

RELATIVE REALIZABILITY OF TROPICAL CURVES IN A PLANE IN THE NON-CONSTANT COEFFICIENT CASE

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ABSTRACT. Let X be a plane in a torus over an algebraically closed field K , with tropicalization the matroidal fan Σ . In this paper we present an algorithm which completely solves the question whether a given one-dimensional balanced polyhedral complex in Σ is relatively realizable, i. e. whether it is the tropicalization of an algebraic curve over $K\{\{t\}\}$ in X . The algorithm implies that the space of all such relatively realizable curves of fixed degree is an abstract polyhedral set.

In the case when X is a general plane in 3-space, we use the idea of this algorithm to prove some necessary and some sufficient conditions for relative realizability. For 1-dimensional polyhedral complexes in Σ that have exactly one bounded edge, passing through the origin, these necessary and sufficient conditions coincide, so that they give a complete non-algorithmic solution of the relative realizability problem.

1. INTRODUCTION

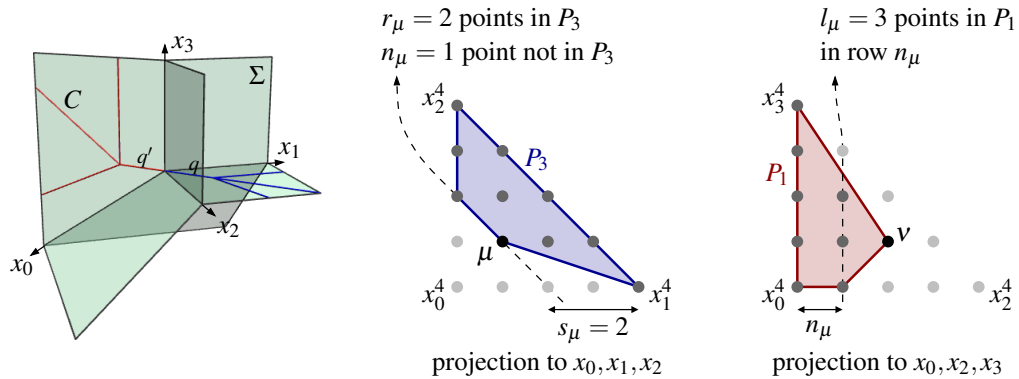
Although tropical geometry is growing in several directions over the last years and a wide spectrum of algebraic concepts has already been transferred to the tropical world, the tropicalization map, and in particular its image, is not yet well understood. There is still no general approach to decide whether a given combinatorial object comes from an algebraic one via the tropicalization map. Results in this direction are currently restricted to rational curves and hypersurfaces in \mathbb{R}^n [Mik04, Spe05, NS06], as well as some partial statements for the case of elliptic curves [Spe07].

The relative case of this realizability question is even less studied: If K is an algebraically closed field, X a variety in an n -dimensional torus over the field $K\{\{t\}\}$ of Puiseux series over K , $\Sigma := \text{Trop}X \subset \mathbb{R}^n$ its tropicalization, and C a weighted polyhedral complex in Σ , is there an algebraic subvariety of X that tropicalizes to C ? So far this question has only been considered in greater detail in the case when X is a plane defined over K and C a 1-dimensional balanced fan (the so-called constant coefficient case). There is then an algorithm to determine which fans are tropicalizations of curves, as well as some necessary and some sufficient conditions for relative realizability [BK11, BS11, GSW13].

In this paper, we will study this relative realizability problem in the case when X is still a plane defined over K , hence Σ a 2-dimensional matroidal fan, but C is now a 1-dimensional balanced polyhedral complex (which is not necessarily a fan). An example of this situation

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is shown in the picture below on the left. After introducing the required background from polyhedral and tropical geometry in Section 2, we will present an algorithm to decide if C is relatively realizable in Section 3. As in the constant coefficient case in [GSW13], this approach is based on projections to coordinate planes, so that one can work with tropical curves in \mathbb{R}^2 and their dual picture of extended Newton polytopes. We show that it is sufficient to consider Puiseux series in $K\{\{t\}\}$ containing only a fixed finite set of powers of t , so that the calculations, presented in Algorithm 3.20, can be performed on a computer. The algorithm is implemented as a library for the computer algebra system Singular, as explained in Remark 3.24 [DGPS, Bir14]. With its help, we cannot only solve the relative realizability problem in matroidal fans algorithmically, but also use its theoretical background to prove non-algorithmic results. For instance, we show in Proposition 3.27 that the space of realizable tropical curves is a polyhedral set in the moduli space of all tropical curves in Σ of fixed degree. Moreover, for a given matroid fan, this theoretical background can be used to explore relations between the projections to the coordinate planes and thus to prove necessary and sufficient non-algorithmic conditions for relative realizability.



The main part of the final Section 4 deals with the special case when X is a general plane in 3-space, so that the corresponding matroid is the uniform matroid of rank 3 on 4 elements, and hence Σ is the fan shown in the picture. As indicated above, we explore relations between the projections to the coordinate planes and use them to prove some necessary and some sufficient non-algorithmic conditions for relative realizability in Σ , both in the constant and the non-constant coefficient case. They are particularly strong in the special situation of the picture when C has exactly one bounded edge, passing through the origin with homogeneous direction $[0, 1, 1, 0]$ and lengths $q, q' \in \mathbb{R}_{\geq 0}$ on the two sides, and hence with unbounded ends contained in two opposite cones of Σ . In this case the necessary and sufficient conditions agree, thus giving a complete non-algorithmic solution of the relative realizability problem.

In fact, these results can be applied to a large class of curves since they are in a certain sense local: By Corollary 4.18, if the recession fan of C is relatively realizable then a tropical curve in Σ containing C as a local part can only be relatively realizable if C is. Example 4.17 shows that this is false without the assumption on the recession fan.

Let us quickly describe the necessary and sufficient conditions for relative realizability in the case of a curve C with one bounded edge as in the picture above. The curve C is then described completely by the two lengths q and q' , together with the Newton polytopes P_3 and P_1 of the projections of C to the x_0, x_1, x_2 -plane and the x_0, x_2, x_3 -plane, respectively, as shown in the picture above on the right. By a row of these polytopes we will mean a line parallel to the x_1 - x_2 -side in the case of P_3 , and to the x_0 - x_3 -side in the case of P_1 . We denote by Δ the standard simplex of size $d := \deg C$, indicated by the dots in the picture.

Now for any vertex μ of one of the polytopes P_3 or P_1 , consider the lattice points of Δ in the row of μ . Let r_μ be the number of these points contained in the polytope, and n_μ the number of these points not in the polytope, so that $s_\mu := d + 1 - r_\mu - n_\mu$ is the row number of μ . Moreover, let l_μ be the number of lattice points in row n_μ of the other of the two polytopes. The condition on μ is then that $l_\mu \geq r_\mu$. We show in Proposition 4.9 that the recession fan of C (i. e. the corresponding fan curve if $q = q' = 0$) is relatively realizable if and only if this condition is satisfied for all vertices of P_3 and P_1 . It is easily checked that this is the case for the curve in the picture above.

Moreover, if the recession fan of C is relatively realizable, Proposition 4.15 states that the (non-constant coefficient) curve C is relatively realizable if and only if for any vertex μ of P_3 with row number $s_\mu \neq 0$ and any vertex ν of P_1 with $n_\nu \neq 0$, the lengths q and q' satisfy the condition

$$\frac{n_\mu}{s_\mu} \leq \frac{q}{q'} \leq \frac{s_\nu}{n_\nu}.$$

For example, for the two vertices μ and ν in the picture above we have $n_\mu = 1$ and $s_\mu = n_\nu = s_\nu = 2$, and so we know that C can only be relatively realizable if $\frac{1}{2} \leq \frac{q}{q'} \leq 1$. The other two relevant vertices of the polytopes give us the same lower and upper bound for $\frac{q}{q'}$, hence we conclude that C is relatively realizable if and only if $\frac{1}{2} \leq \frac{q}{q'} \leq 1$.

It may happen that there are vertices μ of P_3 and ν of P_1 such that $\frac{n_\mu}{s_\mu} > \frac{s_\nu}{n_\nu}$ and so, the tropical curve C is not relatively realizable for any $q, q' \in \mathbb{R}_{>0}$. One may ask if this is only possible if the recession fan of C is not realizable. However, in Examples 4.21 and 4.22 we show that the solvability of the length conditions does not correlate with the relative realizability of the recession fan of C .

2. PRELIMINARIES

As a preparation for the main part of this paper, we need to introduce some basic concepts and statements in tropical geometry to explain our setup. The following section is therefore concerned with some polyhedral theory and the way we want to tropicalize algebraic varieties. We start with purely combinatorial objects, and then explain how to obtain them from algebraic ones.

Convention 2.1 (Polyhedral complexes). By a *polyhedral complex* we will always mean a rational, pure-dimensional polyhedral complex in a vector space $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ associated to a lattice Λ . If $U \subset V$ is a Λ -rational linear subspace of V , we will equip the quotient V/U

with the induced lattice. The vector space \mathbb{R}^n will always be considered with the underlying lattice \mathbb{Z}^n .

Definition 2.2 (Tropical curves). Let C be a 1-dimensional polyhedral complex, together with a weight $\omega(\sigma) \in \mathbb{N}_{>0}$ for each edge σ in C . Then we say that (C, ω) is *balanced* if for all vertices μ in C we have

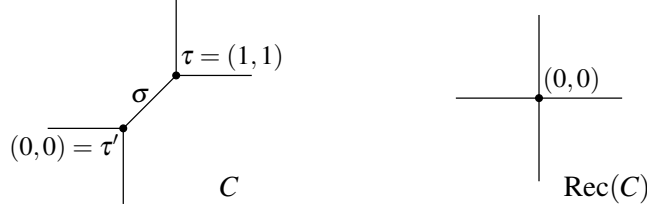
$$\sum_{\sigma} \omega(\sigma) v_{\sigma, \mu} = 0,$$

where the sum is taken over all edges σ adjacent to μ , and $v_{\sigma, \mu} \in \Lambda$ is the primitive direction vector of σ pointing away from μ .

A balanced 1-dimensional polyhedral complex (C, ω) , often just written as C , is called a *tropical curve*. We will identify two such curves if their underlying polyhedral complexes have the same support, and if there is a common refinement such that the induced weights agree.

Definition 2.3 (Recession fan of a tropical curve). Let (C, ω) be a tropical curve in V . For each primitive vector $v \in \Lambda$ let $\omega(v)$ be the sum of all weights of unbounded rays in C with direction v . The collection of all rays $\mathbb{R}v$ with weights $\omega(v)$ for which this weight is positive is called the *recession fan* $\text{Rec}(C)$ of C . The balancing condition at each vertex of C ensures that $\text{Rec}(C)$ is balanced as well.

Example 2.4. We consider the 1-dimensional polyhedral complex C in \mathbb{R}^2 pictured below, with weight 1 on all edges. As the primitive generators of the edges around τ and τ' add up to 0, we see that C is a tropical curve. Its recession fan consists of the four rays in directions $(\pm 1, 0)$ and $(0, \pm 1)$, again each with weight 1.



To consider the tropicalization of algebraic objects, let now K be an algebraically closed field (of any characteristic).

Notation 2.5. By $K\{\{t\}\}$ we denote the field of *generalized Puiseux series* over K in the sense that its elements are formal series of the form $\sum_{k \in \mathbb{R}} a_k t^k$ in a formal variable t such that $\{k \in \mathbb{R} : a_k \neq 0\}$ is a well-ordered subset of \mathbb{R} .

For $n \in \mathbb{N}$, we will often use the notation $K\{\{t\}\}[x]$ instead of $K\{\{t\}\}[x_0, \dots, x_n]$ for the polynomial ring over $K\{\{t\}\}$, and for $v \in \mathbb{N}^{n+1}$ we also write x^v instead of $x_0^{v_0} \cdots x_n^{v_n}$. We denote by e_i for $i = 0, \dots, n$ the i -th unit vector in \mathbb{R}^{n+1} , and set $\mathbf{1} := \sum_{i=0}^n e_i$.

There are several different but equivalent ways to define the tropicalization of an algebraic variety over $K\{\{t\}\}$. We choose the following one because we will use it in our algorithm [MS15, Sections 3.1 and 3.2].

Definition 2.6 (Tropicalization). Let $f = \sum_{\nu} a_{\nu} x^{\nu} \in K\{\{t\}\}[x_0, \dots, x_n]$ be a homogeneous polynomial. The *tropical polynomial associated to f* is defined as the map

$$\text{trop}(f) : \mathbb{R}^{n+1} \rightarrow \mathbb{R}, \quad y \mapsto \min_{\nu} (\text{val}(a_{\nu}) + y \cdot \nu).$$

The *tropical hypersurface* $\text{Trop}(f)$ of f is the locus of non-differentiability of $\text{trop}(f)$, i. e.

$$\text{Trop}(f) = \{y \in \mathbb{R}^{n+1}/\mathbb{R} \cdot \mathbf{1} : \text{the minimum in } \text{trop}(f)(y) \text{ is achieved at least twice}\}.$$

Now let $I \subset K\{\{t\}\}[x]$ be a homogeneous ideal. As a set, we define the *tropicalization* or *tropical variety* of I as

$$\text{Trop}(I) = \bigcap_{\substack{f \in I \\ f \text{ homogeneous}}} \text{Trop}(f) \subset \mathbb{R}^{n+1}/\mathbb{R} \cdot \mathbf{1}.$$

If I is the homogeneous ideal of a projective variety $X \subset \mathbb{P}_{K\{\{t\}\}}^n$, we write $\text{Trop}(I)$ also as $\text{Trop}(X)$. When drawing tropical varieties, we will always identify $\mathbb{R}^{n+1}/\mathbb{R} \cdot \mathbf{1}$ with \mathbb{R}^n by the map $[x_0, x_1, \dots, x_n] \mapsto (x_1 - x_0, \dots, x_n - x_0)$.

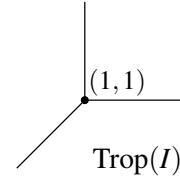
Remark 2.7 (Tropicalization of principal ideals). If $I = (f)$ is principal we have $\text{Trop}(I) = \text{Trop}(f)$: If $g = f \cdot h \in I$ is homogeneous, then $\text{trop}(g)(y) = \text{trop}(f)(y) + \text{trop}(h)(y)$. Hence, if the minimum in $\text{trop}(f)(y)$ is achieved at least twice, then so is the minimum in $\text{trop}(g)(y)$. We therefore have $\text{Trop}(f) \subset \text{Trop}(g)$ and thus $\text{Trop}(I) = \text{Trop}(f)$.

Example 2.8. We want to compute $\text{Trop}(I)$, where

$$I = (x_1^2 + x_2^2 - t^2 x_0^2) \subset K\{\{t\}\}[x_0, x_1, x_2].$$

For $f = x_1^2 + x_2^2 - t^2 x_0^2$ we have $\text{trop}(f)(y) = \min(2y_0 + 2, 2y_1, 2y_2)$, and so by Remark 2.7

$$\text{Trop}(I) = \{[0, y_1, y_1] : y_1 \leq 1\} \cup \{[0, y_1, 1] : y_1 \geq 1\} \cup \{[0, 1, y_2] : y_2 \geq 1\}.$$



Remark 2.9 (Weights on tropical varieties [MS15, Theorem 3.2.5 and Definition 3.4.3]). Let $I \subset K\{\{t\}\}[x]$ be a homogeneous ideal with associated projective variety $X \subset \mathbb{P}_{K\{\{t\}\}}^n$. Using initial ideals, one can give $\text{Trop}(I)$ the structure of a polyhedral complex of dimension $\dim X$ and put weights on its maximal cells in such a way that it becomes a tropical curve in the sense of Definition 2.2 if $\dim X = 1$. In fact, there is also a balancing condition similar to that of Definition 2.2 for higher-dimensional varieties. However, we will not describe these constructions here as we will only need weights for plane curves — in which case they can be obtained easily in terms of Newton subdivisions as we will explain in Proposition 3.8.

Example 2.10 (Tropicalization of linear spaces). Let $X \subset \mathbb{P}_{K\{\{t\}\}}^n$ be a linear space defined over K , so that its ideal $L := I(X) \subset K\{\{t\}\}[x_0, \dots, x_n]$ is generated by homogeneous linear polynomials over K . Assume that X is not contained in any coordinate hyperplane, i. e. that L does not contain any monomial. Then the tropicalization of X can be described in terms of matroid theory [Ox192] as follows.

Let $M = M(L)$ be the matroid defined by L with ground set $\{0, \dots, n\}$. For any flat F of M , we define an associated vector $\nu_F = \sum_{i \in F} [e_i] \in \mathbb{R}^{n+1}/\mathbb{R} \cdot \mathbf{1}$. If \mathcal{F} is a chain $F_0 \subset F_1 \subset \dots \subset F_k$

of flats in M , let $\sigma_{\mathcal{F}}$ be the cone generated by all the associated vectors of flats in \mathcal{F} , which we will write as $\sigma_{\mathcal{F}} = \text{cone}(v_{F_i} : i = 0, \dots, k)$.

The tropicalization $\text{Trop}(X)$ in $\mathbb{R}^{n+1}/\mathbb{R} \cdot \mathbf{1}$ is then given by the collection of all these cones $\sigma_{\mathcal{F}}$, the so-called *Bergman fan* $B(M)$ of M [MS15, Theorem 4.1.11].

Definition 2.11 (Degree of a tropical curve). Let C be a tropical curve in $\mathbb{R}^{n+1}/\mathbb{R} \cdot \mathbf{1}$. We define the *degree* of C to be the degree of its recession fan $\text{Rec}(C)$ in the sense of [GSW13, Definition 2.8], i. e. the intersection product of $\text{Rec}(C)$ with a general tropical linear space of codimension 1.

Remark 2.12. The degree of a curve can also be constructed without referring to intersection theory: for example, we can project the curve to $\mathbb{R}^3/\mathbb{R} \cdot \mathbf{1}$ by forgetting some coordinates, and take the smallest multiple of a standard simplex in which we can fit its dual Newton polygon (see Construction 3.2 and [GSW13, Lemma 4.14]).

We are now ready to describe the main question studied in this paper. Given an algebraic plane $X \subset \mathbb{P}_{K\{\{t\}\}}^n$ defined over K , so that its tropicalization is a 2-dimensional fan as in Example 2.10, we want to know whether a given tropical curve in $\text{Trop}(X)$ can be realized as the tropicalization of an algebraic curve in X . So roughly speaking, we are considering non-constant coefficient tropical curves in a constant coefficient tropical plane. More precisely, we will study the relative realizability of plane curves in the following sense.

Definition 2.13 (Relative realizability of curves). Let $X \subset \mathbb{P}_{K\{\{t\}\}}^n$ be a plane defined over K and not contained in any coordinate hyperplane, with homogeneous ideal $L := I(X) \subset K\{\{t\}\}[x]$ and tropicalization $\Sigma := B(M(L)) \subset \mathbb{R}^{n+1}/\mathbb{R} \cdot \mathbf{1}$ as in Example 2.10. We say that a tropical curve C (with support) in Σ is (*relatively*) *realizable* in X (or L) if there is a homogeneous polynomial $f \in K\{\{t\}\}[x]$ with $\text{Trop}(L + (f)) = C$ (including the weights). If X is clear from the context, we also just say that C is (relatively) realizable.

3. RELATIVE REALIZABILITY

As above, let K be any algebraically closed field, and let $X \subset \mathbb{P}_{K\{\{t\}\}}^n$ be an algebraic plane defined over K which is not contained in any coordinate hyperplane, with homogeneous ideal $L \subset K\{\{t\}\}[x_0, \dots, x_n]$. In this section, we will explain how projections can be used to decide whether or not a tropical curve in $\Sigma = \text{Trop}(X) = B(M(L))$ is relatively realizable in the sense of Definition 2.13. These projections $\Sigma \rightarrow \mathbb{R}^3/\mathbb{R} \cdot \mathbf{1}$ are obtained using matroid theory, and allow us to apply already known statements for tropical curves in \mathbb{R}^2 .

3.1. Projecting to the plane. After introducing the necessary tropical and algebraic projections, we will see that they commute with tropicalization, and that a tropical curve in Σ can be reconstructed from all its images under these projections. This will help us to develop the algorithm mentioned above.

Definition 3.1 (Tropical projections). For a basis $A = (j_0, j_1, j_2)$ of $M(L)$, with $0 \leq j_0, j_1, j_2 \leq n$, we denote by $p^A : \Sigma \rightarrow \mathbb{R}^A/\mathbb{R} \cdot \mathbf{1}$ the projection onto the coordinates of A .

Construction 3.2 (Push-forwards for projections). For the projection $p^A : \Sigma \rightarrow \mathbb{R}^A / \mathbb{R} \cdot \mathbf{1}$ there is an associated push-forward map p_*^A sending tropical curves in Σ to tropical curves in $\mathbb{R}^A / \mathbb{R} \cdot \mathbf{1}$. The exact definition can e.g. be found in [AR10, Construction 7.3]. Informally, for a tropical curve (C, ω) in Σ we subdivide C so that the images of all edges and vertices under p^A form a 1-dimensional polyhedral complex, which we denote by $p_*^A C$. The weight $\omega'(\tau)$ of an edge τ of $p_*^A C$ is given by

$$\omega'(\tau) = \sum_{\sigma} |\mathbb{Z}v_{\tau} / \mathbb{Z}p^A(v_{\sigma})| \cdot \omega(\sigma),$$

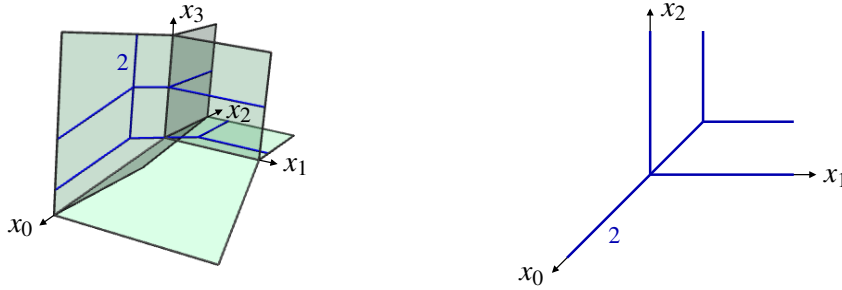
where the sum is taken over all edges σ of C mapping to τ , and v_{σ} resp. v_{τ} denote the primitive integral vectors along these edges. An example can be found in Example 3.3.

By construction, it is obvious that push-forwards along these projections commute with taking recession fans as in Definition 2.3. Moreover, we have $\deg(p_*^A D) = \deg D$ for any tropical fan curve D in Σ by [GSW13, Lemma 3.9]. Hence

$$\deg p_*^A C = \deg \text{Rec } p_*^A C = \deg p_*^A \text{Rec } C = \deg \text{Rec } C = \deg C$$

by Definition 2.11.

Example 3.3. Let $L = (x_0 + x_1 + x_2 + x_3) \subset K\{\{t\}\}[x_0, x_1, x_2, x_3]$, so that $\Sigma = \text{Trop}(L)$ is the Bergman fan of the uniform matroid of rank 3 on 4 elements. Consider the tropical curve C in Σ shown in the picture below, with vertices $[0, 1, 1, 0]$, $[0, 0, 0, 2]$, $[1, 0, 0, 1]$, and $[1, 0, 0, 3]$. For the basis $A = (0, 1, 2)$, its tropical push-forward $p_*^A C$ is shown on the right. If we do not indicate a weight of an edge, the weight is 1.



For the computation of the tropical push-forward $p_*^A C$ we have to refine the polyhedron $\text{conv}([1, 0, 0, 1], [0, 1, 1, 0])$ to the two polyhedra $\text{conv}([1, 0, 0, 1], [0, 0, 0, 0])$ and $\text{conv}([0, 0, 0, 0], [0, 1, 1, 0])$. With this refinement, the polyhedra $\text{conv}([1, 0, 0, 1], [0, 0, 0, 0])$ and $\text{conv}([1, 0, 0, 3], [0, 0, 0, 2])$ of C both map to the polyhedron $\text{conv}([1, 0, 0], [0, 0, 0])$ in $p_*^A C$. As the lattice indices appearing in the formula of Construction 3.2 are both 1, this polyhedron has multiplicity 2. In the same way, the polyhedron $[1, 0, 0] + \text{cone}([1, 0, 0])$ also has multiplicity 2. Since we only consider tropical curves up to refinement, we did not indicate this refinement of the polyhedron $\text{cone}([1, 0, 0])$ of $p_*^A C$ in our picture. The remaining maximal polyhedra of $p_*^A C$ all have weight 1 as there is exactly one maximal polyhedron of C in their preimage under p^A (with weight 1 and induced lattice index 1).

Definition 3.4 (Algebraic projection). Let $A = (j_0, j_1, j_2)$ be a basis of $M(L)$. Moreover, let $R = K\{\{t\}\}[x_0, \dots, x_n]$ and $R^A = K\{\{t\}\}[x_{j_0}, x_{j_1}, x_{j_2}]$. Then the map $R^A \rightarrow R/L$ given by

$x_i \mapsto x_i + L$ is an isomorphism since A is a basis of $M(L)$. Geometrically, it describes the projection isomorphism from X to the plane $\mathbb{P}_{K\{\{t\}\}}^2$ with homogeneous coordinates $x_{j_0}, x_{j_1}, x_{j_2}$.

The ideal L is generated by polynomials in $K[x]$, so there are unique $c_{i,j} \in K$ such that the inverse map of this isomorphism is given by

$$\pi^A : R/L \rightarrow R^A, \quad \bar{x}_i \mapsto c_{i,j_0}x_{j_0} + c_{i,j_1}x_{j_1} + c_{i,j_2}x_{j_2}.$$

For a polynomial $f \in R$, we denote the polynomial $\pi^A(\bar{f}) \in R^A$ by f_A . By construction, it has the property that the zero cycle of f in X maps to the zero cycle of f_A under the projection isomorphism $X \rightarrow \mathbb{P}_{K\{\{t\}\}}^2$ defined by A as described above.

Theorem 3.5. *As above, let $X \subset \mathbb{P}_{K\{\{t\}\}}^n$ be a plane defined over K , with defining ideal L , and let A be a basis of $M(L)$. Then*

$$\text{Trop}(f_A) = p_*^A(\text{Trop}(L + (f)))$$

for every homogeneous polynomial $f \in K\{\{t\}\}[x]$.

Proof. As push-forwards commute with tropicalization by [Gub13, Theorem 13.17], the statement follows since the zero cycle of $L + (f)$ maps to the zero cycle of f_A under the projection determined by A . \square

Theorem 3.6. *Let $C \subset \Sigma$ be a tropical curve, and let $f \in K\{\{t\}\}[x]$ be a homogeneous polynomial. With notations as above, the following are equivalent:*

- (a) $\text{Trop}(L + (f)) = C$,
- (b) for all bases A of $M(L)$ we have $\text{Trop}(f_A) = p_*^A C$.

Proof. Analogously to the proof of [GSW13, Corollary 3.6], cf. [GSW13, Remark 3.8], we can see that if $\text{Trop}(f_A) = p_*^A C$ for all bases A of $M(L)$, then $\text{Trop}(L + (f)) = C$. Using Theorem 3.5, we also know that if $\text{Trop}(L + (f)) = C$, then $\text{Trop}(f_A) = p_*^A C$ for all bases A of $M(L)$. \square

3.2. Plane tropical curves. We now want to collect some facts about tropical curves in $\mathbb{R}^3/\mathbb{R} \cdot \mathbf{1} \cong \mathbb{R}^2$ which will help us with the computations. For simplicity, in this section we will consider tropical curves in \mathbb{R}^2 , i. e. in the non-homogeneous case. In our algorithm, where we use matroid theory, the introduced concepts and facts will be transferred to the homogeneous setting, identifying \mathbb{R}^2 with $\mathbb{R}^3/\mathbb{R} \cdot \mathbf{1}$ by $(y_1, y_2) \mapsto [0, y_1, y_2]$.

Any tropical curve $C \subset \mathbb{R}^2$ is realizable and thus the tropicalization of a single polynomial, see for instance [Mik04, Proposition 2.4]. The combinatorial information about C is encoded in the Newton subdivision of this polynomial. We will therefore now present the concept of Newton subdivisions.

Definition 3.7 (Newton subdivisions).

- (a) Let $f = \sum_{|v| \leq d} a_v x^v \in K\{\{t\}\}[x_1, x_2]$ be a polynomial of degree d . We call $\text{Newt}(f) = \text{conv}(v : a_v \neq 0)$ the *Newton polytope* of f .

If we consider the *extended Newton polytope* $\text{conv}((v, y) : a_v \neq 0, y \geq \text{val}(a_v))$ in $\mathbb{R}^2 \times \mathbb{R}$, then the projection $\mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$ forgetting the last coordinate induces a subdivision of $\text{Newt}(f)$ analogous to [GKZ94, Example 7.2.2]. This subdivision of $\text{Newt}(f)$ is called the *Newton subdivision* of f .

- (b) Let $C \subset \mathbb{R}^2$ be a tropical curve. The *Newton polytope* of C , denoted by $\text{Newt}(C)$, is defined as the Newton polytope of the recession fan $\text{Rec}(C)$ of C , as defined for example in [GSW13, Definition 4.13]. It is the unique polytope such that $\text{Rec}(C)$ is its inner normal fan and there is $d \in \mathbb{N}$ such that $\text{Newt}(C)$ touches all three faces of the standard d -simplex. It has a *Newton subdivision* dual to C .

Proposition 3.8. *Let $f \in K\{\{t\}\}[x_1, x_2]$ be a polynomial. Then the tropicalization $\text{Trop}(f)$ is dual to the Newton subdivision of f . Moreover, the weights on the maximal cells are given by the lattice lengths of the corresponding faces in the Newton subdivision of f .*

Proof. The duality of the Newton subdivision of f and its tropicalization is shown for instance in [MS15, Proposition 3.1.6]. The equality of the weights and the lattice lengths can be found in [MS15, Proposition 3.4.6]. \square

Remark 3.9. In the proof of [MS15, Proposition 3.1.6], one can see that if $\tau = \text{conv}(v, \eta)$ is a 1-dimensional face in the Newton subdivision of f , then the cell of $\text{Trop}(f)$ dual to τ consists of all $y \in \mathbb{R}^2$ such that

$$\text{val}(a_v) + y \cdot v = \text{val}(a_\eta) + y \cdot \eta \leq \text{val}(a_\mu) + y \cdot \mu$$

for all $\mu \in \text{Newt}(f) \cap \mathbb{N}^2$, with the convention that $\text{val}(a_\mu) = \infty$ if $a_\mu = 0$ (so that we do not get a condition from such a lattice point).

Hence, if y is the vertex of $\text{Trop}(f)$ corresponding to a 2-dimensional polytope Q in the Newton subdivision, we have

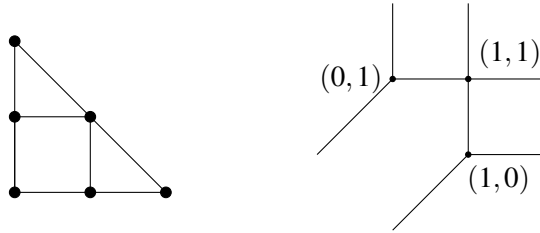
$$\text{val}(a_v) + y \cdot v = \text{val}(a_\mu) + y \cdot \mu$$

for all vertices μ, v of Q , and

$$\text{val}(a_v) + y \cdot v \leq \text{val}(a_\mu) + y \cdot \mu$$

for all vertices v of Q and $\mu \in \text{Newt}(f) \cap \mathbb{N}^2$. Note that these equations suffice to determine y uniquely.

Example 3.10. We consider the polynomial $f = t^2 + tx_1 - 2tx_2 + tx_1^2 - 2tx_2^2 + x_1x_2$ in $K\{\{t\}\}[x_1, x_2]$. Its Newton subdivision and tropicalization are as follows.



Let us now consider the polynomial $g = t^4 + t^2x_1 - 2t^2x_2 + t^2x_1^2 - 2t^2x_2^2 + x_1x_2$. The Newton subdivisions of f and g are equal, but the tropicalization of g has vertices $(2, 0)$, $(0, 2)$ and $(2, 2)$. Hence, we see that the Newton subdivision of a polynomial f does not determine its tropicalization, but only its combinatorial type.

If the Newton polytope is 2-dimensional, we will therefore now mark each 2-dimensional cell of the Newton subdivision with its corresponding vertex in $\text{Trop}(f)$ in order to fix the tropicalization.

Definition 3.11 (Marked subdivision). Let Q be a 2-dimensional lattice polytope in \mathbb{R}^2 and $A = Q \cap \mathbb{Z}^2$. A *marked subdivision* of Q is a family $\{(A_i, y_i) : i \in I\}$ such that

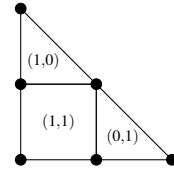
- (a) $y_i \in \mathbb{R}^2$ for all $i \in I$,
- (b) each A_i is a subset of A such that A_i contains exactly the vertices of $Q_i = \text{conv}(A_i)$,
- (c) $\{Q_i : i \in I\}$ forms a subdivision of Q .

Definition 3.12 (Marked Newton subdivisions).

- (a) Let $f = \sum_{|v| \leq d} a_v x^v \in K\{\{t\}\}[x_1, x_2]$ be a polynomial of degree d such that $\text{Newt}(f)$ is 2-dimensional, and let $\{Q_i : i \in I\}$ be the Newton subdivision of f . For any $i \in I$, let $A_i \subset \mathbb{Z}^2$ be the vertices of Q_i and y_i the unique vector in \mathbb{R}^2 such that $\text{val}(a_\mu) + \mu \cdot y_i = \text{val}(a_\nu) + \nu \cdot y_i$ for all $\mu, \nu \in A_i$, which exists by Remark 3.9. The point y_i is then the vertex of $\text{Trop}(f)$ dual to Q_i . The marked subdivision $\text{Subdiv}(f) := \{(A_i, y_i) : i \in I\}$ will be called the *marked Newton subdivision* of f .
- (b) Let $C \subset \mathbb{R}^2$ be a tropical curve which is not a classical line, and let $\{Q_i : i \in I\}$ be its Newton subdivision. The *marked Newton subdivision* $\text{Subdiv}(C)$ of C is defined as the subdivision $\{(A_i, y_i) : i \in I\}$ of $\text{Newt}(C)$ such that A_i is the set of vertices of Q_i , and y_i the vertex of C dual to Q_i for all $i \in I$.

Example 3.13. The marked Newton subdivision of the polynomial f as in Example 3.10 consists of the three pairs

- $(\{(0, 1), (1, 1), (0, 2)\}, (1, 0))$,
- $(\{(0, 0), (1, 0), (1, 1), (0, 1)\}, (1, 1))$, and
- $(\{(1, 0), (1, 1), (2, 0)\}, (0, 1))$.



As expected, the following lemma shows that a tropical curve is completely described by its marked Newton subdivision. We will use this fact in our algorithm by comparing marked Newton subdivisions instead of tropical curves.

Lemma 3.14. *Let C be a tropical curve which is not a classical line, and let f be a polynomial in $K\{\{t\}\}[x_1, x_2]$ such that f is not divisible by a monomial and $\text{Newt}(f)$ is two-dimensional. Then we have $\text{Trop}(f) = C$ if and only if $\text{Subdiv}(f) = \text{Subdiv}(C)$.*

Proof. Assume that $\text{Subdiv}(C)$ is the marked Newton subdivision of f . We have already seen in Proposition 3.8 that the Newton subdivision of f is dual to its tropicalization. Moreover, by definition, C is dual to its marked Newton subdivision, hence C and $\text{Trop}(f)$ are of the same combinatorial type. Additionally they have the same vertices, hence they coincide.

Conversely, assume that $\text{Trop}(f) = C$. As f is not divisible by a monomial, we know by [GSW13, Lemma 4.14] that $\text{Newt}(f) = \text{Newt}(C)$. Moreover, since $\text{Trop}(f)$ is dual to the Newton subdivision of f we know by definition that the subdivisions coincide. In addition, the markings are exactly the vertices of C and $\text{Trop}(f)$, and thus are the same. So we have $\text{Subdiv}(f) = \text{Subdiv}(C)$. \square

Remark 3.15. Using Lemma 3.14, we see that tropicalization preserves degree: We defined the degree of a tropical curve C as the degree of its recession fan, cf. Definition 2.11. Moreover, the Newton polytope of C is defined as the Newton polytope of its recession fan, cf. Definition 3.12. In [GSW13, Lemma 4.14], it is shown that $\text{Newt}(C)$ meets all three faces of $\text{conv}((0,0), (d,0), (0,d))$, where $d = \deg(C)$, and since $\text{Newt}(C) = \text{Newt}(f)$, we know in particular that $\deg(f) = d$.

3.3. Computing Realizability. In this section we will present an algorithm to decide whether or not a given tropical curve in $\Sigma = B(M(L))$ is relatively realizable in L , where as above X is a plane in $\mathbb{P}^n_{K\{\{t\}\}}$ and $L = I(X)$ can be generated by polynomials in $K[x]$.

When working with the computer, we can only consider tropical curves with vertices in \mathbb{Q}^n . The following proposition shows that for the implementation of the relative realizability problem we can in fact restrict to the case of tropical curves with integer vertices.

Proposition 3.16. *Fix $m \in \mathbb{Q}_{>0}$. Let $C \subset \Sigma$ be a tropical curve which is not a classical line, and let C_m be the tropical curve obtained from C by rescaling $y \mapsto my$ (preserving the weights). Moreover, let $f \in K\{\{t\}\}[x]$ be a homogeneous polynomial without monomial factors, and let f_m be its image under the ring homomorphism $K\{\{t\}\}[x] \rightarrow K\{\{t\}\}[x]$ sending t to t^m .*

Then f_m realizes C_m if and only if f realizes C .

Proof. By Theorem 3.6 it suffices to prove the statement for curves in $\mathbb{R}^3/\mathbb{R} \cdot \mathbf{1} \cong \mathbb{R}^2$, where we can use marked Newton subdivisions for the comparison. But if $\{(A_i, y_i) : i \in I\}$ is the marked Newton subdivision of f it follows from Definition 3.12 (a) that the marked Newton subdivision of f_m is just $\{(A_i, my_i) : i \in I\}$. As this corresponds to the curve obtained from C by rescaling by a factor of m , the result follows. \square

As already mentioned, it is the aim of this section to present an algorithm able to decide whether or not a given tropical curve in Σ is relatively realizable. To do so, we will use the idea to compare marked Newton subdivisions. The following proposition shows a way to check if a given marked subdivision is the marked Newton subdivision of a polynomial. As above, we set $\text{val}(0) = \infty$.

Lemma 3.17. *Let $S = \{(A_i, y_i) : i \in I\}$ be the marked Newton subdivision of a tropical curve C in \mathbb{R}^2 which is not a classical line, and let $f = \sum_v a_v x^v \in K\{\{t\}\}[x_1, x_2]$ be a polynomial with $\text{Newt}(f) = \text{Newt}(C)$. Then $\text{Trop}(f) = C$ if and only if the following conditions hold:*

- (a) For all $i \in I$ and all $\mathbf{v}, \boldsymbol{\eta} \in A_i$ we have $\text{val}(a_{\mathbf{v}}) + \mathbf{v} \cdot \mathbf{y}_i = \text{val}(a_{\boldsymbol{\eta}}) + \boldsymbol{\eta} \cdot \mathbf{y}_i$, and
- (b) for any $\mathbf{v} \in \text{Newt}(f) \cap \mathbb{N}^2$ such that $\mathbf{v} \notin A_i$ for all $i \in I$, choose $j \in I$ such that $\mathbf{v} \in \text{conv}(A_j)$ and $\boldsymbol{\eta} \in A_j$ arbitrary, and require that $\text{val}(a_{\mathbf{v}}) + \mathbf{v} \cdot \mathbf{y}_j \geq \text{val}(a_{\boldsymbol{\eta}}) + \boldsymbol{\eta} \cdot \mathbf{y}_j$.

Proof. If $\text{Trop}(f) = C$ then we also have $\text{Subdiv}(f) = \text{Subdiv}(C)$ by Lemma 3.14, and hence the conditions (a) and (b) hold by Remark 3.9.

Conversely, assume that the conditions of the lemma hold. Let us consider the polyhedron

$$Q = \text{conv} \left((\mathbf{v}, \lambda) : \mathbf{v} \in \bigcup_{i \in I} A_i, \lambda \geq \text{val}(a_{\mathbf{v}}) \right) \subset \mathbb{R}^2 \times \mathbb{R}.$$

We first show that projecting Q to \mathbb{R}^2 induces the Newton subdivision of C : We know that C is realizable since it is a plane tropical curve, hence there is a polynomial $g = \sum_{\mathbf{v}} b_{\mathbf{v}} x^{\mathbf{v}} \in K\{\{t\}\}[x_1, x_2]$ such that $\text{Subdiv}(g) = S$. By Remark 3.9, we know that for all $i \in I$ and $\mathbf{v}, \boldsymbol{\eta} \in A_i$, we have $\text{val}(b_{\mathbf{v}}) + \mathbf{y}_i \cdot \mathbf{v} = \text{val}(b_{\boldsymbol{\eta}}) + \mathbf{y}_i \cdot \boldsymbol{\eta}$. We assumed that f fulfills the above conditions, so by (a) there is a constant $c \in \mathbb{R}$ with $\text{val}(a_{\mathbf{v}}) = \text{val}(b_{\mathbf{v}}) + c$ for all $\mathbf{v} \in \bigcup_{i \in I} A_i$. Hence, the induced subdivision of Q equals the Newton subdivision of g and thus the Newton subdivision of C .

We now prove that Q is the extended Newton polytope of f : Due to the form of Q and using (a), we know that $(\mathbf{v}, \lambda) \in Q$ if and only if $\mathbf{v} \in \text{Newt}(f)$ and $\lambda \geq \text{val}(a_{\boldsymbol{\mu}}) + \mathbf{y}_i \cdot (\boldsymbol{\mu} - \mathbf{v})$, where we have $i \in I$ with $\mathbf{v} \in \text{conv}(A_i)$ and $\boldsymbol{\mu} \in A_i$ arbitrary. So, for $\mathbf{v} \in \mathbb{N}^2 \cap \text{Newt}(f)$, we know by (b) that $(\mathbf{v}, \text{val}(a_{\mathbf{v}})) \in Q$. Thus Q is the extended Newton polytope of f . In particular, the Newton subdivision of f is the Newton subdivision of C .

Following the definition of the marked Newton subdivision of a polynomial, we see with (a) that the marking of A_i in the Newton subdivision of f is exactly \mathbf{y}_i . So we have $\text{Subdiv}(f) = S$, and it follows that $\text{Trop}(f) = C$ by Lemma 3.14. \square

For a tropical curve C in $\mathbb{R}^3/\mathbb{R} \cdot \mathbf{1}$ that is not a classical line, Lemma 3.17 can be used to compute all homogeneous polynomials that tropicalize to C . However, if C is a classical line the concept of marked Newton subdivisions is not applicable. So in this case we should describe the polynomials tropicalizing to C separately.

Remark 3.18 (Realizability for classical lines). Let C in \mathbb{R}^2 be a classical line and y any point on C . Its Newton polytope is 1-dimensional, i. e. there are $\mathbf{v}, \boldsymbol{\mu} \in \mathbb{N}^2$ such that $\text{Newt}(C) = \text{conv}(\mathbf{v}, \boldsymbol{\mu}) \subset \mathbb{R}^2$. Let m be the lattice length of $\text{Newt}(C)$. Then the tropical curve C is realizable by exactly the polynomials

$$f = \sum_{i=0}^m a_i x^{(1-\frac{i}{m})\mathbf{v} + \frac{i}{m}\boldsymbol{\mu}} \in K\{\{t\}\}[x_1, x_2]$$

such that

- $\text{val}(a_0) + \mathbf{v} \cdot \mathbf{y} = \text{val}(a_m) + \boldsymbol{\mu} \cdot \mathbf{y}$ and
- $\text{val}(a_i) + ((1 - \frac{i}{m})\mathbf{v} + \frac{i}{m}\boldsymbol{\mu}) \cdot \mathbf{y} \geq \text{val}(a_0) + \mathbf{v} \cdot \mathbf{y}$ for all $i = 1, \dots, m-1$.

We are now ready to present our algorithm to check relative realizability. It uses the matroid projections of Section 3.1. As already mentioned, the concepts of this section can be transferred to the homogeneous case using the identification of \mathbb{R}^2 with $\mathbb{R}^3/\mathbb{R} \cdot \mathbf{1}$ given by $(x_1, x_2) \mapsto [0, x_1, x_2]$. The objects used in the algorithm should be interpreted in this way.

In the following algorithm, we assume that we can perform calculations with Puiseux series. We will explain afterwards in Remark 3.23 that we can restrict ourselves in the computations to Puiseux series with fixed and finitely many t -powers, so that the algorithm can actually be implemented on a computer.

Notation 3.19. Let $f \in K\{\{t\}\}[x]$ be a polynomial, and let B be a basis of $M(L)$. By $a_{B,v}$ we denote the coefficient of x^v in the polynomial f_B of Definition 3.4, i. e. we have $f_B = \sum_{|v|=d} a_{B,v} x^v \in K\{\{t\}\}[x_i : i \in B]$.

Algorithm 3.20 (Computing the relative realizability of curves in a tropical plane).

INPUT: A homogeneous linear ideal L in $K\{\{t\}\}[x]$ corresponding to a plane in $\mathbb{P}_{K\{\{t\}\}}^n$ with generators in $K[x]$, and a tropical curve C in $\Sigma = B(M(L))$

OUTPUT: 1 if the curve is relatively realizable in L , -1 otherwise

ALGORITHM:

- (a) Compute the degree d of C , i. e. the degree of the recession fan of C , as e. g. in [GSW13, Lemma 2.9].
- (b) Compute a basis (j_0, j_1, j_2) of $M(L)$.
- (c) Set

$$f = \sum_{|v|=d} a_v x^v \in K\{\{t\}\}[x_{j_0}, x_{j_1}, x_{j_2}],$$

where we consider the $a_v \in K\{\{t\}\}$ as parameters (note that we need degree d by Remark 3.15, and that any homogeneous polynomial of degree d can be written modulo L in this form).

- (d) For every basis B of $M(L)$, do the following:
 - (i) Compute the tropical push-forward $p_*^B(C)$ (see Construction 3.2).
 - (ii) Compute the polynomial $f_B \in K\{\{t\}\}[x_i : i \in B]$ as in Definition 3.4; its coefficients $a_{B,v}$ are K -linear combinations of the parameters a_v .
 - (iii) For any $v \notin \text{Newt}(p_*^B C)$, collect the condition $a_{B,v} = 0$ (as linear condition on the a_v).
 - (iv) If $p_*^B(C)$ is not a classical line, compute the marked Newton subdivision S of $p_*^B(C)$ and collect the conditions on the valuations of the coefficients $a_{B,v}$ of f_B equivalent to S being the marked Newton subdivision of f_B , i. e. to $p_*^B(C) = \text{Trop}(f_B)$ (see Lemma 3.17):

For any $(A, y) \in S$ with $A = \{v_1, \dots, v_i\}$, we get equalities:

$$\text{val}(a_{B,v_i}) + y \cdot v_i = \text{val}(a_{B,v_1}) + y \cdot v_1 \quad \text{for all } i \in \{2, \dots, t\}.$$

For any lattice point η in $\text{conv}(A)$ which is not contained in A , we get the condition:

$$\text{val}(a_{B,\eta}) + y \cdot \eta \geq \text{val}(a_{B,v_1}) + y \cdot v_1.$$

(v) If $p_*^B(C)$ is a classical line, collect the conditions on f_B from Remark 3.18.

(e) Check if the collected conditions have a common solution. If such a solution exists, C is realizable by Theorem 3.6, so return 1. If there is no common solution, the tropical curve C is not relatively realizable in L , hence set the output to -1 .

Remark 3.21 (Fixing an initial valuation). For any $c \in \mathbb{R}$, we have $L + (f) = L + (t^{-c}f)$. Given a tropical curve C in $B(M(L))$, fixing an initial basis (j_0, j_1, j_2) of $M(L)$ as in Algorithm 3.20 and a vertex η of the Newton subdivision of $\text{Newt}(p_*^{(j_0, j_1, j_2)}C)$, we might thus assume that for any polynomial $f \in K\{\{t\}\}[x_{j_0}, x_{j_1}, x_{j_2}]$ realizing C , the coefficient of x^η in f has valuation zero. This assumption transforms the conditions in Algorithm 3.20 (d) into conditions of the form

$$\begin{aligned} a_{B,v} &= 0 \\ \text{or } \text{val}(a_{B,v}) &= c_v^B \\ \text{or } \text{val}(a_{B,v}) &\geq c_v^B \end{aligned} \quad (*)$$

for some $c_v^B \in \mathbb{R}$ and all bases B of $M(L)$, depending on whether v is contained in the Newton polytope of the projection $p_*^B C$, and whether it is a vertex of a polytope in the corresponding Newton subdivision.

The right sides c_v^B depend on the initial basis (j_0, j_1, j_2) and the lattice point η . However, for two bases A and B and lattice points μ and v in the Newton polytopes of $p_*^A(C)$ and $p_*^B(C)$, respectively, the differences $c_v^B - c_\mu^A$ have no dependence on (j_0, j_1, j_2) and η .

Notation 3.22.

- (a) For a polynomial $f = \sum_{|v| \leq d} a_v x^v \in K\{\{t\}\}[x]$, let a_v^k be the coefficient of t^k in a_v , i. e. we have $a_v = \sum_{k \in \mathbb{R}} a_v^k t^k$.
- (b) Let $A = (j_0, j_1, j_2)$ be a basis of $M(L)$ and η a vertex of the Newton subdivision of p_*^A such that $c_v^B - c_\eta^A \geq 0$ for all bases B of $M(L)$ and lattice points v in the Newton polytope of $p_*^B(C)$, where the c_v^B are as in Remark 3.21. The set of all c_v^B with respect to A and η will in the following be denoted by $\text{RHS}(C)$. It is obviously a finite subset of $\mathbb{R}_{\geq 0}$.

Remark 3.23 (Reducing Puiseux series to finitely many coefficients). In Algorithm 3.20, we start by setting $f = \sum_{|v|=d} a_v x^v$ and consider its coefficients as parameters. We then get conditions on the valuations of $a_{B,v}$, where B is a basis of $M(L)$. However, it is not clear how to work with elements in $K\{\{t\}\}$ and thus how we can check that these conditions have a common solution. We will now see that we can simplify the parameters $a_v \in K\{\{t\}\}$ such that they only contain fixed and finitely many powers of t , namely t^k for $k \in \text{RHS}(C)$.

For any basis B of $M(L)$, $k \in \mathbb{R}$, and $\mu \in \mathbb{N}^2$, note that the $t^k x^\mu$ -coefficient $a_{B,\mu}^k$ of f_B is a K -linear combination of the t^k -coefficients $\{a_v^k : |v| = d\}$ since L is generated by polynomials over K . Hence if f satisfies the conditions $(*)$ of Remark 3.21, the polynomial

$$g = \sum_{|v|=d} b_v x^v \quad \text{with} \quad b_v = \sum_{k \in \text{RHS}(C)} a_v^k t^k$$

has the same t^k -terms as f for all $k \in \text{RHS}(C)$, and thus satisfies the conditions $(*)$ as well. To check relative realizability we can therefore assume from the beginning that $f = \sum_{|v|=d} \sum_{k \in \text{RHS}(C)} a_v^k t^k x^v$, and consider the finitely many coefficients $a_v^k \in K$ as our new set of parameters. The conditions $(*)$ can then be translated into equations $a_{B,v}^k = 0$ (for the condition $a_{B,v} = 0$ or if $k < c_v^B$) and inequalities $a_{B,v}^k \neq 0$ (for the condition $\text{val}(a_{B,v}) = c_v^B$ if $k = c_v^B$), where as explained above the $a_{B,v}^k$ are fixed K -linear combinations of the a_v^k . This finally allows to implement Algorithm 3.20 on a computer.

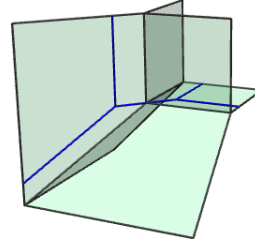
In particular, we see that the conditions on the parameters $\{a_v^k : |v| = d, k \in \text{RHS}(C)\}$ can be decomposed into conditions on the parameters $\{a_v^k : |v| = d\}$ for all $k \in \text{RHS}(C)$. These conditions on the a_v^k for fixed k are parts of the conditions for tropical fan curves. So C is relatively realizable if and only if there are tropical fan curves for each $k \in \text{RHS}(C)$ satisfying the corresponding conditions. In other words, our relative realizability question for non-constant coefficient tropical curves can be decomposed into several relative realizability questions for tropical fan curves, i. e. constant coefficient tropical curves.

Remark 3.24 (Implementation in Singular). The algorithm described above is implemented in the Singular library “realizationMatroidsNC.lib” [Bir14]. Let us explain how to use this library. As in the software package *gfan* by Anders Jensen [Jen07], a tropical curve in $\mathbb{R}^{n+1}/\mathbb{R} \cdot \mathbf{1}$ is given by its vertices, the directions of its recession fan, its edges, and the multiplicities on the edges. More precisely, a tropical curve C is completely described by the following three lists:

- (a) $V = ((1, v_1), \dots, (1, v_s), (0, r_1), \dots, (0, r_t))$, where v_1, \dots, v_s are the vertices of C and r_1, \dots, r_t the primitive vectors of the cones in $\text{Rec}(C)$, both in homogeneous coordinates, and both normalized with minimum 0 over the coordinates.
- (b) $E = ((i, j) : \text{there is a bounded edge between } v_i \text{ and } v_j \text{ or there is an unbounded cell starting at } v_i \text{ in the direction of } r_{j-s})$.
- (c) $M = (m_1, \dots, m_k)$, where m_i is the weight of the maximal cell of C at the i -th entry of E .

For our algorithm, it is not necessary to order the elements in V as above. It is only important that the entries in the tuples in E fit to the positions of the corresponding elements in V . Moreover, the primitive directions may appear only once in V , no matter how many unbounded cells in C leave in this direction.

Let us consider the example curve $C \subset \text{Trop}(L)$ shown on the right: we have $L = (x_0 + x_1 + x_2 + x_3)$, and the curve C has weight 1 on all maximal cells, the unit vectors as unbounded directions, and vertices $[0, 1, 1, 0]$ and $[1, 0, 0, 1]$. The following Singular code then shows how to check the relative realizability of C as in Algorithm 3.20, and how to compute a polynomial realizing C if this is the case. Note that the entries of V are restricted to integers because of Proposition 3.16.



```

> LIB "realizationMatroidsNC.lib";
> ring R = (0,t), (x0,x1,x2,x3), dp;
> ideal L = x0+x1+x2+x3;
> list V = list(intvec(1,0,1,1,0), intvec(1,1,0,0,1),
  intvec(0,0,1,0,0), intvec(0,0,0,1,0),
  intvec(0,0,0,0,1), intvec(0,1,0,0,0));
> list E = list(intvec(1,2), intvec(1,3), intvec(1,4),
  intvec(2,5), intvec(2,6));
> list M = list(1,1,1,1,1);
> list C = list(V,E,M);
> realizable(L,C);
1
> realizablePoly(L,C);
1 (t)*x0+x1+(t+1)*x2

```

Remark 3.25 (Not all projections are necessary). Let C be a tropical curve in Σ , and let \mathcal{A} be a set of bases of $M(L)$ with the following property: For each maximal cone σ of Σ , there is a basis $A \in \mathcal{A}$ so that p^A is injective on σ . In the proof of [GSW13, Corollary 3.6] used in Theorem 3.6, the basis A from [GSW13, Lemma 3.4] for which the projection is evaluated can then always be chosen to lie in \mathcal{A} . Therefore, it is sufficient to consider the projections $p_*^A(C)$ for A in \mathcal{A} to check the relative realizability of C .

3.4. The space of relatively realizable curves. Given a tropical curve in Σ , we can check if it is relatively realizable with our algorithm above. However, one may also want to know if the space of all relatively realizable tropical curves in Σ carries a nice structure. To investigate this question, we first have to study the space of all tropical curves in Σ .

Firstly, consider the space of tropical curves in $\mathbb{R}^n/\mathbb{R} \cdot \mathbf{1}$. Fixing a combinatorial type, including the directions of all edges, any curve of this type can be described by the position of one of its vertices and the lattice lengths of its bounded edges (which might have to satisfy some linear conditions in order for loops to be closed). Requiring the curves to lie in Σ imposes some additional linear conditions. Hence the space of all tropical curves in Σ of a given combinatorial type is an open polyhedron. Its boundary, obtained by shrinking some of the edge lengths to 0, corresponds to curves of different combinatorial types.

For a given degree $d \in \mathbb{N}_{>0}$, let us denote by $T_d(L)$ the space of all tropical curves of degree d in L , obtained by gluing the above polyhedra together according to the degeneration of

the corresponding combinatorial types. Hence $T_d(L)$ can be written as a finite union of polyhedra, glued together at their faces along linear maps. We will refer to such a structure as an *abstract polyhedral set*. Note that it does not come with a natural embedding in a real vector space.

Inside $T_d(L)$, denote by $R_d(L)$ the subset of all relatively realizable tropical curves in Σ . We will prove that $R_d(L)$ is closed in $T_d(L)$, and that it is in fact an abstract polyhedral set itself.

Lemma 3.26. *$R_d(L)$ is closed in $T_d(L)$ for all d .*

Proof. We prove that the space of tropical curves in Σ which are not relatively realizable is open in $T_d(L)$. Let C be a tropical curve in Σ which is not relatively realizable and let c_v^B by the right hand sides of the conditions $\text{val}(a_{B,v}) = c_v^B$ or $\text{val}(a_{B,v}) \geq c_v^B$, respectively, as in Remark 3.21. For simplicity, we set $c_v^B = \infty$ if the corresponding condition is $a_{B,v} = 0$.

Since C is not relatively realizable, there is a basis B of $M(L)$ and a vertex μ of a polytope in the Newton subdivision of $p_*^B(C)$ such that the required condition $a_{B,\mu}^k \neq 0$ for $k = c_\mu^B$ is in contradiction to the required equations $a_{A,v}^k = 0$ for all bases A and lattice points v with $c_v^A > k = c_\mu^B$.

For a small deformation of C , two things can happen:

- A vertex of valence greater than 3 in one (or more) of the projections $p_*^A C$ may be resolved. Correspondingly, the Newton subdivision of $p_*^A C$ is subdivided further. This means that some conditions $\text{val}(a_{A,v}) \geq c_v^A$ change into $\text{val}(a_{A,v}) = c_v^A$. If there was a contradiction with the original conditions, there will still be a contradiction with these stronger conditions.
- The right hand sides of the conditions in Remark 3.21 slightly change. However, if we had $c_v^A > c_\mu^B$ originally, this inequality still holds after a small deformation of these numbers. Hence, the contradiction is still present and thus the deformed curve is not relatively realizable either.

Altogether, this means that the space of tropical curves in Σ that are not relatively realizable is open in $T_d(L)$. \square

Proposition 3.27. *$R_d(L)$ is an abstract polyhedral set for all d .*

Proof. Continuing the notation of the proof of Lemma 3.26, let C be a relatively realizable curve, and let c_v^B be the right hand sides of the conditions in Remark 3.21. Consider the subset S of $T_d(L)$ of all curves \tilde{C} such that

- (a) $p_*^B \tilde{C}$ has the same combinatorial type as $p_*^B C$ for all bases B of $M(L)$, and
- (b) the right hand sides \tilde{c}_v^B satisfy the same equalities and inequalities as the original ones, i. e. for all bases A, B and lattice points μ, v we have $\tilde{c}_\mu^A = \tilde{c}_v^B$ if and only if $c_\mu^A = c_v^B$, and $\tilde{c}_\mu^A < \tilde{c}_v^B$ if and only if $c_\mu^A < c_v^B$.

As these right hand sides depend linearly on the vertices, and thus also on the edge lengths of the curves, this subset $S \subset T_d(L)$ is an open polyhedron. Moreover, by the form of the conditions for relative realizability all curves in S are relative realizable. Hence $R_d(L)$ is a finite union of open polyhedra, and so the proposition follows by Lemma 3.26. \square

4. GENERAL CRITERIA FOR RELATIVE REALIZABILITY

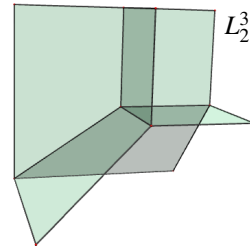
As before, let K be any algebraically closed field. Moreover, let $L \subset K\{\{t\}\}[x]$ be the ideal of a plane $X \subset \mathbb{P}_{K\{\{t\}\}}^n$ not contained in any coordinate hyperplane and with generators in $K[x]$, and let $M(L)$ be the corresponding matroid. As a warm-up, we want to reprove the known fact that if a tropical curve C in $\Sigma = B(M(L))$ is relatively realizable in L , then so is its recession fan, see for instance [MS15, Theorem 3.5.6].

Proposition 4.1. *Let C be a tropical curve in Σ which is relatively realizable in L . Then its recession fan $\text{Rec}(C)$ is also relatively realizable in L .*

Proof. Assume that $\text{Rec}(C)$ is not relatively realizable in L . Since $\text{Rec}(C)$ is a fan, so is any push-forward $p_*^B \text{Rec}(C)$ for a basis B of $M(L)$. In particular, the Newton subdivision of $p_*^B \text{Rec}(C)$ consists of one polytope $\text{Newt}(p_*^B \text{Rec}(C))$. If $p_*^B \text{Rec}(C)$ is not a classical line, then $\text{Newt}(p_*^B \text{Rec}(C))$ is marked with the origin in the sense of Definition 3.12. Hence, the conditions (*) of Remark 3.21 for $\text{Rec}(C)$ are of the form $a_{B,v} = 0$, $\text{val}(a_{B,v}) = 0$, or $\text{val}(a_{B,v}) \geq 0$. The same is true by Remark 3.18 if $p_*^B \text{Rec}(C)$ is a classical line.

As $\text{Rec}(C)$ is not realizable, there must be a contradiction in these conditions. In other words, there must be an $a_{A,v}$ with condition $\text{val}(a_{A,v}) = 0$ which is a linear combination of other $a_{B,\mu}$ with conditions $a_{B,\mu} = 0$. But then v must be a vertex of $\text{Newt}(p_*^A \text{Rec}(C)) = \text{Newt}(p_*^A C)$, and the μ are outside $\text{Newt}(p_*^B \text{Rec}(C)) = \text{Newt}(p_*^B C)$. Hence for C we again have the conditions $\text{val}(a_{A,v}) = c_v^A$ and $a_{B,\mu} = 0$ for the same A, B, v, μ as above, which is still a contradiction. It follows that C is not realizable either. \square

In general, it seems to be difficult to work out non-algorithmic rules to decide whether a given curve in Σ is relatively realizable. However, we will now prove some results in the first interesting case when X is the plane in $\mathbb{P}_{K\{\{t\}\}}^3$ with ideal $L = (x_0 + x_1 + x_2 + x_3)$. In this case, we will denote its tropicalization Σ by L_2^3 , as shown in the picture on the right. Moreover, let us fix the following notation for the projections that we will need.



Notation 4.2 (Projections and Newton polytopes).

- (a) Let f be a homogeneous polynomial of degree d in $K\{\{t\}\}[x] := K\{\{t\}\}[x_0, x_1, x_2, x_3]$.
- We define $f_3 := f_{(0,1,2)} \in K\{\{t\}\}[x_0, x_1, x_2]$ and $f_1 := f_{(0,2,3)} \in K\{\{t\}\}[x_0, x_2, x_3]$, with the notation as in Definition 3.4.
 - For $i \in \{1, 3\}$ let Δ_i be the Newton polytope of f_i .

- We denote the coefficients of f_3 and f_1 by a_v and b_v , respectively. As we always use non-homogeneous coordinates for Newton polytopes, this means that

$$f_3 = \sum_{i+j \leq d} a_{(i,j)} x_0^{d-i-j} x_1^i x_2^j \quad \text{and} \quad f_1 = \sum_{i+j \leq d} b_{(i,j)} x_0^{d-i-j} x_2^i x_3^j.$$

(b) Let C be a tropical curve in L_2^3 .

- We define $C_3 = p_*^{(0,1,2)} C$ and $C_1 = p_*^{(0,2,3)} C$.
- The Newton polytope of C_i will in the following be denoted by P_i , $i = 1, 3$.

In order to prove obstructions to relative realizability, we aim to find relations between the Newton polytopes Δ_1 and Δ_3 , i. e. relations between the polynomials f_1 and f_3 . If the Newton polytopes P_1 and P_3 of a tropical curve C do not satisfy these relations, we can use this to prove that C cannot be relatively realizable.

Notation 4.3. For $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, let $\binom{n}{k} = \frac{1}{k!} n \cdot (n-1) \cdots (n-k+1)$. For $k \in \mathbb{Z}_{<0}$, we set $\binom{n}{k} = 0$. Note that $\binom{n}{k} = 0$ if $n > 0$ and $k > n$.

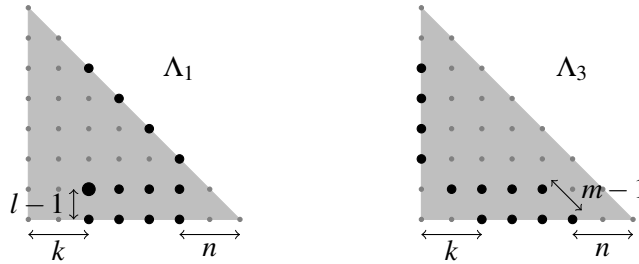
Using induction on a , it is an easy calculation to prove the following lemma.

Lemma 4.4. For $a, b, c \in \mathbb{N}$, we have

$$\sum_{j=0}^a (-1)^j \binom{a}{j} \binom{b+j}{c} = (-1)^a \binom{b}{c-a} \quad \text{and} \quad \sum_{j=0}^a (-1)^j \binom{a}{j} \binom{b-j}{c} = \binom{b-a}{c-a}.$$

Lemma 4.5. Let $f \in K\{\{t\}\}[x]$ be a homogeneous polynomial of degree d . Choose $k, n \in \mathbb{N}$ with $k+n \leq d$, $l \in \mathbb{N}$ with $l \leq n+1$, and $m \in \mathbb{N}$ with $m \leq k+1$.

As in the following picture, we denote by Λ_1 the set of all (i, j) such that $k \leq i \leq d-n$ and either $j < l$ or $j \geq d-i-n+l$. By Λ_3 , we denote the set of all $(d-t-s, s) \in \mathbb{N}^2$ for $t, s \in \mathbb{N}$ with $n \leq t \leq d-k$ and either $s < m$ or $s \geq d-t-k+m$.



Then the coefficients a_v and b_v of f_3 and f_1 as in Notation 4.2 satisfy the relation

$$\sum_{v \in \Lambda_1} \beta_v b_v = \sum_{v \in \Lambda_3} \alpha_v a_v$$

for some $\alpha_v, \beta_v \in \mathbb{Z}$ with $\beta_{(k, l-1)} = 1$.

Proof. As $f_1(x_0, x_2, x_3) = f_3(x_0, -x_0 - x_2 - x_3, x_2)$, it is an easy calculation to see that

$$b_{(i,j)} = \sum_{s=0}^i \sum_{t=0}^{d-i-j} (-1)^{d-t-s} \binom{d-t-s}{i-s} \binom{d-i-t}{j} a_{(d-t-s,s)}.$$

We consider the linear combination

$$S = \sum_{i=k}^{d-n} \sum_{j=0}^{d-i} (-1)^{i-k+l-1-j} \binom{i-m}{i-k} \binom{d-i-n+l-1-j}{d-i-n} b_{(i,j)}.$$

Note that $\binom{d-i-n+l-1-j}{d-i-n} = 0$ if $l \leq j < d-i-n+l$. Hence S is a linear combination of $b_{(i,j)}$ with $(i, j) \in \Lambda_1$. Moreover, the coefficient of $b_{(k,l-1)}$ in S is 1.

Replacing $b_{(i,j)}$ in S by the above equality, we get

$$\begin{aligned} S &= \sum_{i=k}^{d-n} \sum_{j=0}^{d-i} \sum_{s=0}^i \sum_{t=0}^{d-i-j} (-1)^{i-k+l-1-j+d-t-s} \binom{i-m}{i-k} \\ &\quad \cdot \binom{d-i-n+l-1-j}{d-i-n} \binom{d-t-s}{i-s} \binom{d-i-t}{j} a_{(d-t-s,s)}. \end{aligned}$$

Changing the summation order, we get $S = \sum_{s=0}^{d-n} \sum_{t=0}^{\min(d-k,d-s)} \sum_{i=\max(k,s)}^{\min(d-n,d-t)} \sum_{j=0}^{d-i-t} (\dots)$. This expression simplifies by Lemma 4.4 to

$$\begin{aligned} S &= \sum_{s=0}^{d-n} \sum_{t=0}^{\min(d-k,d-s)} \sum_{i=\max(k,s)}^{\min(d-n,d-t)} (-1)^{i-k+l-1+d-t-s} \binom{i-m}{i-k} \\ &\quad \cdot \binom{d-t-s}{i-s} \binom{t+l-n-1}{t-n} a_{(d-t-s,s)}. \end{aligned}$$

We have to check that this expression contains only a_v with $v \in \Lambda_3$. For $t < n$ we have $\binom{t+l-n-1}{t-n} = 0$, and thus $a_{(d-t-s,s)}$ does not appear in S . It is obvious that $a_{(d-t-s,s)}$ also does not appear in S if $t > d-k$. For $i < k$ we have $\binom{i-m}{i-k} = 0$, so we can consider the last sum as starting at $i = s$. For $n \leq t \leq d-k$, the coefficient of $a_{(d-t-s,s)}$ in S is then given by

$$\begin{aligned} &\sum_{i=s}^{d-t} (-1)^{i-k+l-1+d-t-s} \binom{i-m}{i-k} \binom{d-t-s}{i-s} \binom{t+l-n-1}{t-n} \\ &= \binom{t+l-n-1}{t-n} \sum_{i=0}^{d-t-s} (-1)^{i-k+l-1+d-t} \binom{i+s-m}{i+s-k} \binom{d-t-s}{i} \\ &= \binom{t+l-n-1}{t-n} \sum_{i=0}^{d-t-s} (-1)^{i-k+l-1+d-t} \binom{s-m+i}{k-m} \binom{d-t-s}{i} \\ &= \binom{t+l-n-1}{t-n} (-1)^{i-k+l-1+s} \binom{s-m}{t-d+k+s-m}. \end{aligned}$$

This coefficient is zero for all $(d-t-s, s)$ with $n \leq t \leq d-k$ and $m \leq s < d-t-k+m$. Hence the linear combination S only involves a_v for $v \in \Lambda_3$. \square

This dependency between f_1 and f_3 will be used to prove our obstructions to relative realizability: in the proof of Proposition 4.7 we show that for certain k, n, l , and m , all a_v and b_v appearing in the above equation but $b_{(k, l-1)}$ are zero since the corresponding exponents are not contained in the Newton polytopes Δ_1 and Δ_3 , respectively. Using Lemma 4.5, we then see that we must have $b_{(k, l-1)} = 0$.

In [GSW13], we also used relations between f_1 and f_3 to prove obstructions to relative realizability of tropical fan curves. Lemma 4.5 generalizes most of these relations: it implies [GSW13, Lemma 5.9], [GSW13, Lemma 5.15], and [GSW13, Proposition 5.17].

Notation 4.6. Let C be a tropical curve of degree d in L_2^3 . In the following, we will refer to the diagonals in P_3 and the vertical lines in P_1 as *rows*, counting from 0 with the row of lattice length d . As illustrated in the picture in the introduction, for a vertex $\mu = (\mu_1, \mu_2)$ of P_3 we set

$$\begin{aligned} s_\mu &= d - \mu_1 - \mu_2 \text{ (the row number of } \mu), \\ n_\mu &= |\{v \in \mathbb{N}^2 : v_1 + v_2 = \mu_1 + \mu_2, v \notin P_3\}| \text{ (lattice points not in } P_3 \text{ in the row of } \mu), \\ r_\mu &= d + 1 - s_\mu - n_\mu \text{ (lattice points in } P_3 \text{ in the row of } \mu), \\ l_\mu &= |\{v \in \mathbb{N}^2 : v_1 = n_\mu, v \in P_1\}| \text{ (lattice points in } P_1 \text{ in row number } n_\mu). \end{aligned}$$

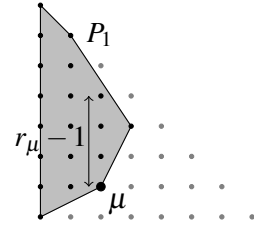
Correspondingly, for a vertex μ of P_1 we define

$$\begin{aligned} s_\mu &= \mu_1, \\ n_\mu &= |\{v \in \mathbb{N}^2 : v_1 = \mu_1, v_2 \leq d - v_1, v \notin P_1\}|, \\ r_\mu &= d + 1 - s_\mu - n_\mu, \\ l_\mu &= |\{v \in \mathbb{N}^2 : d - v_1 - v_2 = n_\mu, v \in P_3\}|. \end{aligned}$$

As a sequel to [GSW13], we will now first restrict our attention to tropical fan curves in L_2^3 and prove a new obstruction to relative realizability, in the case of curves which are only contained in two opposite cones of L_2^3 , say $\text{cone}([e_0], [e_3])$ and $\text{cone}([e_1], [e_2])$. Afterwards, we show that this necessary condition also results in sufficient conditions to relative realizability.

Proposition 4.7. *Let C be a tropical fan curve of degree d in L_2^3 contained only in the two opposite cones $\text{cone}([e_0], [e_3])$ and $\text{cone}([e_1], [e_2])$. If there is a vertex μ of P_1 or P_3 such that $l_\mu < r_\mu$, then C is not relatively realizable.*

Proof. Without loss of generality, we may assume that μ is a “lower vertex” of P_1 , i. e. $(\mu_1, j) \notin P_1$ for all $j < \mu_2$ (otherwise permute the coordinates via $(0, 1, 2, 3) \mapsto (3, 1, 2, 0)$). We want to apply Lemma 4.5. So choose $l = \mu_2 + 1$, $k = \mu_1$ and m such that any lattice point v in P_3 with $v_1 + v_2 = d - n_\mu$ is among the $r_\mu - 1$ points $(d - n_\mu - m, m), \dots, (d - n_\mu - m - r_\mu + 2, m + r_\mu - 2)$.



Let us assume that there is a polynomial $f \in K[x]$ realizing C , so that $\Delta_1 = P_1$ and $\Delta_3 = P_3$. Consider the equation from Lemma 4.5. We see that any coefficient a_v for $v \in \Lambda_3$ and any

coefficient b_ν for $\nu \in \Lambda_1$, except b_μ , that appears in this equation must be zero since its corresponding point is not contained in Δ_3 or Δ_1 , respectively. Hence we must have $b_\mu = 0$. However, this is a contradiction to the fact that μ is a vertex of Δ_1 . This means that such a polynomial cannot exist, hence C is not relatively realizable. \square

Remark 4.8. In the proof of Proposition 4.7 we do not use the complete information that C is only contained in two opposite cones of L_2^3 . Instead, with the notation of Lemma 4.5 we only use that $P_3 \cap \Lambda_3 = \emptyset$, and $P_1 \cap \Lambda_1$ is a single point, which is a vertex of P_1 . So if these conditions are fulfilled for any curve C in L_2^3 , then C is not relatively realizable.

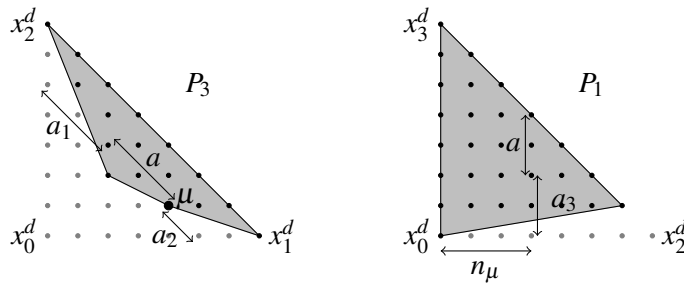
We now show that for this special class of tropical fan curves in L_2^3 , the obstructions of Proposition 4.7 give in fact rise to conditions that are equivalent to relative realizability.

Proposition 4.9. *Let C be a tropical fan curve of degree d in L_2^3 only contained in the two opposite cones $\text{cone}([e_1], [e_2])$ and $\text{cone}([e_0], [e_3])$. Using Notation 4.6, the curve C is relatively realizable if and only if $l_\mu \geq r_\mu$ for all vertices μ of P_3 and P_1 .*

Proof. We already know by Proposition 4.7 that C cannot be realizable if there is a vertex μ of P_1 or P_3 with $l_\mu < r_\mu$. So assume that we have $l_\mu \geq r_\mu$ for any vertex μ of P_1 and P_3 . We will construct a polynomial $f^\mu = \sum_{|v|=d} a_\nu x^\nu \in K[x_0, x_1, x_2, x_3]$ for every vertex μ of P_1 (resp. P_3) such that $\text{Newt}(f_1^\mu) \subset P_1$ and $\text{Newt}(f_3^\mu) \subset P_3$, and the Newton polytope $\text{Newt}(f_1^\mu)$ (resp. $\text{Newt}(f_3^\mu)$) contains the point μ . Summing up these polynomials with generic coefficients, we then get a polynomial f such that $\text{Newt}(f_3) = P_3$ and $\text{Newt}(f_1) = P_1$.

Let μ be a vertex of P_3 . We set $f^\mu = x_0^{a_0} x_1^{a_1} x_2^{a_2} x_3^{a_3} (x_0 + x_3)^a$ with

- $a = d - s_\mu - n_\mu$, i. e. the Newton polytope of $(x_0 + x_3)^a$ contains exactly r_μ points,
- a_1 and a_2 as the number of points on the diagonal $\text{conv}((s_\mu, 0), (0, s_\mu))$ on the left and right side of P_3 , i. e. we have $a_1 + a_2 = n_\mu$,
- by assumption we can choose a part of $\text{conv}((n_\mu, 0), (n_\mu, d - n_\mu)) \cap P_1$ of lattice length a , i. e. it contains r_μ points. Let a_0 and a_3 be the number of lattice points on the left and right side of this part, i. e. we have $a + a_0 + a_3 = d - n_\mu$.



Then

$$f_3^\mu = x_0^{a_0} x_1^{a_1} x_2^{a_2} (-x_0 - x_1 - x_2)^{a_3} (-x_1 - x_2)^a$$

and $f_1^\mu = x_0^{a_0} (-x_0 - x_2 - x_3)^{a_1} x_2^{a_2} x_3^{a_3} (x_0 + x_3)^a,$

so the Newton polytope of f_3^μ is contained in P_3 , and the Newton polytope of f_1^μ is contained in P_1 . Moreover, the coefficient of x^μ in f_3^μ is ± 1 . Analogously, we define the polynomial f^μ for any vertex μ of P_1 .

Let now f be a generic sum of these polynomials f^μ , where μ ranges over all vertices of P_3 and P_1 . Then $\text{Newt}(f_3) = P_3$ and $\text{Newt}(f_1) = P_1$ by construction, which means that $\text{Trop}(f_3) = C_3$ and $\text{Trop}(f_1) = C_1$.

Note that the projection $p^{(0,1,2)}$ is injective on the cones $\text{cone}([e_0], [e_1])$, $\text{cone}([e_0], [e_2])$, and $\text{cone}([e_1], [e_2])$, whereas $p^{(0,2,3)}$ is injective on the cones $\text{cone}([e_0], [e_2])$, $\text{cone}([e_0], [e_3])$, and $\text{cone}([e_2], [e_3])$. So these two projections allow to reconstruct each part of C which is not contained in the interior of $\text{cone}([e_1], [e_3])$. But C does not contain any rays in this interior by assumption, and $\text{Trop}(L + (f))$ does not contain any rays in this region either: $(d, 0)$ and $(0, d)$ are vertices of P_3 , so f contains a term $f^{(d,0)} = (x_0 + x_3)^d$, which implies that $f_{(0,1,3)}$ has a non-zero term x_0^d .

So altogether we conclude that $\text{Trop}(f) = C$, as required. \square

We will now address ourselves to the relative realizability of tropical curves in L_2^3 which are not necessarily fan curves. We start with generalizing Notation 4.6.

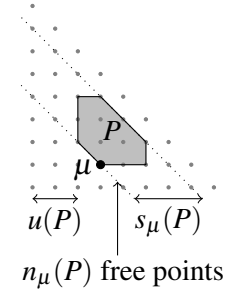
Notation 4.10. Let P be a lattice polytope in \mathbb{R}^2 , and let μ be a vertex of P . We define

$$s_\mu(P) = \max_{v \in P} (v_1 + v_2) - (\mu_1 + \mu_2) \quad (\text{row number of } \mu \text{ in } P),$$

$$n_\mu(P) = |\{v \in \mathbb{N}^2 : v_1 + v_2 = \mu_1 + \mu_2, v \notin P\}|$$

(lattice points not in P in the diagonal of μ),

$$u(P) = \min_{v \in P} v_1 \quad (\text{lattice distance of } P \text{ to the vertical axis}).$$



Proposition 4.11. *Let C be a tropical curve of degree d in L_2^3 that locally around 0 contains a classical line, and assume that this classical line is adjacent to the two vertices $[0, q, q, 0]$ and $[q', 0, 0, q']$ with $q, q' \in \mathbb{R}_{>0}$. Moreover, let $\mu = (\mu_1, \mu_2)$ be a vertex of the polytope P in $\text{Subdiv}(C_3)$ dual to $[0, q, q]$, and let Q be the polytope in $\text{Subdiv}(C_1)$ dual to $[q', 0, q']$.*

Then the curve C can only be relatively realizable if $(n_\mu(P) - u(Q)) \cdot q' \leq s_\mu(P) \cdot q$.

Proof. To simplify the notation, let $s = s_\mu(P)$ and $n = n_\mu(P)$. Assume that C is relatively realizable, say by $f = \sum_{|v|=d} a_v x^v \in K\{\{t\}\}[x_0, x_1, x_2]$. Due to the form of C we know that P has a face τ parallel to $\text{conv}((d, 0), (0, d))$ corresponding to the edge of C through the origin. Let $\eta, \xi \in \mathbb{N}^2$ such that $\tau = \text{conv}(\eta, \xi)$. By Remark 3.21, we can assume that $\text{val}(a_\eta) = 0$. Note that we then have $\text{val}(a_v) \geq 0$ for any lattice point v in $\text{Newt}(f)$: the origin in \mathbb{R}^2 is either a vertex of C_3 or contained in an edge of C_3 . In any case, the polyhedron τ is a

part of the polyhedron in the Newton subdivision of C_3 dual to the origin. With Remark 3.9, we thus see that $\text{val}(a_\xi) = 0$ and $\text{val}(a_\nu) \geq 0$ for all lattice points ν in $\text{Newt}(f)$. Let $A \subset \mathbb{N}^2$ such that $(A, (q, q)) \in \text{Subdiv}(C_3)$, i. e. we have $P = \text{conv}(A)$ and $\mu \in A$. Due to the conditions on f coming from C , we then know that $\text{val}(a_\nu) = (m + s - (\nu_1 + \nu_2)) \cdot q$ for all $\nu = (\nu_1, \nu_2) \in A$ and $\text{val}(a_\nu) > (m + s - (\nu_1 + \nu_2)) \cdot q$ for all $\nu \notin P$, where $m = \mu_1 + \mu_2$. Hence $\text{val}(a_\mu) = sq$, and $\text{val}(a_\nu) > sq$ for all ν with $\nu_1 + \nu_2 < m$, or with $\nu_1 + \nu_2 = m$ and $\nu \notin P$.

Let us now consider the polynomial $g = \sum_{|\nu|=d} a_\nu^{sq} x^\nu \in K[x_0, x_1, x_2]$, cf. Notation 3.22, i. e. the polynomial consisting of only those terms of f corresponding to t^{sq} . With the given arguments, we see that μ is a vertex of $\text{Newt}(g)$ with $n_\mu(\text{Newt}(g)) \geq n$. For simplicity, let $n_\mu(g) = n_\mu(\text{Newt}(g))$.

Let us consider $g_1 = g_{(0,2,3)}$. We want to apply Proposition 4.7 to the polynomial g (in fact to its tropicalization $\text{Trop}(L + (g))$), hence, let us consider the $n_\mu(g)$ -th row of $\text{Newt}(g_1)$. First of all, note that if we have $f_1(x) = f_{(0,2,3)}(x) = \sum_{|\nu|=d} b_\nu x^\nu$ as in Notation 4.2, we see that $g_1 = \sum_{|\nu|=d} b_\nu^{sq} x^\nu$. As above, we know that Q has a face σ parallel to $\text{conv}((0,0), (0,d))$ which is contained in the part of the Newton subdivision of C_1 dual to the origin. Hence, if we write $\sigma = \text{conv}(\eta, \xi)$ then $\text{val}(b_\eta) = \text{val}(b_\xi)$ is minimal in $\text{Newt}(f_1)$. Since the coefficients a_ν are linear combinations of the b_ν and vice versa, this minimal valuation must be zero. So the conditions on $\text{Subdiv}(C_1)$ ensure that $\text{val}(b_\nu) \geq (\nu_1 - u) \cdot q'$ for all lattice points ν in $\text{Newt}(f_1)$. If ν is a lattice point in $\text{Newt}(f_1)$ with $\nu_1 q' > uq' + sq$, we thus see that $\text{val}(b_\nu) > sq$, and so $\nu \notin \text{Newt}(g_1)$.

Hence, if we had $n_\mu(g)q' > uq' + sq$, there would be no points in the $n_\mu(g)$ -th diagonal of $\text{Newt}(g_1)$. By Proposition 4.7, we know that such a polynomial g cannot exist, hence C would not be relatively realizable. Thus, we must have $n_\mu(g)q' \leq uq' + sq$, and we finally get $nq' \leq n_\mu(g)q' \leq uq' + sq$, i. e. $(n - u)q' \leq sq$. \square

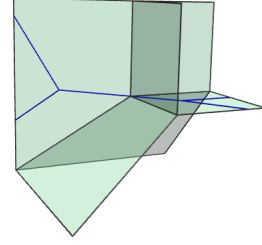
From now on, we will restrict our attention to tropical curves C in L_2^3 with only one bounded edge, which in addition passes through the origin. In this case, as already presented in the introduction, we will be able to give a complete non-algorithmic answer to the relative realizability problem.

Note that the subdivisions of the two Newton polytopes P_3 and P_1 of the projections C_3 and C_1 are trivial in this case. Moreover, the numbers of Notation 4.10 adapted to this situation, as well as to vertices of P_1 instead of P_3 , are exactly the numbers defined in Notation 4.6.

Corollary 4.12. *Let C be a tropical curve of degree d in L_2^3 that has only one bounded edge, passing through the origin with adjacent vertices $[0, q, q, 0]$ and $[q', 0, 0, q']$ for some $q, q' \in \mathbb{R}_{>0}$. Then C can only be relatively realizable if $n_\mu \cdot q' \leq s_\mu \cdot q$ for every vertex μ of P_3 and $n_\mu \cdot q \leq s_\mu \cdot q'$ for every vertex μ of P_1 .*

Proof. The statement for a vertex μ of P_3 follows from Proposition 4.11 with $P = P_3$, $Q = P_1$, and $u(Q) = 0$. The case $\mu \in P_1$ is analogous. \square

Example 4.13. For $q, q' \in \mathbb{R}_{>0}$, consider the tropical curve C in L_2^3 shown on the right. It has vertices $[0, q, q, 0]$ and $[q', 0, 0, q']$, the joining edge has weight 3, and the cones of its recession fan are spanned by the vectors $[1, 0, 0, 0]$ (with weight 2), $[1, 0, 0, 3]$, $[0, 1, 0, 0]$, and $[0, 2, 3, 0]$.



The corresponding Newton polytopes P_3 and P_1 are shown in the picture below.



The only interesting vertices are μ in P_3 and ν in P_1 , as for all other vertices η of P_3 or P_1 we have $s_\eta = n_\eta = 0$. As $n_\mu = s_\nu = 2$ and $n_\nu = s_\mu = 1$, Corollary 4.12 implies that C can only be realizable if $2q' \leq q$ and $q \leq 2q'$, i. e. if $\frac{q}{q'} = 2$. On the other hand, it can be checked using Algorithm 3.20 (or Proposition 4.15 below) that C is in fact realizable if $\frac{q}{q'} = 2$.

Remark 4.14. Let us consider the limit case $q = 0$, i. e. C has a ray at the origin in direction $\text{cone}([1, 0, 0, 1])$ with some weight, but not necessarily in direction $\text{cone}([0, 1, 1, 0])$, and P is considered to be the polyhedron in the subdivision of C_3 dual to the origin. If we follow the proof of Proposition 4.11, we would choose $g = \sum_{|v|=d} a_v^0 x^v$ in $K[x_0, x_1, x_2]$ for any vertex of P since we have $q = 0$. So the Newton polytope of g completely contains P . Hence, in the case $q = 0$, the proof of Proposition 4.11 only works if we choose μ to be a vertex of P such that $s_\mu(P)$ is maximal among the vertices of P . The resulting necessary condition for realizability is $n_\mu(P) \leq u(Q)$. In the other limit case $q' = 0$, the statement of Proposition 4.11 is trivial.

Correspondingly, for Corollary 4.12 let us consider curves with one bounded edge, passing through the origin with direction $[1, 0, 0, 1]$, such that the origin is one of its adjacent vertices (and thus $q = 0$ or $q' = 0$), and the rays adjacent to the two vertices are contained in two opposite cones of L_2^3 . In this case, the necessary condition for realizability is $n_\mu = 0$, where μ is a vertex of P_3 , or P_1 respectively, such that s_μ is maximal. Due to the form of P_3 , we see that if $n_\mu = 0$ for a vertex of P_3 with maximal s_μ , then we have $n_\nu = 0$ for any vertex ν of P_3 . The analogous statement is true for P_1 . Hence, in the degenerated case $q = 0$, or $q' = 0$ the statement of Corollary 4.12 still holds.

Let us now prove that tropical curves in L_2^3 with one bounded edge, passing through the origin, and with realizable recession fan, are realizable if and only if they fulfill the length conditions of Corollary 4.12. The proof is based on the same idea as the proof of Proposition 4.9. Note that the realizability of the recession fan can be checked by Proposition 4.9 since this fan is also contained in two opposite cones of L_2^3 .

Proposition 4.15. *Let C be a tropical curve of degree d in L_2^3 with exactly one bounded edge, passing through the origin, and with vertices $[0, q, q, 0]$ and $[q', 0, 0, q']$. Assume moreover*

that the recession fan of C is relatively realizable. Then C is relatively realizable in L if and only if $n_\mu \cdot q' \leq s_\mu \cdot q$ for every vertex μ of P_3 and $n_\mu \cdot q \leq s_\mu \cdot q'$ for every vertex μ of P_1 .

Proof. We know already by Corollary 4.12 that C is not relatively realizable if one of the length conditions is not fulfilled. So let us prove the other direction of the statement.

With Remark 3.23, the conditions on the coefficients a_v of a polynomial $f = \sum_v a_v x^v \in K\{\{t\}\}[x_0, x_1, x_2]$ realizing C can be decomposed into conditions on the parameters a_v^k for fixed $k \in \text{RHS}(C)$, where $a_v = \sum_{k \in \text{RHS}(C)} a_v^k t^k$. Hence if we find a polynomial $f^k = \sum_v a_v^k x^v$ fulfilling the conditions on the a_v^k for all $k \in \text{RHS}(C)$, then $f = \sum_{k \in \text{RHS}(C)} f^k t^k$ realizes C .

Since C has only two vertices, we have $\text{Subdiv}(C_3) = \{(P_3, (q, q))\}$ and $\text{Subdiv}(C_1) = \{(P_1, (-q', 0))\}$. Assuming that $\text{val}(a_{(d,0)}) = 0$, the conditions on the a_v^k have the following form: For any lattice point v in P_3 we get the condition $a_v^k = 0$ for all $k \in \text{RHS}(C)$ with $k < s_v q$. For any vertex v of P_3 , we get the additional condition $a_v^{s_v q} \neq 0$. Correspondingly, using Notation 4.2 the conditions on the coefficients b_v of f_1 are given by $b_v^k = 0$ for all lattice points v in P_1 and all $k \in \text{RHS}(C)$ with $k < s_v q'$, and $b_v^{s_v q'} \neq 0$ for all vertices v of P_1 .

For any vertex μ of P_3 or P_1 , let f^μ be as defined in the proof of Proposition 4.9. For $k \in \text{RHS}(C)$, we define f^k as a generic sum over f^μ , where μ ranges over all vertices μ of P_3 with $s_\mu q = k$ and all vertices μ of P_1 with $s_\mu q' = k$. As the recession fan of C is relatively realizable and projects to curves dual to the same polytopes P_1 and P_3 , we know by Proposition 4.9 that $l_\mu \geq r_\mu$ for any vertex μ of P_1 and P_3 . Hence, the polynomials f_3^k and f_1^k only contain monomials x^v with $v \in P_3$ and $v \in P_1$, respectively. We claim that f^k fulfills the conditions on the parameters a_v^k : By construction, we have $a_v^{s_v q} \neq 0$ for all vertices v of P_3 and $b_v^{s_v q'} \neq 0$ for all vertices v of P_1 . To show that $a_v^k = 0$ for all lattice points in P_3 with $k < s_v q$ and $b_v^k = 0$ for all lattice points in P_1 with $k < s_v q'$, we show that these conditions are fulfilled for any polynomial f^μ arising in the definition of f^k .

Let μ be a vertex of P_3 with $s_\mu q = k$. Due to the definition of f^μ , it is clear that f^μ only contains monomials x^v with $s_v \leq s_\mu$, and thus it fulfills the conditions $a_v^k = 0$ for $s_v q > k$. So let us consider f_1^μ . We have seen in the proof of Proposition 4.9 that f_1^μ only contains monomials x^v with $s_v \leq n_\mu$. But we have assumed $n_\mu q' \leq s_\mu q = k$, hence the polynomial f^μ only contains monomials with $s_v q' \leq n_\mu q' \leq k$, i. e. the condition $b_v^k = 0$ is fulfilled for all v with $s_v q' > k$.

Analogously, one shows that f^μ with μ a vertex of P_1 and $s_\mu q' = k$ fulfills the above conditions. Setting $f = \sum_{k \in \text{RHS}(C)} f^k t^k$, we thus have $\text{Subdiv}(f_3) = \text{Subdiv}(C_3)$ and $\text{Subdiv}(f_1) = \text{Subdiv}(C_1)$. As in the proof of Proposition 4.9, we see that $\text{Trop}(f)$ does not meet the relative interior of $\text{cone}([e_1], [e_3])$, so f realizes C . \square

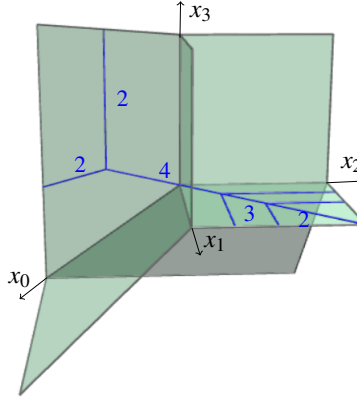
It is well-known that the recession fan of a relatively realizable tropical curve C is also relatively realizable [MS15, Theorem 3.5.6], as well as the star of C at each vertex v of C [MS15, Lemma 3.3.6]. An interesting question is if we can enlarge the star of C at v and prove that C has to be relatively realizable “locally around its edges”. The following definition makes this precise.

Definition 4.16. Let C be a tropical curve in $\mathbb{R}^{n+1}/\mathbb{R} \cdot \mathbf{1}$ and let e be a bounded edge of C with adjacent vertices v_1 and v_2 . We define $C^{(e)}$ to be the unique tropical curve in $\mathbb{R}^{n+1}/\mathbb{R} \cdot \mathbf{1}$ with exactly one bounded edge, adjacent to the two vertices v_1 and v_2 , such that $\text{star}_C(v_i) = \text{star}_{C^{(e)}}(v_i)$ for $i = 1, 2$.

Example 4.17. We consider the polynomial

$$f = (1 - 3t + t^3)x_0^4 + (4 - 8t + 2t^3)x_0^3x_3 + (6 - 6t + t^3)x_0^2x_3^2 + 4x_0x_3^3 + (1 + t)x_3^4 + t^4x_0^2x_2^2$$

in $K\{\{t\}\}[x_0, x_1, x_2, x_3]$. The tropicalization of $L + (f) \subset K\{\{t\}\}[x_0, x_1, x_2, x_3]$ is the tropical curve C shown in the picture below, with vertices $[2, 0, 0, 2]$, $[0, 1, 1, 0]$, and $[0, 2, 2, 0]$.



Let e be the edge of C through the origin. The recession fan of $C^{(e)}$ is described by the set

$$P\left(\text{Rec}(C^{(e)})\right) = \{(0, 0, 0, 2), (4, 0, 0, 2), (0, 1, 0, 0), (0, 0, 1, 0), (0, 3, 3, 0)\}.$$

Applying Proposition 4.7, we see that $\text{Rec}(C^{(e)})$ is not relatively realizable and thus $C^{(e)}$ is not relatively realizable. Therefore, global relative realizability does not imply relative realizability at bounded edges. However, this implication is true if the recession fan of $C^{(e)}$ is relatively realizable:

Corollary 4.18. *Let C be a tropical curve in L_2^3 with a bounded edge e through the origin such that $C^{(e)}$ is only contained in two opposite cones of L_2^3 . Assume moreover that the recession fan of $C^{(e)}$ is relatively realizable. Then C can only be realizable in L if $C^{(e)}$ is relatively realizable.*

Proof. Suppose that $C^{(e)}$ is not relatively realizable. Since the recession fan of $C^{(e)}$ is assumed to be relatively realizable, this means that the length conditions of Proposition 4.15 are contradicted. Without loss of generality, we may assume that the two adjacent vertices of e are of the form $[0, q, q, 0]$ and $[q', 0, 0, q']$ with $q, q' \in \mathbb{R}_{>0}$ and that there is a vertex μ of $P = \text{Newt}(p_*^{(0,1,2)} C^{(e)})$ with $n_\mu(P) \cdot q' > s_\mu(P) \cdot q$ (cf. Notation 4.10).

Let Q be the polytope in the Newton subdivision of $p_*^{(0,1,2)} C$ corresponding to $[0, q, q, 0]$. By definition, it holds $\text{star}_C([0, q, q, 0]) = \text{star}_{C^{(e)}}([0, q, q, 0])$, so there is a vector v with

$Q = P + v$. For the vertex $\eta = \mu + v$ of Q , we have $s_\eta(Q) = s_\mu(P)$ and $n_\eta(Q) \geq n_\mu(P)$, which implies

$$n_\eta(Q) \cdot q' \geq n_\mu(P) \cdot q' > s_\mu(P) \cdot q = s_\eta(Q) \cdot q.$$

Hence C is not relatively realizable by Proposition 4.11. \square

Remark 4.19. In the spirit of Remark 4.14, let us consider the degenerated cases of Proposition 4.15 when $q = 0$ or $q' = 0$. If $q = q' = 0$ then C has the origin as its only vertex, so we can use Proposition 4.9 to check if C is relatively realizable. If only $q = 0$, the curve has still one bounded edge, of direction $[1, 0, 0, 1]$ and weight d , with one vertex being the origin. The necessary conditions in this case are $n_\mu = 0$ for all vertices μ of P_3 . Following the proof of Proposition 4.15 (with the assumption that $\text{Rec}(C)$ is realizable), we see that these conditions are also sufficient.

Remark 4.20 (Result of Proposition 4.15). Let C be a tropical curve as above, of degree d in L_2^3 with exactly one bounded edge, passing through the origin, and with vertices $[0, q, q, 0]$ and $[q', 0, 0, q']$. We also allow the degenerated cases $q = 0$ and/or $q' = 0$. Assume that $\text{Rec}(C)$ is relative realizable.

Consider the case that there are vertices μ, v of P_3 and P_1 , respectively, such that $n_\mu \neq 0$ and $n_v \neq 0$. Note that we also have $s_\mu \neq 0$ and $s_v \neq 0$ since the 0-th row is completely contained in P_3 and P_1 . Among the vertices η of P_3 with $s_\eta \neq 0$, let μ be a vertex such that $\frac{n_\mu}{s_\mu}$ is maximal. Similarly, let v be a vertex of P_1 with $n_v \neq 0$ such that $\frac{s_v}{n_v}$ is minimal. Using Proposition 4.15, we see that C is relatively realizable if and only if

$$q = q' = 0 \quad \text{or} \quad \frac{q}{q'} \in \left[\frac{n_\mu}{s_\mu}, \frac{s_v}{n_v} \right].$$

Note that if $q \neq 0$ (resp. $q' \neq 0$) and C is relatively realizable, then by the length conditions of Proposition 4.15 we must also have $q' \neq 0$ (resp. $q \neq 0$).

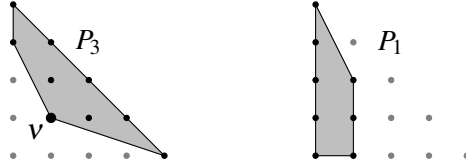
If we have $s_\mu = 0$ for all vertices μ of P_1 , with the argument from above we also have $n_\mu = 0$ for all vertices μ of P_1 . Moreover, using Proposition 4.7, we also have $n_\mu = 0$ for all vertices μ of P_3 . The same is true if we have $s_\mu = 0$ for all vertices μ of P_3 (swap the arguments). In any of these two cases, the tropical curve C is relatively realizable for any $q, q' \in \mathbb{R}_{\geq 0}$ by Proposition 4.15 with Remark 4.19.

Finally, let us consider the case that $n_\mu = 0$ for all vertices μ of P_1 . In this case, the conditions $n_\mu \cdot q \leq \mu_1 \cdot q'$ are trivial. Hence, by Proposition 4.15 with Remark 4.19 the tropical curve C is relatively realizable for all q, q' with either $q' = 0$ or $\frac{q}{q'} \in \left[\frac{n_\mu}{s_\mu}, \infty \right)$, where μ is a vertex of P_3 with $s_\mu \neq 0$ such that $\frac{n_\mu}{s_\mu}$ is maximal. The analogous statement is true if we have $n_\mu = 0$ for all vertices μ of P_3 . If we have $n_\mu = 0$ for all vertices μ of P_3 and P_1 , then C is relatively realizable for all $q, q' \in \mathbb{R}_{\geq 0}$.

Let us now return to the case of a tropical curve C having vertices μ of P_3 and v of P_1 such that $n_\mu \neq 0$ and $n_v \neq 0$. Assume that μ is chosen such that $\frac{n_\mu}{s_\mu}$ maximal and v is chosen such that $\frac{s_v}{n_v}$ is minimal. Finally, let $I = \left[\frac{n_\mu}{s_\mu}, \frac{s_v}{n_v} \right]$.

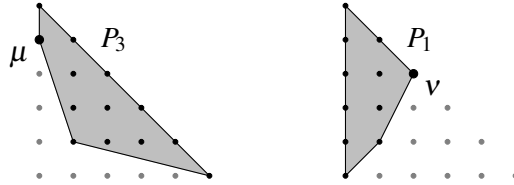
For $q \neq 0$ and $q' \neq 0$, we know by Proposition 4.15 that the tropical curve C with vertices $[0, q, q, 0]$ and $[q', 0, 0, q']$ is relatively realizable if and only if its recession fan is relatively realizable and $\frac{q}{q'} \in I$. However, it may happen that I is empty. In the following two examples, we will see that the relative realizability of the recession fan does not correlate with I being non-empty.

Example 4.21. For $q, q' \in \mathbb{R}_{>0}$, consider the tropical curve C in L_2^3 which has only one bounded edge of weight 4 with adjacent vertices $[0, q, q, 0]$ and $[q', 0, 0, q']$, and whose projections have the following Newton polytopes:



For the vertex v we have $n_v = 2$ and $r_v = 1 > 0 = l_v$. So the recession fan of C is not relatively realizable by Proposition 4.7, and thus C is not relatively realizable either. However, for any vertex μ of P_3 with $s_\mu \neq 0$ we have $\frac{n_\mu}{s_\mu} = 1$, and for any vertex η of P_1 with $n_\eta \neq 0$ we have $\frac{s_\eta}{n_\eta} = 1$. Hence $I = \left[\frac{n_\mu}{s_\mu}, \frac{s_\eta}{n_\eta} \right] = [1, 1]$ is non-empty.

Example 4.22. For $q, q' \in \mathbb{R}_{>0}$ consider the tropical curve C in L_2^3 which has only one bounded edge of weight 5 with adjacent vertices $[0, q, q, 0]$ and $[q', 0, 0, q']$, and whose projections are described by the following Newton polytopes:



For any vertex η of P_3 and P_1 we have $l_\mu \geq r_\mu$, so the recession fan of C is relatively realizable (Proposition 4.9). However, we have $\frac{n_\mu}{s_\mu} = 1 > \frac{2}{3} = \frac{s_v}{n_v}$, so $I = \left[\frac{n_\mu}{s_\mu}, \frac{s_v}{n_v} \right]$ is empty. By Proposition 4.11, there are thus no $q, q' \in \mathbb{R}_{>0}$ such that C is relatively realizable.

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