \in \tau_{ij}$. For a weighted layering with weight function ω we define

$$\Sigma^{sep}_{\omega} := \{p \in \Sigma^{sep} \text{ s.t. there exists } \sigma \in \Sigma \text{ with } p \in \sigma, \omega(\sigma) \neq 0\}$$

We define the k-skeleton $\Sigma^{(k)}$ of Σ to be the set of all pairs

$$\Sigma^{(k)} := \{(\rho, \sigma),\ \rho \leq \sigma \in \Sigma; \dim(\rho) = k\} / \sim$$

where $(\rho, \sigma) \sim (\rho', \sigma')$, if and only if $\rho = \rho' \subseteq \tau(\sigma, \sigma')$. For the sake of notation, we will often write an element of $\Sigma^{(k)}$ as ρ instead of (ρ, σ). The *support* of an element $\xi = (\rho, \sigma) \in \Sigma^{(k)}$ will be $|\xi| := \rho$.

It is easy to see that for a weighted layering, a given codimension one face τ and two equivalent elements $(\tau, \sigma) \sim (\tau, \sigma') \in \Sigma^{(d-1)}$, the layering is balanced at τ in σ, if and only if it is balanced at τ in σ'. Hence we will also simply say that Σ is balanced at $\tau \in \Sigma^{(d-1)}$.

The k-skeleton naturally defines a topological subspace of Σ^{sep}, which by abuse of notation we will also denote by $\Sigma^{(k)}$.

For two elements $\xi \in \Sigma^{(k)}, \xi' \in \Sigma^{(l)}$ with $k \leq l$, we will say that ξ is a *face* of ξ', denoted by $\xi \leq \xi'$, if $\xi \subseteq \xi'$ in Σ^{sep} and $|\xi|$ is a face of $|\xi'|$. This allows us to reformulate our balancing condition for layerings: A weighted layering (Σ, τ, ω) is balanced, if for every $\tau \in \Sigma^{(d-1)}$, we have

$$\sum_{\sigma > \tau} \omega(\sigma) \cdot u_{\sigma/\tau} \in V_\tau,$$

where of course $u_{\sigma/\tau} := u_{|\sigma|/|\tau|}$.

As with tropical varieties, we don't want to distinguish between layerings that only differ by some refinement. We have to make precise what this actually means:

Definition 5.1.5. A weighted d-dim. polyhedral layering $(\Sigma' = \{\sigma'_1, \dots, \sigma'_s\}, \tau', \omega')$ is a *refinement* of a weighted layering $(\Sigma = \{\sigma_1, \dots, \sigma_r\}, \tau, \omega)$ (of the same dimension), if there exists a function

$$C : [s] \smallsetminus \{i : \omega'(\sigma'_i) = 0\} \to [r]$$

(we also write $C(\sigma_i)$ instead of $C(i)$), such that

- $\sigma'_i \subseteq \sigma_{C(i)}$
- $\omega'(\sigma'_i) = \omega(\sigma_{C(i)})$
- $\tau'_{ij} = \tau_{C(i)C(j)} \cap \sigma'_i \cap \sigma'_j$

- The induced map $i_C : \Sigma'^{sep}_{\omega'} \to \Sigma^{sep}_{\omega}$ is bijective (the third property actually ensures that this map is well-defined).

It is easy to see that the properties of a polyhedral layering follow from this.

We say that two weighted polyhedral layerings are *equivalent*, if they have a common refinement.

Of course we now want to make sure that the property of being balanced does not depend on the choice of the representative of a layering. This follows from the following lemma:

Lemma 5.1.6. *Let $(\Sigma, \tau, \omega) \sim (\Sigma', \tau', \omega')$. Then Σ is balanced if and only if Σ' is.*

Proof. This proof is more or less analogous to the one in [GKM, Ex. 2.11(d)], but we include it here for completeness:

We can assume that Σ' is a refinement of Σ. Let τ' be a codimension one face of a cell $\sigma'_i \in \Sigma'$ and let C_τ be the minimal face of $\sigma_{C(i)}$ containing τ'. First assume that $\dim(C_\tau) = \dim(\Sigma)$. Now for all σ'_j with $\tau'_{ij} = \tau'$, we have

$$\tau' = \tau'_{ij} \subseteq \tau_{C(i)C(j)} \leq \sigma_{C(i)}$$

Hence $C(j) = C(i)$. Now bijectivity of i_C implies that there are only two such maximal cells σ'_i, σ'_j, which must have opposite normal vectors and identical weights.

Now if $\dim(C_\tau) = d - 1$, then for any σ'_j with $\tau'_{ij} \geq \tau'$, we must have that C_τ is a face of $\sigma_{C(j)}$. Bijectivity of i_C implies that the map from the set of all these maximal cells σ'_j to the set of their $\sigma_{C(j)}$ is bijective. Since the normal vectors and weights are the same, the balancing condition of Σ' at τ' and of Σ at C_τ are equivalent. □

Example 5.1.7. Let $(\Sigma = \{\sigma_1, \dots, \sigma_r\}, \tau, \omega)$ be a tropical layering. We want to define a refinement $\Sigma' = \{\sigma'_1, \dots, \sigma'_s\}$, such that the *set* $\{\sigma'_1, \dots, \sigma'_r\}$ is the set of maximal cells of a polyhedral complex:

Let $\{g_i \geq \alpha_i, i \in I\}$ be the set of all defining inequalities of all the cells $\sigma \in \Sigma$ and define

$$H_i^+ := \{x \in \mathbb{R}^n : g_i(x) \geq \alpha_i\}$$
$$H_i^- := \{x \in \mathbb{R}^n : g_i(x) \leq \alpha_i\}$$

Now let S_k be the *set* of d-dimensional cones in

$$\{\sigma_k \cap H_i^{+/-}, k = 1, \dots, r; i \in I\}$$

and define

$$\Sigma' := \bigcup_{k=1}^{r} \{(k, \sigma'), \sigma' \in S_k\}$$

Is is easy to see that these cells form a polyhedral complex (with some cells possibly occuring several times). We now set

$$\tau'(\sigma_k \cap H_i, \sigma_l \cap H_j) = \tau_{kl} \cap H_i \cap H_j$$

and $\omega'(\sigma_k \cap H_i) := \omega(\sigma_k)$. This obviously defines a refinement of Σ (and hence a polyhedral layering).

Definition 5.1.8. Let (Σ, τ, ω) be a weighted layering. Choose a refinement $\Sigma' = \{\sigma'_1, \ldots, \sigma'_s\}$ of the σ_i such that they form a polyhedral complex (see previous example) and define a function on the *set* of these cells

$$\omega^{var} : \{\sigma'_i, i = 1, \ldots, s\} \to \mathbb{Z}, \sigma'_i \mapsto \sum_{j:\sigma'_j = \sigma'_i} \omega(\sigma'_j)$$

Then $\Sigma^{var} := (\{\sigma'_i, i = 1, \ldots, s\}, \omega^{var})$ is a weighted polyhedral complex and it is easy to see that, if Σ is balanced, this complex must also be balanced. Furthermore it is obvious that a different refinement of Σ would give an equivalent tropical variety. Hence we call Σ^{var} the *associated tropical variety* of Σ.

Conversely, to each tropical variety (X, ω), we can canonically assign a tropical layering, which we denote $(\Sigma_X, \tau_x, \omega_X)$, where $\Sigma_X = X^{(\dim X)}$, $\tau(\sigma, \sigma') = \sigma \cap \sigma'$ and $\omega_X(\sigma) = \omega(\sigma)$. We call this the *canonical layering* of X.

Definition 5.1.9. Let (Σ, τ, ω) be a d-dimensional weighted layering. Let $\rho \in \Sigma^{(k)}$ for some $k \leq d$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^n / V_\rho$ be the residue class map and write $c_\rho(\sigma) := \mathbb{R}_{\geq 0} \cdot \pi(\sigma - \rho)$ for any polyhedron $\sigma \geq \rho$. We define a weighted fan layering $\mathrm{Star}_\Sigma(\rho) := (\Sigma_\rho, \tau_\rho, \omega_\rho)$ in $\mathbb{R}^n / V_\rho$, where

$$\begin{aligned} \Sigma_\rho &:= \{c_\rho(\sigma); \sigma \in \Sigma^{(\dim \Sigma)}\} \\ \tau_\rho(c_\rho(\sigma), c_\rho(\sigma')) &:= c_\rho(\tau(\sigma, \sigma')) \\ \omega_\rho(c_\rho(\sigma)) &:= \omega(\sigma) \end{aligned}$$

We want to see explicitly that the associativity condition on τ_ρ is satisfied: Let $\sigma, \sigma', \sigma'' > \rho$. We have

$$\begin{aligned} \tau_\rho(c_\rho(\sigma), c_\rho(\sigma')) \cap \tau_\rho(c_\rho(\sigma'), c_\rho(\sigma'')) &= c_\rho(\tau(\sigma, \sigma')) \cap c_\rho(\tau(\sigma', \sigma'')) \\ &\overset{(*)}{=} c_\rho(\tau(\sigma, \sigma') \cap \tau(\sigma', \sigma'')) \\ &\subseteq c_\rho(\tau(\sigma, \sigma'')) \\ &= \tau_\rho(c_\rho(\sigma), c_\rho(\sigma'')) \end{aligned}$$

where $(*)$ can be proven using the following argument: Let σ_1, σ_2 be two polyhedral cells containing ρ as a proper face. Then $c_\rho(\sigma_1) \cap c_\rho(\sigma_2) = c_\rho(\sigma_1 \cap \sigma_2)$. In fact, the inclusion "$\supseteq$" is easy, it remains to see that the other inclusion holds as well: Let $v \in c_\rho(\sigma_1) \cap c_\rho(\sigma_2)$. Hence there are $\alpha, \alpha' > 0$, $p, p' \in \rho$, $q \in \sigma_1, q' \in \sigma_2$, such that $v = \alpha(q - p) = \alpha'(q' - p')$. Since $\rho \subseteq \sigma_1, \sigma_2$, we get two two-dimensional polytopes $\mathrm{conv}\{q, p, p'\}, \mathrm{conv}\{q', p, p'\}$, that are contained in σ_1 and σ_2 respectively. As $q-p, q'-p'$ are linearly equivalent, they must intersect in a common point $q'' \in \sigma_1 \cap \sigma_2 \smallsetminus \rho$ and we can write this point as $q'' = p + \lambda(p' - p) + \delta(q' - p')$, where $\lambda \in [0, 1], \delta > 0$. But this implies

$$v = \frac{\alpha'}{\delta} \left(q'' - \underbrace{(p + \lambda(p' - p))}_{\in \rho} \right) \in c_\rho(\sigma_1 \cap \sigma_2).$$

5.2. Rational functions, divisors and morphisms

Definition 5.2.1 (see also Definition 2.2.1)**.** Let (Σ, τ, ω) be a tropical layering. A *rational function* on Σ is a piecewise affine linear integer function

$$\varphi : |\Sigma| \to \mathbb{R}.$$

Now let φ be a rational function on Σ and choose a refinement of Σ, such that φ is piecewise affine linear on each $\sigma \in \Sigma$. Again we denote by φ_ρ the linear part of $\varphi_{|\rho}$ for any $\rho \in \Sigma^{(k)}, k < \dim \Sigma$. Then we define the *divisor* of φ on Σ to be $\varphi \cdot \Sigma := (\Sigma^{(d-1)}, \tau_\varphi, \omega_\varphi)$, where

$$\tau_\varphi((\rho,\sigma),(\rho',\sigma')) := \tau(\sigma,\sigma') \cap \rho \cap \rho'$$
$$\omega_\varphi(\rho) := \sum_{\sigma > \rho} \omega(\sigma) \cdot \varphi_\sigma(v_{\sigma/\rho}) - \varphi_\rho \left(\sum_{\sigma > \tau} \omega(\sigma) \cdot v_{\sigma/\rho} \right).$$

Here the $v_{\sigma/\rho}$ are arbitrary choices of representatives of normal vectors. As in [R1] it is easy to see that a different choice of these representatives or a different choice of refinement would give an equivalent weighted layering.

Before we check that it is balanced, let us see that the intersections τ_φ are well-defined (it is easy to check that they actually give a polyhedral layering). So assume $(\rho,\sigma) \sim (\rho,\theta)$ and $(\rho',\sigma') \sim (\rho',\theta')$ are elements in $\Sigma^{(d-1)}$. This implies that $\rho \subseteq \tau(\sigma,\theta)$ and $\rho' \subseteq \tau(\sigma',\theta')$. Hence we get

$$\begin{aligned}
\tau_\varphi((\rho,\sigma),(\rho,\sigma')) &= \tau(\sigma,\sigma') \cap \rho \cap \rho' = \tau(\sigma,\sigma') \cap \tau(\sigma',\theta') \cap \rho \cap \rho' \\
&\subseteq \tau(\sigma,\theta') \cap \rho \cap \rho' = \tau(\theta,\sigma) \cap \tau(\sigma,\theta') \cap \rho \cap \rho' \\
&\subseteq \tau(\theta,\theta') \cap \rho \cap \rho' = \tau_\varphi((\rho,\theta),(\rho',\theta'))
\end{aligned}$$

and vice versa for the other inclusion.

Lemma 5.2.2.

1. *Let Σ be a d-dimensional weighted layering. For any $\rho \in \Sigma^{(d-1)}$,* $\mathrm{Star}_\Sigma(\rho)$ *is balanced if and only if* $\mathrm{Star}_\Sigma(\rho)^{var}$ *is balanced.*
2. *Let φ be a rational function on the weighted layering (Σ,τ,ω). Let $\rho \in \Sigma^{(k)}, k < d$ and denote by φ_ρ the induced function on* $\mathrm{Star}_\Sigma(\rho)$*. Then*

$$\varphi_\rho \cdot \mathrm{Star}_\Sigma(\rho) = \mathrm{Star}_{\varphi \cdot \Sigma}(\rho)$$

3. *Under the same assumptions as before, we have*

$$(\varphi_\rho \cdot \mathrm{Star}_\Sigma(\rho))^{var} = \varphi \cdot (\mathrm{Star}_\Sigma(\rho)^{var})$$

Proof. For the first statement it is very easy to see that the balancing conditions on the two objects are equivalent, since each cell occuring in the balancing condition of $\mathrm{Star}_\Sigma(\rho)^{var}$ must also occur in $\mathrm{Star}_\Sigma(\rho)$ (possibly several times).

The second statement can be proven in exactly the same way as in [R1, Prop. 1.2.12].

For the third statement we only have to see that the weights on the maximal cells agree. Let ξ be a codimension one cell of $\mathrm{Star}_\Sigma(\rho)^{var}$. Denote by $\rho_1, \ldots, \rho_k$ the elements of $\mathrm{Star}_\Sigma(\rho)^{(\dim \mathrm{Star}-1)}$ such that $|\rho_i| = \xi$. We denote the weight functions of $\mathrm{Star}_\Sigma(\rho)^{var}$ and $\varphi \cdot (\mathrm{Star}_\Sigma(\rho)^{var})$ by ω_S and $\omega_{\varphi S}$ respectively. Then

$$\begin{aligned}
\omega_{\varphi S}(\xi) &= \sum_{\zeta > \xi} \omega_S(\zeta) \cdot \varphi_\zeta(v_{\zeta/\xi}) - \varphi_\xi \left(\sum_{\zeta>\xi} \omega_S(\zeta) \cdot v_{\zeta/\xi} \right) \\
&= \sum_{\zeta>\xi} \left(\sum_{i=1}^k \sum_{\substack{\sigma > \rho_i \\ |\sigma| = \zeta}} \omega_\rho(\sigma) \right) \varphi_\zeta(v_{\zeta/\xi}) - \varphi_\xi \left(\sum_{\zeta>\xi} \left(\sum_{i=1}^k \sum_{\substack{\sigma > \rho_i \\ |\sigma| = \zeta}} \omega_\rho(\sigma) \right) v_{\zeta/\xi} \right) \\
&= \sum_{\zeta>\xi} \left(\sum_{i=1}^k \sum_{\substack{\sigma > \rho_i \\ |\sigma| = \zeta}} \omega_\rho(\sigma) \varphi_\sigma(v_{\sigma/\rho_i}) \right) - \varphi_\xi \left(\sum_{i=1}^k \left(\sum_{\zeta>\xi} \sum_{\substack{\sigma > \rho_i \\ |\sigma| = \zeta}} \omega_\rho(\sigma) \cdot v_{\sigma/\rho_i} \right) \right)
\end{aligned}$$

Note that the term in the rightmost interior bracket must lie in $V_{\rho_i} = V_\xi$, since it is just the balancing sum for ρ_i reformulated. Hence we can extract the sum from φ_ξ and reformulate:

$$\begin{aligned}
\omega_{\varphi S}(\xi) &= \sum_{i=1}^k \left(\sum_{\sigma > \rho_i} \omega_\rho(\sigma) \varphi_\sigma(v_{\sigma/\rho_i}) - \varphi_{\rho_i} \left(\sum_{\sigma>\rho_i} \omega_\rho(\sigma) \cdot v_{\sigma/\rho_i} \right) \right) \\
&= \sum_{i=1}^k \omega_{\varphi_\rho \mathrm{Star}_\Sigma(\rho)}(\rho_i)
\end{aligned}$$

□

Corollary 5.2.3. *Let Σ be a tropical layering, φ a rational function on Σ. Then $\varphi \cdot \Sigma$ is a tropical layering.*

Proof. We show that $\mathrm{Star}_{\varphi\cdot\Sigma}(\rho)$ is balanced for each cell $\rho \in \Sigma^{(\dim \Sigma - 2)}$. By the previous lemma, this is true if and only if $\mathrm{Star}_{\varphi\cdot\Sigma}(\rho)^{var}$ is balanced. Using the second and third part of the lemma, we get

$$\begin{aligned}
\mathrm{Star}_{\varphi\cdot\Sigma}(\rho)^{var} &= (\varphi_\rho \cdot \mathrm{Star}_\Sigma(\rho))^{var} \\
&= \varphi \cdot (\mathrm{Star}_\Sigma(\rho)^{var})
\end{aligned}$$

and by [R1, Prop. 1.2.13], the latter is balanced. □

Remark 5.2.4. Note that the first statement of lemma 5.2.2 is false in general, if the codimension of ρ is larger then 1. For example, take two copies of the four quadrants of the plane, glued together in the origin p. Assign weights as in figure 5.1 to obtain a weighted layering Σ. Then $\mathrm{Star}_\Sigma(p) = \Sigma$ is not balanced, but $\mathrm{Star}_\Sigma(p)^{var}$ is.

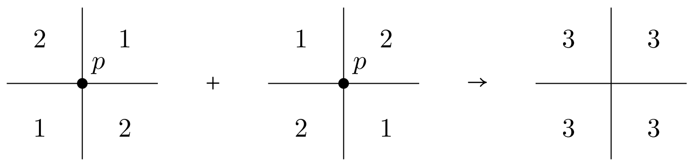

Figure 5.1.: The weighted layering is not balanced around the origin, but the associated weighted complex is.

Corollary 5.2.5. *Let* (Σ, τ, ω) *be a tropical layering,* φ *a rational function on* Σ*. Then*

$$(\varphi \cdot \Sigma)^{var} = \varphi \cdot \Sigma^{var}$$

Proof. This follows directly from the last statement of Lemma 5.2.2. □

Definition 5.2.6. We define a morphism of tropical layerings Σ in $\mathbb{R}^n$,Σ' in $\mathbb{R}^m$ to be a map $f : \Sigma^{sep} \to \Sigma'^{sep}$, such that f is locally induced by an integer affine function, i.e. on $\text{Star}_\Sigma(p)$ it is defined by a *global* linear integer function $f_p : \mathbb{R}^n \to \mathbb{R}^m$ for each $p \in \Sigma^{sep}$.

Example 5.2.7. Let $f : \Sigma \to \Sigma'$ be a morphism of tropical layerings. We define the *push-forward* $f_*\Sigma := (\Sigma_f, \tau_f, \omega_f)$ via

$$\begin{aligned} \Sigma_f &:= \{f(\sigma); \sigma \in \Sigma^{(\dim X)}, f_{|\sigma} \text{ injective}\} \\ \tau_f(f(\sigma), f(\sigma')) &:= f(\sigma \cap \sigma') \\ \omega_f(f(\sigma)) &:= \omega(\sigma) \cdot \left|\Lambda_{f(\sigma)}/f(\Lambda_\sigma)\right| \end{aligned}$$

One can easily check that this is a polyhedral layering. To see that it is balanced, one can easily modify the proof of [GKM, Prop. 2.25].

Now let $f : X \to Y$ be a morphism of tropical varieties. This induces a morphism $f : \Sigma_X \to \Sigma_Y$ between the canonical layerings. It is easy to see that $(f_*\Sigma_X)^{var} = f_*X$.

Part II.

An application: Tropical double Hurwitz cycles

6. Hurwitz numbers and cycles

As already discussed in the introduction, Hurwitz numbers count covers of $\mathbb{P}^1$ with a certain ramification profile. More precisely: Fix $m, d \geq 1, g \geq 0$ and ramification profiles $\alpha_i \in \mathbb{N}^{l_i}, i = 1, \ldots, m$ with $\alpha_i = (\alpha_i^1, \ldots, \alpha_i^{l_i}) \in \mathbb{N}^{l_i}, l_i \geq 1$ and $\sum_{j=1}^{l_i} \alpha_i^j = d$. Also fix distinct points $a_1, \ldots, a_m, b_1, \ldots, b_r$ in $\mathbb{P}^1$, where $r = 2g - 2 + d(2-m) + \sum_{i=1}^m l_i$. Then m-fold Hurwitz numbers $H_{d,g}(\alpha_1, \ldots, \alpha_m)$ count degree d covers of $\mathbb{P}^1$ by smooth genus g curves such that the map has ramification profile α_i over a_i and simple ramification over each b_j.

The following chapters only consider double Hurwitz cycles in genus 0, i.e. $m = 2$ and $g = 0$. Recall that in this setup we fix $(a_1, a_2) = (0, \infty)$ and we write the ramification profile as $x \in \mathbb{Z}^n$ with $\sum_{i=1}^n x_i = 0$. The interpretation is that the positive entries x^+ provide the ramification profile over 0 and the negative part x^- gives the ramification over ∞. Also note that $r = n - 2$ only depends on n. The higher-dimensional cycles can then be defined by letting the simple ramification points move. A *marked* version is obtained by labeling the preimages of the simple ramification points.

We will first define the algebraic Hurwitz cycles in 6.1. After introducing the tropical space of stable maps in 6.2, we will then give the tropical definition of marked and unmarked Hurwitz cycles as subcycles of the moduli space of stable maps and the space of rational curves, respectively in 6.3. We also briefly discuss an algorithm to compute Hurwitz cycles in 6.4. In chapter 7 we will then study some properties of these cycles: In 7.1 we show first that all Hurwitz cycles, marked or unmarked, are connected in codimension one. This is a purely combinatorial statement, which we prove by induction on the number of leaves. This can be used to show that marked Hurwitz cycles are irreducible up to multiplication by a constant for a generic choice of simple ramification image points. For all other cases (unmarked cycles, points in degenerate position, strict irreducibility) we give some negative (computational) examples. In Section 7.2 we prove that all codimension one Hurwitz cycles (with totally degenerate simple ramification images) are divisors of a rational function. For this we introduce the notion of a *push-forward* of a rational function to a smooth target. The function itself can be shown to have a very intuitive geometric meaning: It simply adds up all pairwise distances of images of vertices in the cover defined by an element in the Hurwitz cycle. We conclude the discussion of Hurwitz cycles by studying the “tropicalization” of a different representation of the algebraic cycles and its relation to our original definition.

Note also that all of these questions can now be answered computationally for concrete examples: Checking if a polyhedral complex is connected in codimension one is a simple feat of combinatorics if the codimension one cells are known. We discussed how to check irreducibility in Section 2.3 and saw in 2.2.1 that the inverse divisor problem is a simple linear algebra task.

6.1. Algebraic Hurwitz cycles

We will only briefly cover algebraic Hurwitz cycles. For a more in-depth discussion of its definition and properties, see for example [BCM, GV].

Let $n \geq 4$. We define

$$\mathcal{H}_n := \left\{ x \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i = 0 \right\} \setminus \{0\}.$$

Let $x \in \mathcal{H}_n$ and choose $p_0, \ldots, p_{n-3-k} \in \mathbb{P}^1 \setminus \{0, \infty\}$ (not necessarily distinct). The *double Hurwitz cycle* $\mathbb{H}_k(x)$ is a k-dimensional cycle in the moduli space of rational n-marked curves $\overline{M}_{0,n}$. It parametrizes curves C that *allow* covers $C \xrightarrow{\pi} \mathbb{P}^1$ with the following properties:

- C is a smooth connected rational curve.
- π has ramification profile $(x_i; x_i > 0)$ over 0 and ramification profile $(x_i; x_i < 0)$ over ∞. The corresponding ramification points are the marked points of C.
- π has simple ramification over the p_i and at most simple ramification elsewhere.

The precise definition [BCM, Section 3] actually involves some sophisticated moduli spaces. For the sake of simplicity, we will just cite the following result, that can be taken as a definition throughout this paper.

Lemma 6.1.1 ([BCM, Lemma 3.2])**.**

$$\mathbb{H}_k(x) = st_* \left(\prod_{i=1}^{n-2-k} \psi_i \mathrm{ev}_i^*([pt]) \right),$$

where

- $\overline{M}_{0,n-2-k}$ *is the space of stable maps to* $\mathbb{P}^1$ *with ramification profile* x^+, x^- *over* 0 *and* ∞*.*
- $st : \overline{M}_{0,n-2-k}(x) \to \overline{M}_{0,n}$ *is the morphism forgetting the map and all marked points but the ramification points over* 0 *and* ∞*.*

6.2. Tropical stable maps

To translate the formula of Lemma 6.1.1 to tropical geometry, we need a tropical *space of stable maps*. A precise definition can be found in [GKM, Section 4]. For shortness, we will use their result from Proposition 4.7 as definition and explain the geometric interpretation behind it afterwards.

Definition 6.2.1. Let $m \geq 4, r \geq 1$. For any $\Delta = (v_1, \ldots, v_n), v_i \in \mathbb{R}^r$ with $\sum v_i = 0$ we denote by

$$\mathcal{M}_{0,m}^{\text{trop}}(\mathbb{R}^r, \Delta) := \mathcal{M}_{0,n+m} \times \mathbb{R}^r$$

the *space of stable m-pointed maps of degree* Δ.

Remark 6.2.2. An element of $\mathcal{M}_{0,m}^{\text{trop}}(\mathbb{R}^r, \Delta)$ represents an $(n+m)$-marked abstract curve C together with a continuous, piecewise integer affine linear (with respect to the metric on C) map $h : C \to \mathbb{R}^r$. We label the first n leaves by $\{1, \ldots, n\}$ and require h to have slope $v_1, \ldots, v_n$ on them. The last m leaves we denote by $l_0, \ldots, l_{m-1}$. These are contracted to a point under h. Since we want the image curve to be a tropical curve in $\mathbb{R}^r$, the slope on the bounded edges is already uniquely defined by the condition that the outgoing slopes of h at each vertex have to add up to 0. This defines the map h up to a translation in $\mathbb{R}^r$. The translation is fixed by the $\mathbb{R}^r$-coordinate, which can for example be interpreted as the image of the first contracted end l_0 under h (see figure 6.1 for an example). There are obvious evaluation maps $\text{ev}_i : \mathcal{M}_{0,m}^{\text{trop}}(\mathbb{R}^r, \Delta) \to \mathbb{R}^r, i = 0, \ldots, m-1$, mapping a stable map to $h(l_i)$. [GKM, Proposition 4.8] shows that these are morphisms. Similarly, there is a *forgetful morphism* $\text{ft} : \mathcal{M}_{0,m}^{\text{trop}}(\mathbb{R}^r, \Delta) \to \mathcal{M}_{0,n}^{\text{trop}}$, forgetting the contracted ends and the map h.

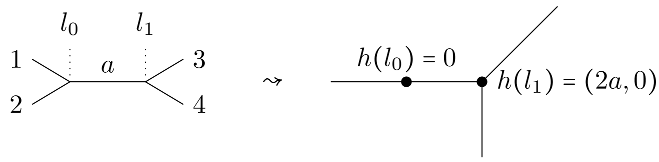

Figure 6.1.: On the left the abstract 6-marked curve $\Gamma = a \cdot v_{\{1,2,l_0\}}$. If we pick $\Delta = ((-1,0), (-1,0), (2,2), (0,-2))$ and fix $h(l_0) = 0$ in $\mathbb{R}^2$, we obtain the curve on the right hand side as $h(\Gamma)$.

6.3. Tropical Hurwitz cycles

We now have all ingredients at hand to "tropicalize" Lemma 6.1.1. Note that a point $q \in \mathbb{R}$ can be considered as the divisor of the tropical polynomial $\max\{x, q\}$, so it can be pulled back along a morphism to $\mathbb{R}$. Also, as $\mathcal{M}_{0,n+m}^{\text{trop}}$ is a subcycle of $\mathcal{M}_{0,m}^{\text{trop}}(\mathbb{R}^r, \Delta) = \mathcal{M}_{0,n+m}^{\text{trop}} \times \mathbb{R}^r$, we can define Psi-classes on the latter: For $i = 0, \ldots m-1$, we define

$$\Psi_i := \psi(l_i) \times \mathbb{R}^r,$$

where $\psi(l_i)$ is the Psi-class of $\mathcal{M}_{0,n+m}^{\text{trop}}$ associated to the leaf l_i we defined in Section 4.3.

Definition 6.3.1. Let $x \in \mathbb{Z}^n \setminus \{0\}$ with $\sum x_i = 0, k \geq 0$ and $N := n - 2 - k$. Choose $p := (p_0, \ldots, p_{N-1}), p_i \in \mathbb{R}$. We define the *tropical marked Hurwitz cycle*

$$\tilde{\mathbb{H}}_k^{\text{trop}}(x, p) := \left(\prod_{i=0}^{N-1} (\Psi_i \text{ev}_i^*(p_i)) \right) \cdot \mathcal{M}_{0,N}^{\text{trop}}(\mathbb{R}, x)$$

We then define the *tropical Hurwitz cycle*

$$\mathbb{H}_k^{\text{trop}}(x, p) := \text{ft}_*(\tilde{\mathbb{H}}_k^{\text{trop}}(x, p)) \subseteq \mathcal{M}_{0,n}^{\text{trop}}.$$

Remark 6.3.2. This definition is obviously the exact analogue of Lemma 6.1.1 and gives us k-dimensional tropical cycles $\tilde{\mathbb{H}}_k^{\text{trop}}(x, p), \mathbb{H}_k^{\text{trop}}(x, p)$. While it formally depends on the choice of the p_j, two different choices p, p' lead to rationally equivalent cycles $\mathbb{H}_k^{\text{trop}}(x, p) \sim \mathbb{H}_k^{\text{trop}}(x, p')$. The reason for this is that any two points in $\mathbb{R}$ are rationally equivalent and this is compatible with pullbacks and taking intersection products. In particular, if we choose all p_i to be equal (e.g. equal to 0), we obtain fans, which we denote by $\tilde{\mathbb{H}}_k^{\text{trop}}(x)$ and $\mathbb{H}_k^{\text{trop}}(x)$. They are obviously the recession fans of $\tilde{\mathbb{H}}_k^{\text{trop}}(x, p), \mathbb{H}_k^{\text{trop}}(x, p)$ for any p.

Example 6.3.3. Let us now see what kind of object these Hurwitz cycles represent. As discussed in Remark 6.2.2, for any fixed x and any n-marked curve C we obtain a map $h : C \to \mathbb{R}$ up to translation. To determine such a map, we have to fix an orientation of each edge and leaf of C and an integer slope along this orientation. In informal terms, the orientation determines, how we position an edge or leaf on $\mathbb{R}$ (the "tip" of the arrow points towards $+\infty$). The slope can then be seen as a stretching factor.

The orientation of each leaf i is chosen so that it "points away" from its vertex if and only if $x_i > 0$. We define its slope to be $|x_i|$. Any bounded edge e induces a split I_e. Its slope is $|x_{I_e}|$, where $x_{I_e} = \sum_{i \in I_e} x_i$. We pick the orientation such that at each vertex the sum of slopes of incoming edges is the sum of slopes of outgoing edges (it is not hard to see that such an orientation exists and must be unique).

As we discussed before, we can fix the translation of h by requiring the image of any of its vertices q to be some $\alpha \in \mathbb{R}$. Denote by $h(C, q, \alpha) : C \to \mathbb{R}$ the corresponding map. Figure 6.2 gives two examples of this construction.

Now choose $p_0, \ldots, p_{N-1} \in \mathbb{R}$. Then $\mathbb{H}_k^{\text{trop}}(x, p)$ is (set-theoretically) the set of all curves C, where we can find vertices $q_0, \ldots, q_{N-1}$ (each vertex q can be picked a number of times equal to $\text{val}(q) - 2$), such that $h(C, q_0, p_0)(q_l) = p_l$ for all l, i.e. all curves that *allow* a cover with fixed images for some of its vertices. E.g. in Figure 6.2, we have

- $C \in \mathbb{H}_1^{\text{trop}}(x, p = (0, 1))$, but $C \notin \mathbb{H}_1^{\text{trop}}(x, p = (0, 0))$.
- $C' \in \mathbb{H}_1^{\text{trop}}(x, p = (0, 0))$, but $C' \notin \mathbb{H}_1^{\text{trop}}(x, p = (0, 1))$.

In particular, if we choose $p_i = 0$ for all i, $\mathbb{H}_k^{\text{trop}}(x, p)$ is the set of all curves, such that $n - 2 - k$ of its vertices have the same image (again, counting higher-valent vertices v with multiplicity $\text{val}(v) - 2$).

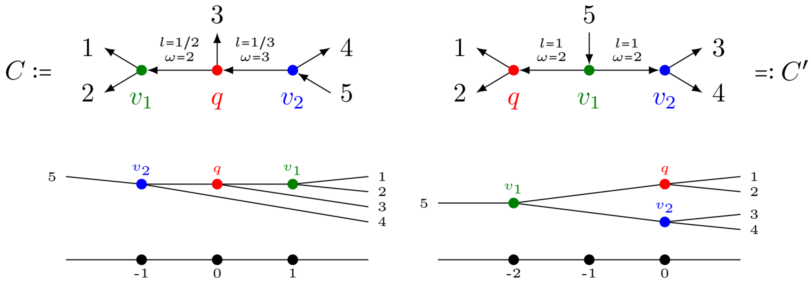

Figure 6.2.: The covers defined by two 5-marked rational curves after fixing the image of a vertex q to be $\alpha = 0$. We chose $x = (1,1,1,1,-4)$ and denoted edge lengths by l, edge slopes by ω.

Of course there may be several possible choices of vertices that are compatible with p. In $\tilde{\mathbb{H}}_k^{\text{trop}}(x,p)$, we fix a choice by attaching the contracted end l_i to the vertex we wish to be mapped to p_i. I.e. $\tilde{\mathbb{H}}_k^{\text{trop}}(x,p)$ is the set of all curves C, such that $l_0,\dots,l_{N-1}$ are attached to vertices and such that in the corresponding cover the vertex with leaf l_i is mapped to p_i. For example, in Figure 6.2 on the left hand side there are two possible choices of vertices that are compatible with $p = (0,1)$. Hence there are two preimages in $\tilde{\mathbb{H}}_k^{\text{trop}}(x,p)$ corresponding to attaching the contracted leaves l_0, l_1 either to q and v_1 or to v_2 and q.

Remark 6.3.4. Let us see how the weight of a cell of $\mathbb{H}_k^{\text{trop}}(x)$ is computed if we choose the p_i to be generic, i.e. pairwise different. Let τ be a maximal cell of $\mathbb{H}_k^{\text{trop}}(x)$ and C the curve corresponding to an interior point of τ. Then τ must lie in the interior of a maximal cell σ of $\mathcal{M}_{0,n}^{\text{trop}}$ and for a generic choice of C there is a unique choice of vertices $q_0,\dots,q_{N-1}$ compatible with the p_i (which fixes a cover). Marking these vertices accordingly, we can consider σ as a cone in $\mathcal{M}_{0,N}^{\text{trop}}(\mathbb{R},x)$. We thus obtain well-defined and linear evaluation maps $\text{ev}_i : \sigma \to \mathbb{R}$, mapping each curve in σ to the image of the vertex q_i. Assume σ is spanned by the rays $v_{I_1},\dots,v_{I_{n-3}}$, then we can write ev_i in cone coordinates as $(a_1^i,\dots,a_{n-3}^i)$, where $a_k^i = \text{ev}_i(v_{I_k})$. It is shown in [BCM, Lemma 4.4] that the weight of τ is then the greatest common divisor of the maximal minors of the matrix $(a_k^i)_{k,i}$.

In the case that all p_i are 0, we use the fact that $\mathbb{H}_k^{\text{trop}}(x,p)$ is the recession fan of the Hurwitz cycle obtained for a generic choice of p_i. By its definition (see section 1.3.2) this means that the total weight of a cell τ is obtained as

$$\omega(\tau) = \sum_{\tau\subseteq\sigma} \sum_{q_i} g_{\sigma,q_i},$$

where the first sum runs over all maximal cones σ of $\mathcal{M}_{0,n}^{\text{trop}}$ containing τ, the second sum runs over all vertex choices $q_0,\dots q_{N-1}$ that are compatible in σ with a *generic* choice of p_i and g_{σ,q_i} is the gcd we obtained in the previous construction. In fact, one can easily see that the same method can be used for computing weights if only some of the p_i are equal.

6.4. Computation

If we approach this naively, we already have everything at hand to compute at least marked Hurwitz cycles: Algorithm 10 tells us how to compute a product of Psi-classes (without having to compute the ambient moduli space, which will be huge!) and then we only have to compute divisors of tropical polynomials on this product. However, this only works for small k, i.e. large codimension. Otherwise, the Psi-class product will already be too large to make this computation feasible.

Also, we will mostly be interested in unmarked Hurwitz cycles and computing push-forwards is, computationally speaking, not desirable, as we discussed in chapter 5. The following approach to compute unmarked cycles directly proves to be more suitable:

Assume we want to compute $\mathbb{H}_k^{\text{trop}}(x, p = (p_0, \dots, p_{N-1}))$ for $x \in \mathbb{Z}^n$. Fix a combinatorial type C of rational n-marked curve, i.e. a maximal cone σ of $\mathcal{M}_{0,n}^{\text{trop}}$. For each choice of distinct vertices $q_0, \dots, q_{N-1}$ of C, we obtain linear evaluation maps on σ, by considering it as a cone of stable maps, where the additional marked ends are attached to the q_i. We can now refine σ by intersecting it with the fan F_i, whose maximal cones are

$$F_i^+ := \{x \in \sigma : \text{ev}_i(x) \geq p_i\}, F_i^- := \{x \in \sigma : \text{ev}_i(x) \leq p_i\}.$$

Iterating over all possible choices of q_i, this will finally give us a subdivision σ' of σ. The part of $\mathbb{H}_k^{\text{trop}}(x, p)$ that lives in σ is now a subcomplex of the k-skeleton of σ': It consists of all k-dimensional cells τ of σ' such that there exists a choice of vertices q_i with the property that the corresponding evaluation maps fulfill $\text{ev}_i(x) = p_i$ for all $x \in \tau$. The weight of such a τ can then be computed using the method described in Remark 6.3.4.

The complete Hurwitz cycle can now be computed by iterating over all maximal cones of $\mathcal{M}_{0,n}^{\text{trop}}$. This gives a feasible algorithm at least for $n \leq 8$ - after that, the moduli space itself becomes too large.

Example 6.4.1. We want to compute (part of) a Hurwitz cycle: We choose $k = 2, x = (2, 2, 6, -5, -4, -1)$ and $(p_0, p_1) = (0, 1)$. Since the complete cycle would be rather large and difficult to visualize (3755 maximal cells living in $\mathbb{R}^9$), we only consider the part of $\mathbb{H}_2^{\text{trop}}(x, p)$ lying in the three-dimensional cone of $\mathcal{M}_{0,6}^{\text{trop}}$ corresponding to the combinatorial type

$$C = v_{\{1,2\}} + v_{\{4,5,6\}} + v_{\{5,6\}}.$$

Figure 6.3 shows the corresponding cover, together with the part of the Hurwitz cycle we computed using the method described above. Each cell of the cycle is obtained by choosing specific vertices of C for the additional marked points p_0 and p_1. The correspondence between these choices and the actual cells, together with the corresponding equation, is laid out in Figure 6.4. While there are of course in theory $4 \cdot 4 = 16$ possible choices, not all of them produce a cell: We only display choices of distinct vertices, such that the image of the vertex for $p_1 = 1$ is larger than the image of the vertex for $p_0 = 0$. This gives $\binom{4}{2} = 6$ valid choices.

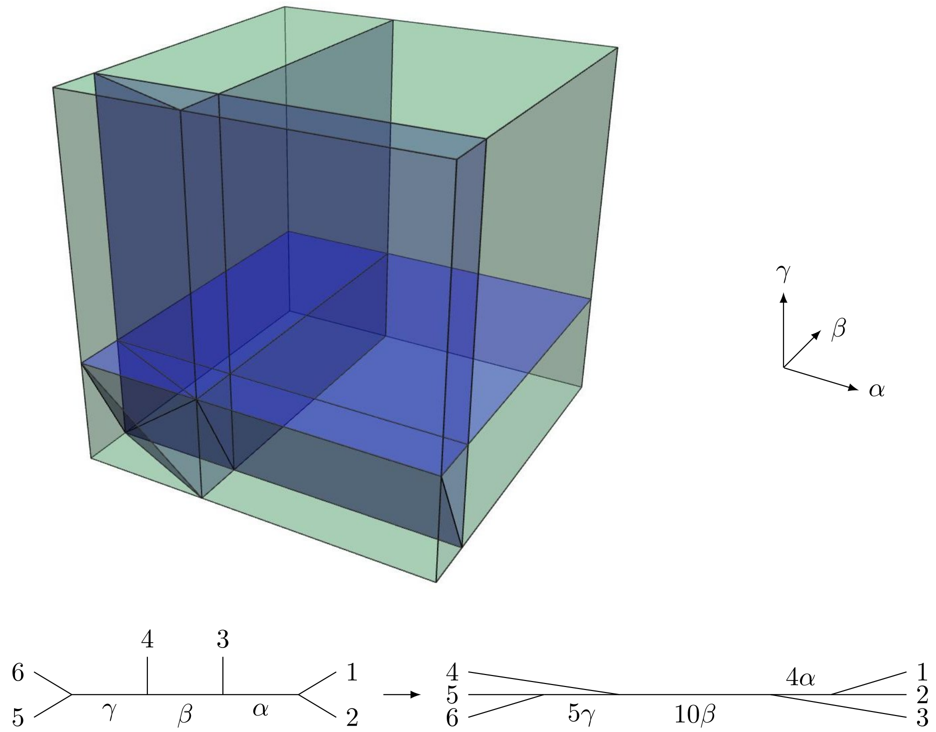

Figure 6.3.: The cube represents the three-dimensional cone in $\mathcal{M}_{0,6}^{\text{trop}}$ that corresponds to the combinatorial type $v_{\{1,2\}} + v_{\{4,5,6\}} + v_{\{5,6\}}$ drawn on the bottom left part of the picture. We denote the length of the interior edges by α, β, γ as indicated. The blue cells represent the Hurwitz cycle living in this cone. The bottom right figure indicates the corresponding cover.

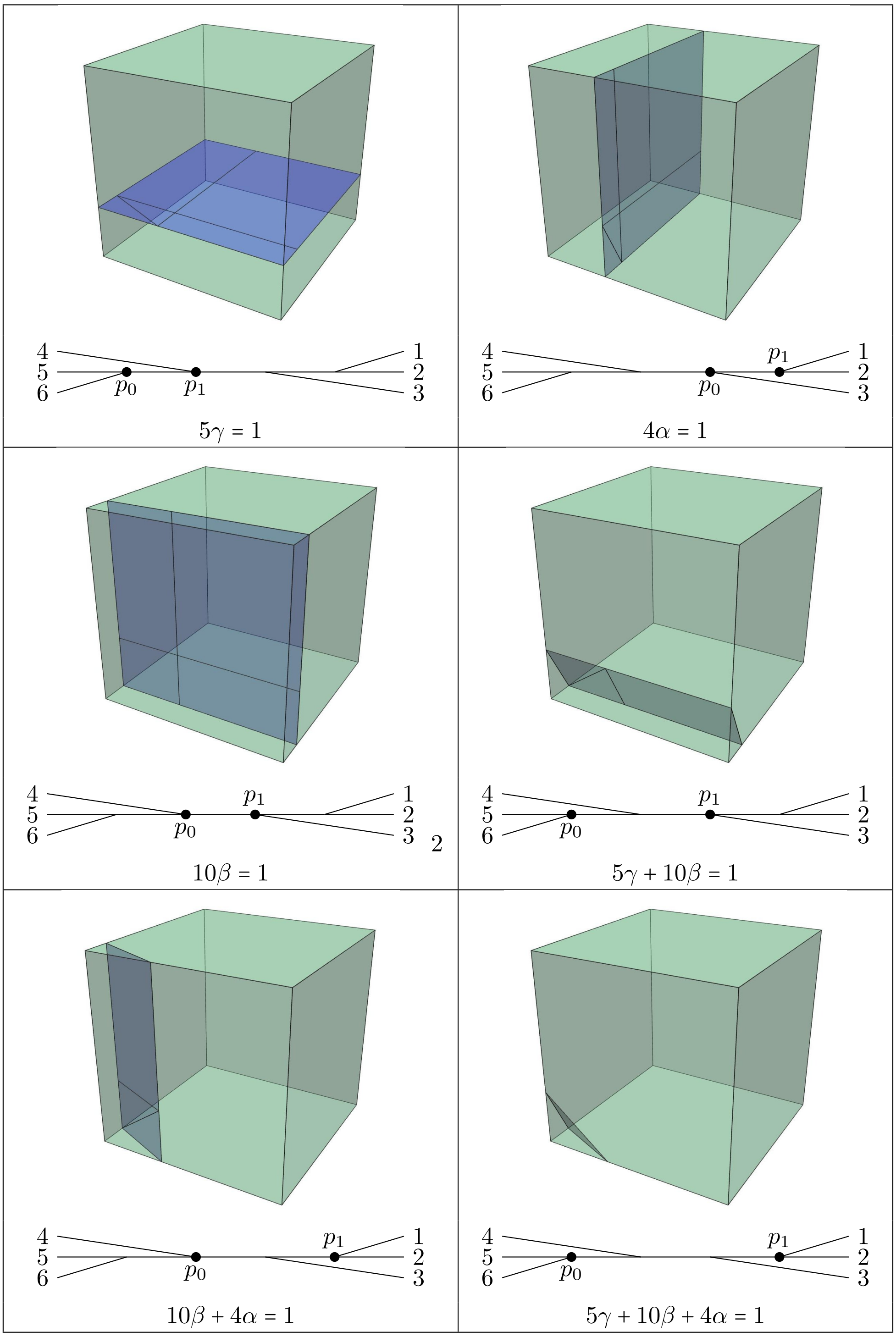

Figure 6.4.: Different choices of vertices yield different cells of $\mathbb{H}_2^{\mathrm{trop}}(x,p)$.

7. Properties of Hurwitz cycles

7.1. Irreducibility of Hurwitz cycles

In this section we want to study whether tropical Hurwitz cycles are irreducible. For this purpose we will first prove that all (marked and unmarked) Hurwitz cycles are connected in codimension one. We will go on to show that for a generic choice of p_j all marked cycles $\tilde{\mathbb{H}}_k^{\text{trop}}(x,p)$ are locally and globally a multiple of an irreducible cycle. Finally we will see that $\mathbb{H}_k^{\text{trop}}(x,p)$ is in general not irreducible.

7.1.1. Connectedness in codimension one

We briefly recall the definition and its relation to the combinatorics of rational curves:

Definition 7.1.1. A tropical cycle X is *connected in codimension one*, if for any two maximal cells σ, σ there exists a sequence of maximal cells $\sigma = \sigma_0, \dots, \sigma_r = \sigma'$, such that two subsequent cells σ_i, σ_{i+1} intersect in codimension one.

This property is obviously necessary for irreducibility: If a cycle X is not connected in codimension one, we can pick any of its codim.-1-connected components and only keep the weights on this component. This will then give a full-dimensional balanced cycle that is properly contained in X.

It is well known that $\mathcal{M}_{0,n}^{\text{trop}}$ is connected in codimension one. In this particular case, the property has a very nice combinatorial description: Maximal cones correspond to rational curves with $n-3$ bounded edges. A codimension one face of a maximal cone is attained by shrinking any of these edges to length 0, thus obtaining a single four-valent vertex. This vertex can then be “drawn apart” in three different ways, thus moving into a maximal cone again. Saying that $\mathcal{M}_{0,n}^{\text{trop}}$ is connected in codimension one means that we can transform any three-valent curve into another by alternatingly contracting edges and drawing apart four-valent vertices.

A similar correspondence holds for Hurwitz covers. An element of a maximal cone of $\tilde{\mathbb{H}}_k^{\text{trop}}(x,p) \subseteq \mathcal{M}_{0,N}^{\text{trop}}(\mathbb{R},x)$ can be considered as an n-marked rational curve C with $N = n - 2 - k$ additional leaves attached to vertices of C. By abuse of notation, throughout this chapter we will also label these additional leaves by $p_0, \dots, p_{N-1}$. By the *valence* of a vertex of an element of $\tilde{\mathbb{H}}_k^{\text{trop}}(x,p)$, we will mean the valence of the vertex in the underlying n-marked curve.

For a generic choice of p, maximal cells of $\tilde{\mathbb{H}}_k^{\text{trop}}(x,p)$ will also correspond to curves with $n-3$ bounded edges and codimension one cells are obtained by shrinking an edge. Hence the problem of connectedness can be formulated in the same manner as for $\mathcal{M}_{0,n}^{\text{trop}}$. However, the requirement that the contracted leaves be mapped to specific points excludes certain combinatorial "moves", as we will shortly see.

Also note that the problem of connectedness does not really change if we allow non-generic points: The combinatorial problem remains essentially the same, we just allow some edge lengths to be 0. Hence we will assume throughout this section that $p_0 < p_1 < \dots < p_{N-1}$.

We will first show connectedness in the case $k = 1$. In this case the Hurwitz cycle is a tropical curve, so saying that $\tilde{\mathbb{H}}_1^{\text{trop}}(x,p)$ is connected in codimension one is the same as requiring that it is path-connected. So we will prove that for each two vertices q, q' of $\tilde{\mathbb{H}}_1^{\text{trop}}(x,p)$ there exists a sequence of edges connecting them.

We will prove this by induction on n, the length of x. For the case $n = 5$ we will simply go through all possible cases explicitly. For $n > 5$, we will first show that any two covers of a special type, called *chain covers*, are connected. Having shown this, we will then introduce a construction that allows us to connect any cover to a chain cover.

The general case is then an easy corollary, since we mark fewer vertices in higher-dimensional Hurwitz cycles, thus obtaining more degrees of freedom.

Remark 7.1.2. Before we start, we want to study more closely why this problem is so difficult. Since $\mathcal{M}_{0,N}^{\text{trop}}$ is connected in codimension one, one would expect to be able to move from one combinatorial type to another without problems. However, the intermediate types need not be valid covers: A vertex of $\tilde{\mathbb{H}}_1^{\text{trop}}(x,p)$ can be considered as a point in a codimension one cone of $\mathcal{M}_{0,n}^{\text{trop}}$, i.e. a curve with one four-valent vertex and only trivalent vertices besides, with an additional marked end attached to every vertex. Moving along an edge of $\tilde{\mathbb{H}}_1^{\text{trop}}(x,p)$ means moving an edge or leaf of that codimension one type along a bounded edge. However, this cannot be done in an arbitrary manner, since not all of these movements will produce valid covers (see figure 7.1 for an example). Note that the p_j already fix the length of all bounded edges of a vertex curve in $\tilde{\mathbb{H}}_1^{\text{trop}}(x,p)$ uniquely. So, we will usually identify each vertex of $\tilde{\mathbb{H}}_1^{\text{trop}}(x,p)$ with the combinatorial type of the corresponding curve.

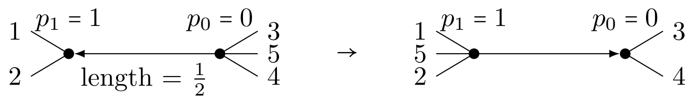

Figure 7.1.: The curve on the left is a vertex of $\tilde{\mathbb{H}}_1^{\text{trop}}(1,1,1,1,-4)$. In $\mathcal{M}_{0,7}$ it corresponds to a ray spanning a cone with the curve on the right. However, the right curve is not an element of $\tilde{\mathbb{H}}_1^{\text{trop}}(1,1,1,1,-4)$ (for any edge length), since the edge direction is not compatible with the vertex ordering.

Recall that the *weight* or slope of an edge e is $x_e := |\sum_{i \in I} x_i|$, where I is the split on $[n]$

induced by e. The *orientation* of e is chosen as in Example 6.3.3: e "points towards I" if and only if $\sum_{i\in I} x_i > 0$.

Now, when moving some leaf along a bounded edge, that edge might change direction. But the direction of the edges is dictated by the order of the p_i, so this is not a valid move. One can easily see the following (see figure 7.2 for an illustration): Moving an edge/leaf i to the other side of a bounded edge e changes the direction of that edge if and only if one of them is incoming and one outgoing (recall that we consider leaves as incoming if they have negative weight) and $|x_i| > x_e$. Note that, even if the direction of an edge does not change, moving an edge might be illegal (see the last diagram in figure 7.2), if the resulting edge configuration does not agree with the order on the p_j.

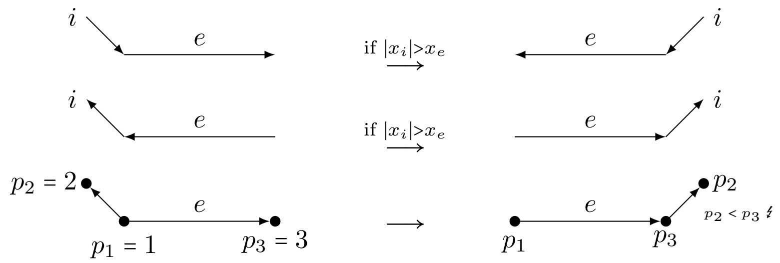

Figure 7.2.: Invalid moves on a Hurwitz cover: In the first two cases, when moving the leaf/edge i along the bounded edge e, the direction of e changes. In the third case the edge direction of e remains the same, but the direction is not compatible with the order of the p_i.

Definition 7.1.3. A *vertex type cover* is any cover corresponding to a vertex of $\tilde{\mathbb{H}}_1^{\text{trop}}(x,p)$.

Lemma 7.1.4. *For $n = 5$, the cycle $\tilde{\mathbb{H}}_1^{\text{trop}}(x,p)$ is connected in codimension one for any p and x.*

Proof. Let q, q' be two vertices of $\tilde{\mathbb{H}}_1^{\text{trop}}(x,p)$ and C, C' the corresponding rational curves. Both curves consist of a single bounded edge connecting contracted ends $p_0 < p_1$ with three leaves on one side and two on the other. We distinguish different cases, depending on how many leaves have to switch sides to go from C to C'.

Assume first that both curves only differ by the placement of one leaf, i.e. we want to move one leaf i from the four-valent vertex in C to the other side of the bounded edge. We can assume without restriction that the four-valent vertex in C is at p_0. Assume that moving i to the other side is an invalid move. Then the direction of the bounded edge would be inverted in C', which is a contradiction to the fact that $p_0 < p_1$.

Now assume that both curves differ by an exchange of two leaves. Again we assume that the four-valent vertex in C (and hence also in C') is at p_0. Denote the leaves in

C at p_0 by i, a, b and the remaining two at p_1 by j, c and assume that C' is obtained by exchanging i and j. If we can move either i in C or j in C', then we are in the case where only one leaf needs to be moved, which we already studied. So assume that i and j cannot be moved in C and C', respectively. By remark 7.1.2, this means that $x_i < -x_e < 0$, where x_e is the weight of the bounded edge in C. Furthermore, $x_i + x_a + x_b = -x_e$, so $x_a + x_b > 0$. We assume without restriction that $x_a > 0$. Hence we can move a along the bounded edge to obtain a valid cover C_1, whose four-valent vertex is at p_1. Since we assumed that we cannot move j in C', we must have $x_j < 0$ (it must be an incoming edge). This implies that we can move it to the left in C_1 to obtain a cover C_2. We now have i, j, b at p_0 and c, a at p_1. We want to show that we can move i to the other side. Assume this is not possible. Then $-x_i > x'_e$, where x'_e is the weight of the bounded edge in C_2. But $x'_e = -x_i - x_j - x_b$. This implies $0 > -x_j - x_b$. Again, since j cannot be moved in C' we have $-x_j > x_a + x_b$. Finally, we obtain that $0 > -x_a + x_b - x_b = x_a > 0$, which is a contradiction. Thus we can move i to the right side to obtain a cover C_3. This cover now only differs from C' by the placement of leaf a, so we are again in the first case (see figure 7.3 for an illustration).

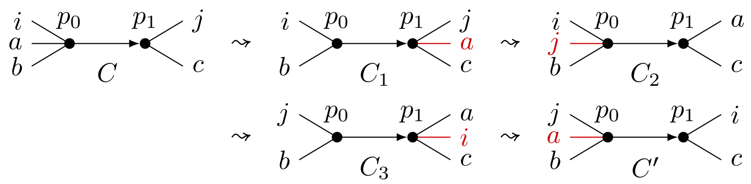

Figure 7.3.: Connecting two curves differing by an exchange of leaves. The leaf we moved in each step is marked by a red line.

Now assume we have to move three leaves (see figure 7.4). That means we have to exchange two leaves i, j from the four-valent vertex in C (again assume it is at p_0) for one leaf k at p_1. Assume we cannot move i in C. In particular, $x_i < 0$. But that means we can move i in C' to obtain a cover C_1. This cover differs from C by the exchange of j and k, so we already know they are connected.

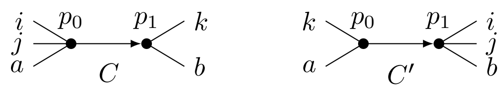

Figure 7.4.: Two vertex types differing by a movement of three leaves. Depending on the direction of i, we can move it either in C or in C'.

Finally, assume that four leaves have to switch sides, i.e. we exchange two leaves i, j at the four-valent at p_0 for the two leaves k, l at p_1. Assume we can move neither i nor j. This means that $x_i, x_j < 0$. But then $x_i + x_j < 0$ as well, so the edge direction would be inverted in C', which is a contradiction. Hence we can move i or j and reduced the problem to the case where only three leaves need to be moved.

It is easy to see that these are all possible cases. In particular, it is impossible to let all five leaves switch sides, since this would automatically invert the direction of the bounded edge. □

As mentioned above, we want to show that for $n > 5$ we can connect each vertex type to a vertex corresponding to a *standard cover*. Let us define this:

Definition 7.1.5. Let $x \in \mathcal{H}_n$. We define an order $<_x$ on $[n]$ by:

$$i <_x j :\iff x_i < x_j \text{ or } (x_i = x_j \text{ and } i < j).$$

A *chain cover for* x is a vertex type cover with the additional property that the vertex marked with p_i is connected to the vertex marked with p_j, if and only if $|i-j| = 1$ (i.e. the p_j are arranged as a single chain in order of their size). Fix an $s \in \{0,\dots,n-4\}$. The *standard cover for* x *at* p_s is the unique chain cover, where the leaves are attached to the p_j according to their size (defined by $<_x$) and p_s is at the four-valent vertex. More precisely: If leaf i is attached to p_k and leaf j is attached to p_l, then $i <_x j \iff p_k < p_l$ (See figure 7.5 for an example of this construction).

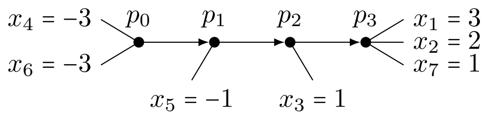

Figure 7.5.: The standard cover for $x = (3,2,1,-3,-1,-3,1)$ at p_3

Lemma 7.1.6. *Each standard cover is a valid Hurwitz cover.*

Proof. We have to show that the edge connecting p_j and p_{j+1} points towards p_{j+1} for all j. Note that the weight and direction of an edge only depend on the split defined by it.

We will say that a leaf *lies behind* p_k, if it is attached to some $p_{k'}$, $k' \geq k$. Denote the leaves lying behind p_{j+1} by $i_1,\dots,i_l$. Their weights are by construction larger than or equal to all weights of remaining leaves. Considering that the sum over all leaves is 0, this implies that $\sum_{s=1}^{l} x_{i_s} > 0$ (if it was 0, then all x_i would have to be 0). Hence the bounded edge points towards p_{j+1}. □

We will also need another construction in our proofs:

Definition 7.1.7. Let C be a vertex type cover and e any bounded edge in C connecting the contracted ends p and q. Removing e, we obtain two path-connected components. For any contracted end r, we write $C_e(r)$ for the component containing r.

Now assume $C_e(p)$ contains the four-valent vertex and at least one other bounded edge. The *split cover at e* is a cover C' obtained in the following way: Remove the edge e and keep only $C_e(p)$. Then attach a leaf to p whose weight is the original weight of e (or its negative, if e pointed towards p). This is obviously a vertex type cover for some $x' = (x'_1, \dots, x'_m)$, where $m < n$ (see figure 7.6 for an example). We denote the leaf replacing e by l_e and call it the *splitting leaf*.

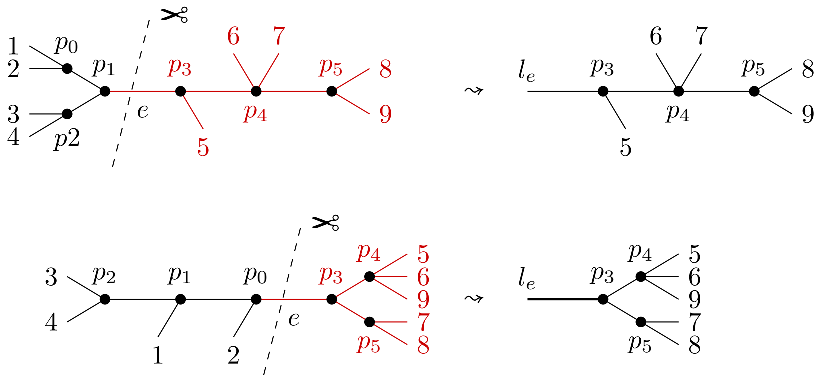

Figure 7.6.: Two Hurwitz covers for $n = 9$. In each case the split cover at the edge marked by e is a cover for $n = 6$ (the labels at the leaves are just indices in this case, not weights).

We now want to see that all chain covers are connected:

Lemma 7.1.8. *Let $x \in \mathcal{H}_n$ and let $p_0, \dots, p_{n-4} \in \mathbb{R}$ with $p_j \leq p_{j+1}$ for all j. Then all chain covers for x are connected to each other.*

Proof. We will show that all chain covers are connected to a standard cover at some p_s. We prove this by induction on n. For $n = 5$, all covers are chain covers and our claim follows from lemma 7.1.4.

So let $n > 5$ and C be any chain cover. We can assume without restriction that the vertices at p_0 and p_{n-4} are trivalent (if they are not, one can easily see that at least one leaf can be moved away). Take any bounded edge e connecting some p_j and p_{j+1}. Suppose there is a leaf k at p_j and a leaf l at p_{j+1}, such that $k >_x l$. This means that exchanging k and l still gives a valid cover. We can assume without restriction that $j > 0$, i.e. e is not the first edge (if $j = 0$, we can use the exact same argument on the last edge).

Let C' be the split cover at the edge connecting p_0 and p_1. This is a cover on $n - 1$ leaves. By induction we know that C' is connected to the cover which only differs from C' by exchanging k and l. Let C'' be any vertex type cover occuring along that path. Since p_0 is smaller than all p_j, we can lift C'' to a cover on n leaves: Simply re-attach the splitting leaf to p_0. (see figure 7.7 for an illustration of the split-and-lift construction in a different case).

Hence we obtain a path between C and and the cover $\tilde{C}$, where k and l have been exchanged. We can apply this procedure iteratively to sort all leaves to obtain a standard cover at some p_s.

Finally, note that all standard covers are connected: One can always move the smallest leaf at the four-valent vertex to the left (except of course at p_0) and the largest leaf to the right. This way the four-valent vertex can be placed at any contracted end. □

Lemma 7.1.9. *Let $x \in \mathcal{H}_n$. Then $\tilde{\mathbb{H}}_1^{\text{trop}}(x,p)$ is connected in codimension one.*

Proof. We prove this by induction on n. The case $n = 5$ was already covered in lemma 7.1.4. Also note that for $n = 4$ the Hurwitz cycle $\tilde{\mathbb{H}}_1^{\text{trop}}(x,p)$ is by definition equal to a Psi-class and hence a fan curve.

So assume $n > 5$ and let q be a vertex of $\tilde{\mathbb{H}}_1^{\text{trop}}(x,p)$ with corresponding rational curve C. We want to show that it is connected to the standard cover on p_0. First, we prove the following technical statement:

1) Let e be a bounded edge connecting p_0 and some p_j, such that $C_e(p_j)$ contains the four-valent vertex. Let C' be the split cover at e with degree $x' = (x'_1, \dots, x'_m)$. Let $P = \{p'_1, \dots, p'_m\}$ be the set of contracted ends in C' and assume $p'_1 < \dots < p'_m$. Then C is connected to the cover C'', obtained in the following way: First, remove all leaves and contracted ends contained in C' from C together with any bounded edges that are attached to them. Then attach all $p \in P$ as an ordered chain to p_0, i.e. p'_1 to p_0, p'_2 to $p'_1, \dots$ etc. Assume the leaves in C' have weights $x_{i_1} \leq \dots \leq x_{i_{m-1}}$. Attach leaf i_1 to p_0, i_2 to p'_1 and so on (see figure 7.7).

We know by induction that C' is connected to the standard cover for x' at any $p \in P$. Choose p, such that the standard cover at p has the splitting leaf attached to the four-valent vertex. Since the splitting leaf has negative weight, we can move it to the smallest contracted end. This gives us a chain cover C_2 connected to C'. As in the proof of lemma 7.1.8, we can lift the connecting path to a path of covers with degree x by attaching p_0 to the splitting leaf. Denote the lift of C_2 by C'_2. This cover has its four-valent vertex at p'_1. Denote by k the smallest leaf at p'_1 with respect to $<_x$ and let ω be the weight of the edge connecting p_0 and p'_1. By definition $\omega = \sum_{i \in I} x_i$, where I is the set of all leaves contained in C'. By construction, k is the minimal element of I with respect to $<_x$. Hence $\omega > k$ and we can move k to p_0 to obtain C''.

We can now use this to prove the following:

2) If p_0 has only one bounded edge attached to it, then C is connected to the standard cover at p_0.

We can assume without restriction that p_0 is not at the four-valent vertex (otherwise, we can move at least one leaf). We now apply the construction described in 1) to the single bounded edge at p_0. This gives us a chain cover for x, which by lemma 7.1.8 is connected to the standard cover.

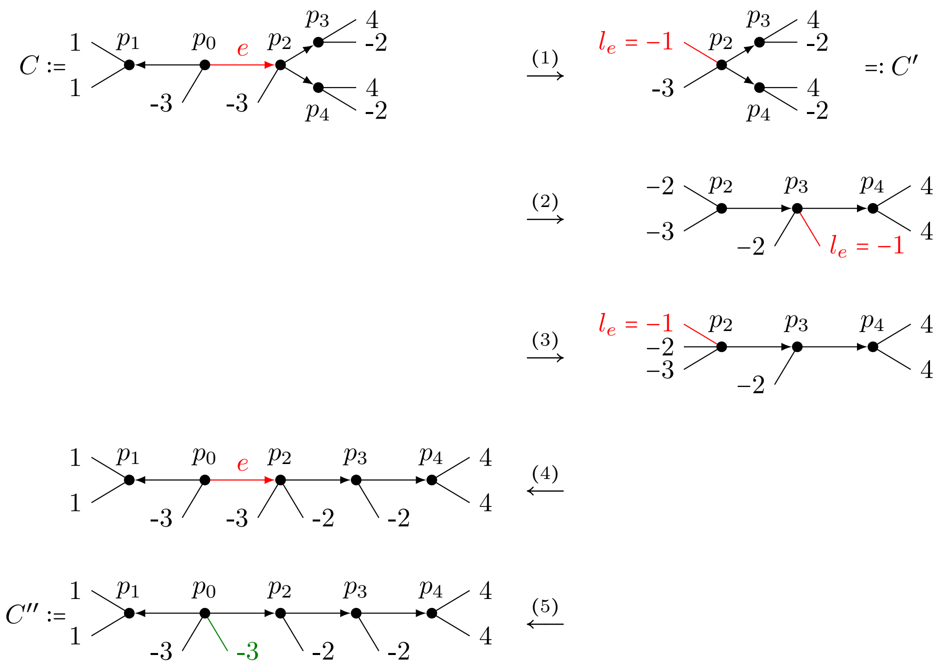

Figure 7.7.: The branch sorting construction:
(1) Take the split cover C' at e.
(2) Move that split cover to a standard cover using induction.
(3) Move the splitting leaf to the smallest p_j.
(4) Consider the lift of this cover.
(5) Move the smallest leaf at $p_1' = p_2$ to p_0 to obtain C''.

It remains to prove the following statement, which implies our theorem:

3) C is always connected to a cover C', in which p_0 has only one bounded edge attached to it.

As any vertex is at most four-valent, p_0 can have at most four bounded edges attached to it. First, assume that only two bounded edges e, e' are attached to p_0 and their other ends are attached to contracted ends $p_e \leq p_{e'}$. If p_0 is four-valent, we can move e' along e to obtain a valid cover in which p_0 has a single bounded edge attached to it. If the four-valent vertex lies behind one of the edges, say e, we apply the construction of 1) to this edge. This way we obtain a cover in which p_0 is still attached to two bounded edges and is also four-valent.

Assume p_0 has three bounded edges e, e', e'' attached to it, connecting it to contracted ends $p_e \leq p_{e'} \leq p_{e''}$. With the same argument as in the case of two bounded edges, we can assume that the vertex at p_0 is four-valent. Now we can move e' along e. Thus we obtain a cover where p_0 has only two bounded edges attached to it. A similar argument covers the case of four bounded edges (see also figure 7.8 for an illustration in the case of two edges).

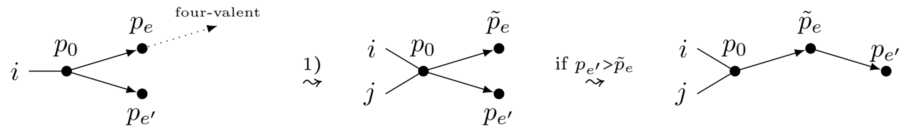

Figure 7.8.: How to reduce the number of bounded edges at p_0: First move the four-valent vertex to p_0 using the construction from 1). Then move one bounded edge along another according to the size of the p_e.

□

Corollary 7.1.10. *For all k, p and x, the cycles $\tilde{\mathbb{H}}_k^{\text{trop}}(x,p)$ and $\mathbb{H}_k^{\text{trop}}(x,p)$ are connected in codimension one.*

Proof. Note that it suffices to show the statement for $\tilde{\mathbb{H}}_k^{\text{trop}}(x,p)$, since $\mathbb{H}_k^{\text{trop}}(x,p) = \text{ft}(\tilde{\mathbb{H}}_k^{\text{trop}}(x,p))$.

The general idea of the proof is that for larger k we mark fewer vertices with contracted ends and thus have more degrees of freedom to "move around", so we can apply induction on k.

Similar to definition 7.1.5 we define a *standard maximal cover* for x on S, with $S \subseteq [n-2], |S| = n-2-k$: We obtain a trivalent curve by attaching the leaves to a chain of $n-3$ bounded edges sorted according to the ordering $<_x$. We number the vertices $\{1,\dots,n-2\}$ (from lowest leaf to highest). We then attach the contracted ends p_j, in order of their size, to the vertices with numbers in S (see figure 7.9 for an example).

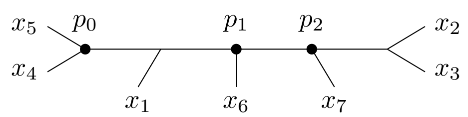

Figure 7.9.: The standard maximal cover in $\mathbb{H}_2^{\text{trop}}(x)$ for $x = (1,2,3,-3,-5,1,1)$ on $S = \{1,3,4\}$

It is easy to see that all standard maximal covers are connected in codimension one: Assume that $(j-1) \notin S \ni j$ (i.e. there is a contracted end at vertex j but none at vertex $(j-1)$). Then the standard cover on $(S \smallsetminus \{j\}) \cup \{j-1\}$ shares a codimension one face with this cover, obtained by shrinking the edge between the two vertices $(j-1), j$ to length 0. In this manner we see that every standard cover is connected to the standard maximal cover on $S = \{1,\dots,n-2-k\}$.

Now we want to see that every maximal cell σ is connected to the maximal cone of a standard cover. The cell σ corresponds to a trivalent curve, with some of the vertices marked with contracted ends $p_0, \dots, p_{n-3-k}$. We now add a further marking $q \in \mathbb{R}$ on an arbitrary vertex such that it is compatible with the edge directions. This gives us an element of $\tilde{\mathbb{H}}_{k-1}^{\text{trop}}(x,p)$. By induction, the corresponding cell is connected to a standard cover on S', with $|S'| = n-3-k$. We can "lift" each intermediate step in the connecting path to a valid cover in $\tilde{\mathbb{H}}_k^{\text{trop}}(x,p)$ simply by forgetting the mark q. Thus we have connected σ to a standard maximal cover. □

7.1.2. Irreducibility

We now want to see when a Hurwitz cycle is irreducible. A key result that we use in this section is

Lemma 7.1.11 ([R1, Lemma 1.2.29])**.** *Let X be a tropical cycle. If X is locally irreducible (i.e.* $\text{Star}_X(\tau)$ *is a multiple of an irreducible cycle for all codimension one cells τ) and X is connected in codimension one, then X is a multiple of an irreducible cycle.*

We just proved that Hurwitz cycles are connected in codimension one. To see whether they are locally irreducible, we will make use of our knowledge of the local structure of $\mathcal{M}_{0,N}^{\text{trop}}$ (Corollary 4.5.4): If τ is a cone of the combinatorial subdivision of $\mathcal{M}_{0,N}^{\text{trop}}$, corresponding to a curve C with vertices $q_1, \dots, q_k$, then

$$\text{Star}_{\mathcal{M}_{0,N}^{\text{trop}}}(\tau) = \mathcal{M}_{0,\text{val}(q_1)}^{\text{trop}} \times \cdots \times \mathcal{M}_{0,\text{val} q_k}^{\text{trop}}.$$

Corollary 7.1.12. *For any $x \in \mathcal{H}_n$ and pairwise different p_j, the cycle $\tilde{\mathbb{H}}_k^{\text{trop}}(x,p)$ is locally at each codimension one face an integer multiple of an irreducible cycle.*

Proof. Let τ be a codimension one cell of $\tilde{\mathbb{H}}_k^{\text{trop}}(x,p)$ and C_τ the corresponding combinatorial type. Since we chose the p_j to be pairwise different, C_τ has exactly one vertex v adjacent to four bounded edges or non-contracted leaves. Depending on whether a contracted end is also attached, the vertex is either four- or five-valent, corresponding to an $\mathcal{M}_4$- or $\mathcal{M}_5$-coordinate.

Denote by $S := \text{Star}_{\tilde{\mathbb{H}}_k^{\text{trop}}(x)}(\tau)$. First let us assume that no contracted end is attached to v. Then there are three maximal cones adjacent to τ, corresponding to the three different possible resolutions of v. The projections of the normal vectors to the $\mathcal{M}_4$-coordinate of v are (multiples of) the three rays of $\mathcal{M}_4$. In particular there is only one possible way to assign weights to these rays so that they add up to 0. Hence the rank of Ω_S is 1, showing that S is a multiple of an irreducible cycle.

Now assume there is a contracted end p at v and four edges/non-contracted ends. Then there are six maximal cones adjacent to τ: Consider v as a four-valent vertex with an additional point for the contracted end. Then we still have three possibilities to resolve v, but in each case we have two possibilities to place the additional point (see figure 7.10). Now label the four ends and p with numbers $1, \dots, 5$ and assume p is labeled with 5. Then the projections of the normal vectors are multiples of the vectors $v_{\{i,j\}} \in \mathcal{M}_5$ with $i, j \neq 5$. The set of these vectors has been studied in [KM2] and it is shown there that there is only one way to assign weights to these rays such that they add up to 0.

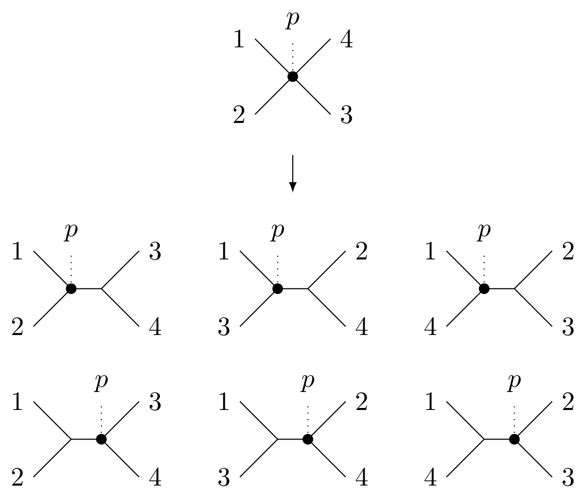

Figure 7.10.: The six possible resolutions of a four-valent vertex with a contracted end.

□

Corollary 7.1.13. *For any $x \in \mathcal{H}_n$ and any pairwise different p_j, $\tilde{\mathbb{H}}_k^{\mathrm{trop}}(x,p)$ is an integer multiple of an irreducible cycle.*

Example 7.1.14. We now want to see that this is the strongest possible statement (see also the subsequent `polymake` example).

- **Non-generic points:** Let $n = 5, k = 1, x = (1,1,1,1,-4)$. If we choose $p_0 = p_1 = 0$, then $\tilde{\mathbb{H}}_1^{\mathrm{trop}}(x,p)$ is not irreducible: One can use `a-tint` to compute that the rank of $\Omega_{\tilde{\mathbb{H}}_1^{\mathrm{trop}}(x)}$ is 3.
- **Strict irreducibility:** Let $x' = (1,1,1,1,1,-5), k' = 1$. For $(p_0, p_1, p_2) = (0,1,2)$, we obviously obtain a cycle with weight lattice $\Omega_{\tilde{\mathbb{H}}_1^{\mathrm{trop}}(x',p)}$ of rank 1. However, the gcd of all weights in this cycle is 2, so it is not irreducible in the strict sense.
- **Unmarked cycles:** Again, choose $x = (1,1,1,1,-4), k' = 1$ and generic points $p_0 = 0, p_1 = 1$. Passing to $\mathbb{H}_1^{\mathrm{trop}}(x,p)$, the rank of $\Lambda_{\mathbb{H}_1^{\mathrm{trop}}}(x,p)$ is 18. One can also see that $\mathbb{H}_1^{\mathrm{trop}}(x,p)$ is not locally irreducible: It contains the two lines $\{\frac{1}{2}v_{\{1,2\}} + \mathbb{R}_{\geq 0}v_{\{3,4\}}\}, \{\frac{1}{2}v_{\{3,4\}} + \mathbb{R}_{\geq 0}v_{\{1,2\}}\}$, which intersect transversely in the vertex $\frac{1}{2}v_{\{1,2\}} + \frac{1}{2}v_{\{3,4\}}$. Locally at this vertex, the curve is just the union of

two lines, which is of course not irreducible. However, one can again use the computer to see that there are also vertices of $\tilde{\mathbb{H}}_1^{\text{trop}}(x,p)$ such that the map induced locally by ft is injective, but such that the image of the local variety at that vertex is not irreducible.

polymake example: Computing Hurwitz cycles.
We compute the Hurwitz cycles from Example 7.1.14. First, we compute the cycle $\tilde{\mathbb{H}}_1^{\text{trop}}((1,1,1,1,-4),p)$ for $p_0 = p_1 = 0$ (If no points are given, they are set to 0). A basis for its weight spaces is given as row vectors of a matrix. We then compute $\tilde{\mathbb{H}}_1^{\text{trop}}((1,1,1,1,1,-5),q)$ for generic points $q = (0,1,2)$ (the first point is always zero in `a-tint`) and display its weight space dimension. Finally we compute $\mathbb{H}_1^{\text{trop}}(1,1,1,1,-4)$ for generic points $(0,1)$ and the dimension of its weight space.

```
atint >  $h1 = hurwitz_marked_cycle(1,(new Vector<Int>(1,1,1,1,-4)));
atint >  print $h1->WEIGHT_SPACE->rows();
3
atint >  $h2 = hurwitz_marked_cycle(1,(new Vector<Int>(1,1,1,1,1,-5)),
         (new Vector<Rational>(1,2)));
atint >  print $h2->WEIGHT_SPACE->rows();
1
atint >  print gcd($h2->TROPICAL_WEIGHTS);
2
atint >  $h3 = hurwitz_cycle(1,(new Vector<Int>(1,1,1,1,-4)),
         (new Vector<Rational>([1])));
atint >  print $h3->WEIGHT_SPACE->rows();
18
```

Remark 7.1.15. It is an open question whether there exists a canonical decomposition of $\mathbb{H}_k^{\text{trop}}(x,p)$ (or $\tilde{\mathbb{H}}_k^{\text{trop}}(x)$ in the non-generic case). Also, we have so far not found a single example of an irreducible Hurwitz cycle $\mathbb{H}_k^{\text{trop}}(x,p)$. If we pick $p = 0$, it is actually obvious that the cycle must be reducible: For any $i = 1,\dots,n$ it contains the Psi-class product $\psi_i^{(n-3-k)}$ as a non-trivial k-dimensional subcycle. This motivates the following:

Conjecture 7.1.16. *Let $n \geq 5$. Then $\mathbb{H}_k^{\text{trop}}(x,p)$ is reducible for any x, p and k.*

7.2. Cutting out Hurwitz cycles

For intersection-theoretic purposes it is very tedious to have a representation of the cycle $\mathbb{H}_k^{\text{trop}}(x)$ only as a push-forward. We would like to find rational functions that successively cut out (recession fans of) Hurwitz cycles directly in the moduli space $\mathcal{M}_{0,n}^{\text{trop}}$. It turns out that there is a very intuitive rational function cutting out the codimension one Hurwitz cycle $\mathbb{H}_{n-4}^{\text{trop}}(x)$ in $\mathcal{M}_{0,n}^{\text{trop}}$. Alas, this seems to be the strongest possible statement already that we can make in this generality. For $n \geq 7$ we can find examples where there is *no rational function at all* that cuts out $\mathbb{H}_{n-5}^{\text{trop}}$ from $\mathbb{H}_{n-4}^{\text{trop}}(x)$. It remains to be seen whether there might be other rational functions or piecewise polynomials cutting out lower-dimensional Hurwitz cycles from $\mathcal{M}_{0,n}^{\text{trop}}$.

Throughout this section we assume $p_i = 0$ for all i, i.e. $\mathbb{H}_k^{\text{trop}}$ is a fan in $\mathcal{M}_{0,n}^{\text{trop}}$.

7.2.1. Push-forwards of rational functions

We already know that $\tilde{\mathbb{H}}_{n-4}^{\text{trop}}(x)$ can by definition be cut out from

$$\text{ev}_0^*(0) \cdot \Psi_0 \cdot \Psi_1 \cdot \mathcal{M}_{0,2}^{\text{trop}}(\mathbb{R}, x) =: \mathcal{M}_x$$

by the rational function $\text{ev}_1^*(0)$ (Note that there is an obvious isomorphism $\mathcal{M}_x \cong \psi_{n+1} \cdot \psi_{n+2} \cdot \mathcal{M}_{0,n+2}^{\text{trop}}$). The forgetful map $\text{ft} : \mathcal{M}_{0,2}^{\text{trop}}(\mathbb{R}, x) \to \mathcal{M}_{0,n}^{\text{trop}}$ now induces a (surjective) morphism of equidimensional tropical varieties (by abuse of notation we also denote it by ft)

$$\text{ft} : \mathcal{M}_x \to \mathcal{M}_{0,n}^{\text{trop}},$$

which is injective on each cone of $\mathcal{M}_x$. We will see that under these conditions, we can actually define the *push-forward* of a rational function:

Definition 7.2.1. Let X, Y be d-dimensional tropical cycles and assume Y is smooth. Let $x \in X$. If $f : X \to Y$ is a morphism, we denote by f_x the induced local map

$$f_x : \text{Star}_X(x) \to \text{Star}_Y(f(x)) =: V_x.$$

We define the *mapping multiplicity* of x to be

$$m_x := f_x^*(f(x)).$$

Note that, since V_x is a smooth fan, any two points in it are rationally equivalent by [FR, Theorem 9.5], so $\deg f_x^*(\cdot)$ is constant on V_x. In particular, to compute m_x, we can replace $f(x)$ by any point y in a sufficiently small neighborhood.

Now let $g : X \to \mathbb{R}$ be a rational function. We define the *push-forward* of g under f to be the function

$$f_* g : Y \to \mathbb{R}, y \mapsto \sum_{x : f(x) = y} m_x g(x).$$

Proposition 7.2.2. *Under the assumptions above, $f_* g$ is a rational function on Y.*

Proof. We can assume without restriction that X and Y have been refined in such a manner, that f maps cells of X to cells of Y and g is affine linear on each cell of X. Let us first see that $f_* g$ is well-defined:

Let $y \in Y$ and denote by τ the minimal cell containing it. We want to see that y has only finitely many preimages $x \in X$ with $m_x \neq 0$. Assume there is a cell ρ in X such that $f(\rho) = \tau$, but $\dim(\rho) > \dim(\tau)$, so $f_{|\rho}$ is not injective. In particular, all maximal cells $\xi > \rho$ map to a cell of dimension strictly less than d. Now let $x \in \operatorname{relint}(\rho)$ with $f(x) = y$. If we pick a point $q \in V_y$ that lies in a maximal cone adjacent to τ, it has no preimage under f_x: All maximal cones in $\operatorname{Star}_X(x)$ are mapped to a lower-dimensional cone. It follows that $m_x = 0$.

We now have to show that $f_* g$ is continuous. Let σ be a maximal cell of Y. Denote by

$$C_\sigma = \{\xi \in X^{(d)}, f(\xi) = \sigma\}.$$

Then for each $y \in \operatorname{relint}(\sigma)$ we have

$$f_* g(y) = \sum_{\xi \in C_\sigma} \omega_X(\xi) \operatorname{ind}(\xi) g(f_{|\xi}^{-1}(y)),$$

where $\operatorname{ind}(\xi) := |\Lambda_\sigma / f(\Lambda_\xi)|$ is the index of f on ξ. Since $f_{|\xi}$ is a homeomorphism, this is just a sum of continuous maps, so $(f_* g)_{|\operatorname{relint}(\sigma)}$ is continuous.

Assume τ is a cell of Y of dimension strictly less than d and contained in some maximal cell σ. Let $s : [0,1] \to \sigma$ be a continuous path with:

- $s([0,1)) \subseteq \operatorname{relint}(\sigma)$
- $s(1) \in \operatorname{relint}(\tau)$

We write $y_t := s(t)$ for $t \in [0,1]$. Then we have to show that $\lim_{t \to 1} f_* g(y_t) = f_* g(y_1)$. If we denote by $s_\xi = (f_{|\xi}^{-1} \circ s)$ the unique lift of s to any $\xi \in C_\sigma$, we have

$$\begin{aligned}
\lim_{t \to 1} f_* g(y_t) &= \lim_{t \to 1} \sum_{\xi \in C_\sigma} \omega_X(\xi) \operatorname{ind}(\xi) g(s_\xi(t)) \\
&= \sum_{\xi \in C_\sigma} \omega_X(\xi) \operatorname{ind}(\xi) \lim_{t \to 1} g(s_\xi(t)) \\
&= \sum_{\xi \in C_\sigma} \omega_X(\xi) \operatorname{ind}(\xi) g(\underbrace{\lim_{t \to 1} s_\xi(t)}_{=: x_\xi}),
\end{aligned}$$

where the last equality is due to the continuity of g. Note that x_ξ lies in the unique face $\rho_\xi < \xi$ such that $f(\rho_\xi) = \tau$.

Conversely, let ρ be any cell of X with $\dim(\rho) = \dim(\tau)$ and $f(\rho) = \tau$. Assume ρ has no adjacent maximal cell mapping to σ. Then, if we let $x := f_{|\rho}^{-1}(y)$, we must again have $m_x = 0$. We define

$$C_\tau := \{\rho \in X^{(\dim \tau)}; f(\rho) = \tau \text{ and there exists } \xi > \rho \text{ with } f(\xi) = \sigma\}.$$

Then we have

$$\begin{aligned}
f_* g(y_1) &= \sum_{\substack{\rho \in X^{(\dim \tau)} \\ f(\rho) = \tau}} m_{f_{|\rho}^{-1}(y_1)} g(f_{|\rho}^{-1}(y_1)) \\
&= \sum_{\rho \in C_\tau} m_{f_{|\rho}^{-1}(y_1)} g(f_{|\rho}^{-1}(y_1)) \qquad (7.2.1)
\end{aligned}$$

If $x_\rho := f_{|\rho}^{-1}(y_1)$, then for small ϵ we have

$$m_{f_{|\rho}^{-1}(y_1)} = \deg f_{x_\rho}^* y_{1-\epsilon} = \sum_{\substack{\xi > \rho \\ f(\xi)=\sigma}} \omega_X(\xi)\mathrm{ind}(\xi).$$

If we plug this into (7.2.1), we see that each $\xi \in C_\sigma$ occurs exactly once (since ξ cannot have two faces ρ mapping to τ due to injectivity), so finally we have $f_* g(y_t) = f_* g(y_1)$. □

Proposition 7.2.3. *Let $f : X \to Y$ be a morphism of d-dimensional tropical cycles. Assume Y is smooth and f is injective on each cell of X. Then*

$$f_* g \cdot Y = f_*(g \cdot X).$$

Proof. By studying this identity locally and dividing out lineality spaces we can assume that:

- Y is a smooth one-dimensional tropical variety.
- $X = \coprod_{i=1}^r X_i$ is a disjoint union of one-dimensional tropical cycles.
- $f_{|X_i} : X_i \to Y$ is a linear map.
- g is affine linear on each ray of X_i.

We write $Z := f_* g \cdot Y$ and $Z' := f_*(g \cdot X)$. We have to show that $\omega_Z(0) = \omega_{Z'}(0)$. We know that

$$\omega_{Z'}(0) = \sum_{i=1}^r \omega_{g \cdot X_i}(0) = \sum_{i=1}^r \sum_{\rho \in X_i^{(1)}} \omega_{X_i}(\rho) g(u_\rho),$$

where u_ρ is the integer primitive generator of ρ. On the other hand we have

$$\begin{aligned} \omega_Z(0) &= \sum_{\sigma \in Y^{(1)}} f_* g(u_\sigma) \\ &= \sum_{\sigma \in Y^{(1)}} \sum_{i=1}^r \sum_{\substack{\rho \in X_i^{(1)} \\ f(\rho)=\sigma}} \omega_{X_i}(\rho)\mathrm{ind}(\rho) g\left(\frac{u_\rho}{\mathrm{ind}(\rho)}\right). \end{aligned}$$

Obviously each ray ρ can occur at most once in this sum and by assumption it occurs at least once. Hence we see that $\omega_Z(0) = \omega_{Z'}(0)$. □

Example 7.2.4. Note that the assumption that f is injective on each cone is necessary: Consider the morphism depicted in figure 7.11: In this case we get $f_*(g \cdot X) = 4$ and $f_* g \cdot Y = 2$.

7.2.2. Cutting out the codimension one cycle

By definition we have

$$\mathbb{H}_{n-4}^{\mathrm{trop}}(x) = \mathrm{ft}_*(\tilde{\mathbb{H}}_{n-4}^{\mathrm{trop}}(x)) = \mathrm{ft}_*(\mathrm{ev}_1^*(0) \cdot \mathcal{M}_x)$$

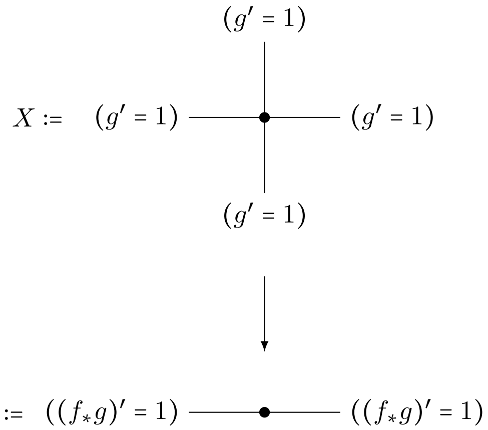

Figure 7.11.: A morphism where the push-forward of a function does not give the same divisor as the push-forward of the divisor of this function. All weights are 1 and the function slopes of g and f_*g are given in brackets.

and we already discussed that $\text{ft} : \mathcal{M}_x \to \mathcal{M}_{0,n}^{\text{trop}}$ is a morphism of $(n-3)$-dimensional tropical varieties which is injective on each cone of $\mathcal{M}_x$. Since $\mathcal{M}_{0,n}^{\text{trop}}$ is smooth, we immediately obtain the following result:

Corollary 7.2.5. *The codimension one Hurwitz cycle can be cut out as*

$$\mathbb{H}_{n-4}^{\text{trop}} = (\text{ft}_*(\text{ev}_1^*(0))) \cdot \mathcal{M}_{0,n}^{\text{trop}}.$$

We now want to describe the rational function $(\text{ft}_*(\text{ev}_{n+2}^*(0)))$ in more intuitive and geometric terms:

Lemma 7.2.6. *Let C be any curve in $\mathcal{M}_{0,n}^{\text{trop}}$. Given $x \in \mathcal{H}_n$ this defines a cover of $\mathbb{R}$ up to translation. Pick any such cover $h : C \to \mathbb{R}$. Let $v_1, \dots, v_r$ be the vertices of C. Then*

$$(\text{ft}_*(\text{ev}_1^*(0)))(C) = \sum_{i \neq j} (\text{val}(v_i) - 2)(\text{val}(v_j) - 2)\,|h(v_i) - h(v_j)|\,.$$

Proof. It suffices to show this for curves in maximal cones. Since $(\text{ft}_*(\text{ev}_1^*(0)))$ is continuous by Proposition 7.2.2, the claim follows for all other cones.

So let C be an n-marked trivalent curve with vertices $v_1, \dots, v_{n-2}$. We obtain all preimages in $\mathcal{M}_x$ by going over all possible choices of vertices v_i, v_j and attaching the additional leaves l_0 to v_i and l_1 to v_j. We denote the corresponding $n+2$-marked curve by $C(i,j)$. Note that ev_1 maps $C(i,j)$ to the image of l_1 under the cover obtained by fixing the image of l_0 to be 0. We immediately see the following:

- $\text{ev}_1(C(i,i)) = 0$.

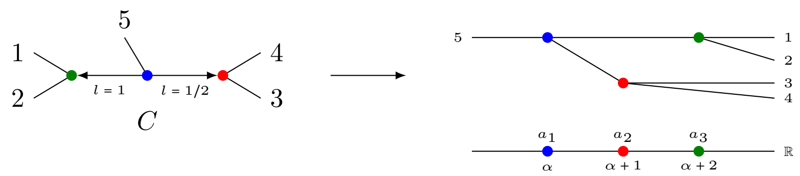

Figure 7.12.: We compute $(\mathrm{ft}_*(\mathrm{ev}_1^*(0)))(C)$ for an example. We choose parameters $x = (1,1,1,1,-4)$ and $C = v_{\{1,2\}} + \frac{1}{2}v_{\{3,4\}}$. In this case Lemma 7.2.6 tells us that the value of the function at C is $|a_2 - a_1| + |a_3 - a_1| + |a_3 - a_2| = 1 + 2 + 1 = 4$.

- $\mathrm{ev}_1(C(i,j)) = -\mathrm{ev}_1(C(j,i))$.
- $|\mathrm{ev}_1(C(i,j))| = |h(v_i) - h(v_j)|$

Since $\mathrm{ev}_1^*(0)(x) = \max\{0, \mathrm{ev}_1(x)\}$ and the forgetful map has index 1, the claim follows.

□

7.3. Hurwitz cycles as linear combinations of boundary divisors

In [BCM], the authors present several different representations of $\mathbb{H}_k(x)$. One is given in

Lemma 7.3.1 ([BCM, Lemma 3.6])**.**

$$\mathbb{H}_k(x) = \sum_{\Gamma \in \mathcal{T}_{n-3-k}} \left(m(\Gamma)\varphi(\Gamma) \prod_{v \in \Gamma^{(0)}} (\mathrm{val}(v) - 2)\Delta_\Gamma \right),$$

where Γ runs over $\mathcal{T}_{n-3-k}$, the set of all combinatorial types of rational n-marked curves with $n-3-k$ bounded edges and Δ_Γ is the stratum of all covers with dual graph Γ. Furthermore, $m(\Gamma)$ is the number of total orderings on the vertices of Γ compatible with edge directions and $\varphi(\Gamma)$ is the product over all edge weights.

There is an obvious, "naive" tropicalization of this: $\mathcal{T}_{n-3-k}$ corresponds to the codimension k skeleton of $\mathcal{M}_{0,n}^{\mathrm{trop}}$. We will write $m(\tau) := m(\Gamma_\tau), x_\tau := \varphi(\Gamma_\tau)$ for any codimension k cone τ and its corresponding combinatorial type Γ_τ. The boundary stratum Δ_Γ we translate like this:

Definition 7.3.2. Let $(\mathcal{X}, w)$ be a simplicial tropical fan. For a d-dimensional cone τ generated by rays $v_1, \dots, v_d$ we define rational functions φ_{v_i} on $\mathcal{X}$ by fixing its value on all rays:

$$\varphi_{v_i}(r) = \begin{cases} 1, & \text{if } r = v_i \\ 0, & \text{otherwise} \end{cases}$$

for all $r \in \mathcal{X}^{(1)}$. We then write $\varphi_\tau := \varphi_{v_1} \cdot \dots \cdot \varphi_{v_d}$ for subsequently applying these d functions. In the case of $\mathcal{X} = \mathcal{M}_{0,n}^{\text{trop}}$ and $v_i = v_I$, we will also write φ_I instead of φ_{v_i}.

As a shorthand notation we will write $\mathcal{C}_k$ for all dimension k cells of $\mathcal{M}_{0,n}^{\text{trop}}$ and $\mathcal{C}^k$ for all codimension k cells (in its combinatorial subdivision).

Now we define the following divisor of a piecewise polynomial (see for example [F] for a treaty of piecewise polynomials. For now it suffices if we define them as sums of products of rational functions):

$$D_k(x) := \sum_{\tau \in \mathcal{C}^k} m(\tau) \cdot x_\tau \cdot \left(\prod_{v \in \Gamma_\tau^{(0)}} (\text{val}(v) - 2) \right) \cdot \varphi_\tau \cdot \mathcal{M}_{0,n}^{\text{trop}},$$

where $\varphi_\tau = \prod_{v_I \in \tau^{(1)}} \varphi_I$ and the sum is to be understood as a sum of tropical cycles.

We can now ask ourselves, what the relation between $D_k(x)$ and $\mathbb{H}_k^{\text{trop}}(x)$ is. They are obviously not equal: $D_k(x)$ is a subfan of $\mathcal{M}_{0,n}^{\text{trop}}$ (in its coarse subdivision), but even if we choose all p_i to be equal to make $\mathbb{H}_k^{\text{trop}}(x)$ a fan, it will still contain rays in the interior of higher-dimensional cones of $\mathcal{M}_{0,n}^{\text{trop}}$.

This also rules out *rational equivalence* (as defined in [AR1]): Two cycles are equivalent, if and only if their recession fans are equal.

But there is another, coarser equivalence on $\mathcal{M}_{0,n}^{\text{trop}}$, that comes from toric geometry. As was shown in [GM], the classical $M_{0,n}$ can be embedded in the toric variety $X(\mathcal{M}_{0,n}^{\text{trop}})$ and we have

$$\begin{aligned} \text{Cl}(X(\mathcal{M}_{0,n}^{\text{trop}})) &\cong \text{Pic}(\overline{M}_{0,n}) \\ D_I &\mapsto \delta_I, \end{aligned}$$

where D_I is the divisor associated to the ray v_I and δ_I is the boundary stratum of curves consisting of two components, each containing the marked points in I and I^c respectively. By [FS2], D_I corresponds to some tropical cycle of codimension one in $\mathcal{M}_{0,n}^{\text{trop}}$ and [R1, Corollary 1.2.19] shows that this is precisely $\varphi_I \cdot \mathcal{M}_{0,n}^{\text{trop}}$. Hence the following is a direct translation of numerical equivalence in $\overline{M}_{0,n}$.

Definition 7.3.3. Two cycles $C, D \subseteq \mathcal{M}_{0,n}^{\text{trop}}$ are *numerically equivalent*, if for all k-dimensional cones $\rho \in \mathcal{C}_k$ we have

$$\varphi_\rho \cdot C = \varphi_\rho \cdot D$$

(considering both sides as elements of $\mathbb{Z}$).

Theorem 7.3.4. *$\mathbb{H}_k^{\text{trop}}(x)$ is numerically equivalent to $D_k(x)$.*

Proof. Note that for a generic choice of p_i,the cycle $\mathbb{H}_k^{\text{trop}}(x)$ does not intersect any cones of codimension larger than k and intersects all codimension k cones transversely.

For the proof we will need the following result from [BCM, Proposition 5.4], describing the intersection multiplicity of $\mathbb{H}_k^{\text{trop}}(x)$ with a codimension k-cell τ:

$$\tau \cdot \mathbb{H}_k^{\text{trop}}(x) = m(\tau) \cdot x_\tau \cdot \prod_{v \in C_\tau^{(0)}} (\text{val}(v) - 2).$$

This implies that for any $\rho \in \mathcal{C}_k$ we have

$$\begin{aligned}
\varphi_\rho \cdot \mathbb{H}_k^{\text{trop}}(x) &= \sum_{\tau \in \mathcal{C}^k} (\tau \cdot \mathbb{H}_k^{\text{trop}}(x)) \cdot \omega_{\varphi_\rho \cdot \mathcal{M}_{0,n}^{\text{trop}}}(\tau) \\
&= \sum_{\tau \in \mathcal{C}^k} m(\tau) \cdot x_\tau \cdot \prod_{v \in C_\tau^{(0)}} (\text{val}(v) - 2) \cdot \omega_{\varphi_\rho \cdot \mathcal{M}_{0,n}^{\text{trop}}}(\tau) \\
&= \sum_{\tau \in \mathcal{C}^k} m(\tau) \cdot x_\tau \cdot \prod_{v \in C_\tau^{(0)}} (\text{val}(v) - 2) \cdot (\varphi_\tau \cdot \varphi_\rho \cdot \mathcal{M}_{0,n}^{\text{trop}}) \\
&= \varphi_\rho \cdot D_k(x),
\end{aligned}$$

where $\omega_{\varphi_\rho \cdot \mathcal{M}_{0,n}^{\text{trop}}}(\tau) = \varphi_\tau \cdot \varphi_\rho \cdot \mathcal{M}_{0,n}^{\text{trop}}$ by [F, Lemma 4.7].

□

Part III.

Appendix

8. The polymake extension a-tint

As a tool for polyhedral computations, polymake is available for Linux and Mac under

www.polymake.org

Since a-tint requires a recent version, it is recommended to download the latest package or - even better - source code for installation. One can also find extensive documentation and some tutorials on this website.

All the algorithms we discussed in the previous sections have been implemented by the author in a-tint, an extension for polymake. It can be obtained under

https://bitbucket.org/hampe/atint

Installation instructions and a user manual can be found under the Wiki link. We include a list of most of the features of the software:

- Creating weighted polyhedral fans/complexes.
- Basic operations on weighted polyhedral complexes: Cartesian product, affine transformations, k-skeleton, computing lattice normals, checking balancing condition.
- Visualization: Display varieties in $\mathbb{R}^2$ or $\mathbb{R}^3$ including (optional) weight labels and coordinate labels.
- Compute degree and recession fan (experimental).
- Rational functions: Arbitrary rational functions (given as complexes with function values) and tropical polynomials (min and max).
- Basic linear arithmetic on rational functions: Compute linear combinations of functions.
- Divisor computation: Compute (k-fold) divisor of a rational function on a tropical variety.
- Intersection products: Compute cycle intersections in $\mathbb{R}^n$.
- Intersection products on matroid fans via rational functions defined by chains of flats (this is very slow).
- Local computations: Compute divisors/intersections locally around a given face/ a given point.
- Functions to create tropical linear spaces.

- Functions to create matroid fans (using a modified version of TropLi by Felipe Rincón [R2]).
- Functions to create the moduli spaces of rational n-marked curves: Globally and locally around a given combinatorial type.
- Computing with rational curves: Convert metric vectors / moduli space elements back and forth to rational curves, do linear arithmetic on rational curves.
- Morphisms: Arbitrary morphisms (given as complexes with values) and linear maps
- Pull-backs: Compute the pull-back of any rational function along any morphism.
- Evaluation maps: Compute the evaluation map ev_i on the labeled version of the moduli spaces $\mathcal{M}_{0,n}^{\mathrm{trop}}$.

8.1. Polyhedral complexes in `polymake` and `a-tint`

In this section we give a short introduction as to how `polymake` and especially `a-tint` represent and handle polyhedral fans and complexes.

8.1.1. Homogeneous coordinates

Recall from Section 1.3.1 that a polyhedron can be defined by giving a list of vertices and rays, as well as generators for its lineality space:

$$\sigma = \mathrm{conv}\{p_1, \dots, p_k\} + \mathbb{R}_{\geq 0} r_1 + \dots + \mathbb{R}_{\geq 0} r_l + L.$$

For computational purposes it is very tedious to have to distinguish between rays and vertices of a polyhedron. `polymake` solves this by representing a polyhedron as a cone in one dimension higher: Mathematically, to a polyhedron $P \subseteq \mathbb{R}^n$, we can associate a cone $C_P \in \mathbb{R}^{n+1}$ via

$$C_P := \overline{\mathbb{R}_{\geq 0} \cdot (\{1\} \times P)}.$$

The polyhedron can then again be retrieved as $P = C_P \cap \{x_0 = 1\}$.

Technically this just means that we prepend an additional coordinate to each ray and vertex of P: A 0 in case of rays (and lineality space generators) and a 1 in case of vertices.

Note that this does in general not induce a one-to-one correspondence between *faces* of C_P and P: Let P be the shifted ray $1 + \mathbb{R}_{\geq 0} \subseteq \mathbb{R}$. Then $C_P = \mathbb{R}_{\geq 0}(1,1) + \mathbb{R}_{\geq 0}(0,1) \subseteq \mathbb{R}^2$ has two one-dimensional faces, generated by its two rays, while P has only one.

However, there is a one-to-one correspondence

$$\{d - \text{dim. faces of } P\} \overset{1:1}{\leftrightarrow} \{(d+1) - \text{dim. faces of } C_P \text{ which intersect } \{x_0 = 1\}\}.$$

8.1.2. Representing a polyhedral complex

We already saw that a polyhedral complex is uniquely determined by its set of maximal cones, since they in turn define all their faces. Hence a complex can be created in `polymake` by providing:

- A list of its rays and vertices, as rows of a matrix R.
- A list of generators for its lineality space, again as rows of a matrix L.
- A list of its maximal cones, each of which is a tuple of the row indices of its rays and vertices in R (starting the count at 0).

We just discussed in the previous section, that we still need to distinguish between fans and complexes if we want to compute faces of polyhedra. `polymake` solves this by using two different data types: `fan::PolyhedralFan` and `fan::PolyhedralComplex`. However, in tropical geometry this distinction is not sensible, since fans and complexes can occur as arguments in the same operation (e.g. if we want to compute an intersection product of two varieties, only one of which is a fan). Hence `a-tint` uses a single data type, `WeightedComplex` (which is derived from `fan::PolyhedralFan`) and adds a boolean property `USES_HOMOGENEOUS_C`, that has to be defined upon creation and determines whether its coordinates should be understood as homogeneous ones.

---

polymake example: Computing a tropical variety.
This first creates the weighted complex consisting of the four orthants of $\mathbb{R}^2$ with weight 1 and checks if it is balanced. It then creates the same complex, but with the vertex shifted to $(1,1)$. `USES_HOMOGENEOUS_C` is `FALSE` by default.

```
atint >   $w = new WeightedComplex(
          RAYS=>[[1,0],[-1,0],[0,1],[0,-1]],
          MAXIMAL_CONES=>[[0,2],[0,3],[1,2],[1,3]],
          TROPICAL_WEIGHTS=>[1,1,1,1]);
atint >   print $w->IS_BALANCED;
1
atint >   $wshift = new WeightedComplex(
          RAYS=>[[1,1,1],[0,1,0],[0,-1,0],[0,0,-1]],
          MAXIMAL_CONES=>[[0,1,3],[0,1,4],[0,2,3],[0,2,4]],
          TROPICAL_WEIGHTS=>[1,1,1,1]);
```

9. Benchmarks

All measurements were taken on a standard office PC with 8 GB RAM and 8x2.8 GHz (though no parallelization took place). Time is given in seconds.

Lattice normal computation

In this Section we want to compare the two methods to compute lattice normals we discussed in Section 2.1. The first method, the *normal* one, is simply computing Hermite normal forms for each pair $\sigma > \tau$ of maximal cells and codimension one faces. The corresponding time taken will be denoted by t_n. The second method is projecting each maximal cell along some lattice-respecting map to reduce codimension. We denote its time by t_p. What we will actually measure is the ratio $r = t_n/t_p$. As it turns out, r behaves rather differently for different types of tropical varieties, so it is very difficult to find a general criterion for deciding which of these methods to pick. There are only a few obvious remarks:

- The ratio r increases with the ambient dimension.
- In low codimension, the normal method is always better, i.e. $r < 1$.
- We can increase r significantly, if we exclude the lattice basis computation for the projection method in the measurement. This sounds rather obvious, but is actually relevant, e.g. in the following situation: Assume we compute a divisor $f \cdot X$ and we already know lattice bases for each maximal cell of X. We then copy these lattice bases for the maximal cells of the divisor. When computing lattice normals for $f \cdot X$, we already have bases we can use for projection. While their span is one dimension larger than the dimension of a maximal cell of $f \cdot X$, we still gain some performance with this operation.
- The ratio depends very much on the combinatorics of the complex in question. The projection method computes Hermite normal forms of large matrices only once for each maximal cell, while the normal method does this for each pair of codimension one cells in maximal cells. Thus the gain of the projection method is much larger if there are many maximal cones with many codimension one cells. In particular, if we did any of the computations below only locally, the ratio would be much smaller.

In Figure 9.1 we compare both methods in the case of the tropical linear spaces L_d^n. This is a d-dimensional unimodular fan in $\mathbb{R}^n$. More precisely, it can be seen as the Bergman fan $B(U_{d+1,n+1})/\langle(1,\dots,1)\rangle$ of a uniform matroid or as the $(n-d)$-fold divisor

$\max\{0, x_1, \ldots, x_n\}^{(n-d)} \cdot \mathbb{R}^n$. Since this computation is actually too fast to properly measure for small n and d, we perform it 1000 times before measuring.

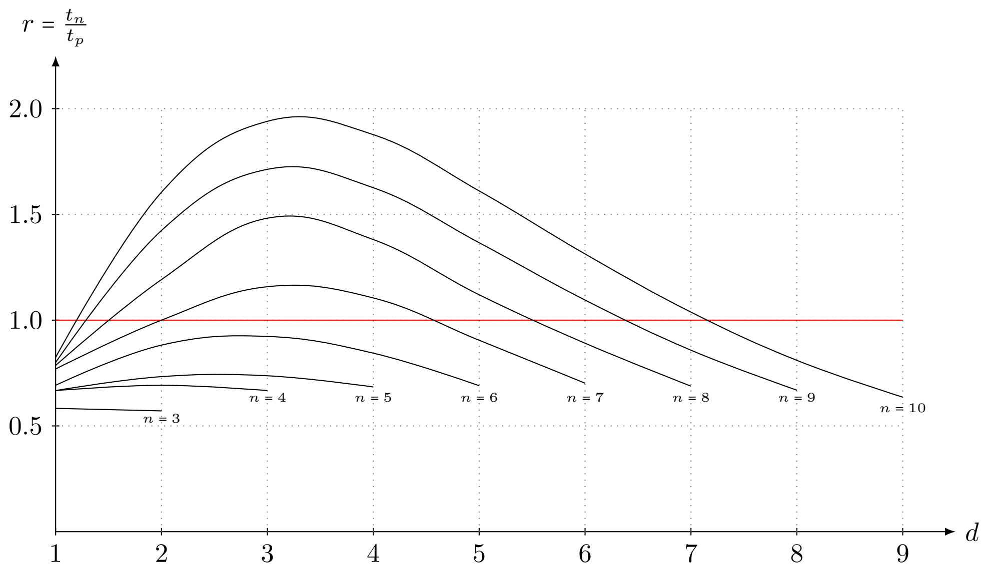

Figure 9.1.: Lattice normal computation for linear spaces L^n_d. The interpolation was done with cubic splines for better visualization.

In Figure 9.2 we measure both methods for $f \cdot L^n_{d+1}$, where f is a random tropical polynomial with 5 terms and integer coefficients. More precisely, for each choice of parameters n and d we compute 100 random polynomials and measure the time taken over all lattice normal computations before computing r. These divisors are in general non-simplicial polyhedral complexes. As one can see, the ratio increases and decreases much faster than in the previous case.

We also include one example (the blue curve), where we copy the lattice bases from L^n_{d+1} as described above. As one can see, in this case the projection method is much faster in higher codimensions, though in lower codimension it behaves much the same way as before.

We want to stress again, that r increases with the ambient dimension. Some (computable) examples for large ambient dimension are the moduli spaces $\mathcal{M}^{\text{trop}}_{0,n}$. Again, where lattice normal computation is normally too fast to measure, we performed the operation 1000 times before computing the ratio. Note also that we only perform the computation locally for $n = 12, 15$. The results are displayed in Table 9.1.

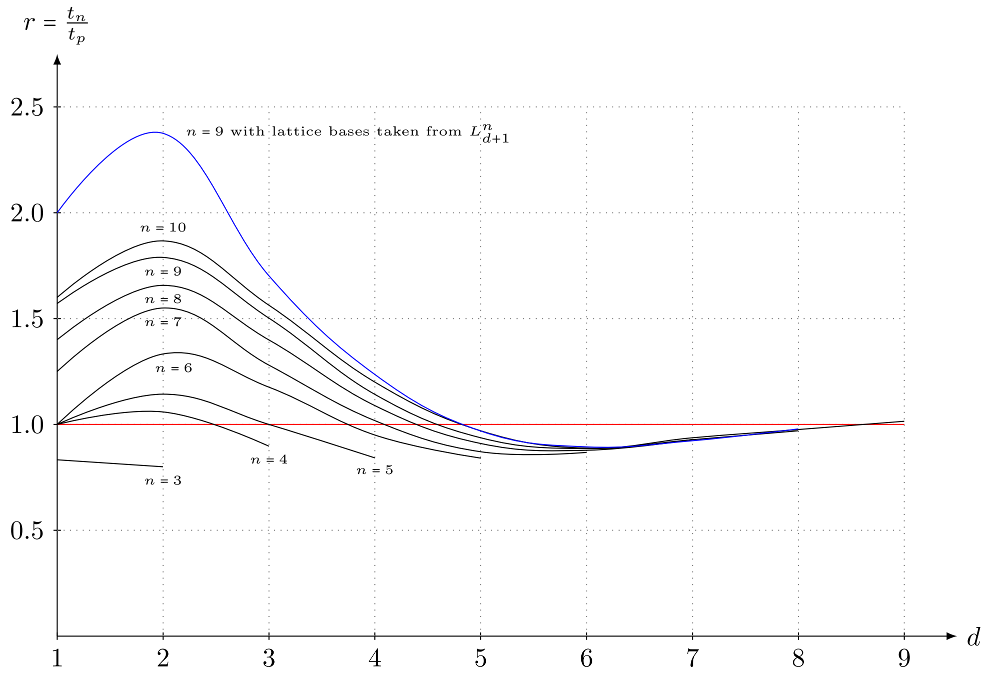

Figure 9.2.: Lattice normal computation for $f \cdot L^n_{d+1}$, f a random tropical polynomial

Conclusion

Using the empirical results obtained above `a-tint` currently uses the following heuristic rules to determine which method it should use:

- If the ambient dimension is larger than 10, use the projection method.
- If the ambient dimension is smaller than 6, use the normal method.
- If the ambient dimension is $n \in \{6, \dots, 10\}$ use the projection method, if and only if the dimension d of the variety fulfills $d \leq \frac{n}{2}$.

Divisor computation

Here we see how the performance of the computation of the divisor of a tropical polynomial on a cycle changes if we change different parameters. In Table 9.2 we take a random tropical polynomial f with l terms, where $l \in \{5, 10, 15\}$. We compute the divisor of this polynomial on X, where X is $L^2_1 \times \mathbb{R}^{k-1}$ in $\mathbb{R}^n$, i.e. X has 3 cones. We do this ten times and take the average.

Note that the normal fan of the Newton polytope of f usually has around l maximal

No. of leaves n (amb. dim. m)	**5 (5)**	**6 (9)**	**7 (14)**	**8 (20)**	**12 (54)**	**15 (90)**
Ratio t_n/t_p	0.73	1.57	2.00	2.65	3.67	4.72

Table 9.1.: Ratio development of lattice normal computation for $\mathcal{M}_{0,n}^{\text{trop}}$. The computation for $n = 12, 15$ was only done locally around a combinatorial type of codimension 4 and 5, respectively.

cones, but is highly non-simplicial: It can have several hundred rays. If, instead of f, we take a polynomial whose Newton polytope is the hypercube in $\mathbb{R}^n$ (here the normal fan is simplicial), then all these computations take less then a second. This is probably due to the fact that convex hull algorithms are very fast on "nice" polyhedra.

		n = 2	**n = 4**	**n = 6**	**n = 8**	**n = 10**
k = 1	$l = 5$	0.3	0.3	0.3	0.3	0.3
	$l = 10$	0.2	0.3	0.5	0.4	0.5
	$l = 15$	0.3	0.5	36.8	1242.6	2327.3
k = 3	$l = 5$		0.2	0.3	0.3	0.3
	$l = 10$		0.4	0.6	0.5	0.5
	$l = 15$		0.7	38.4	1031.7	1860.6
k = 5	$l = 5$			0.3	0.3	0.3
	$l = 10$			1.9	1.8	2.2
	$l = 15$			43.9	1519.7	2313.5
k = 7	$l = 5$				0.4	0.4
	$l = 10$				6.1	7.2
	$l = 15$				2010.9	3149.2
k = 9	$l = 5$					0.4
	$l = 10$					2.4
	$l = 15$					22931

Table 9.2.: Divisor of a random tropical polynomial with $l = 5, 10, 15$ terms on $L \times \mathbb{R}^{k-1} \subseteq \mathbb{R}^n$. The time is given in seconds.

Intersection products

We want to see how the computation of an intersection product compares to divisor computation. If we apply several rational functions $f_1, \dots, f_k$ to $\mathbb{R}^n$, we can compute

$f_1 \cdot \dots \cdot f_k \cdot \mathbb{R}^n$ in two ways: Either as successive divisors $f_1 \cdot (f_2 \cdot (\dots \cdot \mathbb{R}^n))$ or as an intersection product $(f_1 \cdot \mathbb{R}^n) \cdot \dots \cdot (f_k \cdot \mathbb{R}^n)$. Since successive divisors of rational functions appear in many formulas and constructions, it is interesting to see which method is faster. Table 9.3 compares this for $k = 2$. We take f and g to be random tropical polynomials with 5 terms and average over 50 runs. As we can see, the intersection product is significantly faster in low dimensions, but its computation time grows much more quickly: For $n = 8$, the intersection product takes seven times as long as the divisors.

	Successive: $f \cdot (g \cdot \mathbb{R}^n)$	**Product:** $(f \cdot \mathbb{R}^n) \cdot (g \cdot \mathbb{R}^n)$	**Ratio:** $t_{\text{succ}}/t_{\text{prod}}$
$\mathbf{n = 3}$	0.62	0.14	4.43
$\mathbf{n = 4}$	0.68	0.24	2.83
$\mathbf{n = 5}$	1.04	0.38	2.74
$\mathbf{n = 6}$	1.42	0.84	1.69
$\mathbf{n = 7}$	1.5	2.7	0.56
$\mathbf{n = 8}$	1.6	11.66	0.14

Table 9.3.: Comparing successive divisors to intersection products. Time is given in seconds.

Matroid fan computation

Here we compare the computation of matroid fans with different algorithms. Table 9.4 displays the results, in seconds rounded down.

We start with the computation of the moduli space $\mathcal{M}_{0,n}^{\text{trop}}$, first as the Bergman fan $B(K_{n-1})$ of the complete graph on $n-1$ vertices using the TropLi algorithms, then combinatorially as described in Corollary 4.2.8. Finally, we also compute the Bergman fan using the normal fan Algorithm 5 and in its fine subdivision (Definition 3.1.4). Note that `a-tint` computes flats using a brute force algorithm. While the last method is much faster than the normal fan algorithm, it still becomes infeasible rather quickly.

We then compare the performance of the TropLi algorithms [R1] in the case of general matroids. First, we compute the Bergman fan of the uniform matroid $U_{n,k}$. Note that we compute it as a Bergman fan of a matroid without making use of the matrix structure behind it (the uniform matroid is actually realizable). Then we compute two *linear* matroids, i.e. we let TropLi make use of linear algebra to compute fundamental circuits. C_i has as column vectors the vertices of the i-dimensional unit cube (in affine coordinates, i.e. with an additional row of ones on top). Here we see again that the method computing chains of flats is much faster than the normal fan algorithm, but time consumption increases very quickly with matroid complexity.

	TropLi[a]	**Cor.** 4.2.8	**Alg.** 5	**Fine subdiv.**
$\mathbf{M}_{0,6}^{\mathbf{trop}}$	0	0	3	0
$\mathbf{M}_{0,7}^{\mathbf{trop}}$	3	0	3814	5
$\mathbf{M}_{0,8}^{\mathbf{trop}}$	921	0	*(aborted after 24h)*	4052
$\mathbf{U}_{9,6}$	0		3	1
$\mathbf{U}_{10,6}$	0		19	4
$\mathbf{U}_{11,6}$	0		87	12
$\mathbf{C}_3$	0		1	0
$\mathbf{C}_4$	1		29388	18

Table 9.4.: Computation of several Bergman fans. The time is given rounded down to the next second.

[a] We did not use the original TropLi program but an implementation of the algorithms in polymake-C++. In the case of linear matroids the original program is actually much faster. This is probably due to the fact that the data types in polymake are larger and that the linear algebra library used by TropLi is more efficient.

Bibliography

[AZ] M. Aigner and G.M. Ziegler, *Proofs from THE BOOK*, Springer, 1998.

[AR1] L. Allermann and J. Rau, *Tropical rational equivalence on* $\mathbb{R}^r$ (2008), available at arxiv:0811.2860v2.

[AR2] L. Allermann and J. Rau, *First steps in tropical intersection theory*, Math. Z. **264** (2010), no. 3, 633–670, available at arxiv:0709.3705v3.

[AK] F. Ardila and C.J. Klivans, *The Bergman complex of a matroid and phylogenetic trees*, J. Comb. Theory, Ser. B **96** (2006), 38–49, available at arxiv:math/0311370v2.

[ABS] D. Avis, D. Bremner, and R. Seidel, *How good are convex hull algorithms*, Computational Geometry: Theory and Applications **7** (1997), 265-302.

[AF] D. Avis and K. Fukuda, *A Pivoting Algorithm for Convex Hulls and Vertex Enumeration of Arrangements and Polyhedra*, Discrete and Computational Geometry **8** (1992), 295-313.

[BCM] Aaron Bertram, Renzo Cavalieri, and Hannah Markwig, *Polynomiality, Wall Crossings and Tropical Geometry of Rational Double Hurwitz Cycles*, JCTA **120** (2013), no. 7, 1604-1631, available at arxiv:1209.5789v1.

[BY] F. Block and J. Yu, *Tropical convexity via cellular resolutions*, Journal of Algebraic Combinatorics **24** (2006), no. 1, 103-114.

[BG] J.-P. Barthélemy and A. Guénoche, *Trees and Proximity Representations*, Wiley-Interscience, Chichester, 1991.

[BS] E. Brugallé and K. Shaw, *Obstructions to approximating tropical curves in surfaces via intersection theory* (2001), available at arxiv:1110.0533v2.

[B] P. Buneman, *A note on the metric properties of trees*, Journal of combinatorial theory **17** (1974), 48–50.

[C] H. Cohen, *A course in computational algebraic number theory*, 4th ed., Springer Verlag, Berlin, 2000.

[DS] M. Develin and B. Sturmfels, *Tropical convexity*, Documenta Math. **9** (2004), 1-24.

[E] H. Edelsbrunner, *Algorithms in combinatorial geometry*, Springer-Verlag, 1987.

[F] G. François, *Cocycles on tropical varieties via piecewise polynomials*, available at arxiv:1102.4783v2.

[FR] G. François and J. Rau, *The diagonal of tropical matroid varieties and cycle intersections*, available at arxiv:1012.3260v1.

[F] K. Fukuda, *cdd,cddplus and cddlib homepage*, 2002. available at `http://www.ifor.math.ethz.ch/~fukuda/cdd_home/`.

[FS1] E. Feichtner and B. Sturmfels, *Matroid polytopes, nested sets and Bergman fans*, Port. Math. (N.S.) **62** (2005), 437–468, available at arxiv:math/0411260.

[FS2] W. Fulton and B. Sturmfels, *Intersection theory on toric varieties*, Topology **36** (1997), no. 2, available at arxiv:9403002.

[GM] A. Gathmann and H. Markwig, *Kontsevich's formula and the WDVV equations in tropical geometry*, Advances in Mathematics **217** (2008), no. 2, 537-560.

[GKM] A. Gathmann, M. Kerber, and H. Markwig, *Tropical fans and the moduli spaces of tropical curves*, Compos. Math. **145** (2009), no. 1, 173–195, available at arxiv:0708.2268.

[GS] A. Gathmann and F. Schröter, *Broccoli curves and the tropical invariance of Welschinger numbers*, Advances in Mathematics **240** (2013), 520-574, available at arxiv:1104.3118.

[GJ] E. Gawrilow and M. Joswig, *polymake: a Framework for Analyzing Convex Polytopes*, Polytopes — Combinatorics and Computation, 2000, pp. 43-74. polymake is available at `http://www.polymake.org`.

[GM] A. Gibney and D. Maclagan, *Equations for Chow and Hilbert quotients*, Journal of Algebra and Number Theory **4** (2010), no. 7, 855-885.

[GJV] I. Goulden, D.M. Jackson, and R. Vakil, *Towards the geometry of double Hurwitz numbers*, Adv. Math. **198** (2005), 43-92.

[GV] T. Graber and R. Vakil, *Relative virtual localization and vanishing of tautological classes on moduli spaces of curves*, Duke Math. J. **130** (2005), no. 1, 1–37, available at arxiv:math.AG/0309227.

[G] B. Grünbaum, *Convex polytopes*, 2nd ed., Springer-Verlag, 2003.

[HMM] G. Havas, B.S. Majewski, and K.R. Matthews, *Extended gcd and Hermite normal form algorithms via lattice basis reduction*, Experimental Mathematics **7** (1998), 125-136.

[IKS] I. Itenberg, V. Kharlamov, and E. Shustin, *A Caporaso-Harris type formula for Welschinger in- variants of real toric Del Pezzo surfaces*, Comment. Math. Helv. **84** (2009), 87-126, available at arxiv:math/0608549.

[JY] A.N. Jensen and J. Yu, *Stable intersections of tropical varieties*, available at arxiv:1309.7064.

[J1] Anders N. Jensen, *Gfan, a software system for Gröbner fans and tropical varieties*, available at `http://home.imf.au.dk/jensen/software/gfan/gfan.html`.

[J2] P. Johnson, *Hurwitz numbers, ribbon graphs and tropicalization*, Contemp. Math. **580** (2012), 55-72, available at arxiv:1303.1543.

[J3] M. Joswig, *Tropical convex hull computations*, In G.L. Litvinov and S.N. Sergeev, eds., Tropical and Idempotent Mathematics, International Workshop TROPICAL-07, Moscow, Contemporary Mathematics **495** (2009), 193-212.

[KM1] M. Kerber and H. Markwig, *Counting tropical elliptic plane curves with fixed j-invariant*, Commentarii Mathematici Helvetici **84** (2009), no. 2, 387-427, available at arxiv:math/0608472.

[KM2] M. Kerber and H. Markwig, *Intersecting Psi-classes on tropical* $\mathcal{M}_{0,n}$, Int. Math. Res. Notices **2009** (2009), no. 2, 221–240, available at arxiv:0709.3953v2.

[KBE$^+$] L. Khachiyan, E. Boros, K. Elbassioni, V. Gurvich, and K. Makino, *On the complexity of some enumeration problems for matroids*, SIAM J. Discrete Math. **19** (2006), no. 4, 966-984.

[K] J. Kock, *Notes on Psi classes*, available at `http://mat.uab.es/~kock/GW/notes/psi-notes.pdf`.

[M1] Grigory Mikhalkin, *Enumerative tropical algebraic geometry in* $\mathbb{R}^2$, J. Amer. Math. Soc. **18** (2005), 313-377, available at arxiv:math/0312530v4.

[M2] G. Mikhalkin, *Moduli spaces of rational tropical curves*, Proceedings of the 13th Gökova geometry-topology conference, Cambridge, MA, International Press (2007), 39-51, available at arxiv:0704.0839.

[M3] Grigory Mikhalkin, *Tropical geometry and its applications*, Proceedings of the ICM, Madrid, Spain (2006), 827–852, available at arxiv:math/0601041v2.

[M4] G. Mikhalkin, *Moduli spaces of rational tropical curves*, Proceedings of Gokova Geometry-Topology Conference 2006 (2007), 39–51.

[MRTT] T.S. Motzkin, H. Raiffa, G.L. Thompson, and R.M. Thrall, *The double description method*, Contributions to theory of games, Vol. 2, 1953.

[OP] A. Okounkov and R. Pandharipande, *Gromov-Witten theory, Hurwitz numbers, and Matrix models, I* (2001), available at arxiv:math/0101147.

[O] James G. Oxley, *Matroid Theory*, Oxford University Press, Oxford, 1992.

[R1] Johannes Rau, *Tropical intersection theory and gravitational descendants*, PhD thesis, Technische Universität Kaiserslautern, 2009, http://kluedo.ub.uni-kl.de/volltexte/2009/2370.

[R2] J. Rau, *Intersections on tropical moduli spaces* (2008), available at arxiv:0812.3678v1.

[RGST] J. Richter-Gebert, B. Sturmfels, and T. Theobald, *First steps in tropical geometry*, Idempotent Mathematics and Mathematical Physics (G.L. Litvinov and V.P. Maslov, eds.), Proceedings Vienna 2003, American Mathematical Society, Contemp. Math. **377** (2005).

[R1] F. Rincón, *Computing Tropical Linear Spaces*, Journal of Symbolic Computation, to appear, available at arxiv:1109.4130.

[R2] ———, *TropLi: a software tool for computing tropical linear spaces.* `http://math.berkeley.edu/~felipe/tropli/`.

[S1] P.D. Seymour, *A note on hyperplane generation*, Journal of Combinatorial Theory, Series B **1** (1994), no. 1, 88-91.

[S2] K. Shaw, *A tropical intersection product in matroidal fans* (2010), available at arxiv:1010.3967v1.

[S3] D. Speyer, *Tropical linear spaces*, SIAM J. Discrete Math. **22** (2008), 1527–1558, available at arxiv:math/0410455.

[SS] D. Speyer and B. Sturmfels, *The Tropical Grassmannian*, Advances in Geometry **4** (2004), no. 3, 389 – 411, available at math/0304218.

,

[S] B. Sturmfels, *Solving systems of polynomial equations*, CBMS Regional Conferences Series in Mathematics, vol. 97, Published for the Conference Board of the Mathematical Sciences, Washington, DC (2002).

[T] H.R. Tiwary, *On the Hardness of Computing Intersection, Union and Minkowski Sum of Polytopes*, Discrete & Computational Geometry **40** (2008), no. 3, 469-479.

[VDK] Wilberd Van Der Kallen, *Complexity of the Havas,Majewski,Matthews LLL Hermite Normal Form Algorithm*, Journal of Symbolic Computation **30** (2000), no. 3, 329-337.

[VKF] I.A. Volodin, V.E. Kuznecov, and A.T. Fomenko, *The problem of the algorithmic discrimination of the standard three-dimensional sphere*, Uspehi Mat. Nauk **29** (1974), 71-168. Appendix by S. P. Novikov.

[W] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in Diff. Geom. **1** (1991), 243-310.

[Z] G. Ziegler, *Lectures on polytopes*, Springer-Verlag, 1994. Graduate Texts in Mathematics 152.

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