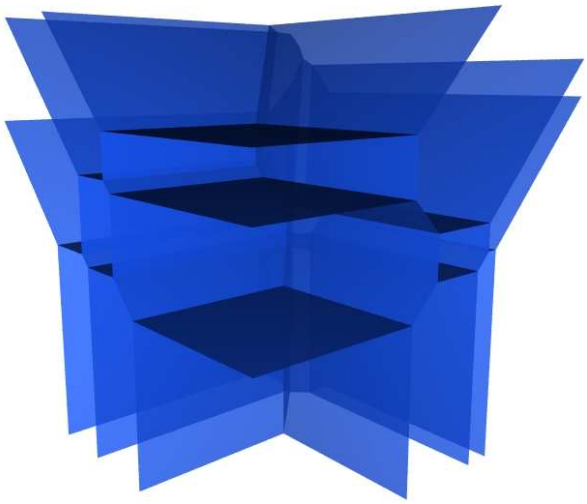
Tropical intersection theory

Lars Allermann



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Contents

Preface

Tropical geometry

Tropical geometry is a rather new field within mathematics. Its roots go back to the work of George M. Bergman [B71] as well as Robert Bieri and John R. J. Groves [BG84], but only in the last ten years tropical geometry became a subject on its own. The general idea of the theory is to map objects in algebraic or symplectic geometry to polyhedral objects using some "tropicalization" process. These latter objects can then be studied by purely combinatorial means, making life much easier in many cases. Nevertheless, in this tropicalization process enough properties of the original objects are preserved such that it is possible to transfer back many tropical results to algebraic or symplectic geometry.
Tropical geometry is a useful tool in many different areas of mathematics, such as real enumerative geometry (e.g. [IKS03], [IKS04], [IKS09], [M05]), symplectic geometry (e.g. [A06]), number theory (e.g. [G07a], [G07b]), combinatorics (e.g. [J08]) as well as algebraic statistics and computational biology (e.g. [PS04]).
There are a number of ways to approach tropical geometry. In this thesis we choose a purely combinatorial point of view on the topic: We set up the beginnings of an extensive tropical intersection theory on its own, without using the existing theory in algebraic geometry. Nevertheless, our definitions and results are highly inspired by the algebro-geometric theory (cf. [F84]).

Results of this thesis

In this thesis we set up the beginnings of a tropical intersection theory covering many concepts and tools of its counterpart in algebraic geometry. For instance:

- We develop notions of tropical varieties and cycles, rational functions and Cartier divisors, intersection products of Cartier divisors with cycles, morphisms of tropical varieties and pull-backs of Cartier divisors and push-forward of cycles as well as rational and numerical equivalence.

- We prove a projection formula for morphisms of tropical varieties.

- For the special case that our ambient cycle is $\mathbb{R}^n$ we prove that the concepts of rational and numerical equivalence agree. Moreover, restricting ourselves to "generic" cycles we study the numerical equivalence of cycles in more detail.

- For the special case that our ambient cycle is a fan we show that every cycle is numerical equivalent to an affine cycle.
- We define an intersection product of cycles in any "smooth" tropical variety and prove some basic properties. We use this intersection product to introduce a concept of pull-back of cycles along morphisms of smooth varieties.
- We prove that under some assumptions the one-to-one correspondence of Weil and Cartier divisors that exists for example on $\mathbb{R}^n$ is preserved by "modifications" as introduced in [M06].
- We introduce the notions of tropical vector bundles and Chern classes of tropical vector bundles and prove some basic properties.

Chapter synopsis

This thesis consists of five chapters: Chapter 1 contains the basics of the theory and is essential for the rest of the thesis. Chapters 2–5 are to a large extent independent of each other and can be read separately.

- **Chapter 1: Foundations of tropical intersection theory**
  In section 1.1 we introduce the concept of affine tropical cycles as balanced weighted fans modulo refinements. After that, in section 1.2, we define Cartier divisors to be piecewise integer affine linear functions modulo globally linear functions and set up an intersection product of Cartier divisors and cycles. In section 1.3 we continue with the definitions of morphisms of tropical cycles, of pull-backs of Cartier divisors and push-forwards of cycles and prove a projection formula. In sections 1.4, 1.5 and 1.6 we generalize these concepts to abstract tropical cycles which are abstract polyhedral complexes modulo refinements with affine cycles as local building blocks. In section 1.7 we introduce a concept of rational equivalence. Finally, in sections 1.8 and 1.9, we set up an intersection product of cycles and prove that every cycle is rationally equivalent to some affine cycle in the special case that our ambient cycle is $\mathbb{R}^n$. We use this result to show that rational and numerical equivalence agree in this case and prove a tropical Bézout's theorem.
- **Chapter 2: Tropical cycles with real slopes and numerical equivalence**
  In section 2.1 we generalize our definitions of tropical cycles to polyhedral complexes with non-rational slopes. We use these cycles with non-rational slopes in section 2.2 to show that if our ambient cycle is a fan then every subcycle is numerically equivalent to some affine cycle. In section 2.3 we restrict ourselves to cycles in $\mathbb{R}^n$ that are "generic" in some sense and study the concept of numerical equivalence in more detail.
- **Chapter 3: Tropical intersection products on smooth varieties**
  In section 3.1 we define an intersection product of tropical cycles on tropical linear spaces L^n_k and on other, related fans. In section 3.2 we use this result to obtain

an intersection product of cycles on any "smooth" tropical variety. Finally, in section 3.3, we use the intersection product to introduce a concept of pull-backs of cycles along morphisms of smooth tropical varieties and prove that this pull-back has all expected properties.

- **Chapter 4: Weil and Cartier divisors under tropical modifications**
  In section 4.1 we introduce "modifications" and "contractions" and study their basic properties. In section 4.2 we prove that under some further assumptions a one-to-one correspondence of Weil and Cartier divisors is preserved by modifications. In particular we can prove that on any smooth tropical variety we have a one-to-one correspondence of Weil and Cartier divisors. Moreover, using the result it is possible to prove that there exists a one-to-one correspondence of Weil and Cartier divisors on the moduli space of n-marked abstract tropical curves $\mathcal{M}_{0,n,\mathrm{trop}}$ (cf. [H07]).

- **Chapter 5: Chern classes of tropical vector bundles**
  In section 5.1 we give definitions of tropical vector bundles and rational sections of tropical vector bundles. We use these rational sections in section 5.2 to define the Chern classes of such a tropical vector bundle. Moreover, we prove that these Chern classes have all expected properties. In section 5.3 we classify all tropical vector bundles on an elliptic curve up to isomorphisms.

Publication of the results

This thesis contains material from my articles [AR07], [AR08] and [A09]. In particular, the first chapter is the outcome of joint work with Johannes Rau and it is virtually impossible to specify the contributions each of us made. As far as it can be told, main contributions of Johannes Rau are contained in sections 1.2, 1.5 and 1.7, whereas sections 1.1, 1.3, 1.4, 1.6 and 1.8 are mainly based on my ideas. Section 1.9 contains important contributions of both of us. Moreover, I omit those parts that are to a large extent the work of Johannes Rau.

Danksagung

Ich danke...

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1 Foundations of tropical intersection theory

This chapter consists of three parts: In the first part (sections 1.1 - 1.3) we start with the introduction of affine tropical cycles as balanced weighted fans modulo refinements and affine tropical varieties as affine cycles with non-negative weights. One would like to define the intersection of two such objects, but in general neither is the set-theoretic intersection of two cycles again a cycle, nor does the concept of stable intersection as introduced in [RGST05] work for arbitrary ambient spaces as can be seen in example 1.2.10. Therefore we introduce the notion of affine Cartier divisors on tropical cycles as piecewise integer affine linear functions modulo globally affine linear functions and define a bilinear intersection product of Cartier divisors and cycles. We then prove the commutativity of this product and a projection formula for push-forwards of cycles and pull-backs of Cartier divisors. In the second part (sections 1.4 - 1.7) we generalize the theory developed in the first part to abstract cycles which are abstract polyhedral complexes modulo refinements with affine cycles as local building blocks. Again, abstract tropical varieties are just cycles with non-negative weights. In both the affine and abstract case a remarkable difference to the classical situation occurs: We can define the mentioned intersection products on the level of cycles, i.e. we can intersect Cartier divisors with cycles and obtain a well-defined cycle — not only a cycle class up to rational equivalence as it is the case in classical algebraic geometry. However, for simplifying the computations of concrete enumerative numbers we introduce a notion of rational equivalence of cycles in section 1.7. In the third part (section 1.8 - 1.9) we finally use our theory to define the intersection product of two cycles with ambient space $\mathbb{R}^n$. Here again it is remarkable that we can define these intersections — even for self-intersections — on the level of cycles. It turns out that this intersection product coincides with the *stable intersection* discussed in [M06] and [RGST05] (see [K09] and [R08]). Afterwards, we study the special case of rational equivalence in $\mathbb{R}^n$ in more detail and show that every tropical cycle in $\mathbb{R}^n$ is equivalent to a uniquely determined affine cycle, called its *degree*. We use this result to prove a tropical Bézout's theorem.

1.1 Affine tropical cycles

In this section we will briefly summarize the definitions and some properties of our basic objects. We refer to [GKM07] for more details (but note that we use a slightly more general definition of fan).
Throughout this paper Λ will always denote a finitely generated free abelian group, i.e.

a group isomorphic to $\mathbb{Z}^r$ for some $r \in \mathbb{N}$, and $V := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ the associated real vector space containing Λ as a lattice. We will denote the dual lattice in the dual vector space by $\Lambda^\vee \subseteq V^\vee$.

Definition 1.1.1 (Cones)
A *cone* in V is a subset $\sigma \subseteq V$ that can be described by finitely many linear integral equalities and inequalities, i.e. a set of the form

$$\sigma = \{x \in V | f_1(x) = 0, \ldots, f_r(x) = 0, f_{r+1}(x) \geq 0, \ldots, f_N(x) \geq 0\}$$

for some linear forms $f_1, \ldots, f_N \in \Lambda^\vee$. We denote by V_σ the smallest linear subspace of V containing σ and by Λ_σ the lattice $V_\sigma \cap \Lambda$. We define the *dimension* of σ to be the dimension of V_σ.

Definition 1.1.2 (Fans)
A *fan* X in V is a finite set of cones in V satisfying the following conditions:

(a) The intersection of any two cones in X belongs to X as well,

(b) every cone $\sigma \in X$ is the disjoint union $\sigma = \dot{\bigcup}_{\tau \in X : \tau \subseteq \sigma} \tau^{\mathrm{ri}}$, where τ^{ri} denotes the relative interior of τ, i.e. the interior of τ in V_τ.

We will denote the set of all k-dimensional cones of X by $X^{(k)}$. The *dimension* of X is defined to be the maximum of the dimensions of the cones in X. The fan X is called *pure-dimensional* if each inclusion-maximal cone in X has this dimension. The union of all cones in X will be denoted $|X| \subseteq V$. If X is a fan of pure dimension k then the cones $\sigma \in X^{(k)}$ are called *facets* of X.

Let X be a fan and $\sigma \in X$ a cone. A cone $\tau \in X$ with $\tau \subseteq \sigma$ is called a *face* of σ. We write this as $\tau \leq \sigma$ (or $\tau < \sigma$ if in addition $\tau \subsetneq \sigma$ holds). Clearly we have $V_\tau \subseteq V_\sigma$ and $\Lambda_\tau \subseteq \Lambda_\sigma$ in this case. Note that $\tau < \sigma$ implies that τ is contained in a proper face (in the usual sense) of σ.

Example 1.1.3
The following figure shows three examples of fans of pure dimension two in $V = \mathbb{R}^2$ according to definition 1.1.2. Note that the third example is not a fan in the sense of [GKM07, definition 2.4] as for example $\sigma_1 \cap \sigma_2$ is not a face of σ_2 according to that definition.

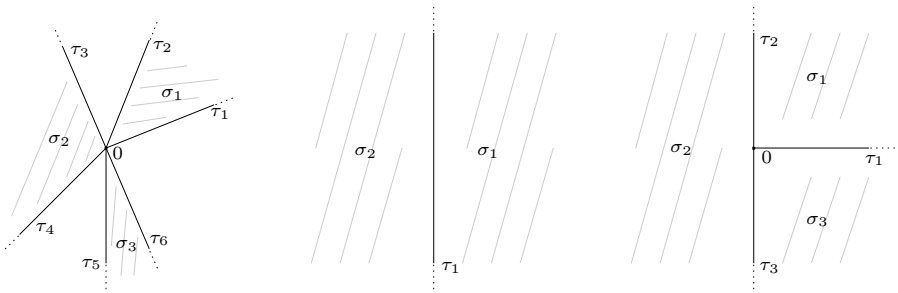


Figure 1.1: Examples of fans in $\mathbb{R}^2$.

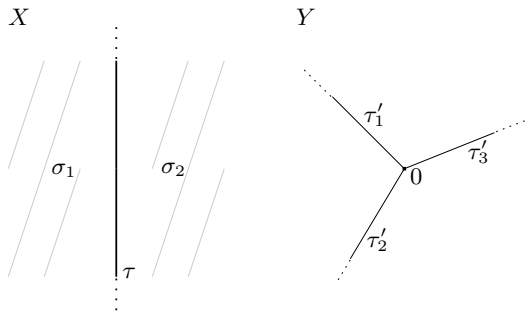


Figure 1.2: A fan X and a subfan $Y \trianglelefteq X$.

Construction 1.1.4 (Normal vectors)
Let $\tau < \sigma$ be cones of some fan X in V with $\dim(\tau) = \dim(\sigma) - 1$. This implies that there is a linear form $f \in \Lambda_\sigma^\vee$ that is zero on τ, non-negative on σ and not identically zero on σ. Let $u_\sigma \in \Lambda_\sigma$ be a vector generating $\Lambda_\sigma/\Lambda_\tau \cong \mathbb{Z}$ with $f(u_\sigma) > 0$. Note that its class $u_{\sigma/\tau} := [u_\sigma] \in \Lambda_\sigma/\Lambda_\tau$ does not depend on the choice of u_σ. We call $u_{\sigma/\tau}$ the *(primitive) normal vector* of σ relative to τ.

Definition 1.1.5 (Subfans)
Let X, Y be fans in V. Y is called a *subfan* of X if for every cone $\sigma \in Y$ there exists a cone $\sigma' \in X$ such that $\sigma \subseteq \sigma'$. In this case we write $Y \trianglelefteq X$ and define a map $C_{Y,X} : Y \to X$ that maps a cone $\sigma \in Y$ to the unique inclusion-minimal cone $\sigma' \in X$ with $\sigma \subseteq \sigma'$.

Definition 1.1.6 (Weighted fans)
A *weighted fan* (X, ω_X) of dimension k in V is a fan X in V of pure dimension k, together with a map $\omega_X : X^{(k)} \to \mathbb{Z}$. The number $\omega_X(\sigma)$ is called the *weight* of the facet $\sigma \in X^{(k)}$. For simplicity we usually write $\omega(\sigma)$ instead of $\omega_X(\sigma)$. Moreover, we want to consider the *empty fan* $\emptyset$ to be a weighted fan of dimension k for all k. Furthermore, by abuse of notation we simply write X for the weighted fan (X, ω_X) if the weight function ω_X is clear from the context.

Definition 1.1.7 (Tropical fans)
A *tropical fan* of dimension k in V is a weighted fan (X, ω_X) of dimension k satisfying the following *balancing condition* for every $\tau \in X^{(k-1)}$:

$$\sum_{\sigma:\tau<\sigma} \omega_X(\sigma) \cdot u_{\sigma/\tau} = 0 \in V/V_\tau.$$

Let (X, ω_X) be a weighted fan of dimension k in V and X^* the fan

$$X^* := \{\tau \in X | \tau \leq \sigma \text{ for some facet } \sigma \in X \text{ with } \omega_X(\sigma) \neq 0\}.$$

$(X^*, \omega_{X^*}) := (X^*, \omega_X|_{(X^*)^{(k)}})$ is called the *non-zero part* of X and is again a weighted fan of dimension k in V (note that $X^* = \emptyset$ is possible). Obviously (X^*, ω_{X^*}) is a tropical fan if and only if (X, ω_X) is one. We call a weighted fan (X, ω_X) *reduced* if all its facets have non-zero weight, i.e. if $(X, \omega_X) = (X^*, \omega_{X^*})$ holds.

Example 1.1.8
The following figure shows three examples of tropical fans in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, according to definition 1.1.7:

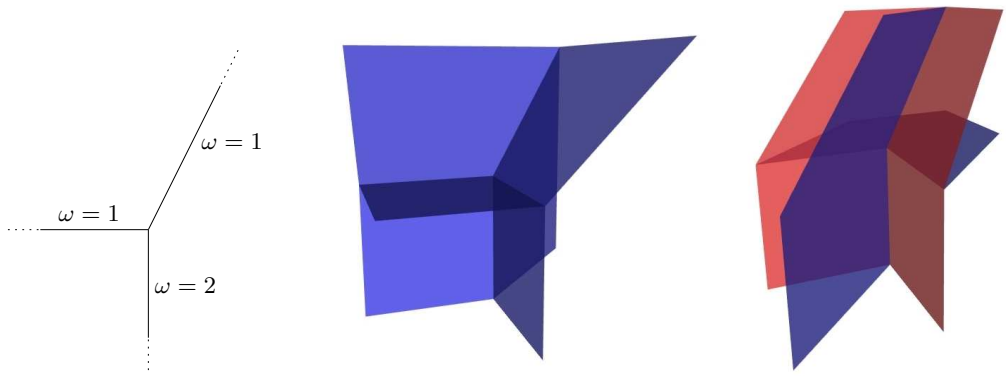


Figure 1.3: Examples of tropical fans in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

The blue cones in the second and third fan are supposed to have weight one, the red cones to have weight minus one.

Remark 1.1.9
Let (X, ω_X) be a tropical fan of dimension k and let $\tau \in X^{(k-1)}$. Let $\sigma_1, \dots, \sigma_N$ be all cones in X with $\sigma_i > \tau$. For all i let $v_{\sigma_i/\tau} \in \Lambda$ be a representative of the primitive normal vector $u_{\sigma_i/\tau} \in \Lambda_{\sigma_i}/\Lambda_\tau$. By the above balancing condition we have $\sum_{i=1}^N \omega_X(\sigma_i) \cdot v_{\sigma_i/\tau} = \lambda_\tau$ for some $\lambda_\tau \in \Lambda_\tau$. Obviously we have $\lambda_\tau = \gcd(\omega_X(\sigma_1), \dots, \omega_X(\sigma_N)) \cdot \widetilde{\lambda}_\tau$ for some further $\widetilde{\lambda}_\tau \in \Lambda_\tau$. We can represent the greatest common divisor by a linear combination $\gcd(\omega_X(\sigma_1), \dots, \omega_X(\sigma_N)) = \alpha_1\omega_X(\sigma_1) + \dots + \alpha_N\omega_X(\sigma_N)$ with $\alpha_1, \dots, \alpha_N \in \mathbb{Z}$ and define

$$\widetilde{v}_{\sigma_i/\tau} := v_{\sigma_i/\tau} - \alpha_i \cdot \widetilde{\lambda}_\tau$$

for all i. Note that $\widetilde{v}_{\sigma_i/\tau}$ is a representative of $u_{\sigma_i/\tau}$, too. Replacing all $v_{\sigma_i/\tau}$ by $\widetilde{v}_{\sigma_i/\tau}$ we can always assume that $\sum_{i=1}^N \omega_X(\sigma) \cdot v_{\sigma/\tau} = 0 \in \Lambda$.

Definition 1.1.10 (Refinements)
Let (X, ω_X) and (Y, ω_Y) be weighted fans in V. We call (Y, ω_Y) a *refinement* of (X, ω_X) if the following holds:

(a) $Y^* \trianglelefteq X^*$,

(b) $|Y^*| = |X^*|$ and

(c) $\omega_Y(\sigma) = \omega_X(C_{Y^*,X^*}(\sigma))$ for every $\sigma \in (Y^*)^{(\dim(Y))}$.

Note that property (b) implies that either $X^* = Y^* = \emptyset$ or $\dim(X) = \dim(Y)$. We call two weighted fans (X, ω_X) and (Y, ω_Y) in V *equivalent* (write $(X, \omega_X) \sim (Y, \omega_Y)$) if they have a common refinement. Note that $(X, \omega_X)$and $(X^*, \omega_X|_{(X^*)^{(\dim(X))}})$ are always equivalent.

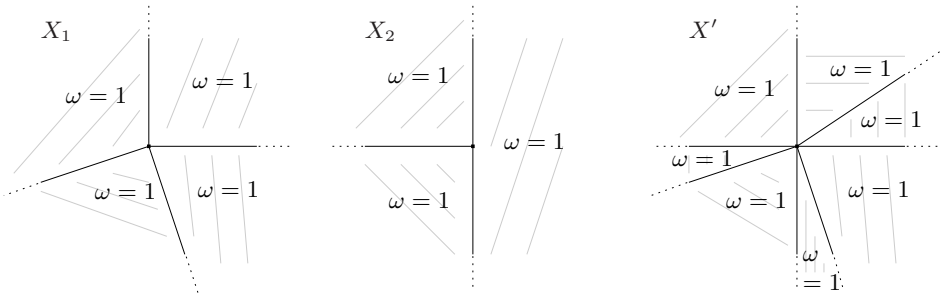


Figure 1.4: Two refinements X_1 and X_2 of $\mathbb{R}^2$ and a common refinement X'.

Remark 1.1.11
Note that for a weighted fan (X, ω_X) of dimension k and a refinement (Y, ω_Y) we have the following two properties:

(a) $|X^*| = |Y^*|$, i.e. the support $|X^*|$ is well-defined on the equivalence class of X,

(b) for every cone $\tau \in Y^{(k-1)}$ there are exactly two cases that can occur: Either we have $\dim C_{Y,X}(\tau) = k$ or we have $\dim C_{Y,X}(\tau) = k-1$. In the first case all cones $\sigma \in Y^{(k)}$ with $\sigma > \tau$ must be contained in $C_{Y,X}(\tau)$. Thus there are precisely two such cones σ_1 and σ_2 with $\omega_Y(\sigma_1) = \omega_Y(\sigma_2)$ and $u_{\sigma_1/\tau} = -u_{\sigma_2/\tau}$. In the second case we have a 1:1 correspondence between cones $\sigma \in Y^{(k)}$ with $\tau < \sigma$ and cones $\sigma' \in X^{(k)}$ with $C_{Y,X}(\tau) < \sigma'$ preserving weights and normal vectors.

Construction 1.1.12 (Refinements)
Let (X, ω_X) be a weighted fan and Y be any fan in V with $|X| \subseteq |Y|$. Let $P := \{\sigma \cap \sigma' | \sigma \in X, \sigma' \in Y\}$. In general P is not a fan in V as can be seen in the following example:

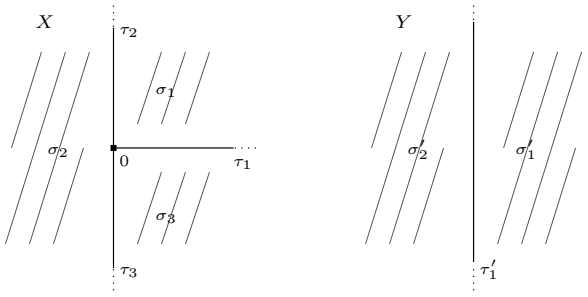


Figure 1.5: Fans X and Y such that $\{\sigma \cap \sigma' | \sigma \in X, \sigma' \in Y\}$ is not a fan.

Here P contains $\tau_1' = \sigma_2 \cap \sigma_1'$, but also $\tau_2 = \sigma_1 \cap \sigma_2'$ and $\tau_3 = \sigma_3 \cap \sigma_2'$. Hence property (b) of definition 1.1.2 is, for instance, not fulfilled for τ_1'. To resolve this, we define

$$X \cap Y := \{\sigma \in P | \nexists\ \tau \in P^{(\dim(\sigma))} \text{ with } \tau \subsetneq \sigma\}.$$

Note that $X \cap Y$ is now a fan in V. We can make it into a weighted fan by setting $\omega_{X\cap Y}(\sigma) := \omega_X(C_{X\cap Y,X}(\sigma))$ for all $\sigma \in (X \cap Y)^{(\dim(X))}$. Then $(X \cap Y, \omega_{X\cap Y})$ is a refinement of (X, ω_X). Note that if (X, ω_X) and (Y, ω_Y) are both weighted fans and

$|X| = |Y|$ we can form both intersections $X \cap Y$ and $Y \cap X$. Of course, the underlying fans are the same in both cases, but the weights may differ since they are always induced by the first fan.

The following setting is a special case of this construction: Let (X, ω_X) be a weighted fan of dimension k in V and let $f \in \Lambda^\vee$ be a non-zero linear form. Then we can construct a refinement of (X, ω_X) as follows:

$$H_f := \{\{x \in V | f(x) \leq 0\}, \{x \in V | f(x) = 0\}, \{x \in V | f(x) \geq 0\}\}$$

is a fan in V with $|H_f| = V$. Thus we have $|X| \subseteq |H_f|$ and by our above construction we get a refinement $(X_f, \omega_{X_f}) := (X \cap H_f, \omega_{X \cap H_f})$ of X.

Obviously we still have to answer the question if the equivalence of weighted fans is indeed an equivalence relation and if this notion of equivalence is well-defined on tropical fans. We will do this in the following lemma:

Lemma 1.1.13
The following two statements hold:

(a) The relation "$\sim$" is an equivalence relation on the set of k-dimensional weighted fans in V.

(b) If (X, ω_X) is a weighted fan of dimension k and (Y, ω_Y) is a refinement then (X, ω_X) is a tropical fan if and only if (Y, ω_Y) is one.

Proof. Recall that a fan and its non-zero part are always equivalent and that a weighted fan X is tropical if and only if its non-zero part X^* is. Thus we may assume that all our fans are reduced and the proof is the same as in [GKM07, section 2]. □

Having done all these preparations we are now able to introduce the most important objects for the succeeding sections:

Definition 1.1.14 (Affine cycles and affine tropical varieties)
Let (X, ω_X) be a tropical fan of dimension k in V. We denote by $[(X, \omega_X)]$ its equivalence class under the equivalence relation "$\sim$" and by $Z_k^{\text{aff}}(V)$ the set of equivalence classes

$$Z_k^{\text{aff}}(V) := \{[(X, \omega_X)] | (X, \omega_X) \text{ tropical fan of dimension } k \text{ in } V\}.$$

The elements of $Z_k^{\text{aff}}(V)$ are called *affine (tropical) k-cycles* in V. A k-cycle $[(X, \omega_X)]$ is called an *affine tropical variety* if $\omega_X(\sigma) \geq 0$ for every $\sigma \in X^{(k)}$. Note that the last property is independent of the choice of the representative of $[(X, \omega_X)]$. Moreover, note that $0 := [\emptyset] \in Z_k^{\text{aff}}(V)$ for every k. We define $|[(X, \omega_X)]| := |X^*|$. This definition is well-defined by remark 1.1.11.

Construction 1.1.15 (Sums of affine cycles)
Let $[(X, \omega_X)]$ and $[(Y, \omega_Y)]$ be k-cycles in V. We would like to form a fan $X + Y$ by taking the union $X \cup Y$, but obviously this collection of cones is in general not a fan. Using appropriate refinements we can resolve this problem: Let $f_1(x) \geq 0, \ldots, f_{N_1}(x) \geq 0$,

$f_{N_1+1}(x) = 0, \ldots, f_N(x) = 0$ and $g_1(x) \geq 0, \ldots, g_{M_1}(x) \geq 0$, $g_{M_1+1}(x) = 0, \ldots, g_M(x) = 0$ be all different equalities and inequalities occurring in the descriptions of all the cones belonging to X and Y, respectively. Using construction 1.1.12 we get refinements

$$\widetilde{X} := X \cap H_{f_1} \cap \cdots \cap H_{f_N} \cap H_{g_1} \cap \cdots \cap H_{g_M}$$

of X and

$$\widetilde{Y} := Y \cap H_{f_1} \cap \cdots \cap H_{f_N} \cap H_{g_1} \cap \cdots \cap H_{g_M}$$

of Y (note that the final refinements do not depend on the order of the single refinements). A cone occurring in $\widetilde{X}$ or $\widetilde{Y}$ is then of the form

$$\sigma = \left\{ \begin{array}{lll} f_i(x) \leq 0, & f_j(x) = 0, & f_k(x) \geq 0, \\ g_{i'}(x) \leq 0, & g_{j'}(x) = 0, & g_{k'}(x) \geq 0 \end{array} \middle| \begin{array}{lll} i \in I, & j \in J, & k \in K, \\ i' \in I', & j' \in J', & k' \in K' \end{array} \right\}$$

for some partitions $I \cup J \cup K = \{1, \ldots, N\}$ and $I' \cup J' \cup K' = \{1, \ldots, M\}$. Now, all these cones σ belong to the fan $H_{f_1} \cap \cdots \cap H_{f_N} \cap H_{g_1} \cap \cdots \cap H_{g_M}$ as well and hence $\widetilde{X} \cup \widetilde{Y}$ fulfills definition 1.1.2. Thus, now we can define the *sum of X and Y* to be $X + Y := \widetilde{X} \cup \widetilde{Y}$ together with weights $\omega_{X+Y}(\sigma) := \omega_{\widetilde{X}}(\sigma) + \omega_{\widetilde{Y}}(\sigma)$ for every facet of $X + Y$ (we set $\omega_{\square}(\sigma) := 0$ if σ does not occur in $\square \in \{\widetilde{X}, \widetilde{Y}\}$). By construction, $(X + Y, \omega_{X+Y})$ is again a tropical fan of dimension k. Moreover, enlarging the sets $\{f_i\}, \{g_j\}$ by more (in)equalities just corresponds to refinements of X and Y and only leads to a refinement of $X + Y$. Thus, replacing the set of relations by another one that also describes the cones in X and Y, or replacing X or Y by refinements keeps the equivalence class $[(X + Y, \omega_{X+Y})]$ unchanged, i.e. taking sums is a well-defined operation on cycles.

This construction immediately leads to the following lemma:

Lemma 1.1.16
$Z_k^{\mathrm{aff}}(V)$ together with the operation "+" from construction 1.1.15 forms an abelian group.

Proof. The class of the empty fan $0 = [\emptyset]$ is the neutral element of this operation and $[(X, -\omega_X)]$ is the inverse element of $[(X, \omega_X)] \in Z_k^{\mathrm{aff}}(V)$. □

Of course we do not want to restrict ourselves to cycles situated in some $\mathbb{R}^n$. Therefore we give the following generalization of definition 1.1.14:

Definition 1.1.17
Let X be a fan in V. An *affine k-cycle in X* is an element $[(Y, \omega_Y)]$ of $Z_k^{\mathrm{aff}}(V)$ such that $|Y^*| \subseteq |X|$. We denote by $Z_k^{\mathrm{aff}}(X)$ the set of k-cycles in X. Note that $\left(Z_k^{\mathrm{aff}}(X), +\right)$ is a subgroup of $(Z_k^{\mathrm{aff}}(V), +)$. The elements of the group $Z_{\dim X - 1}^{\mathrm{aff}}(X)$ are called *Weil divisors* on X. If $[(X, \omega_X)]$ is a cycle in V then $Z_k^{\mathrm{aff}}([(X, \omega_X)]) := Z_k^{\mathrm{aff}}(X^*)$.

1.2 Affine Cartier divisors and their associated Weil divisors

Definition 1.2.1 (Rational functions)
Let C be an affine k-cycle. A *(non-zero) rational function on C* is a continuous piecewise linear function $\varphi : |C| \to \mathbb{R}$, i.e. there exists a representative (X, ω_X) of C such that on each cone $\sigma \in X$ the map φ is the restriction of an integer affine linear function $\varphi|_\sigma = \lambda + c$, $\lambda \in \Lambda_\sigma^\vee$, $c \in \mathbb{R}$. Obviously, c is the same on all faces by $c = \varphi(0)$ and λ is uniquely determined by φ and therefore denoted by $\varphi_\sigma := \lambda$.
The *set of (non-zero) rational functions of C* is denoted by $\mathcal{K}^*(C)$.

Remark 1.2.2 (The zero function and restrictions to subcycles)
The "zero" function can be thought of being the constant function $-\infty$, therefore $\mathcal{K}(C) := \mathcal{K}^*(C) \cup \{-\infty\}$. With respect to the operations max and $+$, $\mathcal{K}(C)$ is a semifield.
Let us note an important difference to the classical case: Let D be an arbitrary subcycle of C and $\varphi \in \mathcal{K}^*(C)$. Then $\varphi|_{|D|} \in \mathcal{K}^*(D)$, whereas in the classical case it might become zero. This will be crucial for defining intersection products not only modulo rational equivalence.

As in the classical case, each non-zero rational function φ on C defines a Weil divisor, i.e. a cycle in $Z^{\mathrm{aff}}_{\dim C-1}(C)$. The idea of course should be to describe the "zeros" and "poles" of φ. A naive approach could be to consider the graph of φ in $V \times \mathbb{R}$ and "intersect it with $V \times \{-\infty\}$ and $V \times \{+\infty\}$". However, our function φ takes values only in $\mathbb{R}$, in fact. On the other hand, the graph of φ is not a tropical object as it is not balanced: Where φ is not linear, our graph gets edges that might violate the balancing condition. So, we first make the graph balanced by adding new faces in the additional direction $(0, -1) \in V \times \mathbb{R}$ and then apply our naive approach. Let us make this precise.

Construction 1.2.3 (The associated Weil divisor)
Let C be an affine k-cycle in $V = \Lambda \otimes \mathbb{R}$ and $\varphi \in \mathcal{K}^*(C)$ a rational function on C. Let furthermore (X, ω) be a representative of C on whose faces φ is affine linear. Therefore, for each cone $\sigma \in X$, we get a cone $\tilde{\sigma} := (\mathrm{id} \times \varphi_\sigma)(\sigma)$ in $V \times \mathbb{R}$ of the same dimension. Obviously, $\Gamma_\varphi := \{\tilde{\sigma} | \sigma \in X\}$ forms a fan which we can make into a weighted fan $(\Gamma_\varphi, \tilde{\omega})$ by $\tilde{\omega}(\tilde{\sigma}) := \omega(\sigma)$. Its support is just the set-theoretic graph of $\varphi - \varphi(0)$ in $|X| \times \mathbb{R}$.
For $\tau < \sigma$ with $\dim(\tau) = \dim(\sigma) - 1$ let $v_{\sigma/\tau} \in \Lambda$ be a representative of the normal vector $u_{\sigma/\tau}$. Then, $\big(v_{\sigma/\tau}, \varphi_\sigma(v_{\sigma/\tau})\big) \in \Lambda \times \mathbb{Z}$ is a representative of the normal vector $u_{\tilde{\sigma}/\tilde{\tau}}$. Therefore, summing around a cone $\tilde{\tau}$ with $\dim \tilde{\tau} = \dim \tau = k - 1$, we get

$$\sum_{\substack{\tilde{\sigma} \in \Gamma_\varphi^{(k)} \\ \tilde{\tau} < \tilde{\sigma}}} \tilde{\omega}(\tilde{\sigma}) \left(v_{\sigma/\tau}, \varphi_\sigma(v_{\sigma/\tau})\right) = \left(\sum_{\substack{\sigma \in X^{(k)} \\ \tau < \sigma}} \omega(\sigma) v_{\sigma/\tau}, \sum_{\substack{\sigma \in X^{(k)} \\ \tau < \sigma}} \varphi_\sigma(\omega(\sigma) v_{\sigma/\tau}) \right).$$

From the balancing condition for (X, ω) it follows that $\sum_{\sigma \in X^{(k)} : \tau < \sigma} \omega(\sigma) v_{\sigma/\tau} \in V_\tau$, which also means $\left(\sum_{\sigma \in X^{(k)} : \tau < \sigma} \omega(\sigma) v_{\sigma/\tau}, \varphi_\tau \left(\sum_{\sigma \in X^{(k)} : \tau < \sigma} \omega(\sigma) v_{\sigma/\tau}\right)\right) \in V_{\tilde{\tau}}$. Therefore,

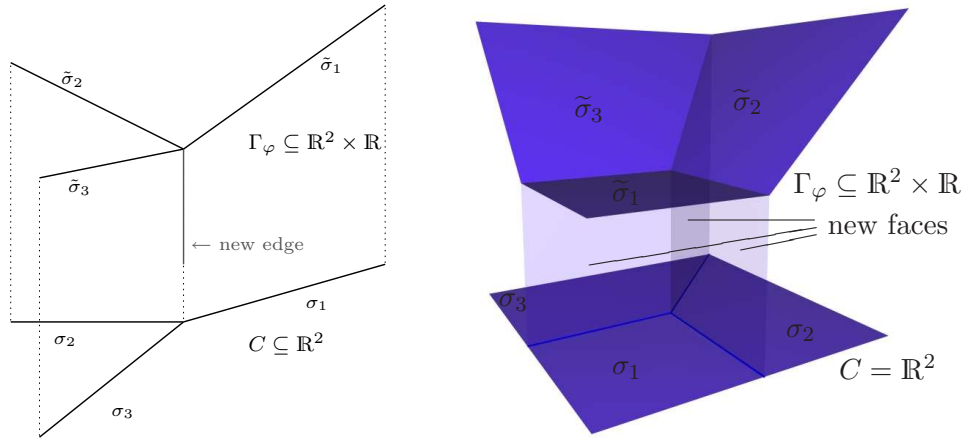


Figure 1.6: Two examples of the construction of a Weil divisor.

modulo $V_{\tilde{\tau}}$, our first sum equals

$$\left(0, \sum_{\substack{\sigma \in X^{(k)} \\ \tau < \sigma}} \varphi_\sigma(\omega(\sigma) v_{\sigma/\tau}) - \varphi_\tau\Big(\sum_{\substack{\sigma \in X^{(k)} \\ \tau < \sigma}} \omega(\sigma) v_{\sigma/\tau}\Big)\right) \in V \times \mathbb{R}.$$

So, in order to "make $(\Gamma_\varphi, \tilde{\omega})$ balanced at $\tilde{\tau}$", we add the cone $\vartheta := \tilde{\tau} + (\{0\} \times \mathbb{R}_{\leq 0})$ with weight $\tilde{\omega}(\vartheta) = \sum_{\sigma \in X^{(k)}: \tau < \sigma} \varphi_\sigma(\omega(\sigma) v_{\sigma/\tau}) - \varphi_\tau\Big(\sum_{\sigma \in X^{(k)}: \tau < \sigma} \omega(\sigma) v_{\sigma/\tau}\Big)$. As obviously $[(0,-1)] = u_{\vartheta/\tilde{\tau}} \in (V \times \mathbb{R})/V_{\tilde{\tau}}$, the above calculation shows that then the balancing condition around $\tilde{\tau}$ holds. In other words, we build the new fan $(\Gamma'_\varphi, \tilde{\omega}')$, where

$$\begin{aligned}
\Gamma'_\varphi &:= \Gamma_\varphi \cup \left\{\tilde{\tau} + (\{0\} \times \mathbb{R}_{\leq 0}) | \tilde{\tau} \in \Gamma_\varphi \setminus \Gamma_\varphi^{(k)}\right\}, \\
\tilde{\omega}'|_{\Gamma_\varphi^{(k)}} &:= \tilde{\omega}, \\
\tilde{\omega}'(\tilde{\tau} + (\{0\} \times \mathbb{R}_{\leq 0})) &:= \sum_{\substack{\sigma \in X^{(k)} \\ \tau < \sigma}} \varphi_\sigma(\omega(\sigma) v_{\sigma/\tau}) - \varphi_\tau\Big(\sum_{\substack{\sigma \in X^{(k)} \\ \tau < \sigma}} \omega(\sigma) v_{\sigma/\tau}\Big) \\
&\qquad \text{if } \dim \tilde{\tau} = k-1.
\end{aligned}$$

This fan is balanced around all $\tilde{\tau} \in \Gamma_\varphi^{(k-1)}$. We will show that it is also balanced at all "new" cones of dimension $k-1$ in proposition 1.2.7.
We now think of intersecting this new fan with $V \times \{-\infty\}$ to get our desired Weil divisor (as our weights are allowed to be negative, we can forget about intersecting also with $V \times \{+\infty\}$). This construction leads to the following definition.

Definition 1.2.4 (Associated Weil divisors)
Let C be an affine k-cycle in $V = \Lambda \otimes \mathbb{R}$ and $\varphi \in \mathcal{K}^*(C)$ a rational function on C. Let furthermore (X, ω) be a representative of C on whose cones φ is affine linear. We

define $\operatorname{div}(\varphi) := \varphi \cdot C := [(\bigcup_{i=0}^{k-1} X^{(i)}, \omega_\varphi)] \in Z^{\mathrm{aff}}_{k-1}(C)$, where

$$\begin{aligned} \omega_\varphi : X^{(k-1)} &\to \mathbb{Z}, \\ \tau &\mapsto \sum_{\substack{\sigma \in X^{(k)} \\ \tau < \sigma}} \varphi_\sigma(\omega(\sigma) v_{\sigma/\tau}) - \varphi_\tau\Big(\sum_{\substack{\sigma \in X^{(k)} \\ \tau < \sigma}} \omega(\sigma) v_{\sigma/\tau} \Big) \end{aligned}$$

and the $v_{\sigma/\tau}$ are arbitrary representatives of the normal vectors $u_{\sigma/\tau}$.
Let D be an arbitrary subcycle of C. By remark 1.2.2, we can define $\varphi \cdot D := \varphi|_{|D|} \cdot D$.

Remark 1.2.5
Obviously, $\omega_\varphi(\tau)$ is independent of the choice of the $v_{\sigma/\tau}$, as a different choice only differs by elements in V_τ.
Our definition does also not depend on the choice of a representative (X, ω): Let (Y, v) be a refinement of (X, ω). For $\tau \in Y^{(k-1)}$, two cases can occur (see also remark 1.1.11): Let $\tau' := C_{Y,X}(\tau)$. If $\dim \tau' = k$, there are precisely two cones at $\tau < \sigma_1, \sigma_2 \in Y^{(k)}$, which then fulfill $C_{Y,X}(\sigma_1) = C_{Y,X}(\sigma_2)$ and therefore $u_{\sigma_1/\tau} = -u_{\sigma_2/\tau}$, $v(\sigma_1) = v(\sigma_2)$ and $\varphi_{\sigma_1} = \varphi_{\sigma_2}$. It follows that $v_\varphi(\tau) = 0$. If $\dim \tau' = k-1$, $C_{Y,X}$ gives a one-to-one correspondence between $\{\sigma \in Y^{(k)} | \tau < \sigma\}$ and $\{\sigma' \in X^{(k)} | \tau' < \sigma'\}$ respecting weights and normal vectors, and we have $\varphi_\sigma = \varphi_{C_{Y,X}(\sigma)}$. It follows that $v_\varphi(\tau) = \omega_\varphi(\tau')$. So the two weighted fans we obtain are equivalent.

Remark 1.2.6 (Affine linear functions and sums)
Let $\varphi \in \mathcal{K}^*(C)$ be globally affine linear, i.e. $\varphi = \lambda|_{|C|} + c$ for some $\lambda \in \Lambda^\vee$, $c \in \mathbb{R}$. Then obviously $\varphi \cdot C = 0$.
Let furthermore $\psi \in \mathcal{K}^*(C)$ be another rational function on C. As $\varphi_\sigma + \psi_\sigma = (\varphi + \psi)_\sigma$ we can conclude that $(\varphi + \psi) \cdot C = \varphi \cdot C + \psi \cdot C$.

Proposition 1.2.7 (Balancing Condition and Commutativity)
The Weil divisor associated to a Cartier divisor as in definition 1.2.4 has the following properties:

(a) Let C be an affine k-cycle in $V = \Lambda \otimes \mathbb{R}$ and $\varphi \in \mathcal{K}^*(C)$ a rational function on C. Then $\operatorname{div}(\varphi) = \varphi \cdot C$ is an equivalence class of tropical fans, i.e. its representatives are balanced.

(b) Let $\psi \in \mathcal{K}^*(C)$ be another rational function on C. Then $\psi \cdot (\varphi \cdot C) = \varphi \cdot (\psi \cdot C)$ holds.

The proof of this statement is to a large extent the work of Johannes Rau, my coauthor of [AR07] and [AR08]. Hence we skip it here and refer to [AR07, proposition 3.7] instead.

Definition 1.2.8 (Affine Cartier divisors)
Let C be an affine k-cycle. The subgroup of globally affine linear functions in $\mathcal{K}^*(C)$ with respect to $+$ is denoted by $\mathcal{O}^*(C)$. We define the *group of affine Cartier divisors of C* to be the quotient group $\operatorname{Div}(C) := \mathcal{K}^*(C)/\mathcal{O}^*(C)$.

Let $[\varphi] \in \mathrm{Div}(C)$ be a Cartier divisor. By remark 1.2.6, the associated Weil divisor $\mathrm{div}([\varphi]) := \mathrm{div}(\varphi)$ is well-defined. We therefore get a bilinear mapping

$$\begin{aligned} \cdot : \mathrm{Div}(C) \times Z_k^{\mathrm{aff}}(C) &\to Z_{k-1}^{\mathrm{aff}}(C) \\ ([\varphi], D) &\mapsto [\varphi] \cdot D = \varphi \cdot D, \end{aligned}$$

called *affine intersection product*.

Example 1.2.9 (Self-intersection of hyperplanes)
Let $\Lambda = \mathbb{Z}^n$ (and thus $V = \mathbb{R}^n$), let $e_1, \ldots, e_n$ be the standard basis vectors in $\mathbb{Z}^n$ and $e_0 := -e_1 - \cdots - e_n$. By abuse of notation our ambient cycle is $\mathbb{R}^n := [(\{\mathbb{R}^n\}, \omega(\mathbb{R}^n) = 1)]$. Let us consider the "linear tropical polynomial"

$$h = x_1 \oplus \cdots \oplus x_n \oplus 0 = \max\{x_1, \ldots, x_n, 0\} : \mathbb{R}^n \to \mathbb{R}.$$

Obviously, h is a rational function in the sense of definition 1.2.1: For each subset $I \subsetneq \{0, 1, \ldots, n\}$ we denote by σ_I the simplicial cone of dimension $|I|$ generated by the vectors $-e_i$ for $i \in I$. Then h is integer linear on all σ_I, namely

$$h|_{\sigma_I}(x_1, \ldots, x_n) = \begin{cases} 0 & \text{if } 0 \notin I, \\ x_i & \text{if there exists an } i \in \{1, \ldots, n\} \setminus I. \end{cases}$$

Let L_k^n be the k-dimensional fan consisting of all cones σ_I with $|I| \leq k$ and weighted with the trivial weight function $\omega_{L_k^n}$. Then L_n^n is a representative of $\mathbb{R}^n$ fulfilling the conditions of definition 1.2.1. We want to show

$$\underbrace{h \cdot \ldots \cdot h}_{k \text{ times}} \cdot \mathbb{R}^n = [L_{n-k}^n]. \qquad (*)$$

This follows inductively from $h \cdot [L_{k+1}^n] = [L_k^n]$, so it remains to compute $\omega_{L_{k+1}^n, h}(\sigma_I)$ for all I with $|I| = k < n$. Let $J := \{0, 1, \ldots, n\} \setminus I$. Obviously, the $(k+1)$-dimensional cones of L_{k+1}^n containing σ_I are precisely the cones $\sigma_{I \cup \{j\}}, j \in J$. Moreover, $-e_j$ is a representative of the normal vector $u_{\sigma_{I\cup\{j\}}/\sigma_I}$. Note also that for all $i \in I', I' \subsetneq \{0, 1, \ldots, n\}$ we have $h_{\sigma_{I'}}(-e_i) = h|_{\sigma_{I'}}(-e_i) = h(-e_i)$. Hence we compute

$$\begin{aligned} \omega_{L_{k+1}^n, h}(\sigma_I) &= \sum_{j \in J} \underbrace{\omega_{L_{k+1}^n}(\sigma_{I\cup\{j\}})}_{=1} h_{\sigma_{I\cup\{j\}}}(-e_j) \\ &\quad + h_{\sigma_I}\Bigg(\underbrace{\sum_{j \in J} \underbrace{\omega_{L_{k+1}^n}(\sigma_{I\cup\{j\}})}_{=1} e_j}_{=-\sum_{i\in I} e_i} \Bigg) \\ &= \sum_{j \in J} h(-e_j) + \sum_{i \in I} h(-e_i) \\ &= h(-e_0) + h(-e_1) + \cdots + h(-e_n) \\ &= 1 + 0 + \cdots + 0 = 1 = \omega_{L_k^n}(\sigma_I), \end{aligned}$$

which implies $h \cdot [L_{k+1}^n] = [L_k^n]$ and also equation $(*)$.
We can summarize this example as follows: Firstly, for a tropical polynomial f, the

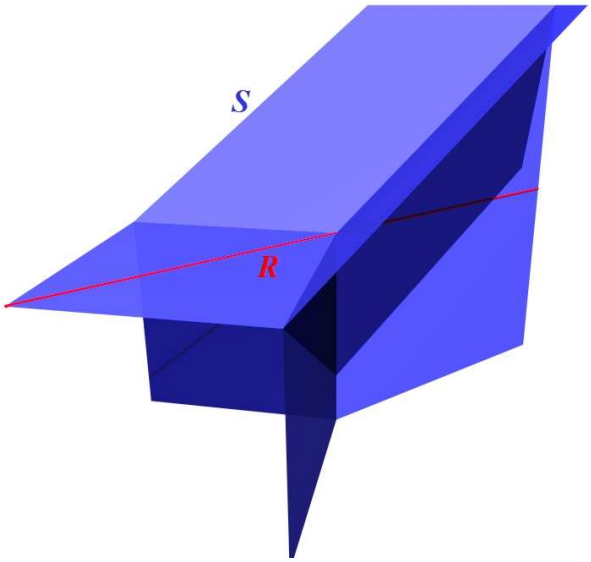


Figure 1.7: The rigid curve R in S.

associated Weil divisor $f \cdot \mathbb{R}^n$ coincides with the locus of non-differentiability $\mathcal{T}(f)$ of f (see [RGST05, section 3]), and secondly, "the k-fold self-intersection of a tropical hyperplane in $\mathbb{R}^n$" is given by its $(n-k)$-skeleton together with trivial weights all equal to 1.

Example 1.2.10 (A rigid curve)
Using notations from example 1.2.9, we consider as ambient cycle the surface $S := [L_2^3] = \max\{x_1, x_2, x_3, 0\} \cdot \mathbb{R}^3$. In this surface, we want to show that the curve $R := [(\mathbb{R} \cdot e_R, \omega_R(\mathbb{R} \cdot e_R) = 1)] \in Z_1^{\text{aff}}(S)$, where $e_R := e_1 + e_2$, has negative self-intersection in the following sense: We construct a rational function φ on S whose associated Weil divisor is R and show that $\varphi \cdot R = \varphi \cdot \varphi \cdot S$ is just the origin with weight -1. This curve and its rigidness were first discussed in [M06, Example 4.11, Example 5.9].
Let us construct φ: First we refine L_2^3 to L_R by replacing $\sigma_{\{1,2\}}$ and $\sigma_{\{0,3\}}$ with $\sigma_{\{1,R\}}$, $\sigma_{\{R\}}$, $\sigma_{\{R,2\}}$, $\sigma_{\{0,-R\}}$, $\sigma_{\{-R\}}$ and $\sigma_{\{-R,3\}}$ (using again the notations from example 1.2.9 and $e_{-R} := -e_R = e_0 + e_3$). We define $\varphi : |S| \to \mathbb{R}$ to be the unique function that is linear on the faces of L_R and fulfills

$$0, -e_1, -e_2, -e_3, -e_{-R} \mapsto 0, \quad -e_0 \mapsto 1 \text{ and } \quad -e_R \mapsto -1.$$

Analogous to 1.2.9, we can compute for $i = 1, 2$

$$\omega_{L_R,\varphi}(\sigma_{\{i\}}) = \varphi(-e_0) + \varphi(-e_3) + \varphi(-e_R) = 1 + 0 - 1 = 0,$$

for $i = 0, 3$

$$\omega_{L_R,\varphi}(\sigma_{\{i\}}) = \varphi(-e_1) + \varphi(-e_2) + \varphi(-e_{-R}) = 0 + 0 + 0 = 0,$$

and finally

$$\omega_{L_R,\varphi}(\sigma_{\{R\}}) = \varphi(-e_1) + \varphi(-e_2) - \varphi(-e_R) = 0 + 0 - (-1) = 1,$$
$$\omega_{L_R,\varphi}(\sigma_{\{-R\}}) = \varphi(-e_0) + \varphi(-e_3) - \varphi(-e_{-R}) = 1 + 0 + 0 = 1,$$

which means $\varphi \cdot S = R$. Now we can easily compute $\varphi \cdot \varphi \cdot S = \varphi \cdot R$ on the representative $\{\sigma_{\{R\}}, \sigma_{\{-R\}}, \{0\}\}$ (with trivial weights) of R:

$$\omega_{R,\varphi}(\{0\}) = \varphi(-e_R) + \varphi(-e_{-R}) = -1 + 0 = -1.$$

Therefore $\varphi \cdot \varphi \cdot S = [(\{0\}, \omega(\{0\}) = -1)]$. Note that we really obtain a cycle with negative weight, not only a cycle class modulo rational equivalence as it is the case in "classical" algebraic geometry.

1.3 Push-forward of affine cycles and pull-back of Cartier divisors

The aim of this section is to construct push-forwards of cycles and pull-backs of Cartier divisors along morphisms of fans and to study the interaction of both constructions. To do this we first of all have to introduce the notion of morphism:

Definition 1.3.1 (Morphisms of fans)
Let X be a fan in $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ and Y be a fan in $V' = \Lambda' \otimes_{\mathbb{Z}} \mathbb{R}$. A *morphism* $f : X \to Y$ is a $\mathbb{Z}$-linear map, i.e. a map from $|X| \subseteq V$ to $|Y| \subseteq V'$ induced by a $\mathbb{Z}$-linear map $\widetilde{f} : \Lambda \to \Lambda'$. By abuse of notation we will usually denote all three maps $f, \widetilde{f}$ and $\widetilde{f} \otimes_{\mathbb{Z}} \mathrm{id}$ by the same letter f (note that the last two maps are in general not uniquely determined by $f : X \to Y$). A morphism of weighted fans is a morphism of fans. A morphism of affine cycles $f : [(X, \omega_X)] \to [(Y, \omega_Y)]$ is a morphism of fans $f : X^* \to Y^*$. Note that in this latter case the notion of morphism does not depend on the choice of the representatives by remark 1.1.11.

Given such a morphism the following construction shows how to build the push-forward fan of a given fan. Afterwards we will show that this construction induces a well-defined operation on cycles.

Construction 1.3.2
We refer to [GKM07, section 2] for more details on the following construction. Let (X, ω_X) be a weighted fan of pure dimension n in $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, let Y be any fan in $V' = \Lambda' \otimes_{\mathbb{Z}} \mathbb{R}$ and let $f : X \to Y$ be a morphism. Passing to an appropriate refinement of (X, ω_X) the collection of cones

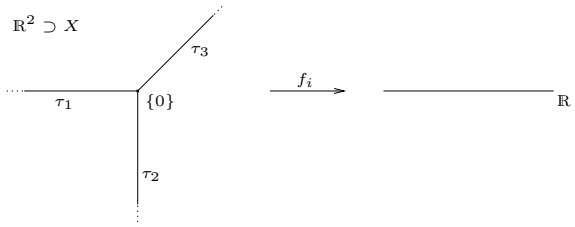
$$f_*X := \{f(\sigma) | \sigma \in X \text{ contained in a maximal cone of } X \text{ on which } f \text{ is injective}\}$$

is a fan in V' of pure dimension n. It can be made into a weighted fan by setting

$$\omega_{f_*X}(\sigma') := \sum_{\sigma \in X^{(n)} : f(\sigma) = \sigma'} \omega_X(\sigma) \cdot |\Lambda'_{\sigma'} / f(\Lambda_\sigma)|$$

for all $\sigma' \in f_*X^{(n)}$. The equivalence class of this weighted fan only depends on the equivalence class of (X, ω_X).

Example 1.3.3
Let X be the fan with cones $\tau_1, \tau_2, \tau_3, \{0\}$ as shown in the figure



and let $\omega_X(\tau_i) = 1$ for $i = 1, 2, 3$. Moreover, let $Y := \mathbb{R}$ be the real line and the morphisms $f_1, f_2 : X \to Y$ be given by $f_1(x, y) := x + y$ and $f_2(x, y) := x$, respectively. Then $(f_1)_*X = (f_2)_*X = \{\{x \leq 0\}, \{0\}, \{x \geq 0\}\}$, but $\omega_{(f_1)_*X}(\{x \leq 0\}) = \omega_{(f_1)_*X}(\{x \geq 0\}) = 2$ and $\omega_{(f_2)_*X}(\{x \leq 0\}) = \omega_{(f_2)_*X}(\{x \geq 0\}) = 1$.

Proposition 1.3.4
Let (X, ω_X) be a tropical fan of dimension n in $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, let Y be any fan in $V' = \Lambda' \otimes_{\mathbb{Z}} \mathbb{R}$ and let $f : X \to Y$ be a morphism. Then f_*X is a tropical fan of dimension n.

Proof. A proof can be found in [GKM07, section 2]. □

By construction 1.3.2 and proposition 1.3.4 the following definition is well-defined:

Definition 1.3.5 (Push-forward of cycles)
Let $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ and $V' = \Lambda' \otimes_{\mathbb{Z}} \mathbb{R}$. Moreover, let $X \in Z_m^{\text{aff}}(V)$, $Y \in Z_n^{\text{aff}}(V')$ and $f : X \to Y$ be a morphism. For $[(Z, \omega_Z)] \in Z_k^{\text{aff}}(X)$ we define

$$f_*[(Z, \omega_Z)] := [(f_*(Z^*), \omega_{f_*(Z^*)})] \in Z_k^{\text{aff}}(Y).$$

Proposition 1.3.6 (Push-forward of cycles)
Let $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ and $V' = \Lambda' \otimes_{\mathbb{Z}} \mathbb{R}$. Let $X \in Z_m^{\text{aff}}(V)$ and $Y \in Z_n^{\text{aff}}(V')$ be cycles and let $f : X \to Y$ be a morphism. Then the map

$$Z_k^{\text{aff}}(X) \longrightarrow Z_k^{\text{aff}}(Y) : C \longmapsto f_*C$$

is well-defined and $\mathbb{Z}$-linear.

Proof. It remains to prove the linearity: Let (A, ω_A) and (B, ω_B) be two tropical fans of dimension k with $A = A^*$, $B = B^*$ and $|A|, |B| \subseteq |X^*|$. We want to show that $f_*(A + B) \sim f_*A + f_*B$. Refining A and B as in construction 1.1.15 we may assume that $A, B \subseteq A + B$. Set $\widetilde{A} := A + B$ and

$$\omega_{\widetilde{A}}(\sigma) := \begin{cases} \omega_A(\sigma), & \text{if } \sigma \in A \\ 0, & \text{else} \end{cases}$$

for all facets $\sigma \in \widetilde{A}$. Analogously, set $\widetilde{B} := A + B$ with according weights. Then $\widetilde{A} \sim A$ and $\widetilde{B} \sim B$. Carrying out a further refinement of $A + B$ like in construction 1.3.2 we can reach that $f_*(A + B) = \{f(\sigma) | \sigma \in A + B$ contained in a maximal cone of $A + B$

on which f is injective}. Using $\widetilde{A} = \widetilde{B} = \widetilde{A} + \widetilde{B} = A + B$ we get $f_*\widetilde{A} = f_*\widetilde{B} = f_*(\widetilde{A} + \widetilde{B}) = f_*(A + B)$ and it remains to compare the weights:

$$\begin{aligned}
\omega_{f_*(\widetilde{A}+\widetilde{B})}(\sigma') &= \sum_{\sigma\in(\widetilde{A}+\widetilde{B})^{(k)}:f(\sigma)=\sigma'} \omega_{\widetilde{A}+\widetilde{B}}(\sigma)\cdot|\Lambda'_{\sigma'}/f(\Lambda_\sigma)| \\
&= \sum_{\sigma\in(\widetilde{A}+\widetilde{B})^{(k)}:f(\sigma)=\sigma'} \left[\omega_{\widetilde{A}}(\sigma)+\omega_{\widetilde{B}}(\sigma)\right]\cdot|\Lambda'_{\sigma'}/f(\Lambda_\sigma)| \\
&= \sum_{\sigma\in\widetilde{A}^{(k)}:f(\sigma)=\sigma'} \omega_{\widetilde{A}}(\sigma)\cdot|\Lambda'_{\sigma'}/f(\Lambda_\sigma)| + \\
&\quad \sum_{\sigma\in\widetilde{B}^{(k)}:f(\sigma)=\sigma'} \omega_{\widetilde{B}}(\sigma)\cdot|\Lambda'_{\sigma'}/f(\Lambda_\sigma)| \\
&= \omega_{f_*\widetilde{A}}(\sigma') + \omega_{f_*\widetilde{B}}(\sigma')
\end{aligned}$$

for all facets σ' of $f_*(A+B)$. Hence $f_*(A+B) \sim f_*(\widetilde{A}+\widetilde{B}) = f_*\widetilde{A} + f_*\widetilde{B} \sim f_*A + f_*B$ as weighted fans. □

Our next step is now to define the pull-back of a Cartier divisor. As promised, our next step will then be to prove a projection formula that describes the interaction between our two constructions.

Proposition 1.3.7 (Pull-back of Cartier divisors)
Let $C \in Z_m^{\mathrm{aff}}(V)$ and $D \in Z_n^{\mathrm{aff}}(V')$ be cycles in $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ and $V' = \Lambda' \otimes_{\mathbb{Z}} \mathbb{R}$, respectively, and let $f : C \to D$ be a morphism. Then there is a well-defined and $\mathbb{Z}$-linear map

$$\mathrm{Div}(D) \longrightarrow \mathrm{Div}(C) : [h] \longmapsto f^*[h] := [h \circ f].$$

Proof. The map $h \mapsto h \circ f$ is obviously $\mathbb{Z}$-linear on rational functions and maps affine linear functions to affine linear functions. Thus it remains to prove that $h \circ f$ is a rational function if h is one: Therefore let (X, ω_X) be any representative of C, let (Y, ω_Y) be a reduced representative of D such that the restriction of h to every cone in Y is affine linear and let $f_V : V \to V'$ be a $\mathbb{Z}$-linear map such that $f_V|_{|C|} = f$. Since $Z := \{f_V^{-1}(\sigma')|\sigma' \in Y\}$ is a fan in V and $|X| \subseteq |Z|$ we can construct the refinement $\widetilde{X} := X \cap Z$ of X such that $h \circ f$ is affine linear on every cone of $\widetilde{X}$. This finishes the proof. □

Proposition 1.3.8 (Projection formula)
Let $C \in Z_m^{\mathrm{aff}}(V)$ and $D \in Z_n^{\mathrm{aff}}(V')$ be cycles in $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ and $V' = \Lambda' \otimes_{\mathbb{Z}} \mathbb{R}$ respectively and let $f : C \to D$ be a morphism. Let $E \in Z_k^{\mathrm{aff}}(C)$ be a cycle and let $\varphi \in \mathrm{Div}(D)$ be a Cartier divisor. Then the following equation holds:

$$\varphi \cdot (f_*E) = f_*(f^*\varphi \cdot E) \in Z_{k-1}^{\mathrm{aff}}(D).$$

Proof. Let $E = [(Z, \omega_Z)]$ and $\varphi = [h]$. We may assume that $Z = Z^*$ and $h(0) = 0$. Replacing Z by a refinement we may additionally assume that f^*h is linear on every cone of Z (cf. definition 1.2.1) and that

$$f_*Z = \{f(\sigma)|\sigma \in Z \text{ contained in a maximal cone of } Z \text{ on which } f \text{ is injective}\}$$

(cf. construction 1.3.2). Note that in this case h is linear on the cones of f_*Z, too. Let $\sigma' \subseteq |D|$ be a cone (not necessarily $\sigma' \in f_*Z$) such that h is linear on σ'. Then there is a unique linear map $h_{\sigma'} : V'_{\sigma'} \to \mathbb{R}$ induced by the restriction $h|_{\sigma'}$. Analogously for $f^*h_\sigma, \sigma \subseteq |C|$. For cones $\tau < \sigma \in Z$ of dimension $k-1$ and k respectively let $v_{\sigma/\tau} \in \Lambda$ be a representative of the primitive normal vector $u_{\sigma/\tau} \in \Lambda_\sigma/\Lambda_\tau$ of construction 1.1.4. Analogously, for $\tau' < \sigma' \in f_*Z$ of dimension $k-1$ and k respectively let $v_{\sigma'/\tau'}$ be a representative of $u_{\sigma'/\tau'} \in \Lambda'_{\sigma'}/\Lambda'_{\tau'}$. Now we want to compare the weighted fans $h \cdot (f_*Z)$ and $f_*(f^*h \cdot Z)$: Let $\tau' \in f_*Z$ be a cone of dimension $k-1$. Then we can calculate the weight of τ' in $h \cdot (f_*Z)$ as follows:

$$\begin{aligned}
\omega_{h\cdot(f_*Z)}(\tau') &= \left(\sum_{\sigma' \in f_*Z : \sigma' > \tau'} \omega_{f_*Z}(\sigma') \cdot h_{\sigma'}(v_{\sigma'/\tau'}) \right) \\
&\quad - h_{\tau'} \left(\sum_{\sigma' \in f_*Z : \sigma' > \tau'} \omega_{f_*Z}(\sigma') \cdot v_{\sigma'/\tau'} \right) \\
&= \left(\sum_{\sigma' \in f_*Z : \sigma' > \tau'} \left(\sum_{\sigma \in Z^{(k)} : f(\sigma) = \sigma'} \omega_Z(\sigma) \cdot |\Lambda'_{\sigma'}/f(\Lambda_\sigma)| \right) \cdot h_{\sigma'}(v_{\sigma'/\tau'}) \right) \\
&\quad - h_{\tau'} \left(\sum_{\sigma' \in f_*Z : \sigma' > \tau'} \left(\sum_{\sigma \in Z^{(k)} : f(\sigma) = \sigma'} \omega_Z(\sigma) \cdot |\Lambda'_{\sigma'}/f(\Lambda_\sigma)| \right) \cdot v_{\sigma'/\tau'} \right) \\
&= \left(\sum_{\sigma \in Z^{(k)} : f(\sigma) > \tau'} \omega_Z(\sigma) \cdot |\Lambda'_{f(\sigma)}/f(\Lambda_\sigma)| \cdot h_{f(\sigma)}(v_{f(\sigma)/\tau'}) \right) \\
&\quad - h_{\tau'} \left(\sum_{\sigma \in Z^{(k)} : f(\sigma) > \tau'} \omega_Z(\sigma) \cdot |\Lambda'_{f(\sigma)}/f(\Lambda_\sigma)| \cdot v_{f(\sigma)/\tau'} \right)
\end{aligned}$$

Now let $\tau' \in f_*(f^*h \cdot Z)$ of dimension $k-1$. The weight of τ' in $f_*(f^*h \cdot Z)$ can be calculated as follows:

$$\begin{aligned}
\omega_{f_*(f^*h\cdot Z)}(\tau') &= \sum_{\substack{\tau \in (f^*h\cdot Z)^{(k-1)} : \\ f(\tau) = \tau'}} \omega_{f^*h\cdot Z}(\tau) \cdot |\Lambda'_{\tau'}/f(\Lambda_\tau)| \\
&= \sum_{\substack{\tau \in (f^*h\cdot Z)^{(k-1)} : \\ f(\tau) = \tau'}} \left(\sum_{\sigma \in Z^{(k)} : \sigma > \tau} \omega_Z(\sigma) f^*h_\sigma(v_{\sigma/\tau}) \right. \\
&\quad \left. - f^*h_\tau \left(\sum_{\sigma \in Z^{(k)} : \sigma > \tau} \omega_Z(\sigma) \cdot v_{\sigma/\tau} \right) \right) \cdot |\Lambda'_{\tau'}/f(\Lambda_\tau)|
\end{aligned}$$

$$= \sum_{\substack{\tau\in(f^*h\cdot Z)^{(k-1)}:\\ f(\tau)=\tau'}} \left(\sum_{\sigma\in Z^{(k)}:\sigma>\tau} \omega_Z(\sigma) h_{f(\sigma)}(f(v_{\sigma/\tau})) - h_{f(\tau)}\left(\sum_{\sigma\in Z^{(k)}:\sigma>\tau} \omega_Z(\sigma)\cdot f\left(v_{\sigma/\tau}\right)\right)\right)\cdot |\Lambda'_{\tau'}/f(\Lambda_\tau)|.$$

Note that $f(v_{\sigma/\tau}) = |\Lambda'_{\sigma'}/(\Lambda'_{\tau'} + \mathbb{Z}f(v_{\sigma/\tau}))|\cdot v_{\sigma'/\tau'} + \lambda_{\sigma,\tau} \in \Lambda'$ for some $\lambda_{\sigma,\tau} \in \Lambda'_{\tau'}$. Since $h_{f(\sigma)}(\lambda_{\sigma,\tau}) = h_{f(\tau)}(\lambda_{\sigma,\tau})$ these parts of the corresponding summands in the first and second interior sum cancel using the linearity of $h_{f(\tau)}$. Moreover, note that $f(v_{\sigma/\tau}) = \lambda_{\sigma,\tau} \in \Lambda'_{\tau'}$ for those $\sigma > \tau$ on which f is not injective and that the whole summands cancel in this case. Thus we can conclude that the sum does not change if we restrict the summation to those $\sigma > \tau$ on which f is injective. Using additionally the equation

$$|\Lambda'_{\sigma'}/f(\Lambda_\sigma)| = |\Lambda'_{\tau'}/f(\Lambda_\tau)|\cdot|\Lambda'_{\sigma'}/(\Lambda'_{\tau'} + \mathbb{Z}f(v_{\sigma/\tau}))|$$

we get

$$\begin{aligned}\omega_{f_*(f^*h\cdot Z)}(\tau') &= \sum_{\substack{\tau\in(f^*h\cdot Z)^{(k-1)}:\\ f(\tau)=\tau'}} \left(\sum_{\substack{\sigma\in Z^{(k)}:\\ \sigma>\tau, f(\sigma)>\tau'}} \omega_Z(\sigma)\cdot|\Lambda'_{f(\sigma)}/f(\Lambda_\sigma)|\cdot h_{f(\sigma)}(v_{f(\sigma)/\tau'}) - h_{\tau'}\left(\sum_{\substack{\sigma\in Z^{(k)}:\\ \sigma>\tau, f(\sigma)>\tau'}} \omega_Z(\sigma)\cdot|\Lambda'_{f(\sigma)}/f(\Lambda_\sigma)|\cdot v_{f(\sigma)/\tau'}\right)\right)\\ &= \left(\sum_{\sigma\in Z^{(k)}:f(\sigma)>\tau'} \omega_Z(\sigma)\cdot|\Lambda'_{f(\sigma)}/f(\Lambda_\sigma)|\cdot h_{f(\sigma)}(v_{f(\sigma)/\tau'})\right)\\ &\quad -h_{\tau'}\left(\sum_{\sigma\in Z^{(k)}:f(\sigma)>\tau'} \omega_Z(\sigma)\cdot|\Lambda'_{f(\sigma)}/f(\Lambda_\sigma)|\cdot v_{f(\sigma)/\tau'}\right).\end{aligned}$$

Note that for the last equation we used again the linearity of $h_{\tau'}$. We have checked so far that a cone τ' of dimension $k-1$ occurring in both $h\cdot(f_*Z)$ and $f_*(f^*h\cdot Z)$ has the same weight in both fans. Thus it remains to examine those cones $f(\tau), \tau \in Z^{(k-1)}$ such that f is injective on τ but not on any $\sigma > \tau$: In this case all vectors $v_{\sigma/\tau}$ are mapped to $\Lambda'_{f(\tau)}$. Again, $h_{f(\sigma)} = h_{f(\tau)}$ and by linearity of $h_{f(\tau)}$ all summands in the sum cancel as above. Hence the the weight of $f(\tau)$ in $f_*(f^*h\cdot Z)$ is 0 and $\varphi\cdot(f_*E) = [h\cdot(f_*Z)] = [f_*(f^*h\cdot Z)] = f_*(f^*\varphi\cdot E)$. □

1.4 Abstract tropical cycles

In this section we will introduce the notion of abstract tropical cycles as spaces that have tropical fans as local building blocks. Then we will generalize the theory from the previous sections to these spaces.

Definition 1.4.1 (Abstract polyhedral complexes)
An *(abstract) polyhedral complex* is a topological space $|X|$ together with a finite set X of closed subsets of $|X|$ and an embedding map $\varphi_\sigma : \sigma \to \mathbb{R}^{n_\sigma}$ for every $\sigma \in X$ such that

(a) X is closed under taking intersections, i.e. $\sigma \cap \sigma' \in X$ for all $\sigma, \sigma' \in X$ with $\sigma \cap \sigma' \neq \emptyset$,

(b) every image $\varphi_\sigma(\sigma)$, $\sigma \in X$ is a rational polyhedron not contained in a proper affine subspace of $\mathbb{R}^{n_\sigma}$,

(c) for every pair $\sigma, \sigma' \in X$ the concatenation $\varphi_\sigma \circ \varphi_{\sigma'}^{-1}$ is integer affine linear where defined,

(d) $|X| = \dot{\bigcup}_{\sigma \in X} \varphi_\sigma^{-1}(\varphi_\sigma(\sigma)^\circ)$, where $\varphi_\sigma(\sigma)^\circ$ denotes the interior of $\varphi_\sigma(\sigma)$ in $\mathbb{R}^{n_\sigma}$.

For simplicity we will usually drop the embedding maps φ_σ and denote the polyhedral complex $(X, |X|, \{\varphi_\sigma | \sigma \in X\})$ by $(X, |X|)$ or just by X if no confusion can occur. The closed subsets $\sigma \in X$ are called the *polyhedra* or *faces of* $(X, |X|)$. For $\sigma \in X$ the open set $\sigma^{\mathrm{ri}} := \varphi_\sigma^{-1}(\varphi_\sigma(\sigma)^\circ)$ is called the *relative interior of* σ. Like in the case of fans the *dimension* of $(X, |X|)$ is the maximum of the dimensions of its polyhedra. $(X, |X|)$ is *pure-dimensional* if every inclusion-maximal polyhedron has the same dimension. We denote by $X^{(n)}$ the set of polyhedra in $(X, |X|)$ of dimension n. Let $\tau, \sigma \in X$. Like in the case of fans we write $\tau \leq \sigma$ (or $\tau < \sigma$) if $\tau \subseteq \sigma$ (or $\tau \subsetneq \sigma$ respectively).

An abstract polyhedral complex $(X, |X|)$ of pure dimension n together with a map $\omega_X : X^{(n)} \to \mathbb{Z}$ is called *weighted polyhedral complex* of dimension n and $\omega_X(\sigma)$ the *weight* of the polyhedron $\sigma \in X^{(n)}$. Like in the case of fans the empty complex $\emptyset$ is a weighted polyhedral complex of every dimension n. If $((X, |X|), \omega_X)$ is a weighted polyhedral complex of dimension n then let

$$X^* := \{\tau \in X | \tau \subseteq \sigma \text{ for some } \sigma \in X^{(n)} \text{ with } \omega_X(\sigma) \neq 0\}, |X^*| := \bigcup_{\tau \in X^*} \tau \subseteq |X|.$$

With these definitions $\big((X^*, |X^*|), \omega_X|_{(X^*)^{(n)}}\big)$ is again a weighted polyhedral complex of dimension n, called the *non-zero part* of $((X, |X|), \omega_X)$. We call a weighted polyhedral complex $((X, |X|), \omega_X)$ *reduced* if $((X, |X|), \omega_X) = ((X^*, |X^*|), \omega_{X^*})$ holds.

Definition 1.4.2 (Subcomplexes and refinements)
Let $(X, |X|, \{\varphi_\sigma\})$ and $(Y, |Y|, \{\psi_\tau\})$ be two polyhedral complexes. We call $(X, |X|, \{\varphi_\sigma\})$ a *subcomplex* of $(Y, |Y|, \{\psi_\tau\})$ if

(a) $|X| \subseteq |Y|$,

(b) for every $\sigma \in X$ exists $\tau \in Y$ with $\sigma \subseteq \tau$ and

(c) the $\mathbb{Z}$-linear structures of X and Y are compatible, i.e. for a pair σ, τ from (b) the maps $\varphi_\sigma \circ \psi_\tau^{-1}$ and $\psi_\tau \circ \varphi_\sigma^{-1}$ are integer affine linear where defined.

We write $(X, |X|, \{\varphi_\sigma\}) \trianglelefteq (Y, |Y|, \{\psi_\tau\})$ in this case. Analogous to the case of fans we define a map $C_{X,Y} : X \to Y$ that maps a polyhedron in X to the inclusion-minimal polyhedron in Y containing it.
We call a weighted polyhedral complex $((X, |X|), \omega_X)$ a *refinement* of $((Y, |Y|), \omega_Y)$ if

(a) $(X^*, |X^*|) \trianglelefteq (Y^*, |Y^*|)$,

(b) $|X^*| = |Y^*|$,

(c) $\omega_X(\sigma) = \omega_Y(C_{X^*,Y^*}(\sigma))$ for all $\sigma \in (X^*)^{(\dim(X))}$.

Definition 1.4.3 (Open fans)
Let $(\widetilde{F}, \omega_{\widetilde{F}})$ be a tropical fan in $\mathbb{R}^n$ and $U \subseteq \mathbb{R}^n$ an open subset containing the origin. The set $F := \widetilde{F} \cap U := \{\sigma \cap U | \sigma \in \widetilde{F}\}$ together with the induced weight function ω_F is called an *open (tropical) fan* in $\mathbb{R}^n$. As in the case of fans let $|F| := \bigcup_{\sigma' \in F} \sigma'$. Note that the open fan F contains the whole information of the entire fan $\widetilde{F}$ as $\widetilde{F} = \{\mathbb{R}_{\geq 0} \cdot \sigma' | \sigma' \in F\}$.

Definition 1.4.4 (Tropical polyhedral complexes)
A *tropical polyhedral complex* of dimension n is a weighted polyhedral complex $((X, |X|), \omega_X)$ of pure dimension n together with the following data: For every polyhedron $\sigma \in X^*$ we are given an open fan F_σ in some $\mathbb{R}^{n_\sigma}$ and a homeomorphism

$$\Phi_\sigma : S_\sigma := \bigcup_{\sigma' \in X^*, \sigma' \supseteq \sigma} (\sigma')^{\mathrm{ri}} \xrightarrow{\sim} |F_\sigma|$$

such that

(a) for all $\sigma' \in X^*, \sigma' \supseteq \sigma$ holds $\Phi_\sigma(\sigma' \cap S_\sigma) \in F_\sigma$ and Φ_σ is compatible with the $\mathbb{Z}$-linear structure on σ', i.e. $\Phi_\sigma \circ \varphi_{\sigma'}^{-1}$ and $\varphi_{\sigma'} \circ \Phi_\sigma^{-1}$ are integer affine linear where defined,

(b) $\omega_X(\sigma') = \omega_{F_\sigma}(\Phi_\sigma(\sigma' \cap S_\sigma))$ for every $\sigma' \in (X^*)^{(n)}$ with $\sigma' \supseteq \sigma$,

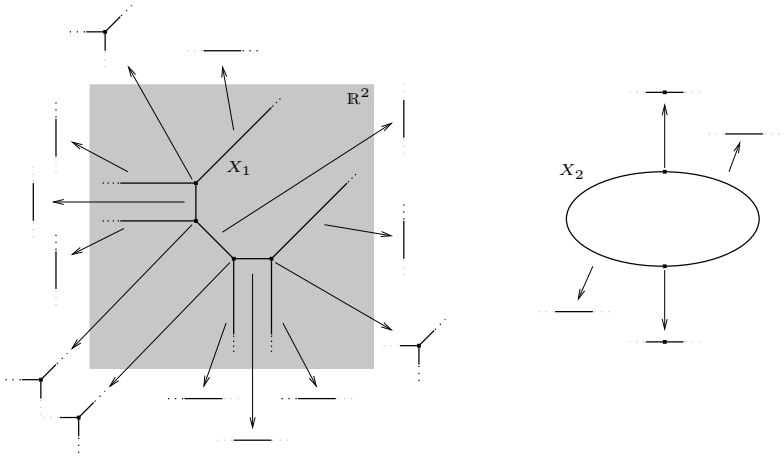
(c) for every pair $\sigma, \tau \in X^*$ there is an integer affine linear map $A_{\sigma,\tau}$ and a commutative diagram

$$\begin{array}{ccc} S_\sigma \cap S_\tau & \xrightarrow[\sim]{\Phi_\tau} & \Phi_\tau(S_\sigma \cap S_\tau)\,. \\ {\scriptstyle \Phi_\sigma}\downarrow{\scriptstyle \sim} & \nearrow_{A_{\sigma,\tau}} & \\ \Phi_\sigma(S_\sigma \cap S_\tau) & & \end{array}$$

For simplicity of notation we will usually drop the maps Φ_σ and write $((X, |X|), \omega_X)$ or just X instead of $(((X, |X|), \omega_X), \{\Phi_\sigma\})$. A tropical polyhedral complex is called *reduced* if the underlying weighted polyhedral complex is.

Example 1.4.5
The following figure shows the topological spaces and the decompositions into polyhedra of two such abstract tropical polyhedral complexes together with the open fan F_σ for

every polyhedron σ:



Remark 1.4.6
The above example X_1 is an instance of a more general construction: Let X be a weighted, balanced polyhedral complex in $\mathbb{R}^n$ as in [M06] or [S05]. Then X can be interpreted as a tropical polyhedral complex in a natural way: For every polyhedron $\sigma \in X$ we have trivial embedding maps $\varphi_\sigma : \sigma \hookrightarrow H_\sigma \cong \mathbb{R}^{\dim(\sigma)}$, where $H_\sigma \subseteq \mathbb{R}^n$ is the smallest affine subspace containing σ. Moreover, for every $\sigma \in X$ we have trivial fan charts $\Phi_\sigma : S_\sigma = \bigcup_{\sigma' \in X, \sigma' \supseteq \sigma} (\sigma')^{\mathrm{ri}} \xrightarrow{\sim} S_\sigma^{\mathrm{tr}}$, where S_σ^{tr} is a translation of S_σ such that σ contains the origin. Then S_σ^{tr} is an open fan sitting in the associated tropical fan $\widetilde{S_\sigma^{\mathrm{tr}}}$.

Construction 1.4.7 (Refinements of tropical polyhedral complexes)
Let $(((X, |X|), \omega_X), \{\Phi_\sigma\})$ be a tropical polyhedral complex and let $((Y, |Y|), \omega_Y)$ be a refinement of its underlying weighted polyhedral complex $((X, |X|), \omega_X)$. Then we can make $((Y, |Y|), \omega_Y)$ into a tropical polyhedral complex as follows: We may assume that X and Y are reduced as we do not pose any conditions on polyhedra with weight zero. Fix some $\tau \in Y$ and let $\sigma := C_{Y,X}(\tau)$. By definition of refinement, for every $\tau' \in Y$ with $\tau' \geq \tau$ there is $\sigma' \in X$, $\sigma' \geq \sigma$ with $\tau' \subseteq \sigma'$. Thus $S_\tau \subseteq S_\sigma$ and we have a map $\Psi_\tau := \Phi_\sigma|_{S_\tau} : S_\tau \xrightarrow{\sim} \Psi_\tau(S_\tau) \subseteq \mathbb{R}^{n_\sigma}$. It remains to give $\Psi_\tau(S_\tau)$ the structure of an open fan: We may assume that $\{0\} \subseteq \Psi_\tau(\tau)$ (otherwise replace Ψ_τ by the concatenation of Ψ_τ with an appropriate translation T_τ, apply T_τ to F_σ^X and Φ_σ and change the maps $A_{\sigma,\sigma'}$ and $A_{\sigma',\sigma}$ accordingly). Let $\widetilde{F}_\sigma^X := \{\mathbb{R}_{\geq 0} \cdot \sigma' | \sigma' \in F_\sigma^X\}$ be the tropical fan associated to F_σ^X and let $\widetilde{F}_\tau^Y$ be the set of cones $\widetilde{F}_\tau^Y := \{\mathbb{R}_{\geq 0} \cdot \Psi_\tau(\tau') | \tau \leq \tau' \in Y\}$. Note that the conditions on the $\mathbb{Z}$-linear structures on X and Y to be compatible and on Φ_σ to be compatible with the $\mathbb{Z}$-linear structure on X assure that $\widetilde{F}_\tau^Y$ is a fan in $\mathbb{R}^{n_\sigma}$. In fact, $\widetilde{F}_\tau^Y$ with the weights induced by Y is a refinement of $(\widetilde{F}_\sigma^X, \omega_{\widetilde{F}_\sigma^X})$. Thus the maps Ψ_τ together with the open fans $\{\varrho \cap \Psi_\tau(S_\tau) | \varrho \in \widetilde{F}_\tau^Y\}$, $\tau \in Y$ fulfill all requirements for a tropical polyhedral complex.

Remark 1.4.8
If not stated otherwise we will from now on equip every refinement of a tropical polyhedral complex coming from a refinement of the underlying weighted polyhedral complex with the tropical structure constructed in 1.4.7.

Definition 1.4.9 (Refinements and equivalence of tropical polyhedral complexes)
Let $C_1 = \big(((X_1, |X_1|), \omega_{X_1}), \{\Phi_{\sigma_1}^{X_1}\}\big)$ and $C_2 = \big(((X_2, |X_2|), \omega_{X_2}), \{\Phi_{\sigma_2}^{X_2}\}\big)$ be tropical polyhedral complexes. We call C_2 a *refinement* of C_1 if

(a) $((X_2, |X_2|), \omega_{X_2})$ is a refinement of $((X_1, |X_1|), \omega_{X_1})$ and

(b) C_2 carries the tropical structure induced by C_1 as in construction 1.4.7, i.e. if $C_2' = \Big(((X_2, |X_2|), \omega_{X_2}), \{\widetilde{\Phi}_{\sigma_2}^{X_2}\}\Big)$ is the tropical polyhedral complex obtained from C_1 and the refinement $((X_2, |X_2|), \omega_{X_2})$ then the maps $\widetilde{\Phi}_{\sigma_2}^{X_2} \circ (\Phi_{\sigma_2}^{X_2})^{-1}$ and $\Phi_{\sigma_2}^{X_2} \circ (\widetilde{\Phi}_{\sigma_2}^{X_2})^{-1}$ are integer affine linear where defined.

We call two tropical polyhedral complexes C_1 and C_2 *equivalent* (write $C_1 \sim C_2$) if they have a common refinement (as tropical polyhedral complexes).

Remark 1.4.10
Note that different choices of translation maps T_τ in construction 1.4.7 only lead to tropical polyhedral complexes carrying the same tropical structure in the sense of definition 1.4.9 (b). In particular definition 1.4.9 does not depend on the choices we made in construction 1.4.7. Note moreover that refinements of $(((X, |X|), \omega_X), \{\Phi_\sigma\})$ and $((Y, |Y|), \omega_Y)$ in construction 1.4.7 only lead to refinements of $(((Y, |Y|), \omega_Y), \{\Psi_\tau\})$.

Construction 1.4.11 (Refinements)
Let $(((X, |X|, \{\varphi_\sigma\}), \omega_X), \{\Phi_\sigma\})$ and $(((Y, |Y|, \{\psi_\tau\}), \omega_Y), \{\Psi_\tau\})$ be reduced tropical polyhedral complexes such that $(Y, |Y|) \trianglelefteq (X, |X|)$ and the tropical structures on X and Y agree, i.e. for every $\tau \in Y$ and $\sigma := C_{Y,X}(\tau) \in X$ the maps $\Psi_\tau \circ \Phi_\sigma^{-1}$ and $\Phi_\sigma \circ \Psi_\tau^{-1}$ are integer affine linear where defined. Moreover let $(((X', |X'|, \{\varphi'_{\sigma'}\}), \omega_{X'}), \{\Phi'_{\sigma'}\})$ be a reduced refinement of $(((X, |X|, \{\varphi_\sigma\}), \omega_X), \{\Phi_\sigma\})$. Like in the case of fans we will construct a refinement $\big(((Y \cap X', |Y \cap X'|, \{\psi_{\tau'}^{Y\cap X'}\}), \omega_{Y\cap X'}), \{\Psi_{\tau'}^{Y\cap X'}\}\big)$ of $(((Y, |Y|, \{\psi_\tau\}), \omega_Y), \{\Psi_\tau\})$ such that $(Y \cap X', |Y \cap X'|) \trianglelefteq (X', |X'|)$ and the tropical structures on $Y \cap X'$ and X' agree:
Fix $\sigma \in X$. Note that the compatibility conditions on the $\mathbb{Z}$-linear structures of X', X and Y, X respectively (cf. definition 1.4.2 (c)) assure that $\varphi_\sigma(\sigma')$, $\sigma' \in X'$ with $\sigma' \subseteq \sigma$ as well as $\varphi_\sigma(\tau)$, $\tau \in Y$ with $\tau \subseteq \sigma$ are rational polyhedra in $\mathbb{R}^{n_\sigma}$. Thus in this case $\varphi_\sigma(\sigma' \cap \tau) = \varphi_\sigma(\sigma') \cap \varphi_\sigma(\tau)$ is a rational polyhedron, too. Let $H_{\sigma',\tau} \cong \mathbb{R}^{n_\tau}$ be the smallest affine subspace of $\mathbb{R}^{n_\sigma}$ containing $\varphi_\sigma(\sigma' \cap \tau)$. We can consider $\varphi_\sigma|_{\sigma'\cap\tau}$ to be a map $\sigma' \cap \tau \to \mathbb{R}^{n_\tau}$. We can hence construct the underlying weighted polyhedral complex of our desired tropical polyhedral complex as follows: Set $P := \{\tau \cap \sigma' | \tau \in Y, \sigma' \in X'\}$, $Y \cap X' := \{\tau \in P | \nexists \widetilde{\tau} \in P^{(\dim(\tau))} : \widetilde{\tau} \subsetneq \tau\}$, $|Y \cap X'| := |Y|$ and $\omega_{Y\cap X'}(\tau) := \omega_Y(C_{Y\cap X',Y}(\tau))$ for all $\tau \in (Y \cap X')^{(\dim(Y))}$. It remains to define the maps $\psi_{\tau'}^{Y\cap X'}$ and $\Psi_{\tau'}^{Y\cap X'}$: For every $\tau' \in Y \cap X'$ choose a triplet $\sigma' \in X', \tau \in Y, \sigma \in X$ such that $\sigma' \cap \tau = \tau'$ and $\sigma', \tau \subseteq \sigma$ and set $\psi_{\tau'}^{Y\cap X'} := \varphi_\sigma|_{\sigma'\cap\tau}$. With these definitions the weighted polyhedral complex $((Y \cap X', |Y \cap X'|, \{\psi_{\tau'}^{Y\cap X'}\}), \omega_{Y\cap X'})$ is a refinement of $((Y, |Y|, \{\psi_\tau\}), \omega_Y)$. Thus we can apply construction 1.4.7 to obtain maps $\{\Psi_{\tau'}^{Y\cap X'}\}$ that endow our weighted polyhedral complex with the tropical structure inherited from $((Y, |Y|, \{\psi_\tau\}), \omega_Y)$. Note that the compatibility property between the tropical structures of Y and X is bequeathed to $Y \cap X'$ and X', too.

Lemma 1.4.12
The equivalence of tropical polyhedral complexes is an equivalence relation.

Proof. Let $C_1 = \big(((X_1, |X_1|), \omega_{X_1}), \{\Phi^{X_1}_{\sigma_1}\}\big)$, $C_2 = \big(((X_2, |X_2|), \omega_{X_2}), \{\Phi^{X_2}_{\sigma_2}\}\big)$ and $C_3 = \big(((X_3, |X_3|), \omega_{X_3}), \{\Phi^{X_3}_{\sigma_3}\}\big)$ be tropical polyhedral complexes such that $C_1 \sim C_2$ via a common refinement $D_1 = \big(((Y_1, |Y_1|), \omega_{Y_1}), \{\Phi^{Y_1}_{\sigma_1}\}\big)$ and $C_2 \sim C_3$ via a common refinement $D_2 = \big(((Y_2, |Y_2|), \omega_{Y_2}), \{\Phi^{Y_2}_{\sigma_2}\}\big)$. We have to construct a common refinement of C_1 and C_3: First of all we may assume that D_1 and D_2 are reduced. Using construction 1.4.11 we get a refinement $D_3 := \big(((Y_1 \cap Y_2, |Y_1 \cap Y_2|), \omega_{Y_1 \cap Y_2}), \{\Phi^{Y_1 \cap Y_2}_{\tau}\}\big)$ of D_1 with $(Y_1 \cap Y_2, |Y_1 \cap Y_2|) \trianglelefteq (Y_2, |Y_2|)$ and a tropical structure that is compatible with the tropical structure on D_2. It is easily checked that D_3 is a refinement of D_2, too. □

Definition 1.4.13 (Abstract tropical cycles)
Let $((X, |X|), \omega_X)$ be an n-dimensional tropical polyhedral complex. Its equivalence class $[((X, |X|), \omega_X)]$ is called an *(abstract) tropical n-cycle*. The set of n-cycles is denoted by Z_n. Since the underlying topological space $|X^*|$ of a tropical polyhedral complex $((X, |X|), \omega_X)$ is by definition invariant under refinements we define $\big|\, [((X, |X|), \omega_X)] \,\big| := |X^*|$. Like in the affine case, an n-cycle $[((X, |X|), \omega_X)]$ is called an *(abstract) tropical variety* if $\omega_X(\sigma) \geq 0$ for all $\sigma \in X^{(n)}$.

Let $C \in Z_n$ and $D \in Z_k$ be two tropical cycles. D is called an *(abstract) tropical cycle in C* or a *subcycle of C* if there exists a representative $(((Z, |Z|), \omega_Z), \{\Psi_\tau\})$ of D and a reduced representative $(((X, |X|), \omega_X), \{\Phi_\sigma\})$ of C such that

(a) $(Z, |Z|) \trianglelefteq (X, |X|)$,

(b) the tropical structures on Z and X agree, i.e. for every $\tau \in Z$ the maps $\Psi_\tau \circ \Phi^{-1}_{C_{Z,X}(\tau)}$ and $\Phi_{C_{Z,X}(\tau)} \circ \Psi^{-1}_\tau$ are integer affine linear where defined.

The set of tropical k-cycles in C is denoted by $Z_k(C)$.

Remark and Definition 1.4.14
(a) Let X be a finite set of rational polyhedra in $\mathbb{R}^n$, $f \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$ a linear form and $b \in \mathbb{R}$. Then let

$$H_{f,b} := \big\{\{x \in \mathbb{R}^n | f(x) \leq b\}, \{x \in \mathbb{R}^n | f(x) = b\}, \{x \in \mathbb{R}^n | f(x) \geq b\}\big\}.$$

Like in the case of fans (cf. construction 1.1.12) we can form sets $P := \{\sigma \cap \sigma' | \sigma \in X, \sigma' \in H_{f,b}\}$ and $X \cap H_{f,b} := \{\sigma \in P | \nexists\, \tau \in P^{(\dim(\sigma))} \text{ with } \tau \subsetneq \sigma\}$.

(b) Again let X be a finite set of rational polyhedra in $\mathbb{R}^n$. Let $\{f_i \leq b_i | i = 1, \ldots, N\}$ be all (integral) inequalities occurring in the description of all polyhedra in X. Then we can construct the set $X \cap H_{f_1,b_1} \cap \cdots \cap H_{f_N,b_N}$. Note that for every collection of polyhedra X this set $X \cap H_{f_1,b_1} \cap \cdots \cap H_{f_N,b_N}$ is a (usual) rational polyhedral complex (i.e. for every polyhedron $\tau \in X$ every face (in the usual sense) of σ is contained in X and the intersection of every two polyhedra in X is a common face of each). Moreover note that the result is independent of the order of the f_i and if $\{g_i \leq c_i | i = 1, \ldots, M\}$ is a different set of inequalities describing the polyhedra in X

then $X \cap H_{f_1,b_1} \cap \cdots \cap H_{f_N,b_N}$ and $X \cap H_{g_1,c_1} \cap \cdots \cap H_{g_M,c_M}$ have a common refinement, namely $X \cap H_{f_1,b_1} \cap \cdots \cap H_{f_N,b_N} \cap H_{g_1,c_1} \cap \cdots \cap H_{g_M,c_M}$.

Construction 1.4.15 (Sums of tropical cycles)
Let $C \in Z_n$ be a tropical cycle. Like in the affine case the set of tropical k-cycles in C can be made into an abelian group by defining the sum of two such k-cycles as follows: Let D_1 and $D_2 \in Z_k(C)$ be the two cycles whose sum we want to construct. By definition there are reduced representatives $\big(((X_1, |X_1|), \omega_{X_1}), \{\Phi_\tau^{X_1}\}\big)$ and $\big(((X_2, |X_2|), \omega_{X_2}), \{\Phi_\tau^{X_2}\}\big)$ of C and reduced representatives $\big(((Y, |Y|), \omega_Y), \{\Phi_\tau^{Y}\}\big)$ of D_1 and $\big(((Z, |Z|), \omega_Z), \{\Phi_\tau^{Z}\}\big)$ of D_2 such that $(Y, |Y|) \trianglelefteq (X_1, |X_1|)$ and the tropical structures on Y and X_1 agree and $(Z, |Z|) \trianglelefteq (X_2, |X_2|)$ and the tropical structures on Z and X_2 agree. As "$\sim$" is an equivalence relation there is a common refinement $\big(((X, |X|, \{\varphi_\tau\}), \omega_X), \{\Phi_\tau^X\}\big)$ of X_1 and X_2 which we may assume to be reduced. Applying construction 1.4.11 to Y and X we obtain the tropical polyhedral complex $\big(((Y \cap X, |Y \cap X|), \omega_{Y\cap X}), \{\Phi_\tau^{Y\cap X}\}\big)$ which is a refinement of Y, has a tropical structure that is compatible with the tropical structure on X and fulfils $(Y \cap X, |Y \cap X|) \trianglelefteq (X, |X|)$. If we further apply construction 1.4.11 to Z and X we get a refinement $\big(((Z \cap X, |Z \cap X|), \omega_{Z\cap X}), \{\Phi_\tau^{Z\cap X}\}\big)$ of Z with analogous properties. Now fix some polyhedron $\sigma \in X$ and let $\tau_1, \ldots, \tau_r \in Y \cap X$ and $\tau_{r+1}, \ldots, \tau_s \in Z \cap X$ be all polyhedra of $Y \cap X$ and $Z \cap X$ respectively that are contained in σ. Note that property (a) of definition 1.4.13 implies that for all $i = 1, \ldots, r$ the image $\varphi_\sigma(\tau_i)$ is a rational polyhedron in $\mathbb{R}^{n_\sigma}$. Like in remark and definition 1.4.14 let $\{f_i \leq b_i | i = 1, \ldots, N\}$ be the set of all integral inequalities occurring in the description of all polyhedra $\varphi_\sigma(\tau_i), i = 1, \ldots, s$ and let $R^\sigma_{Y\cap X} := \{\varphi_\sigma(\tau_i) | i = 1, \ldots, r\} \cap H_{f_1,b_1} \cap \cdots \cap H_{f_N,b_N}$ and $R^\sigma_{Z\cap X} := \{\varphi_\sigma(\tau_i) | i = r+1, \ldots, s\} \cap H_{f_1,b_1} \cap \cdots \cap H_{f_N,b_N}$. Then $P^\sigma_{Y\cap X} := \{\varphi_\sigma^{-1}(\tau) | \tau \in R^\sigma_{Y\cap X}\}$ and $P^\sigma_{Z\cap X} := \{\varphi_\sigma^{-1}(\tau) | \tau \in R^\sigma_{Z\cap X}\}$ are a kind of local refinement of $Y \cap X$ and $Z \cap X$, respectively, but taking the union over all maximal polyhedra $\sigma \in X^{(n)}$ does in general not lead to global refinements as there may be overlaps between polyhedra coming from different σ. We resolve this as follows: For $\sigma \in X^{(n)}$, $\tau \in \bigcup_{i=0}^{n-1} X^{(i)}$ let $P^\sigma_{Y,\tau} := \{\varrho \in P^\sigma_{Y\cap X} | \tau \text{ is the inclusion-minimal polyhedron of } X \text{ containing } \varrho\}$ and $P_{Y,n} := \bigcup_{\sigma\in X^{(n)}} \{\varrho \in P^\sigma_{Y\cap X} | \nexists \widetilde{\tau} \in X^{(n-1)} : \varrho \subseteq \widetilde{\tau}\}$. Analogously for $P^\sigma_{Z,\tau}$ and $P_{Z,n}$. Then let

$$\widetilde{Y} := P_{Y,n} \cup \left(\bigcup_{\tau\in X^{(i)}: i<n} \{ \bigcap_{\sigma\in X^{(n)}:\tau\subseteq\sigma} \tau_\sigma | \tau_\sigma \in P^\sigma_{Y,\tau}\} \right)$$

and

$$\widetilde{Z} := P_{Z,n} \cup \left(\bigcup_{\tau\in X^{(i)}: i<n} \{ \bigcap_{\sigma\in X^{(n)}:\tau\subseteq\sigma} \tau_\sigma | \tau_\sigma \in P^\sigma_{Z,\tau}\} \right).$$

Moreover for every $\tau \in \widetilde{Y} \cup \widetilde{Z}$ choose some $\sigma \in X^{(n)}$ with $\tau \subseteq \sigma$ and let $\psi_\tau := \varphi_\sigma|_\tau$. Note that by construction $(\widetilde{Y}, |Y \cap X|)$ and $(\widetilde{Z}, |Z \cap X|)$ with structure maps $\psi_\tau, \tau \in \widetilde{X}$ or $\tau \in \widetilde{Z}$ respectively and weight functions $\omega_{\widetilde{Y}}$ and $\omega_{\widetilde{Z}}$ induced by $Y \cap X$ and $Z \cap X$ are refinements of $Y \cap X$ and $Z \cap X$ (we need here that $R^\sigma_{Y\cap X}$ and $R^\sigma_{Z\cap X}$ were usual polyhedral complexes in $\mathbb{R}^{n_\sigma}$). Thus we can endow them with the tropical structures inherited from $Y \cap X$ and $Z \cap X$ respectively (cf. construction 1.4.7). As

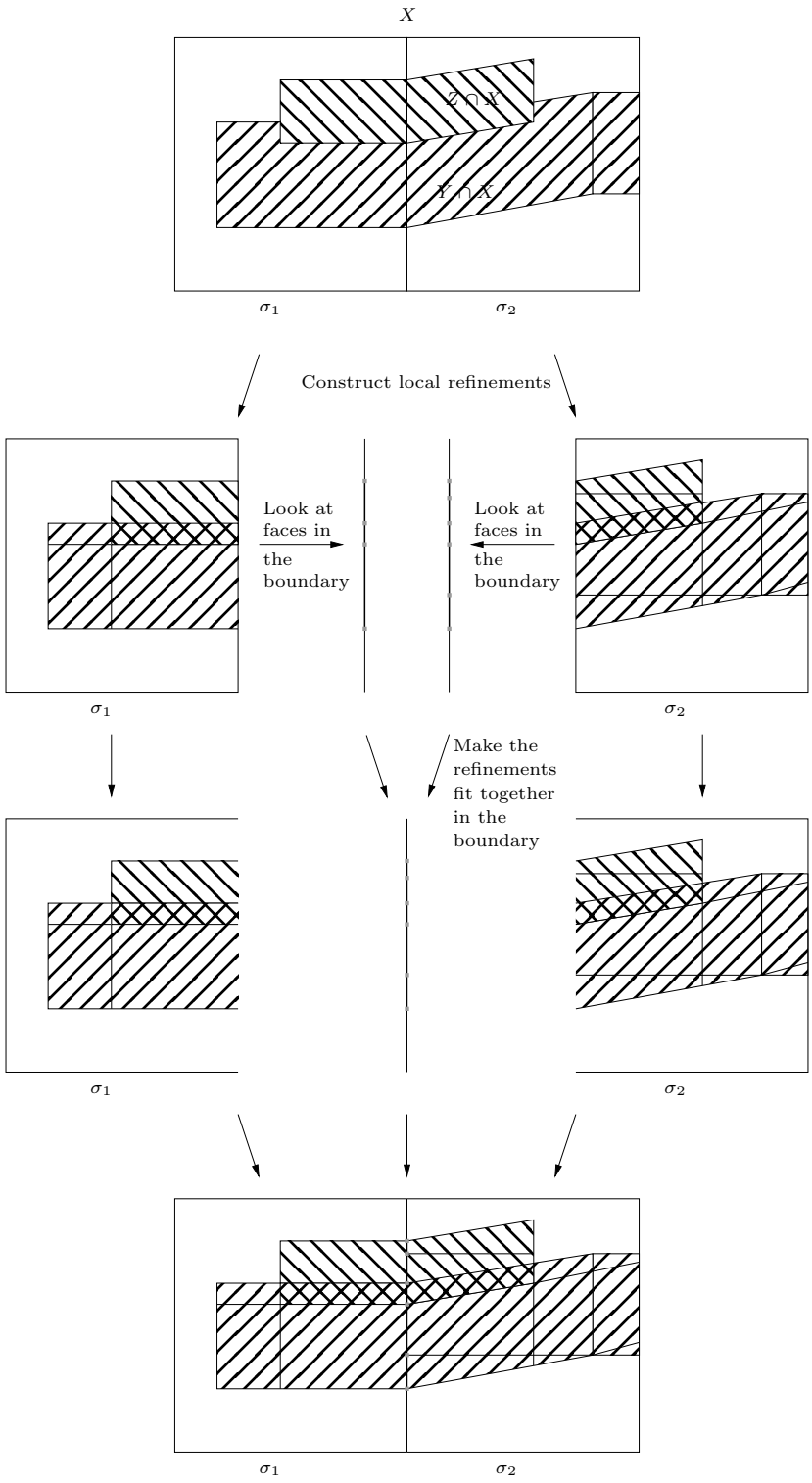


Figure 1.8: An illustration of the process described in construction 1.4.15.

$(\widetilde{X} \cup \widetilde{Y}, |Y \cap X| \cup |Z \cap X|)$ is a polyhedral complex now, we can form

$$((P, |P|), \omega_P) := ((\widetilde{X} \cup \widetilde{Y}, |Y \cap X| \cup |Z \cap X|), \omega_P),$$

where $\omega_P(\sigma) := \omega_{\widetilde{Y}}(\sigma) + \omega_{\widetilde{Z}}(\sigma)$ for all $\sigma \in P^{(k)}$ (we set $\omega_\square(\sigma) := 0$ for $\sigma \notin \square$, $\square \in \{\widetilde{Y}, \widetilde{Z}\}$). Recall that the tropical structures on $\widetilde{Y}$ and $\widetilde{Z}$ are inherited from $Y \cap X$ and $Z \cap X$ and are thus compatible with the tropical structure on X. Thus $\Phi^X_\sigma(S^P_\sigma) \subseteq |F^X_\sigma|$ with weights induced from P is an open fan (the corresponding complete tropical fan is just the sum of the fans coming from $\widetilde{Y}$ and $\widetilde{Z}$). Thus we can set $\widetilde{\Phi}_\sigma := \Phi^X_\sigma|_{S^P_\sigma} : S^P_\sigma \xrightarrow{\sim} \Phi^X_\sigma(S^P_\sigma)$ and can hence define the sum $D_1 + D_2$ to be

$$D_1 + D_2 := \left[\left(((P, |P|), \omega_P), \{\widetilde{\Phi}_\sigma\}\right)\right].$$

Note that the class $[(((P, |P|), \omega_P), \{\widetilde{\Phi}_\sigma\})]$ is independent of the choices we made, i.e. the sum $D_1 + D_2$ is well-defined.

Lemma 1.4.16
Let $C \in Z_n$ be a tropical cycle. The set $Z_k(C)$ together with the operation "+" from construction 1.4.15 forms an abelian group.

Proof. The class of the empty complex $0 = [\emptyset]$ is the neutral element of this operation and $[((Y, |Y|), -\omega_Y)]$ is the inverse element of $[((Y, |Y|), \omega_Y)] \in Z_k(C)$. □

Definition 1.4.17
Let $(((X, |X|, \{\varphi_\sigma\}), \omega_X), \{\Phi_\sigma\})$ and $(((Y, |Y|, \{\psi_\tau\}), \omega_Y), \{\Psi_\tau\})$ be tropical polyhedral complexes. We denote by

$$(((X, |X|, \{\varphi_\sigma\}), \omega_X), \{\Phi_\sigma\}) \times (((Y, |Y|, \{\psi_\tau\}), \omega_Y), \{\Psi_\tau\})$$

their *cartesian product*

$$(((X \times Y, |X| \times |Y|, \{\vartheta_{\sigma\times\tau}\}), \omega_{X\times Y}), \{\Theta_{\sigma\times\tau}\}),$$

where

$$\begin{aligned} X \times Y &:= \{\sigma \times \tau | \sigma \in X, \tau \in Y\}, \\ \vartheta_{\sigma\times\tau} &:= \varphi_\sigma \times \psi_\tau : \sigma \times \tau \longrightarrow \mathbb{R}^{n_\sigma} \times \mathbb{R}^{n_\tau}, \\ \omega_{X\times Y}(\sigma \times \tau) &:= \omega_X(\sigma) \cdot \omega_Y(\tau), \\ \Theta_{\sigma\times\tau} &:= \Phi_\sigma \times \Psi_\tau : S^X_\sigma \times S^Y_\tau \longrightarrow |F^X_\sigma| \times |F^Y_\tau|. \end{aligned}$$

Let $\widetilde{F}^X_\sigma$ and $\widetilde{F}^Y_\tau$ be the entire fans associated with F^X_σ and F^Y_τ from above. Obviously, the product $\widetilde{F}^X_\sigma \times \widetilde{F}^Y_\tau := \{\alpha \times \beta | \alpha \in \widetilde{F}^X_\sigma, \beta \in \widetilde{F}^Y_\tau\}$ with weight function $\omega_{\widetilde{F}^X_\sigma \times \widetilde{F}^Y_\tau}(\alpha \times \beta) := \omega_{\widetilde{F}^X_\sigma}(\alpha) \cdot \omega_{\widetilde{F}^Y_\tau}(\beta)$ is again a tropical fan and thus its intersection with $|F^X_\sigma| \times |F^Y_\tau|$ yields an open fan (cf. definition 1.4.3). Hence the cartesian product

$$(((X \times Y, |X| \times |Y|, \{\vartheta_{\sigma\times\tau}\}), \omega_{X\times Y}), \{\Theta_{\sigma\times\tau}\})$$

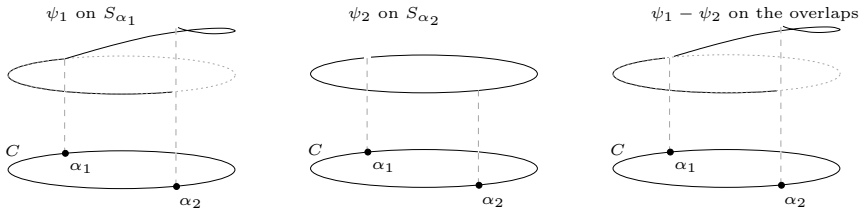


Figure 1.9: The Cartier divisor φ defined in example 1.5.2.

is again a tropical polyhedral complex.

If $C = [(X, \omega_X)]$ and $D = [(Y, \omega_Y)]$ are tropical cycles we define

$$C \times D := [(X, \omega_X) \times (Y, \omega_Y)]$$

for $(X, \omega_X) \times (Y, \omega_Y)$ as defined above. Note that $C \times D$ does not depend on the choice of the representatives X and Y.

1.5 Cartier divisors and their associated Weil divisors

Definition 1.5.1 (Rational functions and Cartier divisors)
Let C be an abstract k-cycle and let U be an open set in $|C|$. A *(non-zero) rational function on U* is a continuous function $\varphi : U \to \mathbb{R}$ such that there exists a representative $(((X, |X|, \{m_\sigma\}_{\sigma \in X}), \omega_X), \{M_\sigma\}_{\sigma \in X})$ of C such that for each face $\sigma \in X$ the map $\varphi \circ m_\sigma^{-1}$ is locally integer affine linear (where defined). The *set of all non-zero rational functions on U* is denoted by $\mathcal{K}_C^*(U)$ or just $\mathcal{K}^*(U)$.
If additionally for each face $\sigma \in X$ the map $\varphi \circ M_\sigma^{-1}$ is locally integer affine linear (where defined), φ is called *regular invertible*. The *set of all regular invertible functions on U* is denoted by $\mathcal{O}_C^*(U)$ or just $\mathcal{O}^*(U)$.
A *representative of a Cartier divisor on C* is a finite set $\{(U_1, \varphi_1), \ldots, (U_l, \varphi_l)\}$, where $\{U_i\}$ is an open covering of $|C|$ and $\varphi_i \in \mathcal{K}^*(U_i)$ are rational functions on U_i that only differ in regular invertible functions on the overlaps, in other words, for all $i \neq j$ we have $\varphi_i|_{U_i \cap U_j} - \varphi_j|_{U_i \cap U_j} \in \mathcal{O}^*(U_i \cap U_j)$.
We define the *sum* of two representatives by $\{(U_i, \varphi_i)\} + \{(V_j, \psi_j)\} = \{(U_i \cap V_j, \varphi_i + \psi_j)\}$, which obviously fulfills again the condition on the overlaps.
We call two representatives $\{(U_i, \varphi_i)\}, \{(V_j, \psi_j)\}$ *equivalent* if $\varphi_i - \psi_j$ is regular invertible (where defined) for all i, j, i.e. $\{(U_i, \varphi_i)\} - \{(V_j, \psi_j)\} = \{(W_k, \gamma_k)\}$ with $\gamma_k \in \mathcal{O}^*(W_k)$.
Obviously, "+" induces a group structure on the set of equivalence classes of representatives with the neutral element $\{(|C|, c_0)\}$, where c_0 is the constant zero function. This group is denoted by $\mathrm{Div}(C)$ and its elements are called *Cartier divisors on C*.

Example 1.5.2
Let us give an example of a Cartier divisor which is not globally defined by a rational function: As abstract cycle C we take the elliptic curve $[X_2]$ from example 1.4.5 (the brackets resemble the fact that, to be precise, we take the equivalence class of the polyhedral complex X_2 with respect to refinements). By α_1, α_2 we denote the two

vertices in X_2. W.l.o.g. we can assume that the maps M_{α_i} map the points α_i exactly to $0 \in \mathbb{R}$. Of course, the stars $S_{\alpha_1}, S_{\alpha_2}$ cover our whole space $|C| = |X_2|$. So we can define the Cartier divisor $\varphi := [\{(S_{\alpha_1}, \psi_1), (S_{\alpha_2}, \psi_2)\}]$, where $\psi_1 := \max(0, x) \circ M_{\alpha_1}$ and $\psi_2 := c_0 \circ M_{\alpha_2}$ with c_0 the constant zero function. Let us check the condition on the overlaps: On one open half of our curve the two functions coincide, whereas on the other open half they differ by a linear function. So we constructed an Cartier divisor which can not be globally defined by one rational function (as ψ_1 can not be completed to a continuous function on $|C|$).

Remark 1.5.3 (Restrictions to subcycles)
Note that, as in the affine case (see remark 1.2.2), we can restrict a non-zero rational function $\varphi \in \mathcal{K}_C^*(U)$ to an arbitrary subcycle $D \subseteq C$, i.e. $\varphi|_{U \cap |D|} \in \mathcal{K}_D^*(U \cap |D|)$. It is also true that a regular invertible function $\varphi \in \mathcal{O}_C^*(U)$ restricted to D is again regular invertible, i.e. $\varphi|_{U \cap |D|} \in \mathcal{O}_D^*(U \cap |D|)$. Hence we can also restrict a Cartier divisor $[\{(U_i, \varphi_i)\}] \in \mathrm{Div}(C)$ to D by setting $[\{(U_i, \varphi_i)\}]\,|_D := [\{(U_i \cap |D|, \varphi_i|_{U_i \cap |D|})\}] \in \mathrm{Div}(D)$.

Construction 1.5.4 (Intersection products)
Let C be an abstract k-cycle and $\varphi = [\{(U_i, \varphi_i)\}] \in \mathrm{Div}(C)$ a Cartier divisor on C. By definition 1.5.1 and lemma 1.4.12, there exists a representative $(((X, |X|, \{m_\sigma\}_{\sigma \in X}), \omega_X), \{M_\sigma\}_{\sigma \in X})$ of C such that for all i and $\sigma \in X$ the map $\varphi_i \circ m_\sigma^{-1}$ is locally integer affine linear (where defined). We can also assume that $X = X^*$, as our functions are defined on $|C| = |X^*|$ at the most. We would like to define the intersection product $\varphi \cdot C$ to be

$$\left[\left(\left(\left(Y, |Y|, \{m_\sigma\}_{\sigma \in Y}\right), \omega_{X,\varphi}\right), \left\{M_\sigma|_{S_\sigma^Y} : S_\sigma^Y \to |F_\sigma^Y|\right\}_{\sigma \in Y}\right)\right],$$

where

$$Y := \bigcup_{i=0}^{k-1} X^{(i)},\ |Y| := \bigcup_{\sigma \in Y} \sigma,\ S_\sigma^Y = \bigcup_{\substack{\sigma' \in Y \\ \sigma \subseteq \sigma'}} (\sigma')^{ri},\ F_\sigma^Y := \bigcup_{i=0}^{k-1} F_\sigma^{(i)}$$

and $\omega_{X,\varphi}$ is an appropriate weight function. So it remains to construct $\omega_{X,\varphi}(\tau)$ for $\tau \in X^{(k-1)}$.
First, we do this pointwise, i.e. we construct $\omega_{X,\varphi}(p)$ for $p \in (\tau)^{\mathrm{ri}}$. Given a $p \in (\tau)^{\mathrm{ri}}$, we pick an i with $p \in U_i$. Let V be the connected component of $M_\tau(U_i \cap S_\tau)$ containing $M_\tau(p)$. Then the function $\varphi_i \circ M_\tau^{-1}|_V$ can be uniquely extended to a rational function $\tilde{\varphi}_i \in \mathcal{K}^*([(\tilde{F}_\tau, \omega_{\tilde{F}_\tau})])$, where $(\tilde{F}_\tau, \omega_{\tilde{F}_\tau})$ is the tropical fan generated by the open fan $(F_\tau, \omega_{F_\tau})$. So, in the affine case, we can compute $\omega_{\tilde{F}_\tau, \tilde{\varphi}_i}(\mathbb{R} \cdot M_\tau(\tau))$ (see construction 1.2.3 and definition 1.2.4) and define $\omega_{X,\varphi}(p) := \omega_{\tilde{F}_\tau, \tilde{\varphi}_i}(\mathbb{R} \cdot M_\tau(\tau))$.
This definition is well-defined, namely if we pick another j with $p \in U_j$ and denote by V' the connected component of $M_\tau(U_j \cap S_\tau)$ containing $M_\tau(p)$, we know by definition of a Cartier divisor that $\varphi_i \circ M_\tau^{-1}|_{V \cap V'} - \varphi_j \circ M_\tau^{-1}|_{V \cap V'}$ is affine linear, hence $\tilde{\varphi}_i - \tilde{\varphi}_j$ is affine linear. By remark 1.2.6 we get $\omega_{\tilde{F}_\tau, \tilde{\varphi}_i}(\mathbb{R} \cdot M_\tau(\tau)) = \omega_{\tilde{F}_\tau, \tilde{\varphi}_j}(\mathbb{R} \cdot M_\tau(\tau))$.
The same argument shows that our definition does not depend on the choice of a representative $\{(U_i, \varphi_i)\}$ of φ.
But as $(\tau)^{\mathrm{ri}}$ is connected, the continuous function $\omega_{X,\varphi} : (\tau)^{\mathrm{ri}} \to \mathbb{Z}$ must be constant.

Hence, we define $\omega_{X,\varphi}(\tau) := \omega_{X,\varphi}(p)$ for some $p \in (\tau)^{\mathrm{ri}}$. With this weight function

$$\Big(\big((Y, |Y|, \{m_\sigma\}_{\sigma\in Y}), \omega_{X,\varphi}\big), \{M_\sigma|_{S_\sigma^Y}\}_{\sigma\in Y}\Big)$$

is a tropical polyhedral complex.
Let us now check if the equivalence class of this complex is independent of the choice of representatives of C. Let therefore $(((X', |X'|, \{m_{\sigma'}\}_{\sigma'\in X'}), \omega_{X'}), \{M_{\sigma'}\}_{\sigma'\in X'})$ be a refinement of $(((X, |X|, \{m_\sigma\}_{\sigma\in X}), \omega_X), \{M_\sigma\}_{\sigma\in X})$ (we can again assume $X' = (X')^*$). Then, for each $\sigma' \in X'$, the map $M_{C_{X',X}(\sigma')} \circ M_{\sigma'}^{-1}$ embeds $F_{\sigma'}$ into a refinement of $F_{C_{X',X}(\sigma')}$. Applying the affine statement here (see remark 1.2.5), we deduce that for each $\tau' \in X'^{(k-1)}$ we have $\omega_{X',\varphi}(\tau') = 0$ (if $\dim C_{X',X}(\tau') = k$) or $\omega_{X',\varphi}(\tau') = \omega_{X,\varphi}(C_{X',X}(\tau'))$ (if $\dim C_{X',X}(\tau') = k-1$).

Definition 1.5.5 (Intersection products)
Let C be an abstract k-cycle and $\varphi = [\{(U_i, \varphi_i)\}] \in \mathrm{Div}(C)$ a Cartier divisor on C. Let furthermore $(((X, |X|, \{m_\sigma\}_{\sigma\in X}), \omega_X), \{M_\sigma\}_{\sigma\in X})$ be a reduced representative of C such that for all i and $\sigma \in X$ the map $\varphi_i \circ m_\sigma^{-1}$ is locally integer affine linear (where defined). The *associated Weil divisor* $\mathrm{div}(\varphi) = \varphi \cdot C$ is defined to be

$$\left[\Big(\big(Y := \bigcup_{i=0}^{k-1} X^{(i)}, \bigcup_{\sigma\in Y} \sigma, \{m_\sigma\}_{\sigma\in Y}\big), \omega_{X,\varphi}\Big), \{M_\sigma|_{S_\sigma^Y}\}_{\sigma\in Y}\Big)\right] \in Z_{k-1}(C),$$

where $S_\sigma^Y = \bigcup_{\substack{\sigma'\in Y\\ \sigma\subseteq\sigma'}} (\sigma')^{ri}$ and $\omega_{X,\varphi}$ is the weight function defined in construction 1.5.4.
Let $D \in Z_l(C)$ be an arbitrary subcycle of C of dimension l. We define the *intersection product of φ with D* to be $\varphi \cdot D := \varphi|_D \cdot D \in Z_{l-1}(C)$.

Example 1.5.6
Let us compute the Weil divisor associated to our Cartier divisor φ on the elliptic curve C constructed in example 1.5.2. In fact, there is nothing to compute: One can see immediately from the picture that $\mathrm{div}(\varphi)$ is just the vertex α_1 with multiplicity 1 (the multiplicity of α_2 is 0 as in order to compute it, one has to use the constant function ψ_2). Let us stress that this single point can not be obtained as the Weil divisor of a (global) rational function, as all such divisors must have "degree 0" (this is defined precisely and proven in remark 1.7.3).

Proposition 1.5.7 (Commutativity)
Let $\varphi, \psi \in \mathrm{Div}(C)$ be two Cartier divisors on C. Then $\psi \cdot (\varphi \cdot C) = \varphi \cdot (\psi \cdot C)$.

Proof. Say $\varphi = [\{(U_i, \varphi_i)\}]$ and $\psi = [\{(V_j, \psi_j)\}]$. Using lemma 1.4.12 we find a reduced representative $(((X, |X|, \{m_\sigma\}_{\sigma\in X}), \omega_X), \{M_\sigma\}_{\sigma\in X})$ of C such that for all i, j and all $\sigma \in X$ the maps $\varphi_i \circ m_\sigma^{-1}$ and $\psi_j \circ m_\sigma^{-1}$ are locally integer affine linear (where defined). For $\theta \in X^{(k-2)}$, $p \in (\theta)^{\mathrm{ri}}$ and i, j with $p \in U_i \cap V_j$ we get (using notations from construction 1.5.4) $\omega_{X,\varphi,\psi}(\theta) = \omega_{X,\varphi,\psi}(p) = \omega_{\tilde{F}_\theta,\tilde{\varphi}_i,\tilde{\psi}_j}(\mathbb{R} \cdot M_\theta(\theta))$ and similarly $\omega_{X,\psi,\varphi}(\theta) = \omega_{\tilde{F}_\theta,\tilde{\psi}_j,\tilde{\varphi}_i}(\mathbb{R} \cdot M_\theta(\theta))$. Using the corresponding statement in the affine case now (see proposition 1.2.7 (b)), we deduce that the two weight functions are equal, which proves the claim. □

1.6 Push-forward of tropical cycles and pull-back of Cartier divisors

Definition 1.6.1 (Morphisms of tropical cycles)
Let $C \in Z_n$ and $D \in Z_m$ be two tropical cycles. A *morphism* $f : C \to D$ of tropical cycles is a continuous map $f : |C| \to |D|$ with the following property: There exist reduced representatives $(((X, |X|), \omega_X), \{\Phi_\sigma\})$ of C and $(((Y, |Y|), \omega_Y), \{\Psi_\tau\})$ of D such that

(a) for every polyhedron $\sigma \in X$ there exists a polyhedron $\widetilde{\sigma} \in Y$ with $f(\sigma) \subseteq \widetilde{\sigma}$,

(b) for every pair $\sigma, \widetilde{\sigma}$ from (a) the map $\Psi_{\widetilde{\sigma}} \circ f \circ \Phi_\sigma^{-1} : |F_\sigma^X| \to |F_{\widetilde{\sigma}}^Y|$ induces a morphism of fans $\widetilde{F}_\sigma^X \to \widetilde{F}_{\widetilde{\sigma}}^Y$ (cf. definition 1.3.1), where $\widetilde{F}_\sigma^X$ and $\widetilde{F}_{\widetilde{\sigma}}^Y$ are the tropical fans associated to F_σ^X and $F_{\widetilde{\sigma}}^Y$ respectively (cf. definition 1.4.3).

First of all we want to show that the restriction of a morphism to a subcycle is again a morphism:

Lemma 1.6.2
Let $C \in Z_n$ and $D \in Z_m$ be two cycles, $f : C \to D$ a morphism and $E \in Z_k(C)$ a subcycle of C. Then the map $f|_{|E|} : |E| \to |D|$ induces a morphism of tropical cycles $f|_E : E \to D$.

Proof. By the definition of a morphism there exist reduced representatives $((X_1, |X_1|), \omega_{X_1})$ of C and $((Y, |Y|), \omega_Y)$ of D such that properties (a) and (b) in definition 1.6.1 are fulfilled. By the definition of a subcycle there exist reduced representatives $((Z_1, |Z_1|), \omega_{Z_1})$ of E and $((X_2, |X_2|), \omega_{X_2})$ of C such that properties (a) and (b) in definition 1.4.13 are fulfilled, i.e. such that $(Z_1, |Z_1|) \trianglelefteq (X_2, |X_2|)$ and the tropical structures on Z_1 and X_2 agree. As "$\sim$" is an equivalence relation there exists a common refinement $((X, |X|), \omega_X)$ of $((X_1, |X_1|), \omega_{X_1})$ and $((X_2, |X_2|), \omega_{X_2})$ which we may assume to be reduced. Applying construction 1.4.11 to Z_1 and X we obtain a refinement $((Z, |Z|), \omega_Z) := ((Z_1 \cap X, |Z_1 \cap X|), \omega_{Z_1 \cap X})$ of $((Z_1, |Z_1|), \omega_{Z_1})$ such that $(Z, |Z|) \trianglelefteq (X, |X|)$ and the tropical structures on Z and X agree. Thus properties (a) and (b) of definition 1.6.1 are fulfilled by Z and Y and the restricted map $f|_{|E|} : |E| \to |D|$ gives us a morphism $f|_E : E \to D$. □

If we are given a morphism and a tropical cycle the following construction shows how to build the push-forward cycle of the given one along our morphism:

Construction 1.6.3 (Push-forward of tropical cycles)
Let $C \in Z_n$ and $D \in Z_m$ be two cycles and let $f : C \to D$ be a morphism. Let $(((X, |X|, \{\varphi_\sigma\}), \omega_X), \{\Phi_\sigma\})$ and $(((Y, |Y|, \{\psi_\sigma\}), \omega_Y), \{\Psi_\tau\})$ be representatives of C and D fulfilling properties (a) and (b) of definition 1.6.1. Consider the collection of polyhedra

$$Z := \{f(\sigma) | \sigma \in X \text{ contained in a maximal polyhedron of } X \text{ on which } f \text{ is injective}\}.$$

In general Z is not a polyhedral complex. We resolve this by subdividing the polyhedra in Z and refining X accordingly:

Fix some polyhedron $\widetilde{\sigma} \in Y^{(m)}$ and let $\tau_1, \dots, \tau_r \in Z$ be all polyhedra that are contained in $\widetilde{\sigma}$. Property (b) of definition 1.6.1 implies that $\{\psi_{\widetilde{\sigma}}(\tau_i) | i = 1, \dots, r\}$ is a set of rational polyhedra in $\mathbb{R}^{n_{\widetilde{\sigma}}}$. As in remark and definition 1.4.14 let $\{g_i(x) \leq b_i | i = 1, \dots, N\}$, $g_i \in \mathrm{Hom}(\mathbb{Z}^{n_{\widetilde{\sigma}}}, \mathbb{Z})$, $b_i \in \mathbb{R}$ be all inequalities occurring in the description of all polyhedra in $\{\psi_{\widetilde{\sigma}}(\tau_i) | i = 1, \dots, r\}$ and let

$$\begin{aligned} R_{\widetilde{\sigma}} &:= \{\psi_{\widetilde{\sigma}}(\tau_i) | i = 1, \dots, r\} \cap H_{G_1, b_1} \cap \dots \cap H_{G_N, b_N}, \\ P_{\widetilde{\sigma}} &:= \{\psi_{\widetilde{\sigma}}^{-1}(\tau) | \tau \in R_{\sigma_i}\}. \end{aligned}$$

As in construction 1.4.15 $P_{\widetilde{\sigma}}$ can be seen as a kind of local refinement of Z. But here again taking the union over all maximal polyhedra $\widetilde{\sigma} \in Y^{(m)}$ does in general not lead to a global refinement as there may be overlaps between polyhedra coming from different $\widetilde{\sigma}$. We fix this as follows (cf. construction 1.4.15): For $\widetilde{\sigma} \in Y^{(m)}$ and $\widetilde{\tau} \in \bigcup_{i=0}^{m-1} Y^{(i)}$ let

$$P_{Z,\widetilde{\tau}}^{\widetilde{\sigma}} := \{\varrho \in P_{\widetilde{\sigma}} | \widetilde{\tau} \text{ is the inclusion minimal polyhedron of } Y \text{ containing } \varrho\}$$

and

$$P_{Z,m} := \bigcup_{\widetilde{\sigma} \in Y^{(m)}} \{\varrho \in P_{\widetilde{\sigma}} | \nexists \widetilde{\tau} \in Y^{(m-1)} : \varrho \subseteq \widetilde{\tau}\}.$$

Then

$$\widetilde{Z} := P_{Z,m} \cup \left(\bigcup_{\widetilde{\tau} \in Y^{(i)} : i < m} \{ \bigcap_{\widetilde{\sigma} \in Y^{(m)} : \widetilde{\tau} \subseteq \widetilde{\sigma}} \tau_{\widetilde{\sigma}} | \tau_{\widetilde{\sigma}} \in P_{Z,\widetilde{\tau}}^{\widetilde{\sigma}} \} \right)$$

is the set of polyhedra (without any overlaps now) that shall induce our wanted refinement of X: Let

$$T := \{\sigma \in X^{(n)} | f \text{ is injective on } \sigma\},$$
$$Q_0 := \{\tau \in X | \nexists \sigma \in T : \tau \subseteq \sigma\}$$

and

$$Q_1 := \left(\bigcup_{\sigma \in T} \{(f|_\sigma)^{-1}(\tau) | \tau \in \widetilde{Z}, \tau \subseteq f(\sigma)\} \right).$$

Then define $\widetilde{X} := Q_0 \cup Q_1$.
Let $\tau \in Q_1$ and choose $\sigma \in T$ with $\tau \subseteq \sigma$. Property (b) of definition 1.6.1 implies that $\psi_{\widetilde{\sigma}} \circ f \circ \varphi_\sigma^{-1}$ is integer affine linear where defined. Hence $\varphi_\sigma(\tau)$ is a rational polyhedron in $\mathbb{R}^{n_\sigma}$. Denote by $H_{\sigma,\tau}$ the smallest affine subspace of $\mathbb{R}^{n_\sigma}$ containing $\varphi_\sigma(\tau)$. We can consider $\varrho_\tau := \varphi_\sigma|_\tau$ to be a map $\varrho_\tau : \tau \to H_{\sigma,\tau} \cong \mathbb{R}^{n_\tau}$. Note that by construction $(\widetilde{X}, |X|, \{\varrho_\tau\})$ is a polyhedral complex. We endow it with the weight function $\omega_{\widetilde{X}}$ and tropical structure $\{\Phi_\tau^{\widetilde{X}}\}$ induced by X. Now we are able to define

$$f_*X := \{f(\sigma) | \sigma \in \widetilde{X} \text{ contained in a maximal polyhedron of } \widetilde{X} \text{ on which } f \text{ is injective}\}$$

and $|f_*X| := \bigcup_{\tau \in f_*X} \tau$. For every polyhedron $\tau \in f_*X$ let $\sigma_\tau \in Y$ be the inclusion-minimal polyhedron containing τ. Then define $\vartheta_\tau := \psi_{\sigma_\tau}|_\tau : \tau \to H_{\sigma_\tau,\tau} \cong \mathbb{R}^{n_\tau}$, where $H_{\sigma_\tau,\tau} \subseteq \mathbb{R}^{n_{\sigma_\tau}}$ is the smallest affine subspace containing the rational polyhedron $\psi_{\sigma_\tau}(\tau) \in \widetilde{Z}$. Note that this makes $(f_*X, |f_*X|, \{\vartheta_\tau\})$ into a polyhedral complex. Moreover, note that property (b) of definition 1.6.1 still holds for $\widetilde{X}$ and Y. Hence we

can assign weights and tropical fans to f_*X as follows: Let $\sigma \in f_*X$, let $\widetilde{\sigma} \in Y$ be the inclusion-minimal polyhedron containing it and let $\tau_1, \ldots, \tau_r \in \widetilde{X}$ be all polyhedra with $f(\tau_i) = \sigma$ that are contained in a maximal polyhedron of $\widetilde{X}$ on which f is injective. Then let $\Psi_{\widetilde{\sigma}}(S_{\widetilde{\sigma}}) = F^Y_{\widetilde{\sigma}}$ and $\Phi^{\widetilde{X}}_{\tau_i}(S_{\tau_i}) = F^{\widetilde{X}}_{\tau_i}$ respectively be the corresponding open fans and $\widetilde{F}^Y_{\widetilde{\sigma}}$, $\widetilde{F}^{\widetilde{X}}_{\tau_i}$ be the associated tropical fans. Property (b) of definition 1.6.1 implies that $f_*\widetilde{F}^{\widetilde{X}}_{\tau_i} \subseteq |\widetilde{F}^Y_{\widetilde{\sigma}}|$ is again a tropical fan (note that we do not need to refine $\widetilde{F}^{\widetilde{X}}_{\tau_i}$ to construct this push-forward). Thus we can define

$$\left(\widetilde{F}^{f_*X}_{\sigma}, \omega_{\widetilde{F}^{f_*X}_{\sigma}}\right) := \left(\bigcup_{i=1}^{r} f_*\widetilde{F}^{\widetilde{X}}_{\tau_i}, \sum_{i=1}^{r} \omega_{f_*\widetilde{F}^{\widetilde{X}}_{\tau_i}}\right) \quad \text{and} \quad F^{f_*X}_{\sigma} := \widetilde{F}^{f_*X}_{\sigma} \cap \Psi_{\widetilde{\sigma}}(S_\sigma)$$

(here again we assume that $\omega_{f_*\widetilde{F}^{\widetilde{X}}_{\tau_i}}(\tau) = 0$ if $\tau \notin f_*\widetilde{F}^{\widetilde{X}}_{\tau_i}$). Moreover we define

$$\Theta_\sigma := \Psi_{\widetilde{\sigma}}|_{S_\sigma} : S_\sigma \to |F^{f_*X}_{\sigma}|.$$

Then the map Θ_σ, $\sigma \in f_*X$ is 1:1 on polyhedra and we can endow the maximal polyhedra of f_*X with weights $\omega_{f_*X}(\cdot)$ coming from $F^{f_*X}_{\sigma}$ in this way. These weights are obviously well-defined by property (c) of the tropical polyhedral complex Y (cf. definition 1.4.4) and the maps Θ_σ for different $\sigma \in f_*X$ are obviously compatible. Hence we can define

$$f_*C := \left[\left(\left((f_*X, |f_*X|, \{\vartheta_\tau\}), \omega_{f_*X}\right), \{\Theta_\tau\}\right)\right] \in Z_n(D).$$

Note that the class $[(((f_*X, |f_*X|, \{\vartheta_\tau\}), \omega_{f_*X}), \{\Theta_\tau\})]$ is independent of the choices we made. Thus construction 1.6.3 immediately leads to the following

Corollary 1.6.4 (Push-forward of tropical cycles)
Let $C \in Z_n$ and $D \in Z_m$ be two cycles and let $f : C \to D$ be a morphism. Then for all k there is a well-defined and $\mathbb{Z}$-linear map

$$Z_k(C) \longrightarrow Z_k(D) : E \longmapsto f_*E := (f|_E)_*E.$$

Proof. The linearity can be proven similar to the affine case (cf. proposition 1.3.6). □

Our next aim is to define the pull-back of Cartier divisors. But first we need the following

Lemma 1.6.5
Let $C \in Z_n$ and $D \in Z_m$ be two tropical cycles and let $f : C \to D$ be a morphism. By definition there exist reduced representatives $(((X, |X|, \{\varphi_\sigma\}), \omega_X), \{\Phi_\sigma\})$ of C and $(((Y, |Y|, \{\psi_\tau\}), \omega_Y), \{\Psi_\tau\})$ of D such that properties (a) and (b) in definition 1.6.1 are fulfilled. Let $(((Y_1, |Y_1|, \{\psi'_{\tau'}\}), \omega_{Y_1}), \{\Psi'_{\tau'}\})$ be a refinement of Y. Then there is a refinement $(((X_1, |X_1|, \{\varphi'_{\sigma'}\}), \omega_{X_1}), \{\Phi'_\sigma\})$ of X such that properties (a) and (b) of definition 1.6.1 are fulfilled for X_1 and Y_1.

Proof. Let $X_1 := \{\sigma \cap f^{-1}(\tau) | \sigma \in X, \tau \in Y_1\}$. By property (b) of definition 1.6.1 all $\varphi_\sigma(\sigma \cap f^{-1}(\tau))$ are rational polyhedra in $\mathbb{R}^{n_\sigma}$. For every $\sigma' \in X_1$ choose $\sigma \in X$ such that $\sigma' = \sigma \cap f^{-1}(\tau)$ for some $\tau \in Y_1$. Then we can define $\varphi'_{\sigma'} := \varphi_\sigma|_{\sigma'} : \sigma' \to H_{\sigma,\sigma'} \cong \mathbb{R}^{n_{\sigma'}}$, where $H_{\sigma,\sigma'}$ is the smallest affine subspace of $\mathbb{R}^{n_\sigma}$ containing $\varphi_\sigma(\sigma')$. Moreover let $|X_1| := |X|$. Note that with these settings $(X_1, |X_1|, \{\varphi'_{\sigma'}\})$ is a polyhedral complex. We can endow it with the weight function ω_{X_1} and the tropical structure $\{\Phi'_{\sigma'}\}$ induced by X. Together with Y_1 the tropical polyhedral complex $(((X_1, |X_1|, \{\varphi'_{\sigma'}\}), \omega_{X_1}), \{\Phi'_{\sigma'}\})$ fulfills the requirements (a) and (b) of definition 1.6.1. □

Proposition 1.6.6 (Pull-back of Cartier divisors)
Let $C \in Z_n$ and $D \in Z_m$ be tropical cycles and let $f : C \to D$ be a morphism. Then there is a well-defined and $\mathbb{Z}$-linear map

$$\mathrm{Div}(D) \longrightarrow \mathrm{Div}(C) : [\{(U_i, h_i)\}] \longmapsto f^*[\{(U_i, h_i)\}] := [\{(f^{-1}(U_i), h_i \circ f)\}].$$

Proof. We have to show that $h \circ f \in \mathcal{K}^*_C(f^{-1}(U))$ for $h \in \mathcal{K}^*_D(U)$ and that $h \circ f \in \mathcal{O}^*_C(f^{-1}(U))$ for $h \in \mathcal{O}^*_D(U)$. Then the rest is obvious.
So let $h \in \mathcal{K}^*_D(U)$. Then there exists a representative $(((Y, |Y|, \{\psi_\sigma\}), \omega_Y), \{\Psi_\tau\})$ of D such that for every polyhedron $\sigma \in Y$ the map $h \circ \psi_\sigma^{-1}$ is locally integer affine linear. Moreover, since f is a morphism there exist representatives $(((X, |X|, \{\varphi_\sigma\}), \omega_X), \{\Phi_\tau\})$ of C and $(((Y', |Y'|, \{\psi'_{\sigma'}\}), \omega_{Y'}), \{\Psi'_{\tau'}\})$ of D such that properties (a) and (b) of definition 1.6.1 are fulfilled, i.e. $f(\sigma) \subseteq \widetilde{\sigma} \in Y'$ for all $\sigma \in X$ and the maps $\Psi_{\widetilde{\sigma}} \circ f \circ \Phi_\sigma^{-1}$ induce morphisms of fans. By lemma 1.6.5 we may assume that $Y = Y'$. Now let $\sigma \in X$ and choose some $\widetilde{\sigma} \in Y$ such that $f(\sigma) \subseteq \widetilde{\sigma}$. Property (b) of definition 1.6.1 implies that $\psi_{\widetilde{\sigma}} \circ f \circ \varphi_\sigma^{-1}$ and $\Psi_{\widetilde{\sigma}} \circ f \circ \Phi_\sigma^{-1}$ are integer affine linear. Thus $h \circ f \circ \varphi_\sigma^{-1} = (h \circ \psi_{\widetilde{\sigma}}^{-1}) \circ (\psi_{\widetilde{\sigma}} \circ f \circ \varphi_\sigma^{-1})$ is locally integer affine linear and $h \circ f \in \mathcal{K}^*_C(f^{-1}(U))$. If additionally $h \circ \Psi_{\widetilde{\sigma}}^{-1}$ is locally integer affine linear then so is $h \circ f \circ \Phi_\sigma^{-1} = (h \circ \Psi_{\widetilde{\sigma}}^{-1}) \circ (\Psi_{\widetilde{\sigma}} \circ f \circ \Phi_\sigma^{-1})$. Hence $h \circ f \in \mathcal{O}^*_C(f^{-1}(U))$ for $h \in \mathcal{O}^*_D(U)$. □

Our last step in this chapter is to state the analogon of the projection formula from 1.3.8:

Proposition 1.6.7 (Projection formula)
Let $C \in Z_n$ and $D \in Z_m$ be two cycles and $f : C \to D$ be a morphism. Let $E \in Z_k(C)$ be a subcycle of C and $d \in \mathrm{Div}(D)$ be a Cartier divisor. Then the following holds:

$$d \cdot (f_* C) = f_*(f^* d \cdot C) \in Z_{k-1}(D).$$

Proof. The claim follows from the constructions of f_*C and f^*d, from definition 1.5.5 and proposition 1.3.8. □

1.7 Rational equivalence

In this section we will introduce a concept of rational equivalence and prove that this equivalence is compatible with push-forwards of cycles and pull-backs of Cartier divisors.

Definition 1.7.1 (Rational equivalence)
Let C be a cycle. We denote by $R(C) := \{(|C|, h) | h \text{ bounded}\} \subseteq \mathrm{Div}(C)$ the subgroup of all Cartier divisors globally given by a bounded rational function. Then we define the *Picard group* of C to be

$$\mathrm{Pic}(C) := \mathrm{Div}(C)/R(C).$$

Moreover, let D be a subcycle of C. We call D *rationally equivalent to zero on* C if there exist a tropical cycle C' of dimension $\dim(D)+1$, a morphism $f : C' \to C$ and a bounded rational function h on C' such that

$$f_*(h \cdot C') = D.$$

Let D' be another subcycle of C. Then we call D and D' rationally equivalent, denoted by $D \sim D'$, if $D - D'$ is rationally equivalent to zero. We define the k-th *Chow group* of C to be

$$A_k(C) := Z_k(C)/\sim .$$

To prove that this equivalence is not too strong for applications in enumerative geometry we need the following lemma:

Lemma 1.7.2
Let C be a one-dimensional abstract tropical cycle, $\varphi \in R(C)$ a bounded rational function on C and $(((X, |X|, \{m_\sigma\}_{\sigma\in X}), \omega_X), \{M_\sigma\}_{\sigma\in X})$ a reduced representative of C such that for all $\sigma \in X$ the map $\varphi \circ m_\sigma^{-1} =: \varphi_\sigma$ is integer affine linear. Then

$$\sum_{\{p\}\in X^{(0)}} \omega_\varphi(\{p\}) = 0,$$

i.e. $\varphi \cdot C$ is of degree zero.

The proof of this statement is to a large extent the work of Johannes Rau, my coauthor of [AR07] and [AR08]. Hence we skip it here and refer to [AR07, lemma 8.3] instead.

Remark 1.7.3
Note that this property $\sum_{\{p\}\in X^{(0)}} \omega_\varphi(\{p\}) = 0$ is preserved by pushing forward. As a consequence, for any tropical cycle C there is a well-defined morphism

$$\deg : A_0(C) \longrightarrow \mathbb{Z} : [\lambda_1 P_1 + \ldots + \lambda_r P_r] \longmapsto \lambda_1 + \ldots + \lambda_r.$$

For $D \in A_0(C)$ the number $\deg(D)$ is called the *degree* of D.

Lemma 1.7.4
Let D be a cycle in C that is rationally equivalent to zero. Then the following holds:

(a) Let E be another cycle. Then $D \times E$ is also rationally equivalent to zero.

(b) Let $\varphi \in \mathrm{Div}(C)$ be a Cartier divisor on C. Then $\varphi \cdot D$ is also rationally equivalent to zero.

(c) Let $g : C \to \widetilde{C}$ be a morphism. Then $g_*(D)$ is also rationally equivalent to zero.

Proof. Let $f : C' \to C$ be a morphism and h a bounded function on C' such that $f_*(h \cdot C') = D$. Then $f \times \mathrm{id} : C' \times E \to C \times E$ provides (a), restricting f to $f : f^*(\varphi) \cdot C' \to C$ provides (b) and composing f with g provides (c). □

Remark 1.7.5
In particular, the above lemma implies that there is a well-defined map

$$\mathrm{Pic}(C)^d \times A_d(C) \longrightarrow \mathbb{Z} : (([\varphi_1], \ldots, [\varphi_d]), D) \longmapsto \deg([\varphi_1 \cdot \ldots \cdot \varphi_d \cdot D]),$$

where C is our ambient cycle. This map is of particular interest when dealing with enumerative questions.

Of course, our notion of rational equivalence should also be compatible with the pull-back of Cartier divisors. This is indeed the case:

Lemma 1.7.6 (Pull-back of rational equivalence)
Let C, D be tropical cycles and let $f : C \to D$ be a morphism between them. Then the pull-back map $\mathrm{Div}(D) \to \mathrm{Div}(C), \varphi \mapsto f^*\varphi$ *induces a well-defined map on the quotients* $\mathrm{Pic}(D) \to \mathrm{Pic}(C) : [\varphi] \mapsto [f^*\varphi]$.

Proof. We only have to show that for each element $(|D|, \psi) \in R(D)$ the pull-back Cartier divisor $f^*(|D|, \psi)$ lies in $R(C)$. But this follows from the trivial fact that the composition $\psi \circ f$ of a bounded function ψ and an arbitrary map f is again bounded. □

1.8 Intersection of cycles in $\mathbb{R}^n$

So far we are only able to intersect Cartier divisors with cycles. Our aim in this section is now to define the intersection of two cycles with ambient cycle $\mathbb{R}^n$ (with trivial structure maps). We do this as follows:

Remark 1.8.1
We can express the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$

$$[(\triangle, 1)] = [(\{(x, x) | x \in \mathbb{R}^n\}, 1)] \in Z_n(\mathbb{R}^n \times \mathbb{R}^n)$$

as a product of Cartier divisors, namely

$$[(\triangle, 1)] = \psi_1 \cdots \psi_n \cdot (\mathbb{R}^n \times \mathbb{R}^n),$$

where $\psi_i = [\{(\mathbb{R}^n \times \mathbb{R}^n, \max\{0, x_i - y_i\})\}] \in \mathrm{Div}(\mathbb{R}^n \times \mathbb{R}^n)$, $i = 1, \ldots, n$. We will use this fact to define the intersection product of any two cycles in $\mathbb{R}^n$.

Definition 1.8.2
Let $\pi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n : (x, y) \mapsto x$. Then we define the intersection product of cycles in $\mathbb{R}^n$ by

$$\begin{array}{rcl} Z_{n-k}(\mathbb{R}^n) \times Z_{n-l}(\mathbb{R}^n) & \longrightarrow & Z_{n-k-l}(\mathbb{R}^n) \\ (C, D) & \longmapsto & C \cdot D := \pi_*(\triangle \cdot (C \times D)), \end{array}$$

where π_* denotes the push-forward as defined in 1.6.4 and $\triangle \cdot (C \times D) := \psi_1 \cdots \psi_n \cdot (C \times D)$ with $\psi_1, \ldots, \psi_n$ as defined in remark 1.8.1.

Having defined this intersection product of arbitrary cycles in $\mathbb{R}^n$ we will prove now some basic properties. But as a start we need the following lemmas:

Lemma 1.8.3
Let $C \in Z_k(\mathbb{R}^n)$ be a cycle with representative (X, ω_X) and let $\psi_1, \ldots, \psi_n$ be the Cartier divisors defined in remark 1.8.1. Then (X_j, ω_{X_j}) with

$$X_j := \big\{(\mathbb{R}^n \times \sigma) \cap \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n | x_i = y_i \text{ for } i = j, \ldots, n\} | \sigma \in X\big\},$$

$$\omega_{X_j}\big((\mathbb{R}^n \times \sigma) \cap \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n | x_i = y_i \text{ for } i = j, \ldots, n\}\big) := \omega_X(\sigma)$$

is a representative of $\psi_j \cdots \psi_n \cdot (\mathbb{R}^n \times C)$.

Proof. We use induction on j. For $j = n+1$ there is nothing to show. Now let the above representative be correct for some $j+1$. We have to show that X_j is a tropical polyhedral complex and that it represents $\psi_j \cdots \psi_n \cdot (\mathbb{R}^n \times C)$: Note that

$$\begin{aligned} &\dim\left((\mathbb{R}^n \times \sigma) \cap \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n | x_i = y_i \text{ for } i = j, \ldots, n\}\right) \\ &< \dim\left((\mathbb{R}^n \times \sigma) \cap \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n | x_i = y_i \text{ for } i = j+1, \ldots, n\}\right) \end{aligned} \qquad (*)$$

for all $\sigma \in X$. Hence X_j is a tropical polyhedral complex. Moreover note that

$$\widetilde{X}_{j+1} := \{\sigma \cap \{x_j - y_j = 0\}, \sigma \cap \{x_j - y_j \leq 0\}, \sigma \cap \{x_j - y_j \geq 0\} | \sigma \in X_{j+1}\}$$

with weights induced by X_{j+1} is a refinement of X_{j+1} such that $\max\{0, x_j - y_j\}$ is linear on every face of $\widetilde{X}_{j+1}$. By $(*)$ there are exactly two types of faces of codimension one in $\widetilde{X}_{j+1}$:

(i) $(\mathbb{R}^n \times \sigma) \cap \{x_i - y_i = 0 \text{ for } i = j, \ldots, n\}$ with $\sigma \in X$, $\operatorname{codim}(\sigma) = 0$,

(ii) $(\mathbb{R}^n \times \sigma) \cap \{x_i - y_i = 0 \text{ for } i = j+1, \ldots, n;\ x_j - y_j \leq 0\}$ or
$(\mathbb{R}^n \times \sigma) \cap \{x_i - y_i = 0 \text{ for } i = j+1, \ldots, n;\ x_j - y_j \geq 0\}$ with $\sigma \in X$,
$\operatorname{codim}(\sigma) = 1$,

where the faces of the second type are not contained in $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n | x_j = y_j\}$. Hence $\max\{0, x_j - y_j\}$ is linear on a neighborhood of every face of type (ii) and thus these faces get weight zero in $\max\{0, x_j - y_j\} \cdot \widetilde{X}_{j+1}$. The faces of type (i) are weighted by $\omega_{X_{j+1}}((\mathbb{R}^n \times \sigma) \cap \{x_i - y_i = 0 \text{ for } i = j+1, \ldots, n\})$ in $\max\{0, x_j - y_j\} \cdot \widetilde{X}_{j+1}$ since $x_1 - y_1, \ldots, x_n - y_n$ are part of a lattice basis of $(\mathbb{Z}^n \times \mathbb{Z}^n)^\vee$. Thus $\max\{0, x_j - y_j\} \cdot \widetilde{X}_{j+1} = X_j$ and X_j is a representative of $\psi_j \cdots \psi_n \cdot (\mathbb{R}^n \times C)$. □

Corollary 1.8.4
Let $C \in Z_k(\mathbb{R}^n)$ be a cycle. Then we have the equation:

$$\mathbb{R}^n \cdot C = C.$$

Proof. Let (X, ω_X) be a representative of C, let $\pi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n : (x, y) \mapsto x$ and let $\psi_1, \ldots, \psi_n$ be the Cartier divisors defined in remark 1.8.1. By lemma 1.8.3 we know that $X_1 = \{\{(x, x)|x \in \sigma\}|\sigma \in X\}$ with $\omega_{X_1}(\{(x, x)|x \in \sigma\}) = \omega_X(\sigma)$ is a representative of $\psi_1 \cdots \psi_n \cdot (\mathbb{R}^n \times C)$. Hence

$$\mathbb{R}^n \cdot C = \pi_*(\psi_1 \cdots \psi_n \cdot \mathbb{R}^n \times C) = [\pi_*(X_1, \omega_{X_1})] = [(X, \omega_X)] = C.$$

□

Lemma 1.8.5
Let $C \in Z_k(\mathbb{R}^n)$ and $D \in Z_l(\mathbb{R}^m)$ be abstract cycles, $\varphi \in \mathrm{Div}(\mathbb{R}^n)$ a Cartier divisor and $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n : (x, y) \mapsto x$. Then:

$$(\varphi \cdot C) \times D = \pi^*\varphi \cdot (C \times D).$$

Proof. We prove the statement for affine cycles C, D and an affine Cartier divisor φ. The general case then follows by applying the statement locally.
Choose arbitrary representatives Y of D and h of φ and choose a representative X of C such that h is linear on every face of X. This implies that π^*h is linear on every face of $X \times Y$, too. In $X \times Y$ we have two types of faces of codimension one:

(i) $\sigma \times \tau$ with $\sigma \in X, \tau \in Y, \mathrm{codim}(\sigma) = 1, \mathrm{codim}(\tau) = 0$,

(ii) $\sigma \times \tau$ with $\sigma \in X, \tau \in Y, \mathrm{codim}(\sigma) = 0, \mathrm{codim}(\tau) = 1$.

For the second type the adjacent facets are exactly all $\sigma \times \widetilde{\tau}$ with $\widetilde{\tau} > \tau$. We get $\omega_h(\sigma \times \tau) = 0$ in $h \cdot (X \times Y)$ as π^*h is linear on $\sigma \times |Y|$. For the first type the adjacent facets are exactly all $\widetilde{\sigma} \times \tau$ with $\widetilde{\sigma} > \sigma$ and the weights can be calculated exactly like for $h \cdot X$. This finishes the proof. □

Let C and D be cycles in $\mathbb{R}^n$. Assume that C can be expressed as a product of Cartier divisors, i.e. there are $\varphi_1, \ldots, \varphi_r \in \mathrm{Div}(\mathbb{R}^n)$ such that $C = \varphi_r \cdots \varphi_1 \cdot \mathbb{R}^n$. The obvious questions are now how $C \cdot D$ relates to $\varphi_r \cdots \varphi_1 \cdot D$ and whether $\varphi_r \cdots \varphi_1 \cdot D$ depends on the choice of the Cartier divisors φ_i. To answer this question we first prove a somewhat stronger statement:

Lemma 1.8.6
Let $C \in Z_k(\mathbb{R}^n)$ and $D \in Z_l(\mathbb{R}^n)$ be cycles and $\varphi \in \mathrm{Div}(\mathbb{R}^n)$ a Cartier divisor. Then we have the equality:

$$(\varphi \cdot C) \cdot D = \varphi \cdot (C \cdot D).$$

Proof. Let $\pi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n : (x, y) \mapsto x$ be like above. The following holds:

$$\begin{aligned}
(\varphi \cdot C) \cdot D &= \pi_*(\triangle \cdot ((\varphi \cdot C) \times D)) \\
&\overset{1.8.5}{=} \pi_*(\pi^*\varphi \cdot \triangle \cdot (C \times D)) \\
&\overset{1.6.7}{=} \varphi \cdot \pi_*(\triangle \cdot (C \times D)) \\
&= \varphi \cdot (C \cdot D).
\end{aligned}$$

□

Corollary 1.8.7
Let $C \in Z_k(\mathbb{R}^n)$ be a cycle such that there are Cartier divisors $\varphi_1, \dots, \varphi_r \in \mathrm{Div}(\mathbb{R}^n)$ with $\varphi_r \cdots \varphi_1 \cdot \mathbb{R}^n = C$ and let $D \in Z_l(\mathbb{R}^n)$ be any cycle. Then

$$\varphi_r \cdots \varphi_1 \cdot D = C \cdot D.$$

Proof. Applying lemma 1.8.6 and lemma 1.8.3 we obtain

$$C \cdot D = (\varphi_r \cdots \varphi_1 \cdot \mathbb{R}^n) \cdot D = \varphi_r \cdots \varphi_1 \cdot (\mathbb{R}^n \cdot D) = \varphi_r \cdots \varphi_1 \cdot D.$$

□

Remark 1.8.8
Note that corollary 1.8.7 in particular implies that our definition of the intersection product on $\mathbb{R}^n$ (cf. definition 1.8.2) is independent of the choice of the Cartier divisors describing the diagonal $\triangle$.

Theorem 1.8.9
Let $C, C' \in Z_k(\mathbb{R}^n)$, $D \in Z_l(\mathbb{R}^n)$ and $E \in Z_m(\mathbb{R}^n)$ be cycles. Then the following equations hold:

(a) $C \cdot D = D \cdot C$,

(b) $(C + C') \cdot D = C \cdot D + C' \cdot D$,

(c) $(C \cdot D) \cdot E = C \cdot (D \cdot E)$.

Proof. (a): Let $\pi^i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n : (x_1, x_2) \mapsto x_i$ and $\pi^{12} : (\mathbb{R}^n \times \mathbb{R}^n)^2 \to \mathbb{R}^n \times \mathbb{R}^n : (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2)$. Moreover, let $\triangle_{\mathbb{R}^n}$ be the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$ and $\triangle_{\mathbb{R}^n \times \mathbb{R}^n}$ be the diagonal in $(\mathbb{R}^n \times \mathbb{R}^n)^2$. Then we can conclude by corollary 1.8.7 that

$$\begin{array}{rcll} |\psi_1 \cdots \psi_n \cdot (C \times D)| & = & |\triangle_{\mathbb{R}^n} \cdot (C \times D)| & \\ & = & |\pi^{12}_*(\triangle_{\mathbb{R}^n \times \mathbb{R}^n} \cdot (\triangle_{\mathbb{R}^n} \times (C \times D)))| & \subseteq \ |\triangle_{\mathbb{R}^n}| \end{array}$$

and hence

$$\begin{array}{rcl@{\quad}rcl} C \cdot D & = & \pi^1_*(\psi_1 \cdots \psi_n \cdot (C \times D)) & = & \pi^1_*(\triangle_{\mathbb{R}^n} \cdot (C \times D)) \\ & = & \pi^2_*(\triangle_{\mathbb{R}^n} \cdot (C \times D)) & = & \pi^1_*(\triangle_{\mathbb{R}^n} \cdot (D \times C)) \\ & = & \pi^1_*(\psi_1 \cdots \psi_n \cdot (D \times C)) & = & D \cdot C. \end{array}$$

(b): Follows immediately by bilinearity of the intersection product

$$\mathrm{Div}(\mathbb{R}^n \times \mathbb{R}^n) \times Z_p(\mathbb{R}^n \times \mathbb{R}^n) \xrightarrow{\cdot} Z_{p-1}(\mathbb{R}^n \times \mathbb{R}^n),$$

linearity of the push-forward and the fact that $(C + C') \times D = C \times D + C' \times D$.
(c): We will show that $\triangle \cdot (C \times \pi_*(\triangle \cdot (D \times E))) = \triangle \cdot (\pi_*(\triangle \cdot (C \times D)) \times E)$:
Let $\pi^{12} : (\mathbb{R}^n)^3 \to (\mathbb{R}^n)^2 : (x, y, z) \mapsto (x, y)$, $\pi^{13} : (\mathbb{R}^n)^3 \to (\mathbb{R}^n)^2 : (x, y, z) \mapsto (x, z)$ and $\pi^{23} : (\mathbb{R}^n)^3 \to (\mathbb{R}^n)^2 : (x, y, z) \mapsto (y, z)$. An easy calculation shows that

$$\triangle \cdot (C \times \pi_*(\triangle \cdot (D \times E))) = \triangle \cdot \pi^{12}_*(C \times (\triangle \cdot (D \times E))) \tag{1.1}$$

and

$$\triangle \cdot (\pi_*(\triangle \cdot (C \times D)) \times E) = \triangle \cdot \pi_*^{13}((\triangle \cdot (C \times D)) \times E). \tag{1.2}$$

Now let $\psi_1, \ldots, \psi_n$ be the Cartier divisors defined in remark 1.8.1. We label these Cartier divisors with pairs of letters ψ_i^{xy} to point out the coordinates they are acting on. We obtain

$$\begin{aligned}
&\triangle \cdot (C \times \pi_*(\triangle \cdot (D \times E))) \\
&\overset{(1.1)}{=} \triangle \cdot \pi_*^{12}(C \times (\triangle \cdot (D \times E))) \\
&= \psi_1^{xy} \cdots \psi_n^{xy} \cdot \pi_*^{12}(C \times (\psi_1^{yz} \cdots \psi_n^{yz} \cdot (D \times E))) \\
&\overset{1.6.7}{=} \pi_*^{12}((\pi^{12})^* \psi_1^{xy} \cdots (\pi^{12})^* \psi_n^{xy} \cdot (C \times (\psi_1^{yz} \cdots \psi_n^{yz} \cdot (D \times E)))) \\
&\overset{1.8.5}{=} \pi_*^{12}((\pi^{23})^* \psi_1^{yz} \cdots (\pi^{23})^* \psi_n^{yz} \cdot (\pi^{12})^* \psi_1^{xy} \cdots (\pi^{12})^* \psi_n^{xy} \cdot (C \times D \times E)) \\
&\overset{1.8.7}{=} \pi_*^{13}((\pi^{12})^* \psi_1^{xy} \cdots (\pi^{12})^* \psi_n^{xy} \cdot (\pi^{13})^* \psi_1^{xz} \cdots (\pi^{13})^* \psi_n^{xz} \cdot (C \times D \times E)) \\
&\overset{1.8.5}{=} \pi_*^{13}((\pi^{13})^* \psi_1^{xz} \cdots (\pi^{13})^* \psi_n^{xz} \cdot (\psi_1^{xy} \cdots \psi_n^{xy} \cdot (C \times D)) \times E) \\
&\overset{1.6.7}{=} \psi_1^{xz} \cdots \psi_n^{xz} \cdot \pi_*^{13}((\psi_1^{xy} \cdots \psi_n^{xy} \cdot (C \times D)) \times E) \\
&= \triangle \cdot \pi_*^{13}((\triangle \cdot (C \times D)) \times E) \\
&\overset{(1.2)}{=} \triangle \cdot (\pi_*(\triangle \cdot (C \times D)) \times E).
\end{aligned}$$

This proves (c). □

It remains to show that our intersection product is well-defined modulo rational equivalence. If this is the case the intersection product induced on $A_*(\mathbb{R}^n)$ clearly inherits the properties of the intersection product on $Z_*(\mathbb{R}^n)$ we have proven in this section.

Proposition 1.8.10
The intersection product $Z_{n-k}(\mathbb{R}^n) \times Z_{n-l}(\mathbb{R}^n) \overset{\cdot}{\longrightarrow} Z_{n-k-l}(\mathbb{R}^n)$ induces a well-defined and bilinear map

$$A_{n-k}(\mathbb{R}^n) \times A_{n-l}(\mathbb{R}^n) \overset{\cdot}{\longrightarrow} A_{n-k-l}(\mathbb{R}^n) : ([C], [D]) \longmapsto [C] \cdot [D] := [C \cdot D].$$

Proof. The intersection product $C \cdot D$ is defined to be

$$\pi_*(\max\{x_1, y_1\} \cdots \max\{x_r, y_r\} \cdot (C \times D)),$$

where the x_i (resp. y_i) are the coordinates of the first (resp. second) factor of $\mathbb{R}^n \times \mathbb{R}^n$ and $\pi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is the projection onto the first factor (cf. definition 1.8.2). Thus we can apply lemma 1.7.4 (a) – (c) and the claim follows. □

1.9 Rational equivalence on $\mathbb{R}^n$

In this section we will prove that every tropical cycle in $\mathbb{R}^n$ is rationally equivalent to a uniquely determined affine cycle, called its *degree*. We will use this equivalence to prove a tropical version of Bézout's theorem.
We start the section with the easiest example of rationally equivalent cycles in $\mathbb{R}^n$, namely translations:

Definition 1.9.1
Let X be a tropical polyhedral complex in $\mathbb{R}^n$ and let $v \in \mathbb{R}^n$. We denote by $X(v)$ the translation

$$X(v) := \{\sigma + v | \sigma \in X\}$$

of X along v. If $[X] = C \in Z_k(\mathbb{R}^n)$ then $C(v) := [X(v)]$. Note that the class $C(v)$ is independent of the representative X.

Lemma 1.9.2
Let C be a subcycle of $\mathbb{R}^n$ and let $v \in \mathbb{R}^n$ be an arbitrary vector. Then the equation

$$C(v) \sim C$$

holds.

Proof. Consider the cycle $C \times \mathbb{R}$ in $\mathbb{R}^n \times \mathbb{R}$ with morphism

$$\begin{aligned} f : \mathbb{R}^n \times \mathbb{R} &\to \mathbb{R}^n, \\ (x,t) &\mapsto x + t \cdot e_i, \end{aligned}$$

where e_i is the i-th unit vector in $\mathbb{R}^n$. For $\mu \in \mathbb{R}$ let h_μ be the bounded function

$$h_\mu(x,t) = \begin{cases} 0 & t \leq 0 \\ t & 0 \leq t \leq \mu \\ \mu & t \geq \mu. \end{cases}$$

Then $f_*(h_\mu \cdot C \times \mathbb{R}) = C - C(\mu \cdot e_i)$, which proves the claim. □

Definition 1.9.3
Let C be a cycle in $\mathbb{R}^n$ of codimension k. Then we define d_C to be the map

$$\begin{aligned} d_C : \ Z_k(\mathbb{R}^n) &\longrightarrow \mathbb{Z}, \\ D &\longmapsto \deg(C \cdot D). \end{aligned}$$

Lemma 1.9.4
Let $C = [(X, \omega_X)]$ be a d-dimensional affine cycle in $\mathbb{R}^n$. Then there always exists a representative $(X', \omega_{X'})$ of C and a complete simplicial fan Θ such that $X' \subseteq \Theta$.

Proof. Let $X_0 := X = \{\sigma_1, \ldots, \sigma_N\}$ and let $\sigma_i = \{x \in \mathbb{R}^n | f_1^{\sigma_i}(x) \geq 0, \ldots, f_{k_{\sigma_i}}^{\sigma_i}(x) \geq 0\}$. Moreover, let $Y_0 := \{\mathbb{R}^n\}$ and for $f \in \Lambda^\vee$ let

$$H_f := \{\{x | f(x) \geq 0\}, \{x | f(x) = 0\}, \{x | f(x) \leq 0\}\}.$$

For all $i = 1, \ldots, N$ we construct refinements

$$X_i := X_{i-1} \cap H_{f_1^{\sigma_i}} \cap \ldots \cap H_{f_{k_{\sigma_i}}^{\sigma_i}}$$

and

$$Y_i := Y_{i-1} \cap H_{f_1^{\sigma_i}} \cap \ldots \cap H_{f_{k_{\sigma_i}}^{\sigma_i}}$$

as described in [GKM07, 2.5(e)]. This construction yields fans X_N and Y_N with $X_N^{(k)} \subseteq Y_N^{(k)}$ for all k and $|X| = |X_N|$. Moreover, Y_N is a complete fan in $\mathbb{R}^n$. We can make Y_N into a simplicial fan by further subdividing its cones: Let $\Theta := Y_N$. If $\sigma \in \Theta^{(p)}$ is generated by vectors $v_1, \ldots, v_q$ then remove σ and add all cones $\mathbb{R}_{\geq 0} v_{i_1} + \ldots + \mathbb{R}_{\geq 0} v_{i_k}$ for $1 \leq k \leq p$ and $1 \leq i_1 < \ldots < i_k \leq q$ to Θ. Finally, we take $(X', \omega_{X'}) := (X, \omega_X) \cap \Theta$ as described in [GKM07, 2.11(b)]. □

Lemma 1.9.5
Let C_1 and C_2 be affine cycles in $\mathbb{R}^n$ with $d_{C_1} = d_{C_2}$. Then $C_1 = C_2$.

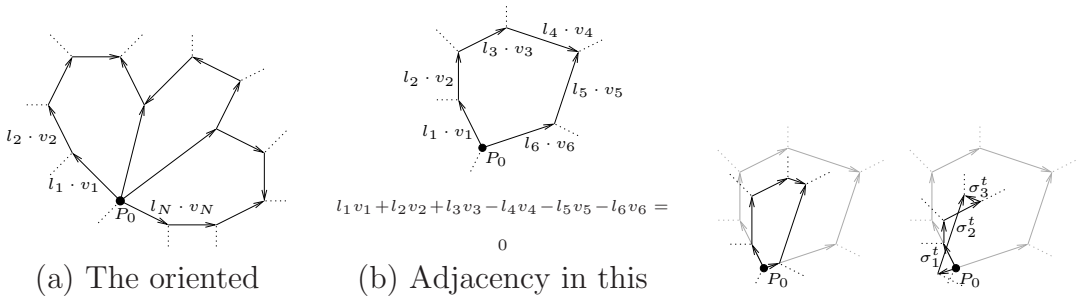
The proof of this statement is to a large extent the work of Johannes Rau, my coauthor of [AR07] and [AR08]. Hence we skip it here and refer to [AR08, lemma 6] instead.
Combining this statement with proposition 1.8.10 we can immediately conclude the following:

Corollary 1.9.6
Let C_1 and C_2 be affine cycles in $\mathbb{R}^n$ with $C_1 \sim C_2$. Then $C_1 = C_2$.

Theorem 1.9.7
Let C be a cycle in $\mathbb{R}^n$. Then there exists an affine cycle $\delta(X)$ in $\mathbb{R}^n$ with

$$X \sim \delta(X).$$

Proof. Let (X_1, ω_{X_1}) be a representative of $C_1 := C$. Refining (X_1, ω_{X_1}) we may assume that every polyhedron $\sigma \in X_1$ is the convex hull of its 1-skeleton (see for example [Z95, 1.2 and 2.2]) and that every polyhedron $\sigma \in X_1$ contains at least one vertex $\sigma \supseteq P_\sigma \in X_1^{(0)}$.
The 1-skeleton of X_1 is a finite graph Γ with edges $X_1^{(1)} = \{\varepsilon_1, \ldots, \varepsilon_N\}$ and vertices $X_1^{(0)} = \{P_0, \ldots, P_M\}$. By lemma 1.9.2 we may assume that P_0 is the origin. On every edge ε_i of this graph we choose an orientation and a primitive direction vector $v_i \in \Lambda_{\varepsilon_i}$ respecting this orientation (see figure 1.10 (a)). Then for $i = 1, \ldots, N$ let $l_i \cdot ||v_i||$ be the length of the edge ε_i (we set $l_i = \infty$ if ε_i is unbounded).
Adjacency of the bounded edges in the graph Γ yields a system of linear equations in the variables l_i having the entries of the vectors v_i as coefficients (see figure 1.10 (b)). As the system is solved by the given lengths $l_i \in \mathbb{R}_{>0}$ and all vectors v_i are integral there exists a positive and integral solution $l_1', \ldots, l_N'$. Using these numbers l_i' we construct a polyhedral complex X_1^t, $t \in \mathbb{R}$ as follows: We keep the position of the point P_0 fixed and for all $i = 1, \ldots, N$ with $l_i < \infty$ we change the length of the edge ε_i to $l_i + t \cdot l_i'$. For all unbounded edges ε_i we just keep the directions and the lengths unchanged. For a given polyhedron $\sigma \in X_1$ this process yields a deformation σ^t of



(a) The oriented graph constructed from Γ.

(b) Adjacency in this graph yields linear equations.

(c) Two examples of the shrinking process.

Figure 1.10: Constructions described in the proof of theorem 1.9.7.

σ which is not necessarily a polyhedron, but that can be decomposed into polyhedra $\sigma_1^t, \dots, \sigma_{p_t}^t$ (see figure 1.10 (c)). If such a polyhedron σ_j^t is of dimension $\dim(C)$, then we define its weight to be

$$\widetilde{\omega_{X_1^t}}(\sigma_j^t) := (-1)^{\delta(\sigma_j^t)} \cdot \omega_{X_1}(\sigma),$$

where $\delta(\sigma_j^t)$ is defined as follows: $\delta(\sigma_j^t) := \sum d(\sigma_j^{t'})$, where the sum is taken over all values $t' \in \mathbb{R}$ between 0 and t such that at least one of the lengths $l_{i_q} + t' \cdot l'_{i_q}$ occurring in the boundary of σ_j^t is zero and $d(\sigma_j^{t'}) := \dim(\sigma_j^t) - \dim(\sigma_j^{t'})$. We denote by $\widetilde{X_1^t}$ the set of all polyhedra σ_j^t for $\sigma \in X_1$ and by $\widetilde{\omega_{X_1^t}}$ the weight function on the polyhedra of maximal dimension. Refining and possibly merging some of the σ_j^t (we have to add up the weights of all merged polyhedra) yields a tropical polyhedral complex $(X_1^t, \omega_{X_1^t})$. Note that $(X_1^0, \omega_{X_1^0}) = (X_1, \omega_{X_1})$. Furthermore, for $\sigma \in X_1$ we can consider the set

$$\widetilde{\sigma} := \bigcup_{t \in \mathbb{R}} \left(\bigcup_{j=1}^{p_t} \sigma_j^t \times \{t\} \right) \subseteq \mathbb{R}^n \times \mathbb{R}.$$

This set naturally splits up into polyhedra $\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_{s_\sigma}$. If a polyhedron $\widetilde{\sigma}_i$ is of maximal dimension we associate the weight $\widetilde{\omega_{X_1^t}}(\sigma_j^t)$ to it, where σ_j^t is a polyhedron containing a point in the relative interior of $\widetilde{\sigma}_i$ (this weight is obviously well-defined). We denote by $\widetilde{Z}$ the set $\{\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_{s_\sigma} | \sigma \in X_1\}$ and by $\widetilde{\omega_Z}$ the weight function on the polyhedra of maximal dimension. The choice of the weights $\widetilde{\omega_Z}(\widetilde{\sigma}_i)$ ensures that refining some of the $\widetilde{\sigma}_i$ yields a tropical polyhedral complex (Z, ω_Z).
Now, for $\mu \in \mathbb{R}$ let φ_μ be the rational function defined by

$$\varphi_\mu : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} : (x,t) \mapsto \max\{0,t\} - \max\{\mu,t\}.$$

Let $\widetilde{\sigma}_i \in Z$ be a polyhedron of maximal dimension and let (possibly after a refinement of X_1^t) $\sigma_j^t \subseteq \widetilde{\sigma}_i$ be a polyhedron of X_1^t of maximal dimension. As every polyhedron in X_1 contains at least one vertex, this property also holds for X_1^t and we can choose a vertex $P_{\sigma_j^t} \subseteq \sigma_j^t$. Let $P_{\sigma_j^{t+1}}$ be the translation of $P_{\sigma_j^t}$ in X_1^{t+1}. We have

$$P_{\sigma_j^{t+1}} - P_{\sigma_j^t} = \sum_{j=1}^{k} \pm l'_{i_j} v_{i_j}$$

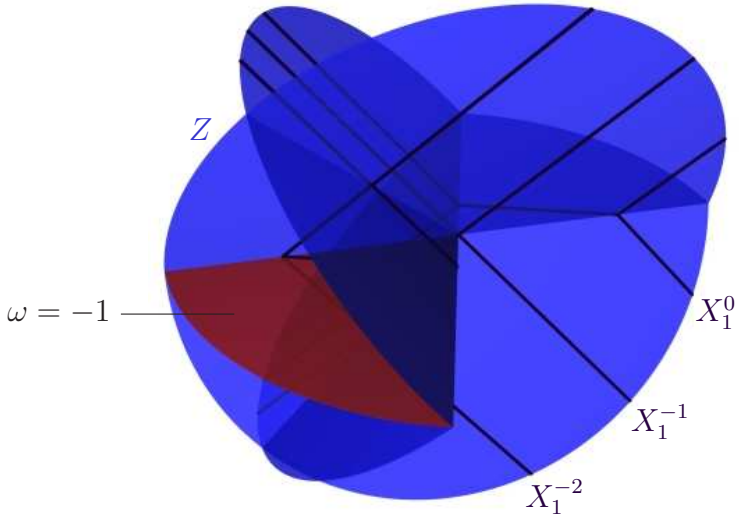


Figure 1.11: An example of a cycle Z as constructed in theorem 1.9.7.

for some $i_j \in \{1, \ldots, N\}$. Hence

$$\begin{pmatrix} \sum_{j=1}^{k} \pm l'_{i_j} v_{i_j} \\ 1 \end{pmatrix} \in \Lambda \times \mathbb{Z}$$

is a generator of $(\Lambda \times \mathbb{Z})_{\tilde{\sigma_i}}/(\Lambda \times \mathbb{Z})_{\sigma^t}$ and we can deduce that

$$\varphi_\mu \cdot [(Z, \omega_Z)] = [(X_1^0, \omega_{X_1^0})] - [(X_1^\mu, \omega_{X_1^\mu})].$$

Now let $t_0 \in \mathbb{R}_{<0}$ be the largest value such that there exists an edge that has been shrunk to length 0, i.e. an edge $\varepsilon'_i \in (X_1^{t_0})^{(1)}$ with length $l_i + t_0 \cdot l'_i = 0$. We conclude that

$$\varphi_{t_0} \cdot [(Z, \omega_Z)] = C_1 - C_2,$$

where $C_2 := [(X_1^{t_0}, \omega_{X_1^{t_0}})]$ can be seen as the cycle $C = C_1$ with at least one bounded polyhedron shrunk to one dimension less.
We repeat the whole process until all bounded polyhedra are shrunk to a point, i.e. until we obtain an affine cycle C_p. By construction we have

$$C = C_1 \sim C_2 \sim \ldots \sim C_p,$$

which proves the claim. □

Definition 1.9.8
Let C be a cycle in $\mathbb{R}^n$. We define the *recession cycle or degree of* C, denoted by $\delta(C)$, to be the affine cycle equivalent to C. This affine cycle exists by theorem 1.9.7 and is unique by lemma 1.9.5.

Remark 1.9.9
Let σ be a polyhedron in $\mathbb{R}^n$. We define the *recession cone of* σ to be

$$\mathrm{rc}(\sigma) := \{v \in \mathbb{R}^n | x + \mathbb{R}_{\geq 0} v \subseteq \sigma \forall x \in \sigma\} = \{v \in \mathbb{R}^n | \exists x \in \sigma \text{ s.t. } x + \mathbb{R}_{\geq 0} v \subseteq \sigma\}.$$

The two sets coincide as σ is closed and convex.
Let C be a d-dimensional cycle in $\mathbb{R}^n$ with representative (X, ω_X) and let

$$\widetilde{R(C)} := \{\mathrm{rc}(\sigma) | \sigma \in X\}.$$

By removing all cones of $\widetilde{R(C)}$ that are not contained in a d-dimensional cone and by subdividing the remaining cones we can make this set into a fan $R(C)$ of pure dimension d. To every cone $\sigma \in R(C)^{(d)}$ we associate the weight

$$\omega_{R(C)}(\sigma) := \sum_{\substack{\sigma' \in X \\ \sigma \subseteq \mathrm{rc}(\sigma')}} \omega_X(\sigma').$$

The proof of theorem 1.9.7 indeed shows that

$$\delta(C) = [(R(C), \omega_{R(C)})]$$

holds.

Theorem 1.9.10
Let C, D be two tropical cycles in $\mathbb{R}^n$. Then the following are equivalent:

i) $C \sim D$

ii) $d_C = d_D$

iii) $\delta(C) = \delta(D)$

Proof. i) $\Rightarrow$ ii) follows from proposition 1.8.10. iii) $\Rightarrow$ i) is an immediate consequence of theorem 1.9.7. ii) $\Rightarrow$ iii) follows from theorem 1.9.7, i) $\Rightarrow$ ii) and lemma 1.9.5. □

Remark 1.9.11
In other words, the above theorem says: The notions of rational equivalence, numerical equivalence and "having the same degree" coincide.

Theorem 1.9.12 (General Bézout's theorem)
Let C, D be two tropical cycles in $\mathbb{R}^n$. Then

$$\delta(C \cdot D) = \delta(C) \cdot \delta(D).$$

Proof. We apply theorem 1.9.7 and proposition 1.8.10 and get

$$\delta(C \cdot D) \sim C \cdot D \sim \delta(C) \cdot \delta(D).$$

By corollary 1.9.6 two rationally equivalent affine cycles are equal. □

Our last step in this section is to prove a version of Bézout's theorem for a special class of tropical cycles in $\mathbb{R}^n$ called $\mathbb{P}^n$*-generic cycles*. But first we need some further definitions:

Definition 1.9.13
Let $C \in Z_k(\mathbb{R}^n)$ be a tropical cycle and let L^n_k be the tropical fan defined in example 1.2.9. Then we define the *degree of* C to be the number

$$\deg(C) := \deg(C \cdot L^n_{\operatorname{codim} X}),$$

where the second map $\deg : Z_0(\mathbb{R}^n) \to \mathbb{Z} : \lambda_1 P_1 + \ldots + \lambda_r P_r \mapsto \lambda_1 + \ldots + \lambda_r$ is the usual degree map. Then the map $\deg : Z_k(\mathbb{R}^n) \to \mathbb{Z}$ is obviously linear by definition. Moreover, we define the degree of $[C] \in A_k(\mathbb{R}^n)$ to be $\deg([C]) := \deg(C)$. Note that $\deg([C])$ is well-defined by remark 1.7.3 and proposition 1.8.10.

Definition 1.9.14 ($\mathbb{P}^n$-generic cycles)
Let $C \in Z_k(\mathbb{R}^n)$ be a tropical cycle. C is called $\mathbb{P}^n$*-generic* if for one (and thus for every) representative X of C holds: For every face $\sigma \in X^{(k)}$ there exists a polytope $P_\sigma \subseteq \mathbb{R}^n$ of some dimension $r \in \{0, \ldots, k\}$ and a cone $\widetilde{\sigma} \in (L^n_k)^{(k-r)}$ such that $\sigma \subseteq P_\sigma + \widetilde{\sigma}$.

Remark 1.9.15
Note that that the above definition of $\mathbb{P}^n$-generic cycles is equivalent to the condition that the recession cycle $\delta(C)$ of our cycle $C \in Z_k(\mathbb{R}^n)$ is contained in the tropical fan L^n_k. By [GKM07, example 2.18] this implies that C is $\mathbb{P}^n$-generic if and only if $\delta(C) = \lambda \cdot L^n_k$ holds for some integer λ. In this case we can easily conclude that $\deg(C) = \deg(\delta(C)) = \lambda$.

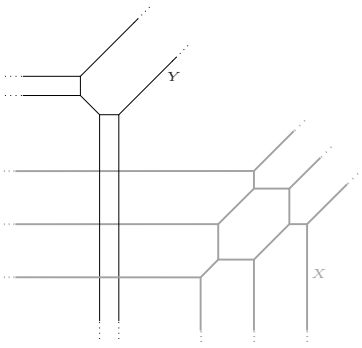


Figure 1.12: The intersection of two $\mathbb{P}^n$-generic cycles of degrees 2 and 3.

Theorem 1.9.16 (Bézout's theorem)
Let $C \in Z_k(\mathbb{R}^n)$ and $D \in Z_{n-k}(\mathbb{R}^n)$ be two tropical cycles of complementary dimensions. Moreover, assume that C is $\mathbb{P}^n$-generic. Then the following equation holds:

$$\deg(C \cdot D) = \deg(C) \cdot \deg(D).$$

Proof. By proposition 1.8.10, theorem 1.9.7 and remark 1.9.15 we can conclude that

$$\begin{aligned} \deg(C \cdot D) &= \deg(\delta(C) \cdot D) \\ &= \deg((\deg(C) \cdot L^n_k) \cdot D) \\ &= \deg(C) \cdot \deg(L^n_k \cdot D) \\ &= \deg(C) \cdot \deg(D). \end{aligned}$$

□

2 Tropical cycles with real slopes and numerical equivalence

In this chapter we introduce a generalized definition of tropical fans and tropical cycles that allows non-rational slopes. Obviously, these objects cannot be realized as images of algebraic varieties over the field of Puiseux series under the valuation, but they are quite useful for some theoretical aspects of tropical geometry. For example, using tropical cycles with real slopes we give a proof that in any tropical fan any cycle is numerically equivalent to its degree cycle.

2.1 Tropical cycles with real slopes

We start this section giving the necessary generalizations of the first definitions in [GKM07] and chapter 1. In the following, let V always be a real vector space of finite dimension.

Definition 2.1.1
A *polyhedron* in V is a non-empty subset $\sigma \subseteq V$ of the form

$$\sigma = \{x \in V | f_1(x) = b_1, \ldots, f_r(x) = b_r, f_{r+1}(x) \geq b_{r+1}, \ldots, f_N(x) \geq b_N\}$$

for some linear forms $f_1, \ldots, f_N \in V^\vee$ and numbers $b_1, \ldots, b_N \in \mathbb{R}$. A *cone* in V is a polyhedron for which all the numbers $b_1, \ldots, b_N$ are zero.

Definition 2.1.2
A *polyhedral complex* X in V is a finite set of polyhedra in V such that the following properties are fulfilled:

(a) All faces of the polyhedra in X are again in X and

(b) the intersection of two polyhedra in X is either a face of each or empty.

A *fan* in V is a polyhedral complex consisting of cones only. As before we denote by $|X|$ the union of all polyhedra in X.

Definition 2.1.3
A *marked fan* in V is a pure-dimensional simplicial fan X in V together with a generating vector v_σ for every edge in X, i.e. a vector $v_\sigma \in \sigma \setminus \{0\}$ for every $\sigma \in X^{(1)}$.
As in [GKM07, construction 2.13], for all pairs $\sigma \in X^{(\dim(X))}$ and $\tau \in X^{(\dim(X)-1)}$ with

$\tau < \sigma$ there is exactly one edge $\sigma' \in X^{(1)}$ that is a face of σ but not of τ. We denote the associated vector $v_{\sigma'}$ by $v_{\sigma/\tau}$ and call it the *normal vector* of σ relative to τ.

Definition 2.1.4
A *tropical fan (with real slopes)* in V is a pair (X, ω_X) where X is a marked fan and $\omega_X : X^{(\dim(X))} \to \mathbb{R}$ is a weight function on the cones of maximal dimension such that the balancing condition

$$\sum_{\sigma:\tau<\sigma} \omega_X(\sigma) \cdot v_{\sigma/\tau} = 0 \text{ in } V/V_\tau$$

is fulfilled for all $\tau \in X^{(\dim(X)-1)}$. As in definition 1.1.7 we denote by $(X^*, \omega_{X^*}) := (\{\tau \in X | \tau \subseteq \sigma \text{ for some } \sigma \in X^{(\dim(X))} \text{ with } \omega_X(\sigma) \neq 0\}, \omega_X|_{(X^*)^{(\dim(X))}})$ the *non-zero part* of X.

Remark and Definition 2.1.5
Let (X, ω_X) be a tropical fan with real slopes, let $\sigma_1 \in X^{(1)}$ be an edge of X and let $\lambda \in \mathbb{R}_{>0}$ be a positive number. Then let $X' := X$ and

$$\omega_{X'}(\sigma) := \begin{cases} \omega_X(\sigma), & \text{if } \sigma_1 \not\subseteq \sigma \\ \frac{1}{\lambda} \cdot \omega_X(\sigma), & \text{if } \sigma_1 \subseteq \sigma. \end{cases}$$

Moreover, replace in X' the vector v_{σ_1} by $\lambda \cdot v_{\sigma_1}$. We would like to call (X, ω_X) and $(X', \omega_{X'})$ the same tropical fan, but according to definition 2.1.4 this is not the case. Hence, in the following we will always identify tropical fans that arise one from the other by operations as above.

We also have to take care of this ambiguity when defining refinements:

Definition 2.1.6
Let (X, ω_X) and (Y, ω_Y) be tropical fans with real slopes in V. Analogously to definition 1.1.5 we say that Y is a *subfan* of X if for every cone $\sigma \in Y$ there exists a cone $\widetilde{\sigma} \in X$ such that $\sigma \subseteq \widetilde{\sigma}$. In this case we define a map $C_{Y,X} : Y \to X$ that maps a cone $\sigma \in Y$ to the inclusion-minimal cone $\widetilde{\sigma} \in X$ with $\sigma \subseteq \widetilde{\sigma}$.
We say that (Y, ω_Y) is a *refinement* of (X, ω_X) if

(a) Y^* is a subfan of X^*,

(b) $|Y^*| = |X^*|$ and

(c) $\omega_X(C_{Y,X}(\sigma)) \cdot |\det(A_{\sigma, C_{Y,X}(\sigma)})| = \omega_Y(\sigma)$ for every $\sigma \in (Y^*)^{(\dim(Y))}$,

where $A_{\sigma, C_{Y,X}(\sigma)}$ is a matrix expressing the generating vectors of $C_{Y,X}(\sigma)$ in terms of the generating vectors of σ.

Remark 2.1.7
Using the above definitions we can carry over the other basic definitions from section 1.1, in particular the definition of *affine tropical cycles*, to our context. Note that in some cases, e.g. for taking sums of cycles, it is necessary to further refine the tropical fans our constructions in section 1.1 yield to obtain simplicial fans. Moreover, we can

define open marked fans and open tropical fans in analogy to definition 1.4.3. Using rational functions with arbitrary slopes it is obviously possible to define intersection products of Cartier divisors and tropical cycles with real slopes in exactly the same way as we did in section 1.2. What is missing so far is the notion of morphisms:

Definition 2.1.8
Let (X, ω_X) and (Y, ω_Y) be tropical fans with real slopes in finite-dimensional real vector spaces V and V', respectively. A *morphism* $f : X \to Y$ is just the restriction of an $\mathbb{R}$-linear map $\widetilde{f} : V \to V'$ to $|X|$. As before we can refine X in such a way that

$$f_*X := \{f(\sigma) | \sigma \in X \text{ is contained in a maximal cone of } X \text{ on which } f \text{ is injective}\}$$

is a fan in V'. We can make f_*X into a tropical fans as follows: For every cone $\sigma' \in f_*X^{(1)}$ we choose a generating vector $v_{\sigma'} \in \sigma' \setminus \{0\}$. Moreover, for all cones $\sigma' \in f_*X^{(\dim(X))}$ we set

$$\omega_{f_*X}(\sigma') := \sum_{\sigma \in X : f(\sigma) = \sigma'} \omega_X(\sigma) \cdot |\det(A_{\sigma', f(\sigma)})|,$$

where $A_{\sigma', f(\sigma)}$ is a matrix expressing the images $f(v_{\tau_i})$ of the generating vectors v_{τ_i} of σ in terms of the chosen generating vectors of σ'.

In the following we will define marked polyhedral complexes and tropical polyhedral complexes with real slopes in analogy to marked and tropical fans. This will lead us to (embedded) tropical cycles with real slopes as spaces "looking locally like affine cycles".

Definition 2.1.9
A *marked polyhedral complex* in V is a pure-dimensional polyhedral complex X in V fulfilling the following conditions:

(a) For every polyhedron $\sigma \in X$ there exists a vertex $P_\sigma \in X^{(0)}$ with $P_\sigma \in \sigma$,

(b) for every vertex $P \in X^{(0)}$ and every edge $\sigma \in X^{(1)}$ with $P \in \sigma$ we are given a vector $v_\sigma^P \in V$ such that the set $\{\sigma \in X | P \in \sigma\}$ together with the set of vectors $\{v_\sigma^P | P \in \sigma \in X^{(1)}\}$ is an open marked fan in V (after a suitable translation),

(c) for all vertices $P_1, P_2 \in X^{(0)}$ with $P_1 \neq P_2$ and every edge $\sigma \in X^{(1)}$ with $P_1, P_2 \in \sigma$ holds $v_\sigma^{P_1} = -v_\sigma^{P_2}$ and

(d) for every pair of vertices $P_1, P_2 \in X^{(0)}$ and all polyhedra $\sigma \in X$ with $P_1, P_2 \in \sigma$ the base change from $v_{\tau_1}^{P_1}, \ldots, v_{\tau_r}^{P_1}$ to $v_{\varrho_1}^{P_2}, \ldots, v_{\varrho_r}^{P_2}$ has determinant ± 1, where $\tau_1, \ldots, \tau_r \in X^{(1)}$ and $\varrho_1, \ldots, \varrho_r \in X^{(1)}$ are all the edges contained in σ that contain P_1 respectively P_2.

Definition 2.1.10
A *tropical polyhedral complex (with real slopes)* in V is a pair (X, ω_X) where X is a marked polyhedral complex and $\omega_X : X^{(\dim(X))} \to \mathbb{R}$ is a weight function on the polyhedra of maximal dimension such that for all vertices $P \in X^{(0)}$ these weights make the open marked fan $\{\sigma \in X | P \in \sigma\}$ into a tropical fan. As before, we denote by $(X^*, \omega_{X^*}) := (\{\tau \in X | \tau \subseteq \sigma \text{ for some } \sigma \in X^{(\dim(X))} \text{ with } \omega_X(\sigma) \neq 0\}, \omega_X|_{(X^*)^{(\dim(X))}})$ the *non-zero part* of X.

Definition 2.1.11
Let (X, ω_X) and (Y, ω_Y) be tropical polyhedral complexes with real slopes in V. Exactly as in definition 2.1.6 we say that Y is a *subcomplex* of X if for every polyhedron $\sigma \in Y$ there exists a polyhedron $\widetilde{\sigma} \in X$ such that $\sigma \subseteq \widetilde{\sigma}$. Again, we can define a map $C_{Y,X} : Y \to X$ that maps a polyhedron $\sigma \in Y$ to the inclusion-minimal polyhedron $\widetilde{\sigma} \in X$ with $\sigma \subseteq \widetilde{\sigma}$.
We say that (Y, ω_Y) is a *refinement* of (X, ω_X) if

(a) Y^* is a subcomplex of X^*,

(b) $|Y^*| = |X^*|$ and

(c) $\omega_X(C_{Y,X}(\sigma)) \cdot |\det(A_{\sigma, C_{Y,X}(\sigma)})| = \omega_Y(\sigma)$ for every $\sigma \in (Y^*)^{(\dim(Y))}$,

where the matrix $A_{\sigma, C_{Y,X}(\sigma)}$ arises as follows: Pick vertices $P \in \sigma$ and $Q \in C_{Y,X}(\sigma)$. Then $A_{\sigma, C_{Y,X}(\sigma)}$ is a matrix expressing the generating vectors $v^Q_{\tau_i}$ of $C_{Y,X}(\sigma)$ in the open marked fan $\{\widetilde{\sigma} \in X | Q \in \widetilde{\sigma}\}$ in terms of the generating vectors $v^P_{\varrho_i}$ of σ in the open marked fan $\{\widetilde{\sigma} \in Y | P \in \widetilde{\sigma}\}$. By property (d) of definition 2.1.9 condition (c) is independent of the choice of the vertices P, Q.

Remark 2.1.12
As before, using these new definitions we can carry over the definitions and results from sections 1.4 – 1.6, in particular the definition of a *tropical cycle* and the notion of a *morphism of cycles*, to our context. Here again it is necessary in some cases to further refine the polyhedral complexes we obtain by our constructions in chapter 1 to end up with simplicial complexes. Note that properties (c) and (d) of definition 2.1.9 make sure that we can define the intersection of a Cartier divisor with a cycle by taking local intersection products as we did in definition 1.5.5.

Example 2.1.13
We want to consider tropical curves in $\mathbb{R}^2$, i.e. tropical cycles of dimension one according to remark 2.1.12, and their moduli space in analogy to [GKM07, section 4]. Therefore we fix a degree $\triangle_0 := \{v_1, v_2, v_3\}, v_i \in \mathbb{R}^2, \sum_{i=1}^3 v_i = 0$ and say a curve with real slopes has *degree* d if its degree (in the sense of [GKM07, definition 4.1]) is

$$\{v_1, \dots, v_1, v_2, \dots, v_2, v_3, \dots, v_3\},$$

where each v_i occurs exactly d times. Given $n := 3d-1$ points $P_1, \dots, P_n \in \mathbb{R}^2$ we want to calculate the number of tropical curves with real slopes of degree d passing through $P_1, \dots, P_n$ (counted with multiplicities) as an intersection product on the corresponding moduli space $\mathcal{M}^{\text{lab}}_{0,n,\text{trop}}(\mathbb{R}^2, d) = \mathcal{M}_{0,3d+n,\text{trop}} \times \mathbb{R}^2$:
Let the matrix A be given by

$$A = (-v_1| - v_2)^{-1}.$$

Hence we have maps

$$\mathcal{M}_{0,3d+n,\text{trop}} \times \mathbb{R}^2 \xrightarrow{\text{ev}_i} \mathbb{R}^2 \xrightarrow{A} \mathbb{R}^2.$$

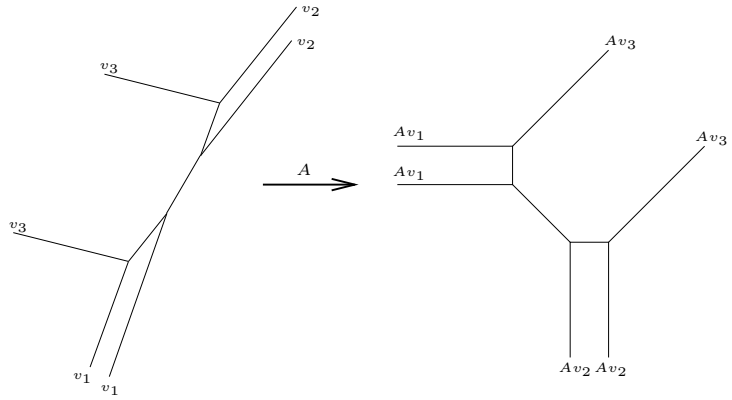


Figure 2.1: A curve with real slopes mapped to an ordinary degree-2-curve.

Note that for a curve C of degree d in $\mathbb{R}^2$ we have that $A(C)$ is an ordinary degree-d-curve, i.e. $A(C)$ has degree d in the sense of [GKM07, definition 4.1]. Let $P_i = (a_1^{(i)}, a_2^{(i)})$. We can write P_i as

$$P_i = \max\{a_1^{(i)}, x\} \cdot \max\{a_2^{(i)}, y\} \cdot \mathbb{R}^2,$$

where x, y are the coordinates on $\mathbb{R}^2$. Moreover, we have

$$\begin{aligned} A^* \max\{a_1^{(i)}, x\} \cdot A^* \max\{a_2^{(i)}, y\} \cdot \mathbb{R}^2 &= |\det(A)| \cdot P_i \\ &= |\det(A)| \cdot \max\{a_1^{(i)}, x\} \cdot \max\{a_2^{(i)}, y\} \cdot \mathbb{R}^2. \end{aligned}$$

Using this equation and the projection formula we can conclude that

$$\begin{aligned} \mathcal{N}_d &= \deg\left(\prod_{i=1}^{n} \mathrm{ev}_i^*\, A^* \max\{a_1^{(i)}, x\} \cdot \mathrm{ev}_i^*\, A^* \max\{a_2^{(i)}, y\} \cdot \mathcal{M}_{0,n+3d,\mathrm{trop}} \times \mathbb{R}^2\right) \\ &= \deg\left(\prod_{i=1}^{n} |\det(A)| \cdot \mathrm{ev}_i^* \max\{a_1^{(i)}, x\} \cdot \mathrm{ev}_i^* \max\{a_2^{(i)}, y\} \cdot \mathcal{M}_{0,n+3d,\mathrm{trop}} \times \mathbb{R}^2\right) \\ &= |\det(A)|^n \cdot \deg\left(\prod_{i=1}^{n} \mathrm{ev}_i^* \max\{a_1^{(i)}, x\} \cdot \mathrm{ev}_i^* \max\{a_2^{(i)}, y\} \cdot \mathcal{M}_{0,n+3d,\mathrm{trop}} \times \mathbb{R}^2\right) \\ &= |\det(A)|^n \cdot \widetilde{\mathcal{N}_d}, \end{aligned}$$

where $\mathcal{N}_d$ is the number of ordinary degree-d-curves passing through $P_1, \ldots, P_n$ (counted with multiplicities) and $\widetilde{\mathcal{N}_d}$ is exactly the number we want to calculate. Hence we obtain the equation

$$\widetilde{\mathcal{N}_d} = \frac{1}{|\det(A)|^n} \cdot \mathcal{N}_d = |\det(v_1|v_2)|^n \cdot \mathcal{N}_d.$$

Remark 2.1.14
In the following, the notions of tropical fan, tropical polyhedral complex and tropical cycle will always refer to the objects introduced in chapter 1. We will state explicitly if a fan or cycle is a tropical fan or tropical cycle with real slopes according to definition 2.1.4 or remark 2.1.12, respectively.

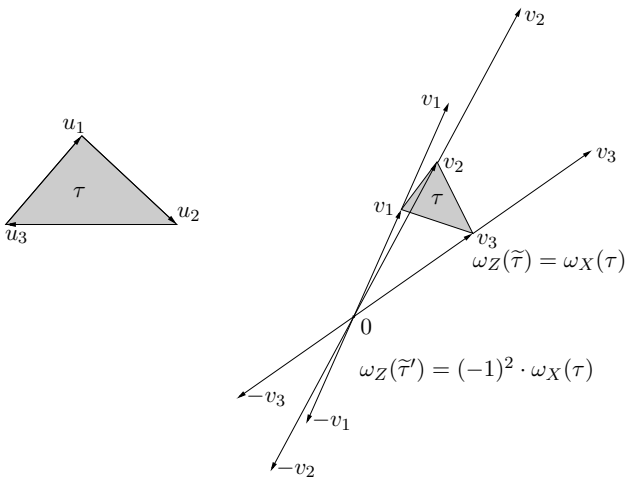


Figure 2.2: Cones $\widetilde{\tau}, \widetilde{\tau}'$ with normal vectors v_i over a polyhedron τ.

2.2 Numerical equivalence on fans

Definition 2.2.1
Let Y be a tropical fan in some vector space V and let $C, D \in Z_k(Y)$ be tropical cycles. We call C and D *numerically equivalent* if $\deg(\varphi_1 \cdots \varphi_k \cdot C) = \deg(\varphi_1 \cdots \varphi_k \cdot D)$ holds for all Cartier divisors $\varphi_1, \ldots, \varphi_k$ on Y.

Proposition 2.2.2
Let Y be a tropical fan in $\mathbb{R}^n$ and let $C \in Z_k(Y)$ be a tropical cycle. Then C and its degree cycle $\delta(C) \in Z_k(Y)$ are numerically equivalent (see definition 1.9.8 for the definition of $\delta(C)$).

Proof. Let X be a representative of C. We perform the same construction as in the proof of theorem 1.9.7, but with some modifications: Like before, we translate our chosen vertex P_0 to the point $P_0' := 0$, but instead of replacing the lengths l_i by $l_i + t \cdot l_i'$ we replace l_i by $t \cdot l_i$. Note that we do not change the directions and lengths of unbounded edges. After a suitable refinement of X the cones arising from the polyhedra $\tau \in X$ for $t \geq 0$ in this way are simplicial. Moreover, for $t \leq 0$ a polyhedron $\tau \in X$ in general yields a union of simplicial cones. We denote this set of cones by Z. We can make Z into a tropical polyhedral complex by further refining it and equipping it with weights and normal vectors as follows: Let first $\tau \in X$ be a bounded polyhedron. Performing operations as described in remark and definition 2.1.5 we may assume that every normal vector associated to an edge E of τ with endpoints P_E and Q_E is just $P_E - Q_E$ (cf. figure 2.2). If $\widetilde{\tau}$ (for $t \geq 0$) and $\widetilde{\tau}'$ (for $t \leq 0$) are the cones in Z arising from τ and $P_1, \ldots, P_r \in X^{(0)}$ are all vertices in τ we choose $\pm(P_1, 1), \ldots, \pm(P_r, 1) \in \mathbb{R}^n \times \mathbb{R}$ as generating vectors of $\widetilde{\tau}$ and $\widetilde{\tau}'$, respectively. If $\tau \in X$ is an unbounded polyhedron we have all vectors $(v_\varrho, 0) \in \mathbb{R}^n \times \mathbb{R}$, where v_ϱ is the primitive normal vector of an unbounded edge in τ in our original complex X, as additional primitive normal vectors generating $\widetilde{\tau}$. In both cases we set $\omega_Z(\widetilde{\tau}) := \omega_X(\tau)$ if $\widetilde{\tau}$ is a cone of maximal dimension. Moreover, if τ is unbounded, $\widetilde{\tau}'$ is in general not a cone but a union of cones that contains the primitive normal vectors $(v_\varrho, 0) \in \mathbb{R}^n \times \mathbb{R}$, too, but due to

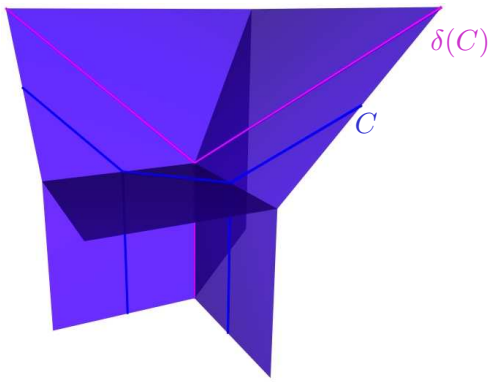


Figure 2.3: A curve C and its degree cycle $\delta(C)$.

the necessary refinements we need some additional generating vectors in this case. Let $\widetilde{\tau}_i'$ be one such cone (or $\widetilde{\tau}_i' = \widetilde{\tau}'$ if τ is bounded). Then we set $\omega_Z(\widetilde{\tau}_i') := (-1)^{d(\tau)}\omega_X(\tau)$, where $d(\tau) := \dim((\widetilde{\tau}_i')^t) - \dim(\mathrm{rc}((\widetilde{\tau}_i')^t))$ and $(\widetilde{\tau}_i')^t$ is the intersection of $(\widetilde{\tau}_i')$ with $\mathbb{R}^n \times \{t\}$ for some $t < 0$ (cf. theorem 1.9.7, remark 1.9.9 and figure 2.2). Performing a refinement of overlapping cones and adding up their weights as in theorem 1.9.7 the above choices make Z into a tropical polyhedral complex with real slopes (Z, ω_Z) according to definition 2.1.10 in $\mathbb{R}^n \times \mathbb{R}$ such that moreover $\max\{1,t\} \cdot [(Z,\omega_Z)] = C(-P_0)$ holds, where t is the coordinate of the additional factor $\mathbb{R}$ and $C(-P_0)$ is the translation of C by the vector $-P_0$. Additionally we have $\max\{0,t\} \cdot [(Z,\omega_Z)] = \delta(C)$ in this case. Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be the map given by $(x,t) \mapsto x + t \cdot P_0$. Then f maps $|Z| \cap \{0 \leq t \leq 1\}$ to Y. We can conclude the following:

$$\begin{aligned}
& \deg(\varphi_1 \cdots \varphi_k \cdot (C - \delta(C))) \\
= \; & \deg(\varphi_1 \cdots \varphi_k \cdot f_*(\max\{1,t\} - \max\{0,t\} \cdot [(Z,\omega_Z)])) \\
= \; & \deg(f_*(f^*\varphi_1 \cdots f^*\varphi_k \cdot (\max\{1,t\} - \max\{0,t\}) \cdot [(Z,\omega_Z)])) \\
= \; & \deg(f^*\varphi_1 \cdots f^*\varphi_k \cdot (\max\{1,t\} - \max\{0,t\}) \cdot [(Z,\omega_Z)]) \\
= \; & \deg\left(\widetilde{f^*\varphi_1} \cdots \widetilde{f^*\varphi_k} \cdot (\max\{1,t\} - \max\{0,t\}) \cdot [(Z,\omega_Z)]\right) \\
= \; & \deg\left((\max\{1,t\} - \max\{0,t\}) \cdot \widetilde{f^*\varphi_1} \cdots \widetilde{f^*\varphi_k} \cdot [(Z,\omega_Z)]\right) \\
= \; & 0,
\end{aligned}$$

where $\widetilde{f^*\varphi_i}$ is any continuation of $f^*\varphi_i : |Z| \cap \{0 \leq t \leq 1\} \to \mathbb{R}$ on the whole cycle Z. Hence, using the linearity of deg and the intersection product, we can conclude that $\deg(\varphi_1 \cdots \varphi_k \cdot C) = \deg(\varphi_1 \cdots \varphi_k \cdot \delta(C))$ for all Cartier divisors $\varphi_1, \ldots, \varphi_k$ on Y and thus that C and $\delta(C)$ are numerically equivalent. □

2.3 Chow groups via numerical equivalence

In analogy to $\mathbb{P}^n$-generic cycles (cf. definition 1.9.14) we can define generic cycles corresponding to other toric compactifications of the algebraic torus:

Definition 2.3.1 (F-generic cycles)
Let F be a complete fan in $\mathbb{R}^n$ corresponding to a compact and smooth toric variety, i.e. $|F| = \mathbb{R}^n$ and every cone of F is generated by part of a lattice basis. Let $C \in Z_k(\mathbb{R}^n)$ be a tropical cycle. C is called *F-generic* if for one (and thus for every) representative X of C holds: For every face $\sigma \in X^{(k)}$ there exists a polytope $P_\sigma \subseteq \mathbb{R}^n$ of some dimension $r \in \{0, \dots, k\}$ and a cone $\widetilde{\sigma} \in F^{(k-r)}$ such that $\sigma \subseteq P_\sigma + \widetilde{\sigma}$.
We denote by $Z_k^F(\mathbb{R}^n)$ the group of all k-dimensional and F-generic cycles in $\mathbb{R}^n$.

Remark 2.3.2
The above definition of F-generic cycles is equivalent to the condition that the recession cycle $\delta(C)$ of our cycle $C \in Z_k(\mathbb{R}^n)$ is contained in the k-skeleton of our given fan F. Hence, for every F-generic cylce C there is a representative Y of $\delta(C)$ such that $Y \subseteq \bigcup_{i=0}^{k} F^{(i)}$.

Example 2.3.3
Let F_1 and F_2 be the complete fans in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, defining $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, respectively, as a toric variety. The following figure shows examples of F_1- and F_2-generic cycles:

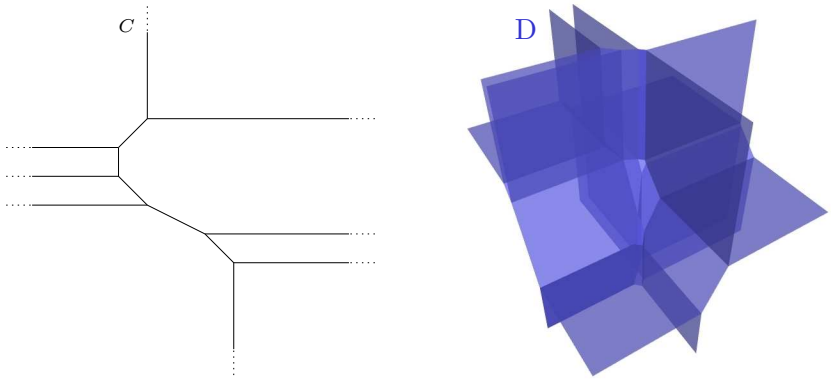


Figure 2.4: An F_1-generic curve C and an F_2-generic cycle D.

For the rest of this section F will always denote a complete fan in $\mathbb{R}^n$ corresponding to a compact and smooth toric variety as in definition 2.3.1.

Definition 2.3.4 (F-numerical equivalence)
Let $C, C' \in Z_k(\mathbb{R}^n)$ be arbitrary tropical cycles in $\mathbb{R}^n$. We say that C and C' are *F-numerical equivalent*, denoted by $C \sim_{\text{num}} C'$, if the maps $d_C : Z_k^F(\mathbb{R}^n) \to \mathbb{Z} : D \mapsto \deg(C \cdot D)$ and $d_{C'} : Z_k^F(\mathbb{R}^n) \to \mathbb{Z} : D \mapsto \deg(C' \cdot D)$ coincide (cf. definition 1.9.3). We define the k-th tropical Chow group of $\mathbb{R}^n$ induced by numerical equivalence to be the group

$$A_k^F(\mathbb{R}^n) := Z_k(\mathbb{R}^n)/\sim_{\text{num}} .$$

Theorem 2.3.5
Let $X(F)$ be the toric variety associated to F and $A_k(X(F))$ and $A^k(X(F))$ its k-th Chow group and k-th Chow cohomology group, respectively. Then we have an isomorphism:

$$A_k^F(\mathbb{R}^n) \cong A^k(X(F)),$$

i.e. the "classical" Chow cohomology groups and the tropical Chow groups are naturally isomorphic. Hence we also have

$$A_k^F(\mathbb{R}^n) \cong A_k(X(F)),$$

but there is no canonical map giving this last isomorphism.

Proof. As $X(F)$ is a smooth toric variety we can deduce that $A^k(X(F))$ is free. Hence, by [FS97, theorem 2.1] there are Minkowski weights $c_1, \dots, c_r$ on $F^{(n-k)}$ freely generating $A^k(X(F))$. We define $X(c_i)$ to be the tropical fan with cones $X(c_i) := \bigcup_{i=0}^{n-k} F^{(i)}$ and weight function c_i. Moreover, [F84, corollary 17.4] and [FS97, proposition 1.4] imply that the map

$$\begin{aligned} I: \qquad A^k(X(F)) &\longrightarrow \mathrm{Hom}(A_k(X(F)), \mathbb{Z}) \longrightarrow \mathrm{Hom}(A^{n-k}(X(F)), \mathbb{Z}) \\ c &\longmapsto \deg((c \cup \cdot) \cap [X(F)]) \end{aligned}$$

is an isomorphism. As $A^{n-k}(X(F))$ is a free group, we can conclude that $A^k(X(F)) \cong A^{n-k}(X(F))$. Thus, there are again r Minkowski weights $d_1, \dots, d_r$ on $F^{(k)}$ freely generating the group $A^{n-k}(X(F))$ and associated tropical fans $X(d_i) := \bigcup_{i=0}^{k} F^{(i)}$ with weight functions d_i. The isomorphism I is then given by

$$\begin{aligned} I: \qquad A^k(X(F)) &\longrightarrow \mathrm{Hom}(A^{n-k}(X(F)), \mathbb{Z}) \\ \sim\downarrow & \qquad\qquad \downarrow\sim \\ \mathbb{Z}^r &\longrightarrow \mathbb{Z}^r \\ (\alpha_1, \dots, \alpha_r) &\longmapsto (\textstyle\sum_{j=1}^r \gamma_{1j}\alpha_j, \dots, \sum_{j=1}^r \gamma_{rj}\alpha_j), \end{aligned}$$

where

$$\gamma_{ij} := \deg((d_i \cup c_j) \cap [X(F)]) = (d_i \cup c_j)(\{0\}).$$

The last equation follows from [FS97, proposition 3.1]. By [R08, theorem 1.9] we can deduce that moreover

$$\gamma_{ij} = \deg(X(d_i) \cdot X(c_j))$$

holds.
Now, let $C \in Z_{n-k}(\mathbb{R}^n)$ be a tropical cycle and let $\delta_i := \deg(X(d_i) \cdot C) \in \mathbb{Z}$. As the map I is an isomorphism, the matrix $(\gamma_{ij})_{ij}$ is invertible over $\mathbb{Z}$ and there are $\alpha_1, \dots, \alpha_r \in \mathbb{Z}$ with $\sum_{j=1}^r \gamma_{ij}\alpha_j = \delta_i$ for all $i = 1, \dots, r$. We claim that $C \sim_{\mathrm{num}} \alpha_1 X(c_1) + \ldots + \alpha_r X(c_r)$: Let $D \in Z_k^F(\mathbb{R}^n)$ be an F-generic cycle. By remark 2.3.2 there exists a representative Y of $\delta(D)$ such that $Y \subseteq \bigcup_{i=0}^{k} F^{(i)}$. The weight function ω_Y defines a Minkowski weight $c(Y)$ on F^{n-k}. Hence, there are coefficients $\beta_1, \dots, \beta_r \in \mathbb{Z}$ such that $c(Y) = \beta_1 d_1 + \ldots + \beta_r d_r$ and equivalently $Y = \beta_1 X(d_1) + \ldots + \beta_r X(d_r)$. Thus we obtain

$$\deg(Y \cdot C) = \deg((\beta_1 X(d_1) + \ldots + \beta_r X(d_r)) \cdot C) = \beta_1\delta_1 + \ldots + \beta_r\delta_r.$$

Moreover, we can deduce that

$$\begin{aligned} & \deg(Y \cdot (\alpha_1 X(c_1) + \ldots + \alpha_r X(c_r))) \\ = \; & \deg((\beta_1 X(d_1) + \ldots + \beta_r X(d_r)) \cdot (\alpha_1 X(c_1) + \ldots + \alpha_r X(c_r))) \\ = \; & \textstyle\sum_{i=1}^r \left(\beta_i \left(\sum_{j=1}^r \alpha_j\gamma_{ij}\right)\right) \\ = \; & \textstyle\sum_{i=1}^r \beta_i\delta_i. \end{aligned}$$

Thus we have $C \sim_{\text{num}} \alpha_1 X(c_1) + \ldots + \alpha_r X(c_r)$.
Now, let $\alpha_1, \ldots, \alpha_r \in \mathbb{Z}$ be given such that $\alpha_1 X(c_1) + \ldots + \alpha_r X(c_r) \sim_{\text{num}} 0$. In particular, this implies that

$$\deg((\alpha_1 X(c_1) + \ldots + \alpha_r X(c_r)) \cdot X(d_i)) = \sum_{j=1}^{r} \alpha_j \gamma_{ij} = 0$$

for all $i = 1, \ldots, r$. As $(\gamma_{ij})_{ij}$ is invertible we can deduce that $\alpha_1 = \ldots = \alpha_r = 0$.
Altogether we can conclude that the map

$$\langle X(c_1), \ldots, X(c_r) \rangle_{\mathbb{Z}} \to A_k^F(\mathbb{R}^n) : \sum_{i=1}^{r} \alpha_i X(c_i) \mapsto \left[\sum_{i=1}^{r} \alpha_i X(c_i) \right]_{\sim_{\text{num}}}$$

is an isomorphism. Hence we obtain

$$A_k^F(\mathbb{R}^n) \cong \langle X(c_1), \ldots, X(c_r) \rangle_{\mathbb{Z}} \cong \langle c_1, \ldots, c_r \rangle_{\mathbb{Z}} \cong A^k(X(F)) \cong A_k(X(F)),$$

which finishes the proof. □

3 Tropical intersection products on smooth varieties

In this chapter we define an intersection product of tropical cycles on tropical linear spaces L^n_k, i.e. on tropical fans of the type $\max\{0, x_1, \ldots, x_n\}^{n-k} \cdot \mathbb{R}^n$, in analogy to section 1.8. Afterwards we use this result to obtain an intersection product of cycles on every smooth tropical variety, i.e. on every tropical variety that arises from gluing such tropical linear spaces. In contrast to classical algebraic geometry these products always yield well-defined cycles, not cycle classes only. Using these intersection products we are able to define the pull-back of a tropical cycle along a morphism between smooth tropical varieties.

3.1 Intersection products on tropical linear spaces

In this section we will give a proof that tropical linear spaces L^n_k admit an intersection product. Therefore we show at first that the diagonal in the Cartesian product $L^n_k \times L^n_k$ of such a linear space with itself is a sum of products of Cartier divisors. Given two cycles C and D we can then intersect the diagonal with $C \times D$ and define the product $C \cdot D$ to be the projection thereof.
Throughout the section $e_1, \ldots, e_n$ will always be the standard basis vectors in $\mathbb{R}^n$ and $e_0 := -e_1 - \ldots - e_n$.
We begin the section with our basic definitions:

Definition 3.1.1 (Tropical linear spaces)
For $I \subsetneq \{0, 1, \ldots, n\}$ let σ_I be the cone generated by the vectors e_i, $i \in I$. We denote by L^n_k the tropical fan consisting of all cones σ_I with $I \subsetneq \{0, 1, \ldots, n\}$ and $|I| \leq k$, whose maximal cones all have weight one (cf. example 1.2.9). This fan L^n_k is a representative of the tropical linear space $\max\{0, x_1, \ldots, x_n\}^{n-k} \cdot \mathbb{R}^n$.

Definition 3.1.2
Let $C \in Z_k(\mathbb{R}^n)$ be a tropical cycle and let the map $i : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ be given by $x \mapsto (x, x)$. Then the push-forward cycle

$$\triangle_C := i_*(C) \in Z_k(\mathbb{R}^n \times \mathbb{R}^n)$$

is called the *diagonal* of $C \times C$.

In order to express the diagonal in $L^n_k \times L^n_k$ by means of Cartier divisors we first have to refine $L^n_k \times L^n_k$ in such a way that the diagonal is a subfan of this refinement:

Definition 3.1.3
Let F_k^n be the refinement of $L_k^n \times L_k^n$ that arises recursively from $L_k^n \times L_k^n$ as follows: Let $M := (L_k^n \times L_k^n)^{(2k)}$ be the set of maximal cones in $L_k^n \times L_k^n$. If a cone $\sigma \in M$ is generated by

$$\left(\frac{-e_i}{0}\right), \left(\frac{0}{-e_i}\right), v_3, \ldots, v_{2k}$$

for some i and vectors

$$v_j \in \left\{ \left(\frac{-e_\mu}{-e_\mu}\right), \left(\frac{-e_\mu}{0}\right), \left(\frac{0}{-e_\mu}\right) \middle| \mu = 0, \ldots, n \right\}$$

then replace the cone σ by the two cones spanned by

$$\left(\frac{-e_i}{-e_i}\right), \left(\frac{-e_i}{0}\right), v_3, \ldots, v_{2k}$$

and

$$\left(\frac{-e_i}{-e_i}\right), \left(\frac{0}{-e_i}\right), v_3, \ldots, v_{2k},$$

respectively. Repeat this process until there are no more cones in M that can be replaced. The fan F_k^n is then the set of all faces of all cones in M.

The next lemma and the following corollary provide technical tools needed in the proofs of the subsequent theorems:

Lemma 3.1.4
Let F be a complete and smooth fan in $\mathbb{R}^n$ (in the sense of toric geometry) and let $h_1, \ldots, h_r$, $r \leq n$, be rational functions on $\mathbb{R}^n$ that are linear on every cone of F. Then the intersection product $h_1 \cdots h_r \cdot F$ is given by

$$h_1 \cdots h_r \cdot F = \left(\bigcup_{i=0}^{n-r} F^{(i)}, \omega_{h_1 \cdots h_r} \right)$$

with some weight function $\omega_{h_1 \cdots h_r}$ on the cones of dimension $n-r$. For a cone $\sigma \in F^{(n-r)}$ such that all functions $h_1, \ldots, h_r$ are identically zero on σ the weight $\omega_{h_1 \cdots h_r}(\sigma)$ is given by

$$\omega_{h_1 \cdots h_r}(\sigma) = \sum_{\substack{r_{u_1}, \ldots, r_{u_r} \in F^{(1)}: \\ r_{u_1}, \ldots, r_{u_r}, \sigma \text{ span a cone in } F^{(n)}}} h_1(u_1) \cdots h_r(u_r),$$

where r_{u_i} denotes the ray generated by the primitive lattice vector u_i.

Proof. We prove the claim by induction on r: For $r = 1$ the above formula is just the definition of the intersection product (see definition 1.2.4). For $r > 1$ we have the equation

$$\omega_{h_1 \cdots h_r}(\sigma) = \sum_{\substack{r_{u_1} \in F^{(1)}: \\ r_{u_1}, \sigma \text{ span a cone } \widetilde{\sigma} \in F^{(n-r+1)}}} \omega_{h_2 \cdots h_r}(\widetilde{\sigma}) \cdot h_1(u_1).$$

Using the induction hypothesis we can conclude that

$$\begin{aligned}
&\omega_{h_1\cdots h_r}(\sigma)\\
=&\sum_{\substack{r_{u_1}\in F^{(1)}:\\ r_{u_1},\sigma \text{ span a cone } \widetilde{\sigma}\in F^{(n-r+1)}}}\left(\sum_{\substack{r_{u_2},\ldots,r_{u_r}\in F^{(1)}:\\ r_{u_2},\ldots,r_{u_r},\widetilde{\sigma} \text{ span a cone in } F^{(n)}}} h_2(u_2)\cdots h_r(u_r)\right)\cdot h_1(u_1)\\
=&\sum_{\substack{r_{u_1},\ldots,r_{u_r}\in F^{(1)}:\\ r_{u_1},\ldots,r_{u_r},\sigma \text{ span a cone in } F^{(n)}}} h_1(u_1)\cdots h_r(u_r).
\end{aligned}$$

□

Corollary 3.1.5
Under the above assumptions the weight of the cone σ can equivalently be written as

$$\omega_{h_1\cdots h_r}(\sigma)=\sum_{\substack{\varrho\in F^{(n)}:\\ \varrho \text{ is generated by } r_{u_1},\ldots,r_{u_r},\sigma}} \operatorname{perm}\left((h_i(u_j)_{i,j=1,\ldots,r}\right),$$

where $\operatorname{perm}(A)$ *denotes the permanent of the matrix* A.

Proof. Using lemma 3.1.4 we can conclude that

$$\begin{aligned}
\omega_{h_1\cdots h_r}(\sigma) &= \sum_{\substack{u_1,\ldots,u_r\in F^{(1)}:\\ u_1,\ldots,u_r,\sigma \text{ span a cone in } F^{(n)}}} h_1(u_1)\cdots h_r(u_r)\\
&= \sum_{\substack{\varrho\in F^{(n)},\\ \varrho \text{ is generated by } u_1,\ldots,u_r,\sigma}} \sum_{p\in S_r}\left(h_1(u_{p(1)})\cdots h_r(u_{p(r)})\right)\\
&= \sum_{\substack{\varrho\in F^{(n)}:\\ \varrho \text{ is generated by } u_1,\ldots,u_r,\sigma}} \operatorname{perm}\left((h_i(u_j)_{i,j=1,\ldots,r}\right).
\end{aligned}$$

□

Remark 3.1.6
Note that the above assumption that all rational functions are identically zero on σ is not really a restriction: We can always achieve this by adding suitable globally linear functions to the rational functions h_i which does not change the intersection product and in particular not the weight $\omega_{h_1\cdots h_r}(\sigma)$.

Notation 3.1.7
Let F be a simplicial fan in $\mathbb{R}^n$ and let u be a generator of a ray r_u in F. By abuse of notation we also denote by u the unique rational function on $|F|$ that is linear on every cone in F, that has the value one on u and that is identically zero on all rays of F other than r_u.
If not stated otherwise, vectors considered as Cartier divisors will from now on always denote rational functions on the complete fan F_n^n.

Notation 3.1.8
Let C be a tropical cycle and let $h_1, \dots, h_r \in \mathrm{Div}(C)$ be Cartier divisors on C. If

$$P(x_1, \dots, x_r) = \sum_{i_1+\dots+i_r \leq d} \alpha_{i_1,\dots,i_r} x_1^{i_1} \cdots x_r^{i_r}$$

is a polynomial in variables $x_1, \dots, x_r$ we denote by $P(h_1, \dots, h_r) \cdot C$ the intersection product

$$P(h_1, \dots, h_r) \cdot C := \sum_{i_1+\dots+i_r \leq d} \left(\alpha_{i_1,\dots,i_r} h_1^{i_1} \cdots h_r^{i_r} \cdot C\right).$$

In the following theorem we give a description of the diagonal $\triangle_{L^n_{n-k}}$ by means of Cartier divisors on our fan F^n_n:

Theorem 3.1.9
The fan

$$\left(\begin{pmatrix} -e_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -e_0 \end{pmatrix}\right) \dots \left(\begin{pmatrix} -e_n \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -e_0 \end{pmatrix}\right) \cdot \left(\begin{pmatrix} -e_0 \\ 0 \end{pmatrix} + \begin{pmatrix} -e_0 \\ -e_0 \end{pmatrix}\right)^k \cdot F^n_n$$

is a representative of the diagonal $\triangle_{L^n_{n-k}}$.

Proof. First of all, note that

$$\begin{pmatrix} -e_0 \\ 0 \end{pmatrix} + \begin{pmatrix} -e_0 \\ -e_0 \end{pmatrix}$$

is a representation of the tropical polynomial $\max\{0, x_1, \dots, x_n\}$, where $x_1, \dots, x_n$ are the coordinates of the first factor of $\mathbb{R}^n \times \mathbb{R}^n$. Applying lemma 1.8.5 we obtain

$$\left[\left(\begin{pmatrix} -e_0 \\ 0 \end{pmatrix} + \begin{pmatrix} -e_0 \\ -e_0 \end{pmatrix}\right)^k \cdot F^n_n\right] = [L^n_{n-k} \times \mathbb{R}^n].$$

By lemma 1.8.3 we can conclude that $\triangle_{\mathbb{R}^n} \cdot [L^n_{n-k} \times \mathbb{R}^n] = i_*([L^n_{n-k}]) = \triangle_{L^n_{n-k}}$ and hence it suffices to show that $[X] = \triangle_{\mathbb{R}^n}$ for

$$X := \left(\begin{pmatrix} -e_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -e_0 \end{pmatrix}\right) \dots \left(\begin{pmatrix} -e_n \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -e_0 \end{pmatrix}\right) \cdot F^n_n$$

to prove the claim. Therefore, let $\sigma = \langle r_1, \dots, r_n \rangle_{\mathbb{R}_{\geq 0}} \in X^{(n)}$ be a cone not contained in $|\triangle_{\mathbb{R}^n}|$. We will show that the weight of σ in X has to be zero. W.l.o.g. we assume that

$$r_1 \notin D := \left\{\begin{pmatrix} -e_1 \\ -e_1 \end{pmatrix}, \dots, \begin{pmatrix} -e_n \\ -e_n \end{pmatrix}\right\}.$$

Moreover, let

$$T := \left\{\begin{pmatrix} -e_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} -e_n \\ 0 \end{pmatrix}\right\} \text{ and } B := \left\{\begin{pmatrix} 0 \\ -e_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ -e_n \end{pmatrix}\right\}.$$

We distinguish between two cases:

1. First, we assume that
$$r_i \notin \left\{ \begin{pmatrix} -e_0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -e_0 \end{pmatrix} \right\}, i = 1, \ldots, n.$$
Changing the given rational functions by globally linear functions we can rewrite the above intersection product as $X = \varphi_1 \cdots \varphi_n \cdot F_n^n$, where
$$\varphi_i = \begin{cases} \begin{pmatrix} -e_i \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -e_0 \end{pmatrix}, & \text{if } \begin{pmatrix} -e_i \\ 0 \end{pmatrix} \notin \{r_1, \ldots, r_n\} \\ \begin{pmatrix} 0 \\ -e_i \end{pmatrix} + \begin{pmatrix} -e_0 \\ 0 \end{pmatrix}, & \text{else.} \end{cases}$$
All occurring rational functions are identically zero on σ now and we can apply corollary 3.1.5: If the weight of σ in X is non-zero there must be at least one cone $\widetilde{\sigma} = \langle r_1, \ldots, r_n, v_1, \ldots, v_n \rangle_{\mathbb{R}_{\geq 0}} \in F_n^n$ such that $\operatorname{perm}((\varphi_i(v_j))_{i,j}) \neq 0$. We study three subcases:
   a) More than one vector r_i is contained in T (or in B): If $\begin{pmatrix} -e_i \\ 0 \end{pmatrix}, \begin{pmatrix} -e_j \\ 0 \end{pmatrix} \in \{r_1, \ldots, r_n\}$ for some $i \neq j$ then $\varphi_i = \begin{pmatrix} 0 \\ -e_i \end{pmatrix} + \begin{pmatrix} -e_0 \\ 0 \end{pmatrix}$ and $\varphi_j = \begin{pmatrix} 0 \\ -e_j \end{pmatrix} + \begin{pmatrix} -e_0 \\ 0 \end{pmatrix}$. Hence we need two vectors out of $\begin{pmatrix} 0 \\ -e_i \end{pmatrix}, \begin{pmatrix} 0 \\ -e_j \end{pmatrix}, \begin{pmatrix} -e_0 \\ 0 \end{pmatrix}$ among the v_μ such that $\operatorname{perm}((\varphi_i(v_j))_{i,j})$ can be non-zero. But there is no cone in F_n^n containing $\begin{pmatrix} -e_i \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -e_i \end{pmatrix}$ or $\begin{pmatrix} -e_j \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -e_j \end{pmatrix}$. (Analogously for B.)
   b) There are vectors $r_i \in T$ and $r_j \in B$: Then we need both vectors $\begin{pmatrix} -e_0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -e_0 \end{pmatrix}$ among the v_μ such that $\operatorname{perm}((\varphi_i(v_j))_{i,j})$ can be non-zero. But again there is no cone in F_n^n containing these two vectors.
   c) $r_1 \in T$ (or $r_1 \in B$) and $r_j \in D$ for $j \neq 1$: Like in (a) we need $v_1 = \begin{pmatrix} -e_0 \\ 0 \end{pmatrix}$ and $v_2, \ldots, v_n \in \left\{ \begin{pmatrix} -e_2 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} -e_n \\ 0 \end{pmatrix} \right\}$ such that $\operatorname{perm}((\varphi_i(v_j))_{i,j})$ can be non-zero. But there is no cone in F_n^n containing $\begin{pmatrix} -e_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} -e_n \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -e_0 \\ 0 \end{pmatrix}$. (Analogously for B.)
2. Now we assume that
$$r_1 = \begin{pmatrix} -e_0 \\ 0 \end{pmatrix} \quad \left(\text{or } r_1 = \begin{pmatrix} 0 \\ -e_0 \end{pmatrix} \right).$$

Again, by changing the given rational functions by globally linear functions we can rewrite the intersection product as $X = \varphi_1 \cdots \varphi_n \cdot F_n^n$, where

$$\varphi_i = \begin{cases} \left(\frac{-e_i}{0}\right) + \left(\frac{0}{-e_0}\right), & \text{if } \left(\frac{-e_i}{0}\right) \notin \{r_1, \ldots, r_n\} \\ \text{some equivalent rational function not involving } r_1, \ldots, r_n, & \text{else.} \end{cases}$$

Again we reached that all rational functions are identically zero on σ and we can apply corollary 3.1.5: If $\left(\frac{-e_i}{0}\right) \notin \{r_1, \ldots, r_n\}$ then $\varphi_i = \left(\frac{-e_i}{0}\right) + \left(\frac{0}{-e_0}\right)$ and we need $\left(\frac{-e_i}{0}\right)$ or $\left(\frac{0}{-e_0}\right)$ among the v_μ such that perm $((\varphi_i(v_j))_{i,j})$ can be non-zero. But as there is no cone in F_n^n containing $\left(\frac{0}{-e_0}\right)$ and $\left(\frac{-e_0}{0}\right)$ we must have $\left(\frac{-e_i}{0}\right) \in \{v_1, \ldots, v_n\}$. Hence all the vectors $\left(\frac{-e_1}{0}\right), \ldots, \left(\frac{-e_n}{0}\right)$ and $\left(\frac{-e_0}{0}\right)$ must be contained in $\{r_1, \ldots, r_n, v_1, \ldots, v_n\}$, but there is no such cone in F_n^n. (Analogously for $r_1 = \left(\frac{0}{-e_0}\right)$.)

Our last step in this proof is to show that at least one cone in the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$ occurs with weight one in X. As the diagonal is irreducible we can then conclude by [GKM07, lemma 2.21] that the whole diagonal occurs with weight one. We have a look at the cone

$$\sigma = \langle r_1, \ldots, r_n \rangle_{\mathbb{R}_{\geq 0}} = \left\langle \left(\frac{-e_1}{-e_1}\right), \ldots, \left(\frac{-e_n}{-e_n}\right) \right\rangle_{\mathbb{R}_{\geq 0}}.$$

As all the rational functions

$$\varphi_i = \left(\frac{-e_i}{0}\right) + \left(\frac{0}{-e_0}\right)$$

are already zero on σ we can immediately apply corollary 3.1.5: There is exactly one cone

$$\begin{aligned} \widetilde{\sigma} &= \langle r_1, \ldots, r_n, v_1, \ldots, v_n \rangle_{\mathbb{R}_{\geq 0}} \\ &= \left\langle \left(\frac{-e_1}{-e_1}\right), \ldots, \left(\frac{-e_n}{-e_n}\right), \left(\frac{-e_1}{0}\right), \ldots, \left(\frac{-e_n}{0}\right) \right\rangle_{\mathbb{R}_{\geq 0}} \end{aligned}$$

in F_n^n containing σ such that the permanent perm $((\varphi_i(v_j))_{i,j})$ is non-zero and for this cone we have perm $((\varphi_i(v_j))_{i,j}) = 1$. This finishes the proof. □

Our next step is to derive a description of the diagonal $\triangle_{L_{n-k}^n}$ on $L_{n-k}^n \times L_{n-k}^n$ from our description on F_n^n:

Theorem 3.1.10
The intersection product in theorem 3.1.9 can be rewritten as

$$\left(\sum_{i=1}^{r} h_{i,1}\dots h_{i,n-k}\right)\cdot\left(\left(\frac{0}{-e_0}\right)+\left(\frac{-e_0}{-e_0}\right)\right)^k\cdot\left(\left(\frac{-e_0}{0}\right)+\left(\frac{-e_0}{-e_0}\right)\right)^k\cdot F_n^n$$

for some Cartier divisors $h_{i,j}$ on F_n^n.

We have to prepare the proof of the theorem by the following lemma:

Lemma 3.1.11
Let $C\in Z_l(L_{n-k}^n)$ be a subcycle of L_{n-k}^n. Then the following intersection products are zero:

(a) $\left(\frac{-e_0}{0}\right)\cdot\left(\frac{0}{-e_0}\right)\cdot(C\times\mathbb{R}^n)$,

(b) $v_{i_1}\cdots v_{i_{n-k+r}}\cdot(C\times\mathbb{R}^n)$,

(c) $\left(\frac{0}{-e_0}\right)\cdot\left(\frac{-e_0}{-e_0}\right)^s\cdot v_{i_1}\cdots v_{i_{n-k-s+r}}\cdot(C\times\mathbb{R}^n)$,

where $r,s>0$ and the vectors

$$v_{i_j}\in\left\{\left(\frac{-e_0}{0}\right),\dots,\left(\frac{-e_n}{0}\right),\left(\frac{-e_0}{-e_0}\right),\dots,\left(\frac{-e_n}{-e_n}\right)\right\}$$

are pairwise distinct.

Proof. (a) and (b): In both cases, a cone that can occur in the intersection product with non-zero weight has to be contained in a cone of F_n^n that is contained in $|L_{n-k}^n\times\mathbb{R}^n|$ and that contains the vectors $\left(\frac{-e_0}{0}\right),\left(\frac{0}{-e_0}\right)$ or $v_{i_1},\dots,v_{i_{n-k+r}}$, respectively. But there are no such cones.
(c): By (a) and lemma 1.8.6 we can rewrite the intersection product as

$$\begin{aligned}
&\left(\frac{0}{-e_0}\right)\cdot\left(\frac{-e_0}{-e_0}\right)^s\cdot v_{i_1}\cdots v_{i_{n-k-s+r}}\cdot(C\times\mathbb{R}^n)\\
=\;&\left(\frac{0}{-e_0}\right)\cdot\left(\left(\frac{-e_0}{0}\right)+\left(\frac{-e_0}{-e_0}\right)\right)^s\cdot v_{i_1}\cdots v_{i_{n-k-s+r}}\cdot(C\times\mathbb{R}^n)\\
=\;&\left(\frac{0}{-e_0}\right)\cdot v_{i_1}\cdots v_{i_{n-k-s+r}}\cdot\left[\left(\left(\frac{-e_0}{0}\right)+\left(\frac{-e_0}{-e_0}\right)\right)^s\cdot C\right]\times\mathbb{R}^n\\
=\;&\left(\frac{0}{-e_0}\right)\cdot v_{i_1}\cdots v_{i_{n-k-s+r}}\cdot[\max\{0,x_1,\dots,x_n\}^s\cdot C]\times\mathbb{R}^n,
\end{aligned}$$

which is zero by (b) as $\max\{0,x_1,\dots,x_n\}^s\cdot C$ is contained in L_{n-k-s}^n. □

Proof of theorem 3.1.10. By theorem 3.1.9 we have the representation

$$\begin{aligned}\triangle_{L^n_{n-k}} &= \left(\left(\tfrac{-e_1}{0}\right)+\left(\tfrac{0}{-e_0}\right)\right)\dots\left(\left(\tfrac{-e_n}{0}\right)+\left(\tfrac{0}{-e_0}\right)\right)\cdot\underbrace{\left(\left(\tfrac{-e_0}{0}\right)+\left(\tfrac{-e_0}{-e_0}\right)\right)^k\cdot[F^n_n]}_{=[L^n_{n-k}\times\mathbb{R}^n]}\\ &= \left(\left(\tfrac{-e_1}{0}\right)\cdots\left(\tfrac{-e_n}{0}\right)+\ldots+\left(\tfrac{0}{-e_0}\right)^n\right)\cdot[L^n_{n-k}\times\mathbb{R}^n].\end{aligned}$$

By lemma 3.1.11 (b) all the summands containing $\left(\frac{0}{-e_0}\right)^s$ with a power $s<k$ are zero. Hence we can rewrite the intersection product as

$$\begin{aligned}\triangle_{L^n_{n-k}} &= \left[\left(\frac{-e_1}{0}\right)\cdots\left(\frac{-e_k}{0}\right)+\ldots+\left(\frac{0}{-e_0}\right)^{n-k}\cdot\left(\left(\frac{0}{-e_0}\right)+\left(\frac{-e_0}{-e_0}\right)\right)^k-A\right]\\ &\quad\cdot[L^n_{n-k}\times\mathbb{R}^n],\end{aligned}$$

where A contains all the summands we added too much. Thus all the summands of A are of the form

$$\alpha\cdot v_1\cdots v_{n-s-r}\cdot\left(\frac{0}{-e_0}\right)^s\cdot\left(\frac{-e_0}{-e_0}\right)^t$$

for some integer α, vectors $v_i\in\left\{\left(\frac{-e_1}{0}\right),\dots,\left(\frac{-e_n}{0}\right)\right\}$ and powers $1\le t\le k$, $0\le s\le n$. By lemma 3.1.11 (b) and (c) such a summand applied to $[L^n_{n-k}\times\mathbb{R}^n]$ is zero if $s<k$ and only those summands remain in A that have $t\ge 1, s\ge k$. Let

$$S:=\alpha\cdot v_1\cdots v_{n-s-r}\cdot\left(\frac{0}{-e_0}\right)^s\cdot\left(\frac{-e_0}{-e_0}\right)^t$$

be one of the remaining summands. By lemma 3.1.11 (a) we obtain the equation

$$\begin{aligned}&\alpha\cdot v_1\cdots v_{n-s-t}\cdot\left(\tfrac{0}{-e_0}\right)^s\cdot\left(\tfrac{-e_0}{-e_0}\right)^t\cdot[L^n_{n-k}\times\mathbb{R}^n]\\ =\;&\left(\sum_{j=0}^{t}\binom{t}{j}\cdot\alpha\cdot v_1\cdots v_{n-s-t}\cdot\left(\tfrac{0}{-e_0}\right)^s\cdot\left(\tfrac{-e_0}{-e_0}\right)^j\cdot\left(\tfrac{-e_0}{0}\right)^{t-j}\right)\cdot[L^n_{n-k}\times\mathbb{R}^n]\\ =\;&\left(\alpha\cdot v_1\cdots v_{n-s-t}\cdot\left(\tfrac{0}{-e_0}\right)^s\cdot\left(\left(\tfrac{-e_0}{0}\right)+\left(\tfrac{-e_0}{-e_0}\right)\right)^t\right)\cdot[L^n_{n-k}\times\mathbb{R}^n]\\ =\;&\Bigg[\left(\left(\tfrac{0}{-e_0}\right)+\left(\tfrac{-e_0}{-e_0}\right)\right)^k\\ &\cdot\left(\alpha\cdot v_1\cdots v_{n-s-t}\cdot\left(\tfrac{0}{-e_0}\right)^{s-k}\cdot\left(\left(\tfrac{-e_0}{0}\right)+\left(\tfrac{-e_0}{-e_0}\right)\right)^t\right)-B_S\Bigg]\cdot[L^n_{n-k}\times\mathbb{R}^n],\end{aligned}$$

where B_S contains again all the summands we added too much. Thus all the summands of B_S are of the form

$$S':=\beta\cdot\binom{t}{j}\cdot v_1\cdots v_{n-s-t}\cdot\left(\frac{0}{-e_0}\right)^{s-s'}\cdot\left(\frac{-e_0}{-e_0}\right)^{s'}\cdot\left(\frac{-e_0}{0}\right)^{t'}\cdot\left(\frac{-e_0}{-e_0}\right)^{t-t'}$$

for some integer β and powers $1\le s'\le k$, $0\le t'\le t$. If $s-s'<k$ we group all corresponding summands together as

$$\beta\cdot v_1\cdots v_{n-s-t}\cdot\left(\frac{0}{-e_0}\right)^{s-s'}\cdot\left(\frac{-e_0}{-e_0}\right)^{s'}\cdot\left(\left(\frac{-e_0}{0}\right)+\left(\frac{-e_0}{-e_0}\right)\right)^t.$$

This product applied to $[L^n_{n-k} \times \mathbb{R}^n]$ is zero by lemma 3.1.11 (b) and (c). Moreover, all summands S' with $s - s' \geq k$ and $t' > 0$ yield zero on $[L^n_{n-k} \times \mathbb{R}^n]$ by lemma 3.1.11 (a). Thus only those summands S' are left in B_S that are of the form

$$S' = \beta' \cdot v_1 \cdots v_{n-s-t} \cdot \left(\frac{0}{-e_0}\right)^{s-s'} \cdot \left(\frac{-e_0}{-e_0}\right)^{t+s'}$$

with $s - s' \geq k$ and $s' \geq 1$. Applying this process inductively to all summands with $t = 1, \ldots, n-k-1$ in which we could not factor out $\left(\left(\frac{0}{-e_0}\right) + \left(\frac{-e_0}{-e_0}\right)\right)^k$, yet, we can by and by increase the power of $\left(\frac{-e_0}{-e_0}\right)$ in all remaining summands until finally only one summand

$$\gamma \cdot \left(\frac{0}{-e_0}\right)^k \cdot \left(\frac{-e_0}{-e_0}\right)^{n-k}$$

is left. But

$$\begin{aligned}
&\gamma \cdot \left(\frac{0}{-e_0}\right)^k \cdot \left(\frac{-e_0}{-e_0}\right)^{n-k} \cdot [L^n_{n-k} \times \mathbb{R}^n] \\
&= \gamma \cdot \left(\left(\frac{0}{-e_0}\right) + \left(\frac{-e_0}{-e_0}\right)\right)^k \cdot \left(\left(\frac{-e_0}{0}\right) + \left(\frac{-e_0}{-e_0}\right)\right)^{n-k} \cdot [L^n_{n-k} \times \mathbb{R}^n]
\end{aligned}$$

as

$$\begin{aligned}
&\left(\frac{0}{-e_0}\right)^i \cdot \left(\frac{-e_0}{-e_0}\right)^{k-i} \cdot \left(\left(\frac{-e_0}{0}\right) + \left(\frac{-e_0}{-e_0}\right)\right)^{n-k} \cdot [L^n_{n-k} \times \mathbb{R}^n] \\
&= \left(\frac{0}{-e_0}\right)^i \cdot \left(\frac{-e_0}{-e_0}\right)^{k-i} \cdot [L^n_0 \times \mathbb{R}^n] \\
&= 0
\end{aligned}$$

for all $i < k$ by lemma 3.1.11 (b) and

$$\left(\frac{0}{-e_0}\right)^k \cdot \left(\frac{-e_0}{0}\right)^j \cdot \left(\frac{-e_0}{-e_0}\right)^{n-k-j} \cdot [L^n_{n-k} \times \mathbb{R}^n] = 0$$

for all $j > 0$ by lemma 3.1.11 (a). This proves the claim. □

Corollary 3.1.12
The Cartier divisors $h_{i,j}$ from theorem 3.1.10 provide the following description of the diagonal $\triangle_{L^n_{n-k}}$:

$$\triangle_{L^n_{n-k}} = \sum_{i=1}^{r} h_{i,1} \ldots h_{i,n-k} \cdot [L^n_{n-k} \times L^n_{n-k}].$$

Proof. Let $x_1, \ldots, x_n$ be the coordinates of the first and $y_1, \ldots, y_n$ be coordinates of the second factor of $\mathbb{R}^n \times \mathbb{R}^n$. Applying lemma 1.8.5 we can conclude that

$$\begin{aligned}
&\left[\left(\left(\frac{0}{-e_0}\right) + \left(\frac{-e_0}{-e_0}\right)\right)^k \cdot \left(\left(\frac{-e_0}{0}\right) + \left(\frac{-e_0}{-e_0}\right)\right)^k \cdot F^n_n\right] \\
&= \left[\max\{0, x_1, \ldots, x_n\}^k \cdot \max\{0, y_1, \ldots, y_n\}^k \cdot F^n_n\right] \\
&= [L^n_{n-k} \times L^n_{n-k}]
\end{aligned}$$

and hence by theorem 3.1.9 and theorem 3.1.10 that

$$\sum_{i=1}^{r} h_{i,1} \dots h_{i,n-k} \cdot [L^n_{n-k} \times L^n_{n-k}] = \triangle_{L^n_{n-k}}.$$

□

Remark 3.1.13
As lemma 3.1.11 does not only hold on $L^n_{n-k} \times \mathbb{R}^n$ but also on any $C \times \mathbb{R}^n$ with C a subcycle of L^n_{n-k}, the proof of theorem 3.1.10 indeed shows that

$$\begin{aligned} & \left(\sum_{i=1}^{r} h_{i,1} \dots h_{i,n-k}\right) \cdot \left(\left(\frac{0}{-e_0}\right) + \left(\frac{-e_0}{-e_0}\right)\right)^k \cdot (C \times \mathbb{R}^n) \\ = \quad & \left(\left(\frac{-e_1}{0}\right) + \left(\frac{0}{-e_0}\right)\right) \dots \left(\left(\frac{-e_n}{0}\right) + \left(\frac{0}{-e_0}\right)\right) \cdot (C \times \mathbb{R}^n) \end{aligned}$$

for all cycles $C \in Z_l(L^n_{n-k})$. Using corollary 1.8.7 we can conclude that

$$\begin{aligned} & \left(\sum_{i=1}^{r} h_{i,1} \dots h_{i,n-k}\right) \cdot \left(\left(\frac{0}{-e_0}\right) + \left(\frac{-e_0}{-e_0}\right)\right)^k \cdot (C \times \mathbb{R}^n) \\ = \quad & \triangle_{\mathbb{R}^n} \cdot (C \times \mathbb{R}^n) \\ = \quad & \triangle_C \end{aligned}$$

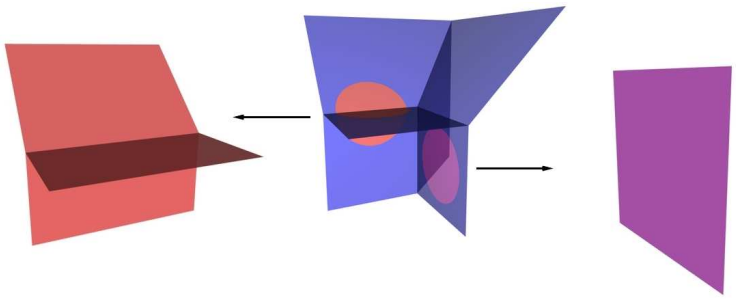
for all such cycles C.

Corollary 3.1.14
Let $\sigma \in L^n_{n-k}$, let $x \in \sigma$ and let $U \subseteq S_\sigma = \bigcup_{\sigma' \in L^n_{n-k} : \sigma' \supseteq \sigma} (\sigma')^{ri}$ be an open subset of $|L^n_{n-k}|$ containing x. Moreover, let F be the open fan $F := \{-x + \sigma \cap U | \sigma \in L^n_{n-k}\}$ and $\widetilde{F}$ the associated tropical fan. Then there are Cartier divisors $h'_{i,j}$ on $\widetilde{F} \times \widetilde{F}$ such that

$$\triangle_{[\widetilde{F}]} = \sum_{i=1}^{r} h'_{i,1} \dots h'_{i,n-k} \cdot [\widetilde{F} \times \widetilde{F}].$$

Proof. To obtain the Cartier divisors $h'_{i,j}$ we just have to restrict the Cartier divisors $h_{i,j}$ from corollary 3.1.12 to the open set $U \times U$, translate them suitably and extend them from $F \times F$ to the associated tropical fan $\widetilde{F} \times \widetilde{F}$. □

Example 3.1.15
The following figure shows two fans associated to open subsets of L^3_2 as in corollary 3.1.14:



Lemma 3.1.16
Let $C \in Z_k(\mathbb{R}^n)$ and $D \in Z_l(\mathbb{R}^n)$ be tropical cycles such that there exist representations of the diagonals $\triangle_C$ and $\triangle_D$ as sums of products of Cartier divisors on $C \times C$ and $D \times D$, respectively. Then there also exists a representation of $\triangle_{C\times D}$ as a sum of products of Cartier divisors on $(C \times D)^2$.

Proof. Let $\triangle_C = \sum_{i=1}^r \varphi_{i,1} \dots \varphi_{i,k} \cdot (C \times C)$ and $\triangle_D = \sum_{i=1}^s \psi_{i,1} \dots \psi_{i,l} \cdot (D \times D)$. Moreover, let $\pi_x, \pi_y : (\mathbb{R}^n)^4 \to (\mathbb{R}^n)^2$ be given by $(x_1, y_1, x_2, y_2) \mapsto (x_1, x_2)$ and $(x_1, y_1, x_2, y_2) \mapsto (y_1, y_2)$, respectively. Then we have the equation

$$\triangle_{C\times D} = \left(\sum_{i=1}^r \pi_x^* \varphi_{i,1} \dots \pi_x^* \varphi_{i,k} \right) \cdot \left(\sum_{i=1}^s \pi_y^* \psi_{i,1} \dots \pi_y^* \psi_{i,l} \right) \cdot (C \times D)^2.$$

□

Now we are ready to define intersection products on all spaces on which we can express the diagonal by means of Cartier divisors:

Definition 3.1.17 (Intersection products)
Let $C \in Z_k(\mathbb{R}^n)$ be a tropical cycle and assume that there are Cartier divisors $\varphi_{i,j}$ on $C \times C$ such that

$$\triangle_C = \sum_{i=1}^r \varphi_{i,1} \dots \varphi_{i,k} \cdot (C \times C).$$

Moreover, let $\pi : C \times C \to C$ be the morphism given by $(x, y) \mapsto x$. Then we define the intersection product of subcycles of C by

$$\begin{array}{rcl} Z_{k-l}(C) \times Z_{k-l'}(C) & \longrightarrow & Z_{k-l-l'}(C) \\ (D_1, D_2) & \longmapsto & D_1 \cdot D_2 := \pi_* \left(\sum_{i=1}^r \varphi_{i,1} \dots \varphi_{i,k} \cdot (D_1 \times D_2) \right). \end{array}$$

We use the rest of this section to prove that this intersection product is independent of the used representation of the diagonal and that it has all the properties we expect — at least for those spaces we are interested in:

Lemma 3.1.18
Let $C \in Z_k(\mathbb{R}^n)$ be a tropical cycle, $D \in Z_{k-l}(C)$, $E \in Z_{k-l'}(C)$ be subcycles, let $\varphi \in \mathrm{Div}(C)$ be a Cartier divisor and $\pi : C \times C \to C$ the morphism given by $(x, y) \mapsto x$. Then the following equation holds:

$$(\varphi \cdot D) \times E = \pi^* \varphi \cdot (D \times E).$$

Proof. The proof is exactly the same as for lemma 1.8.5. □

Corollary 3.1.19
Let $C \in Z_k(\mathbb{R}^n)$ be a tropical cycle that admits an intersection product as in definition 3.1.17, let $D \in Z_{k-l}(C)$, $E \in Z_{k-l'}(C)$ be subcycles and let $\varphi \in \mathrm{Div}(C)$ be a Cartier divisor. Then the following equation holds:

$$(\varphi \cdot D) \cdot E = \varphi \cdot (D \cdot E).$$

Proof. The proof is exactly the same as for lemma 1.8.6. □

Proposition 3.1.20
Let $D \in Z_l(L^n_{n-k})$ be a subcycle. Then the equation

$$D \cdot [L^n_{n-k}] = [L^n_{n-k}] \cdot D = D$$

holds on L^n_{n-k}.

Proof. Let $\pi_i : L^n_{n-k} \times L^n_{n-k} \to L^n_{n-k}$ be the morphism given by $(x_1, x_2) \mapsto x_i$. By remark 3.1.13 we get the equation

$$\begin{aligned} D \cdot [L^n_{n-k}] &= (\pi_1)_* \left(\sum_{i=1}^{r} h_{i,1} \dots h_{i,n-k} \cdot \left(D \times [L^n_{n-k}] \right) \right) \\ &= (\pi_1)_* \left(\left(\sum_{i=1}^{r} h_{i,1} \dots h_{i,n-k} \right) \cdot \left(\left(\frac{0}{-e_0} \right) + \left(\frac{-e_0}{-e_0} \right) \right)^k \cdot (D \times \mathbb{R}^n) \right) \\ &= (\pi_1)_* \left(\triangle_{\mathbb{R}^n} \cdot (D \times \mathbb{R}^n) \right) \\ &= (\pi_1)_* \left(\triangle_D \right) \\ &= D. \end{aligned}$$

Furthermore, let $\phi : L^n_k \times L^n_k \to L^n_k \times L^n_k$ be the morphism induced by $(x, y) \mapsto (y, x)$. Obviously we have the equality

$$\left(\sum_{i=1}^{r} h_{i,1} \dots h_{i,n-k} \right) \cdot [L^n_{n-k} \times L^n_{n-k}] = \left(\sum_{i=1}^{r} \phi^* h_{i,1} \dots \phi^* h_{i,n-k} \right) \cdot [L^n_{n-k} \times L^n_{n-k}].$$

If $\pi_{ij} : (L^n_{n-k})^4 \to (L^n_{n-k})^2$ is the morphism given by $(x_1, x_2, x_3, x_4) \mapsto (x_i, x_j)$ and

$$\triangle := \left(\sum_{i=1}^{r} \pi_{13}^* h_{i,1} \dots \pi_{13}^* h_{i,n-k} \right) \cdot \left(\sum_{i=1}^{r} \pi_{24}^* h_{i,1} \dots \pi_{24}^* h_{i,n-k} \right)$$

we can conclude by proposition 1.6.7 and lemma 1.8.5 that

$$\begin{aligned} & \left(\sum_{i=1}^{r} \phi^* h_{i,1} \dots \phi^* h_{i,n-k} \right) \cdot (D \times [L^n_{n-k}]) \\ = & \left(\sum_{i=1}^{r} \phi^* h_{i,1} \dots \phi^* h_{i,n-k} \right) \cdot \Big((D \times [L^n_{n-k}]) \cdot ([L^n_{n-k} \times L^n_{n-k}]) \Big) \\ = & \left(\sum_{i=1}^{r} \phi^* h_{i,1} \dots \phi^* h_{i,n-k} \right) \cdot (\pi_{12})_* \Big(\triangle \cdot \Big((D \times [L^n_{n-k}]) \times ([L^n_{n-k} \times L^n_{n-k}]) \Big) \Big) \\ = & \left(\sum_{i=1}^{r} \phi^* h_{i,1} \dots \phi^* h_{i,n-k} \right) \cdot (\pi_{12})_* \left(\triangle_{D \times [L^n_{n-k}]} \right) \\ = & \left(\sum_{i=1}^{r} \phi^* h_{i,1} \dots \phi^* h_{i,n-k} \right) \cdot (\pi_{34})_* \left(\triangle_{D \times [L^n_{n-k}]} \right) \\ = & (\pi_{34})_* \left(\left(\sum_{i=1}^{r} \pi_{34}^* \phi^* h_{i,1} \dots \pi_{34}^* \phi^* h_{i,n-k} \right) \cdot \triangle \cdot \Big((D \times [L^n_{n-k}]) \times ([L^n_{n-k} \times L^n_{n-k}]) \Big) \right) \\ = & (\pi_{34})_* \left(\triangle \cdot (D \times [L^n_{n-k}]) \times \left(\left(\sum_{i=1}^{r} \phi^* h_{i,1} \dots \phi^* h_{i,n-k} \right) \cdot [L^n_{n-k} \times L^n_{n-k}] \right) \right) \\ = & (\pi_{34})_* \left(\triangle \cdot (D \times [L^n_{n-k}]) \times \left(\left(\sum_{i=1}^{r} h_{i,1} \dots h_{i,n-k} \right) \cdot [L^n_{n-k} \times L^n_{n-k}] \right) \right) \\ = & \left(\sum_{i=1}^{r} h_{i,1} \dots h_{i,n-k} \right) \cdot (D \times [L^n_{n-k}]). \end{aligned}$$

Hence we can deduce that

$$\begin{aligned} D \cdot [L^n_{n-k}] &= (\pi_1)_* (\triangle_D) \\ &= (\pi_2)_* (\triangle_D) \\ &= (\pi_2)_* \left(\left(\sum_{i=1}^{r} h_{i,1} \dots h_{i,n-k} \right) \cdot (D \times [L^n_{n-k}]) \right) \\ &= (\pi_2)_* \left(\left(\sum_{i=1}^{r} \phi^* h_{i,1} \dots \phi^* h_{i,n-k} \right) \cdot (D \times [L^n_{n-k}]) \right) \\ &= (\pi_1)_* \left(\left(\sum_{i=1}^{r} h_{i,1} \dots h_{i,n-k} \right) \cdot ([L^n_{n-k}] \times D) \right) \\ &= [L^n_{n-k}] \cdot D. \end{aligned}$$

This proves the claim. □

Remark 3.1.21
We can prove in the same way that $[L^n_{n-k} \times L^m_{m-l}] \cdot D = D$ holds for all subcycles D of $L^n_{n-k} \times L^m_{m-l}$ and even that $[L^{n_1}_{n_1-k_1} \times \ldots \times L^{n_r}_{n_r-k_r}] \cdot D = D$ holds for all $r \geq 1$ and all subcycles D of $L^{n_1}_{n_1-k_1} \times \ldots \times L^{n_r}_{n_r-k_r}$. Moreover, restricting the intersection products to open subsets of $|L^n_k|$ or $|L^{n_1}_{n_1-k_1} \times \ldots \times L^{n_r}_{n_r-k_r}|$, respectively, implies that $X \cdot D = D$ also holds for all subcycles $D \in Z_l(X)$ if $X \in \{[\widetilde{F}], [\widetilde{F_1} \times \ldots \times \widetilde{F_r}]\}$ where $\widetilde{F}$, $\widetilde{F}_i$ are tropical fans associated to an open subsets of some $|L^n_k|$ like in corollary 3.1.14.

Proposition 3.1.22
Let $C \in Z_k(\mathbb{R}^n)$ be a tropical cycle that admits an intersection product as in definition 3.1.17 and let $D, D' \in Z_l(C)$, $E \in Z_{l'}(C)$ be subcycles. Then the following equation holds:

$$(D + D') \cdot E = D \cdot E + D' \cdot E.$$

Proof. The proof is exactly the same as for theorem 1.8.9. □

Proposition 3.1.23
Let $C \in Z_k(\mathbb{R}^n)$ be a tropical cycle that admits an intersection product as in definition 3.1.17 and let $D \in Z_l(C)$ be a subcycle of C. Moreover, let $E \in Z_{l'}(C)$ be a subcycle such that there are Cartier divisors $\psi_{i,j} \in \mathrm{Div}(C)$ with

$$\sum_{i=1}^{r} \psi_{i,1} \dots \psi_{i,k-l'} \cdot C = E.$$

If additionally $C \cdot D = D$ holds then

$$\sum_{i=1}^{r} \psi_{i,1} \dots \psi_{i,k-l'} \cdot D = E \cdot D.$$

Proof. The proof is the same as for corollary 1.8.7. □

Remark 3.1.24
The meaning of proposition 3.1.23 is the following: If $X \in Z_k(\mathbb{R}^n)$ is a tropical cycle

such that the diagonal $\triangle_X$ can be written as a sum of products of Cartier divisors as in definition 3.1.17 and additionally $(X \times X) \cdot Y = Y$ is fulfilled for all subcycles Y of $X \times X$ then we can apply proposition 3.1.23 with $C := X \times X$ and $E := \triangle_X$ to deduce that the definition of the intersection product is independent of the choice of the Cartier divisors describing the diagonal. In particular we have well-defined intersection products on L_k^n, $L_{k_1}^{n_1} \times \ldots \times L_{k_r}^{n_r}$, $\widetilde{F}$ and $\widetilde{F_1} \times \ldots \times \widetilde{F_r}$ for all tropical fans $\widetilde{F}$, $\widetilde{F_i}$ associated to an open subset of some $|L_k^n|$ like in corollary 3.1.14.

Theorem 3.1.25
Let $C \in Z_k(\mathbb{R}^n)$ be a tropical cycle that admits an intersection product as in definition 3.1.17 such that additionally $(C \times C) \cdot D = D$ is fulfilled for all subcycles D of $C \times C$. Moreover, let $E, E' \in Z_l(C)$, $F \in Z_{l'}(C)$ and $G \in Z_{l''}(C)$ be subcycles. Then the following equations hold:

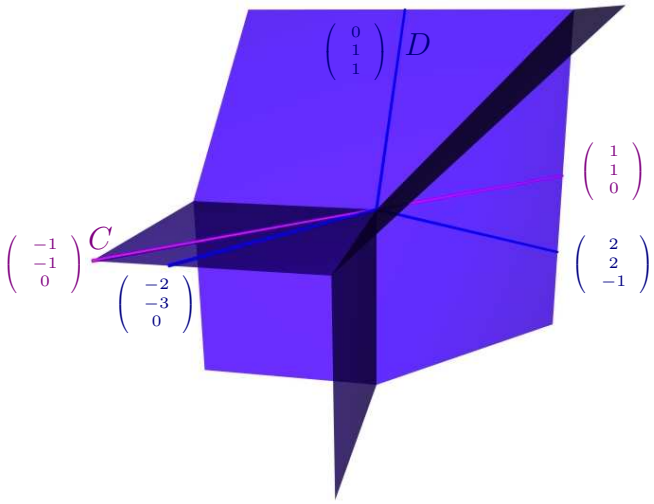
(a) $E \cdot F = F \cdot E$,

(b) $(E \cdot F) \cdot G = E \cdot (F \cdot G)$.

Proof. The proof is exactly the same as for theorem 1.8.9 (a) and (c). □

We finish this section with an example showing that even curves intersecting in the expected dimension can have negative intersections:

Example 3.1.26
Let $C, D \in Z_1(L_2^3)$ be the curves shown in the figure. We want to compute the intersection $C \cdot D$. By proposition 3.1.23 the easiest way to achieve this is to write one of the curves as $\psi \cdot [L_2^3]$ for some Cartier divisor ψ on L_2^3.



Let F be the refinement of L_2^3 arising by dividing the cones $\langle -e_1, -e_2\rangle_{\mathbb{R}_{\geq 0}}$ and $\langle -e_0, -e_3\rangle_{\mathbb{R}_{\geq 0}}$ into cones $\langle -e_1, -e_1 - e_2\rangle_{\mathbb{R}_{\geq 0}}$, $\langle -e_2, -e_1 - e_2\rangle_{\mathbb{R}_{\geq 0}}$ and $\langle -e_0, -e_0 - e_3\rangle_{\mathbb{R}_{\geq 0}}$, $\langle -e_3, -e_0 - e_3\rangle_{\mathbb{R}_{\geq 0}}$, respectively. Then

$$\psi := \begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}-1\\-1\\0\end{pmatrix}$$

defines a rational function on F. As shown in example 1.2.10 we have $\psi \cdot [L_2^3] = C$. Hence we can calculate

$$\begin{aligned} C \cdot D = \psi \cdot D &= \left(\psi \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix} + \psi \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \psi \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \psi \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right) \cdot \{0\} \\ &= (-2+0+1-0) \cdot \{0\} \\ &= -1 \cdot \{0\}. \end{aligned}$$

3.2 Intersection products on smooth tropical varieties

In this section we use our results from section 3.1 to define an intersection product on smooth tropical varieties, i.e. on varieties with tropical linear spaces as local building blocks:

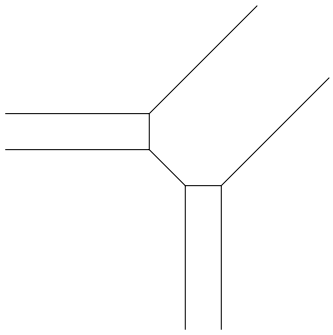
Definition 3.2.1 (Smooth tropical varieties)
An abstract tropical variety C is called a *smooth variety* if it has a representative $(((X, |X|), \omega_X), \{\Phi_\sigma\})$ such that all the maps

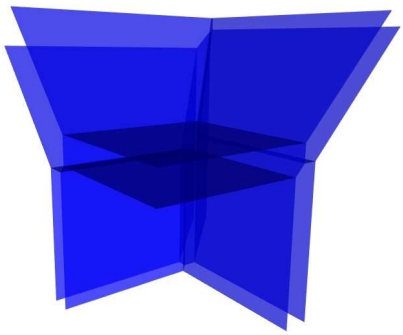
$$\Phi_\sigma : S_\sigma = \bigcup_{\sigma' \in X^*, \sigma' \supset \sigma} (\sigma')^{ri} \xrightarrow{\sim} |F_\sigma| \subseteq |\widetilde{F_\sigma}|$$

(cf. definition 1.4.4) map into tropical fans $\widetilde{F_\sigma} = \widetilde{F_1^\sigma} \times \ldots \times \widetilde{F_{r_\sigma}^\sigma}$ where the $\widetilde{F_i^\sigma}$ are tropical fans associated to open subsets of some $|L_{k_{\sigma,i}}^{n_{\sigma,i}}|$ as in corollary 3.1.14.

Remark 3.2.2
Note that the existence of such a representative $(((X, |X|), \omega_X), \{\Phi_\sigma\})$ for C implies that all representatives of C have the requested property.

Example 3.2.3
The following figures show two examples of smooth tropical varieties:





Definition 3.2.4
Let C be an abstract tropical cycle, D a subcycle of C with representative X and $U \subseteq |C|$ an open subset. We denote by $X \cap U$ the *open tropical polyhedral complex*

$$X \cap U := (\{\sigma \cap U | \sigma \in X\}, |X| \cap U)$$

and by $[X \cap U]$ its equivalence class modulo refinements. As this class only depends on the class of X we can define $D \cap U := [X \cap U]$.

Remark 3.2.5
If we are given an open covering $\{U_1, \ldots, U_r\}$ of C and open tropical polyhedral complexes $D_1 \cap U_1, \ldots, D_r \cap U_r$ such that $D_i \cap U_i \cap U_j = D_j \cap U_i \cap U_j$ we can glue $D_1 \cap U_1, \ldots, D_r \cap U_r$ to obtain a cycle $D \in Z_*(C)$.

Definition 3.2.6 (Intersection products)
Let C be a smooth tropical variety and let $(((X, |X|), \omega_X), \{\Phi_\sigma\})$ be a representative of C like in definition 3.2.1. Moreover, let D, E be subcycles of C. We construct local intersection products as follows: For every $\sigma \in X$ we can regard $(D \cap S_\sigma)$ and $(E \cap S_\sigma)$ as open tropical cycles in $\widetilde{F_\sigma}$ via the map Φ_σ. Let $\widetilde{D \cap S_\sigma}$ and $\widetilde{E \cap S_\sigma}$ be any tropical cycles in $\widetilde{F_\sigma}$ restricting to $D \cap S_\sigma$ and $E \cap S_\sigma$. As we have an intersection product on $\widetilde{F_\sigma}$ by remark 3.1.24 we can define the intersection

$$(D \cdot_\sigma E) \cap S_\sigma := \left((\widetilde{D \cap S_\sigma}) \cdot (\widetilde{E \cap S_\sigma}) \right) \cap S_\sigma.$$

Note that $(D \cdot_\sigma E) \cap S_\sigma$ does not depend on the choice of the cycles $\widetilde{D \cap S_\sigma}$ and $\widetilde{E \cap S_\sigma}$. Since $\{S_\sigma | \sigma \in X\}$ is an open covering of $|C|$ and the local intersection products $(D \cdot_\sigma E) \cap S_\sigma$, $\sigma \in X$ are compatible by the following lemma we can glue them to obtain a global intersection cycle $D \cdot E \in Z_*(C)$.

Lemma 3.2.7
For the local intersection products in definition 3.2.6 holds:

$$(D \cdot_\sigma E) \cap S_\sigma \cap S_{\sigma'} = (D \cdot_{\sigma'} E) \cap S_\sigma \cap S_{\sigma'}.$$

Proof. By definition we have an integer linear map

$$|\widetilde{F_1}| \supseteq \Phi_\sigma(S_\sigma \cap S_{\sigma'}) \xrightarrow{f} \Phi_{\sigma'}(S_\sigma \cap S_{\sigma'}) \subseteq |\widetilde{F_2}|$$

with integer linear inverse f^{-1}, where $\widetilde{F_1}, \widetilde{F_2}$ are the tropical fans generated by $\Phi_\sigma(S_\sigma \cap S_{\sigma'})$ and $\Phi_{\sigma'}(S_\sigma \cap S_{\sigma'})$, respectively. Let C_1, C_2 be subcycles of $\widetilde{F_1}$. We have to show that

$$C_1 \cdot C_2 = (f^{-1})_*(f_*(C_1) \cdot f_*(C_2)).$$

If π is the respective projection on the first factor we obtain by proposition 3.1.23 and remark 3.1.24 the equation

$$\begin{aligned}
(f^{-1})_*(f_*(C_1) \cdot f_*(C_2)) &= (f^{-1})_* \left(\pi_* \left(\triangle_{\widetilde{F_2}} \cdot (f_*(C_1) \times f_*(C_2)) \right) \right) \\
&= \pi_* \left((f^{-1} \times f^{-1})_* \left(\triangle_{\widetilde{F_2}} \cdot (f_*(C_1) \times f_*(C_2)) \right) \right) \\
&= \pi_* \left((f^{-1} \times f^{-1})_* \left((f \times f)_*(\triangle_{\widetilde{F_1}}) \cdot (f \times f)_*(C_1 \times C_2) \right) \right) \\
&= \pi_* \left(\triangle_{\widetilde{F_1}} \cdot C_1 \times C_2 \right) \\
&= C_1 \cdot C_2.
\end{aligned}$$

□

Remark 3.2.8
Lemma 3.2.7 also implies that further refinements of the representative $(((X, |X|), \omega_X), \{\Phi_\sigma\})$ of C do not change the result $D \cdot E$. Hence the intersection product is well-defined.

Our last step consists in proving basic properties of our intersection product:

Theorem 3.2.9
Let C be a smooth tropical variety, let $D, D' \in Z_l(C)$, $E \in Z_{l'}(C)$ and $F \in Z_{l''}(C)$ be subcycles and let $\varphi \in \mathrm{Div}(C)$ be a Cartier divisor on C. Then the following equations hold in $Z_*(C)$:

(a) $C \cdot D = D$,

(b) $D \cdot E = E \cdot D$,

(c) $(D + D') \cdot E = D \cdot E + D' \cdot E$,

(d) $(D \cdot E) \cdot F = D \cdot (E \cdot F)$,

(e) $\varphi \cdot (D \cdot E) = (\varphi \cdot D) \cdot E$.

If moreover $D = (\sum_{i=1}^{r} \varphi_{i,1} \cdots \varphi_{i,l}) \cdot C$ for some Cartier divisors $\varphi_{i,j} \in \mathrm{Div}(C)$ then

$$D \cdot E = \sum_{i=1}^{r} \varphi_{i,1} \cdots \varphi_{i,l} \cdot E$$

holds.

Proof. The statements follow immediately from the definition of the intersection product and the corresponding statements in section 3.1. □

3.3 Pull-backs of cycles on smooth varieties

We will now use the intersection product defined in section 3.2 to introduce pull-backs of tropical cycles along morphisms between smooth tropical varieties.

Definition 3.3.1 (Pull-back)
Let X and Y be smooth tropical varieties of dimension m and n, respectively, and let $f : X \to Y$ be a morphism of tropical cycles. Moreover, let $\pi : X \times Y \to X$ be the projection onto the first factor and let $\gamma_f : X \to X \times Y$ be the morphism given by $x \mapsto (x, f(x))$. We denote by $\Gamma_f := (\gamma_f)_* X$ the graph of f. For a cycle $C \in Z_{n-k}(Y)$ we define its *pull-back* $f^*C \in Z_{m-k}(X)$ to be

$$f^*C := \pi_* \left(\Gamma_f \cdot (X \times C) \right).$$

The easiest non-trivial, but nevertheless important example of a pull-back is the following:

Example 3.3.2
Let C and D be smooth tropical cycles and let $p : C \times D \to D$ be the projection on the second factor. We want to calculate the pull-back p^*E for a cycle $E \in Z_k(D)$: The map γ_p from definition 3.3.1 is then just given by $\gamma_p : C \times D \to C \times D \times D : (x, y) \mapsto (x, y, y)$ and the map $\pi : C \times D \times D \to C \times D$ is the projection to the first two factors. Hence we can conclude that $\Gamma_p = C \times \triangle_D$. Moreover, let $\pi^1 : C \times D \times D \to C$ be the projection to the first and $\pi^2 : C \times D \times D \to D$ be the projection to the second factor. We obtain by definition 3.3.1:

$$\begin{aligned} p^*E &= \pi_*(\Gamma_p \cdot (C \times D \times E)) \\ &= \pi_*((C \times \triangle_D) \cdot (C \times D \times E)) \\ &= \pi^1_*(C \cdot C) \times \pi^2_*(\triangle_D \cdot (D \times E)) \\ &= C \times E. \end{aligned}$$

The pull-back has the following basic properties:

Theorem 3.3.3
Let X, Y and Z be smooth tropical varieties and let $f : X \to Y$ and $g : Y \to Z$ be morphisms of tropical cycles. Moreover, let $C, C' \in Z_*(Y)$ and $D \in Z_*(X)$ be subcycles. Then the following holds:

(a) $f^*Y = X$,

(b) $\mathrm{id}_Y^* C = C$,

(c) if $C = \varphi_1 \cdots \varphi_r \cdot Y$ *then* $f^*C = f^*\varphi_1 \cdots f^*\varphi_r \cdot X$,

(d) $C \cdot f_*D = f_*(f^*C \cdot D)$,

(e) $(g \circ f)^*C = f^*g^*C$,

(f) $f^*(C \cdot C') = f^*C \cdot f^*C'$.

Proof. Throughout the proof, let $\pi^X, \pi_X, \pi^1, \pi_1, \pi^Y, \pi_Y, \pi^2, \pi_2, \pi^{X,Y}, \pi_{X,Y}, \pi^{1,2}, \pi_{1,2}$ and so forth be the projections to the respective factors.
(a) and (b): By definition of the pull-back follows

$$f^*Y = \pi^X_*(\Gamma_f \cdot (X \times Y)) = \pi^X_*(\Gamma_f) = X$$

and

$$\mathrm{id}_Y^* C = \pi^1_*(\Gamma_{\mathrm{id}_Y} \cdot (Y \times C)) = \pi^1_*(\triangle_Y \cdot (Y \times C)) = Y \cdot C = C.$$

(c): We have

$$\begin{aligned} f^*C &= \pi^X_* \left(\Gamma_f \cdot (X \times (\varphi_1 \cdots \varphi_r \cdot Y))\right) \\ &= \pi^X_* \left(\pi_2^*\varphi_1 \cdots \pi_2^*\varphi_r \cdot \Gamma_f \cdot (X \times Y)\right) \\ &= \pi^X_* \left(\pi_2^*\varphi_1 \cdots \pi_2^*\varphi_r \cdot \Gamma_f\right). \end{aligned}$$

By definition of the intersection product (see definitions 1.2.4 and 1.5.5) this last line is equal to

$$f^*\varphi_1 \cdots f^*\varphi_r \cdot X.$$

(d): Let $\pi_X : X \times Y \to X$ be the projection on X. By example 3.3.2 we know that $\pi_X^* D = D \times Y$. As the diagonal $\triangle_X$ can locally be expressed by Cartier divisors we can apply proposition 1.6.7 and statement (c) locally to deduce that for all subcycles E of $X \times Y$ holds

$$\begin{aligned} D \cdot \pi_*^X E &= \pi_*^1(\triangle_X \cdot (D \times \pi_*^X E)) \\ &= \pi_*^1(\triangle_X \cdot (\mathrm{id} \times \pi^X)_*(D \times E)) \\ &= \pi_*^1((\mathrm{id} \times \pi^X)_*((\mathrm{id} \times \pi^X)^* \triangle_X \cdot (D \times E))) \\ &= \pi_*^1((\mathrm{id} \times \pi^X)_*((\triangle_X \times Y) \cdot (D \times E))) \\ &= \pi_*^1(\pi_*^{1,2}((\triangle_X \times Y) \cdot (D \times E))) \\ &= \pi_*^1(\pi_*^{1,2}(\triangle_{X \times Y} \cdot (D \times Y \times E))) \\ &= \pi_*^1((D \times Y) \cdot E) \\ &= \pi_*^X(\pi_X^* D \cdot E). \end{aligned}$$

This implies that

$$\begin{aligned} f^*C \cdot D &= D \cdot \pi_*^X(\Gamma_f \cdot (X \times C)) \\ &= \pi_*^X(\pi_X^* D \cdot \Gamma_f \cdot (X \times C)) \\ &= \pi_*^X((D \times Y) \cdot \Gamma_f \cdot (X \times C)) \\ &= \pi_*^X(\Gamma_f \cdot (D \times C)). \end{aligned}$$

Moreover, it is easy to check that $(f \times \mathrm{id})^* \triangle_Y = \Gamma_f$. As above we can conclude that

$$\begin{aligned} C \cdot f_* D &= \pi_*^1(\triangle_Y \cdot (C \times f_* D)) \\ &= \pi_*^1((\mathrm{id} \times f)_*((\mathrm{id} \times f)^* \triangle_Y \cdot (C \times D))) \\ &= f_*(\pi_*^X((\mathrm{id} \times f)^* \triangle_Y \cdot (C \times D))) \\ &= f_*(\pi_*^X((f \times \mathrm{id})^* \triangle_Y \cdot (D \times C))) \\ &= f_*(\pi_*^X(\Gamma_f \cdot (D \times C))) \\ &= f_*(f^*C \cdot D). \end{aligned}$$

(e): Let $\Phi : X \to X \times Y \times Z$ be given by $x \mapsto (x, f(x), g(f(x)))$. An easy calculation shows that $(\Gamma_f \times Z) \cdot (X \times \Gamma_g) = \Phi_* X$. Hence we can conclude by statement (d) that

$$\begin{aligned} f^* g^* C &= \pi_*^X \left(\Gamma_f \cdot \left(X \times \pi_*^Y (\Gamma_g \cdot (Y \times C)) \right) \right) \\ &= \pi_*^X \left(\pi_*^{X,Y} ((\Gamma_f \times Z) \cdot (X \times \Gamma_g) \cdot (X \times Y \times C)) \right) \\ &= \pi_*^X ((\Gamma_f \times Z) \cdot (X \times \Gamma_g) \cdot (X \times Y \times C)) \\ &= \pi_*^X (\Phi_* X \cdot (X \times Y \times C)) \\ &= \pi_*^X (\Gamma_{g \circ f} \cdot (X \times C)) \\ &= (g \circ f)^* C. \end{aligned}$$

(f): Let $\Phi : X \to X \times Y \times Y$ be given by $x \mapsto (x, f(x), f(x))$ and let $\pi^{1,2}, \pi^{1,3} : X \times Y \times Y \to X \times Y$ be the projections to the respective factors. An easy calculation shows that

$$(\Gamma_f \times Y) \cdot (X \times \Gamma_{\mathrm{id}_Y}) = \Phi_* X = \pi_{1,2}^* \Gamma_f \cdot \pi_{1,3}^* \Gamma_f.$$

Hence we can deduce that

$$\begin{aligned}
f^*(C \cdot C') &= \pi_*^X (\Gamma_f \cdot (X \times (C \cdot C'))) \\
&= \pi_*^X (\Gamma_f \cdot (X \times \pi_*^1(\Gamma_{\mathrm{id}_Y} \cdot C \times C'))) \\
&= \pi_*^X (\Gamma_f \cdot \pi_*^{1,2}((X \times \Gamma_{\mathrm{id}_Y}) \cdot (X \times C \times C'))) \\
&= \pi_*^X (\pi_*^{1,2}((\Gamma_f \times Y) \cdot (X \times \Gamma_{\mathrm{id}_Y}) \cdot (X \times C \times C'))) \\
&= \pi_*^X (\pi_*^{1,3}((\Gamma_f \times Y) \cdot (X \times \Gamma_{\mathrm{id}_Y}) \cdot (X \times C \times C'))) \\
&= \pi_*^X \left(\pi_*^{1,3}(\pi_{1,2}^* \Gamma_f \cdot \pi_{1,3}^* \Gamma_f \cdot (X \times C \times C'))\right) \\
&= \pi_*^X (\Gamma_f \cdot \pi_*^{1,3}((\Gamma_f \times Y) \cdot (X \times C \times C'))) \\
&= \pi_*^X \left(\Gamma_f \cdot (\pi_*^X (\Gamma_f \cdot (X \times C)) \times C')\right) \\
&= \pi_*^X (\Gamma_f \cdot (f^* C \times C')) \\
&= f^* C \cdot f^* C'.
\end{aligned}$$

□

We finish the section with another important example:

Example 3.3.4
Let D be a smooth tropical variety and let $C \in Z_k(D)$ be a smooth tropical subvariety. Moreover, let $\iota : C \to D$ be the inclusion map. We want to calculate the pull-back $\iota^* E$ for a cycle $E \in Z_l(D)$: Let $\pi^C : C \times D \to C$ and $\pi^D : C \times D \to D$ be the projections to the first and second factor and let $\gamma_\iota : C \to C \times D$ be given by $x \mapsto (x, x)$. Hence we can deduce that $\Gamma_\iota = (\gamma_\iota)_* C = \triangle_C$ and by example 3.3.2 that $(\pi^D)^* E = C \times E$. Thus we can conclude by theorem 3.3.3 (d):

$$\begin{aligned}
\iota^* E &= \pi_*^C (\Gamma_\iota \cdot (C \times E)) \\
&= \pi_*^C (\triangle_C \cdot (C \times E)) \\
&= \pi_*^D (\triangle_C \cdot (C \times E)) \\
&= \pi_*^D (\triangle_C \cdot (\pi^D)^* E) \\
&= \pi_*^D (\triangle_C) \cdot E \\
&= C \cdot E,
\end{aligned}$$

where $C \cdot E$ is the intersection product on D.

4 Weil and Cartier divisors under tropical modifications

In this chapter we study "contractions" (the concept of contractions was introduced by G. Mikhalkin in [M06]) and "modifications" — the inverse operation of contractions — of affine tropical cycles. We will prove that under some further assumptions these modifications preserve the 1:1-correspondence of Weil and Cartier divisors that exists, for example, on $\mathbb{R}^n$. In particular we can conclude that on the moduli spaces $\mathcal{M}_{0,n,\text{trop}}$ of n-marked abstract tropical curves Weil and Cartier divisors agree. Applying our results locally we can moreover prove that there is a 1:1-correspondence of Weil and Cartier divisors on smooth tropical varieties as introduced in chapter 3.

4.1 Modifications and contractions

In this section we introduce *modifications* and *contractions* as our main objects of study.

Definition 4.1.1 (Modifications and contractions)
Let $C \in Z_m^{\text{aff}}(V)$ be an m-cycle in V and $h \in \mathcal{K}^*(C)$, $h : |C| \to \mathbb{R}$ a rational function on C. Let (X, ω_X) be a representative of C such that h is linear on every face of X, i.e. for every face $\sigma \in X$ we have $h|_\sigma = h_\sigma + c$ for some $h_\sigma \in \Lambda^\vee$ and $c \in \mathbb{R}$. Let $V' := (\Lambda \times \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. Then we define the *modification of X along h* to be the tropical fan

$$(\Gamma_{X,h}, \omega_{\Gamma_{X,h}})$$

in V', where

$$\begin{aligned}
&\Gamma_{X,h} := \{(\text{id} \times h_\sigma)(\sigma) | \sigma \in X\} \cup \{(\text{id} \times h_\sigma)(\sigma) + (\{0_V\} \times \mathbb{R}_{\leq 0}) | \sigma \in X \setminus X^{(m)}\},\\
&\omega_{\Gamma_{X,h}}((\text{id} \times h_\sigma)(\sigma)) := \omega_X(\sigma) \text{ for all } \sigma \in X,\\
&\omega_{\Gamma_{X,h}}((\text{id} \times h_\sigma)(\sigma) + (\{0_V\} \times \mathbb{R}_{\leq 0})) := \omega_h(\sigma) \text{ for all } \sigma \in X^{(m-1)}
\end{aligned}$$

(cf. construction 1.2.3). Conversely, X is called the *contraction of $\Gamma_{X,h}$ along h*. Note that the equivalence class $[\Gamma_{X,h}]$ only depends on C and not on the choice of the representative X. Hence we can define the *modification of C along h* to be

$$\Gamma_{C,h} := [\Gamma_{X,h}] \in Z_m^{\text{aff}}(V')$$

and call C the *contraction of $\Gamma_{C,h}$ along h*.

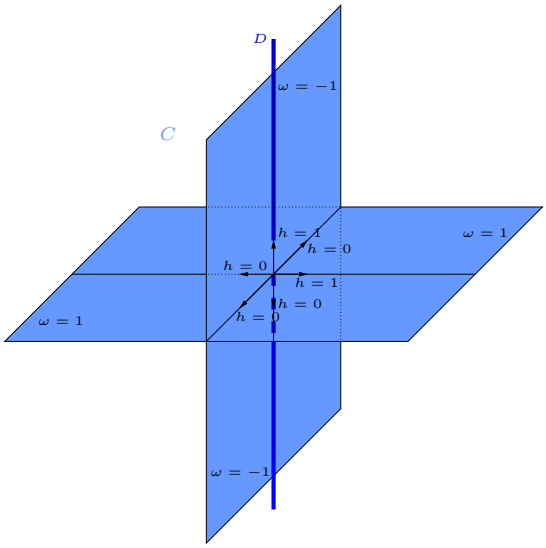


Figure 4.1: $D \in Z^{\text{aff}}_{m-1}(C)$, but $\Gamma_{D,h} \notin Z^{\text{aff}}_{m-1}(\Gamma_{C,h})$.

Definition 4.1.2
For $C \in Z^{\text{aff}}_m(V)$ we define a map

$$\gamma : Z^{\text{aff}}_{m-1}(C) \longrightarrow Z^{\text{aff}}_{m-1}(\Gamma_{C,h}) : D \longmapsto \Gamma_{D,h}.$$

Note that for some cycles $C \in Z^{\text{aff}}_m(V)$ this map is not well-defined as $D \in Z^{\text{aff}}_{m-1}(C)$ does in general not imply that $\Gamma_{D,h}$ is a subcycle of $\Gamma_{C,h}$ (the simplest example of such cycles C and D is drawn in figure 4.1). But the following lemma shows that the latter implication in fact is true for the cases we are interested in:

Lemma 4.1.3
Let $C \in Z^{\text{aff}}_m(V)$ be a cycle such that $\text{Div}(C) \xrightarrow{\sim} Z^{\text{aff}}_{m-1}(C) : \varphi \mapsto \varphi \cdot C$ is an isomorphism. Then holds:

(a) γ is well-defined, i.e. $\Gamma_{D,h} \in Z^{\text{aff}}_{m-1}(\Gamma_{C,h})$ for all $D \in Z^{\text{aff}}_{m-1}(C)$,

(b) $\gamma(D_1 + D_2) = \gamma(D_1) + \gamma(D_2)$.

Proof. (a): We fix a Weil divisor $D \in Z^{\text{aff}}_{m-1}(C)$ and have to prove that $|\Gamma_{D,h}| \subseteq |\Gamma_{C,h}|$. Therefore it suffices to show that $|h \cdot D| \subseteq |h \cdot C|$. By assumption we know that $D = \varphi \cdot C$ for some Cartier divisor $\varphi \in \text{Div}(C)$ and hence

$$|h \cdot D| = |h \cdot \varphi \cdot C| = |\varphi \cdot h \cdot C| \subseteq |h \cdot C|.$$

(b): Is obvious by definition of γ. □

Definition 4.1.4
Let $C \in Z^{\text{aff}}_m(V)$. The projection $p : \Lambda \times \mathbb{Z} \to \Lambda$ induces a morphism $p : \Gamma_{C,h} \to C$ and thus a homomorphism

$$p_* : Z^{\text{aff}}_{m-1}(\Gamma_{C,h}) \to Z^{\text{aff}}_{m-1}(C).$$

Remark 4.1.5
Note that $p_* \circ \gamma = \mathrm{id}_{Z^{\mathrm{aff}}_{m-1}(C)}$ if $\mathrm{Div}(C) \xrightarrow{\sim} Z^{\mathrm{aff}}_{m-1}(C)$.

The next proposition shows how the intersection products on C and $\Gamma_{C,h}$ are related:

Proposition 4.1.6
Let $C \in Z^{\mathrm{aff}}_m(V)$ be a cycle such that $\mathrm{Div}(C) \xrightarrow{\sim} Z^{\mathrm{aff}}_{m-1}(C)$. Moreover let $\varphi \in \mathrm{Div}(C)$ be a Cartier divisor and let $D \in Z^{\mathrm{aff}}_{m-1}(C)$ be a Weil divisor. The following equation holds:

$$\varphi \cdot D = p_*(p^*\varphi \cdot \gamma(D)).$$

Proof. Using the projection formula (cf. proposition 1.3.8) we can conclude that

$$\varphi \cdot D = \varphi \cdot (p_* \circ \gamma)(D) = p_*(p^*\varphi \cdot \gamma(D)).$$

□

Proposition 4.1.7
Let $C \in Z^{\mathrm{aff}}_m(V)$ be a cycle in $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ and let $D \in Z^{\mathrm{aff}}_n(V')$ be a cycle in $V' = \Lambda' \otimes_{\mathbb{Z}} \mathbb{R}$. Moreover let $h_C \in \mathcal{K}^*(C)$ and $h_D \in \mathcal{K}^*(D)$ be rational functions. For simplicity of notation we write h_C and h_D for the pull-backs of h_C and h_D along the occurring projection maps as well. Then the following equation holds:

$$\Gamma_{\Gamma_{C\times D,h_C},h_D} = \Gamma_{C,h_C} \times \Gamma_{D,h_D}.$$

Proof. An easy calculation shows that $\Gamma_{C\times D,h_C} = \Gamma_{C,h_C} \times D$. Hence $\Gamma_{\Gamma_{C\times D,h_C},h_D} = \Gamma_{C,h_C\times D,h_D} = \Gamma_{C,h_C} \times \Gamma_{D,h_D}$. □

Corollary 4.1.8
Let $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ and $V' = \Lambda' \otimes_{\mathbb{Z}} \mathbb{R}$. Then let $\widetilde{C} \in Z^{\mathrm{aff}}_m(V \times \mathbb{R}^k)$ and $\widetilde{D} \in Z^{\mathrm{aff}}_n(V' \times \mathbb{R}^l)$ be cycles that arise from $C \in Z^{\mathrm{aff}}_m(V)$ and $D \in Z^{\mathrm{aff}}_n(V')$ respectively by a finite series of modifications. Then $\widetilde{C} \times \widetilde{D} \in Z^{\mathrm{aff}}_{m+n}(V \times V' \times \mathbb{R}^{k+l})$ arises from $C \times D$ by a finite series of modifications.

4.2 Cartier divisors and Weil divisors

Our aim in this section is to prove that a 1:1-correspondence between Weil divisors and Cartier divisors on an affine cycle $C \in Z^{\mathrm{aff}}_m(V)$ implies a 1:1-correspondence between Weil divisors and Cartier divisors on many modifications $\Gamma_{C,h}$ of C. To prove this statement we need some preparations:

Definition 4.2.1
Let $C \in Z^{\mathrm{aff}}_m(V)$ be a cycle. If $C \neq 0$ we define the number $\mathrm{lcm}(C)$ to be the lowest common multiple

$$\mathrm{lcm}\left\{\omega_X(\sigma) | \sigma \in (X^*)^{(m)}\right\} \in \mathbb{Z}_{>0}$$

of all non-zero weights of the facets of X for some (and thus for every) representative (X, ω_X) of C. If $C = 0$ we define $\mathrm{lcm}(C) := 0$.

Definition 4.2.2
Let $C \in Z_m^{\text{aff}}(V)$ be a cycle. C is called *locally irreducible* if for some (and thus for every) reduced representative (X, ω_X) of C holds: For every cone $\tau \in X^{(m-1)}$ the equality

$$\sum_{\sigma > \tau} \lambda_\sigma \cdot u_{\sigma/\tau} = 0 \in \Lambda/\Lambda_\tau, \quad \lambda_\sigma \in \mathbb{Z}$$

(where $u_{\sigma/\tau}$ denotes the primitive normal vector of σ relative to τ) implies that there exists $\lambda \in \mathbb{Q}$ such that $\lambda_\sigma = \lambda \cdot \omega_X(\sigma)$ for all $\sigma > \tau$.

Example 4.2.3
The following figure shows an example of a tropical cycle C which is not locally irreducible as the condition in definition 4.2.2 is not fulfilled around τ. But note that the condition is fulfilled for all other cones of codimension one.

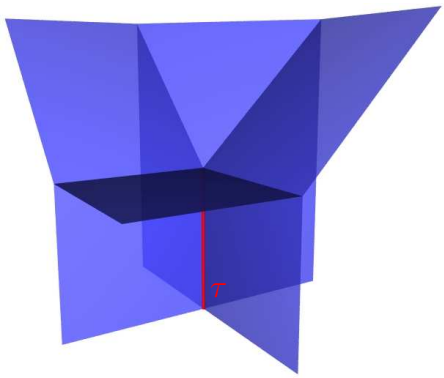


Figure 4.2: A cycle C which is not locally irreducible.

Lemma 4.2.4
Let $C \in Z_m^{\text{aff}}(V)$ be a cycle such that $\text{Div}(C) \xrightarrow{\sim} Z_{m-1}^{\text{aff}}(C)$, let $h \in \mathcal{K}^*(C)$ be a rational function and let $D \in \ker(p_*) \subseteq Z_{m-1}^{\text{aff}}(\Gamma_{C,h})$. If $m \leq 2$ or $\Gamma_{C,h}$ is locally irreducible then there exists a Cartier divisor $\varphi_D \in \text{Div}(\Gamma_{C,h})$ such that $\varphi_D \cdot \Gamma_{C,h} = \text{lcm}(\Gamma_{C,h}) \cdot D$.

Proof. Note that for $m = 1$ the statement is trivial. Thus we may assume that $m \geq 2$. Moreover we can choose representatives $X = X^*$ of C and $\Gamma_Y = \Gamma_Y^*$ of D such that

$$p_*(\Gamma_Y) = \{p(\sigma) | \sigma \in \Gamma_Y \text{ contained in a maximal cone of } \Gamma_Y \text{ on which } p \text{ is injective}\}$$

and such that $Y := p_*(\Gamma_Y) \subseteq X$ (cf. constructions 1.1.15 and 1.3.2). Using these representatives we ensure that above every cone $\sigma \in Y^{(m-1)}$ there is a finite number of facets $\sigma_1, \ldots, \sigma_{r(\sigma)} \subseteq \varrho(\sigma) := (\text{id} \times h_\sigma)(\sigma) + (\{0_V\} \times \mathbb{R}_{\leq 0})$ projecting injectively onto σ as shown in figure 4.3, plus possibly some additional facets in the boundary of $\varrho(\sigma)$ on which p is not injective. Our assumption $D \in \ker(p_*)$ implies that

$$|\Gamma_Y| \subseteq \bigcup_{\sigma \in X^{(m-1)}} (\text{id} \times h_\sigma)(\sigma) + (\{0_V\} \times \mathbb{R}_{\leq 0})$$

and thus that, in fact, every facet of Γ_Y is of one of those two types mentioned above. Now we construct a piecewise linear function $\varphi : |\Gamma_Y| \to \mathbb{R}$ as follows: First, we set

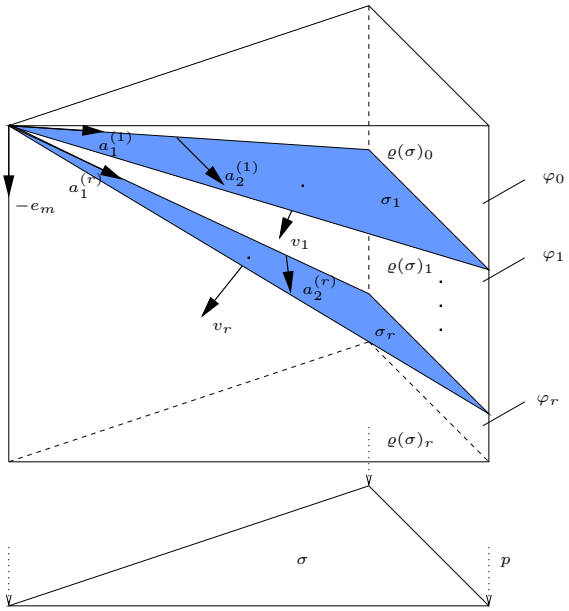


Figure 4.3: $\sigma_1, \ldots, \sigma_r \subseteq (\mathrm{id} \times h_\sigma)(\sigma) + (\{0_V\} \times \mathbb{R}_{\leq 0})$.

$\varphi|_{(\mathrm{id}\times h_\sigma)(\sigma)} := 0$ for all $\sigma \in X^{(m)}$ and $\varphi|_{\varrho(\sigma)} := 0$ for all $\sigma \in X^{(m-1)} \setminus Y^{(m-1)}$. Then we fix a cone $\sigma \in Y^{(m-1)}$. Choosing a basis $b_1, \ldots, b_{m-1}$ of Λ_σ we obtain a basis $b_1, \ldots, b_{m-1}, b_m := (0_\Lambda, 1)$ of $\Lambda_{\varrho(\sigma)} = \Lambda_\sigma \times \mathbb{Z}$ and can hence consider the face $\varrho(\sigma)$ in $\mathbb{R}^m \cong V'_{\varrho(\sigma)}$ with lattice $\mathbb{Z}^m$ (see figure 4.3): Let $\sigma_1, \ldots, \sigma_r$, $r = r(\sigma)$, be all facets of Γ_Y contained in $\varrho(\sigma)$, let $\sigma_0 := (\mathrm{id} \times h_\sigma)(\sigma)$ and let $\varrho(\sigma)_i$ denote the m-dimensional cone in $\varrho(\sigma)$ bounded by σ_i and σ_{i+1}, respectively the cone below σ_r for $i = r$ (note that $\varrho(\sigma)_0$ might be $(m-1)$-dimensional if $\sigma_0 = \sigma_1$). We will construct a linear function φ_i on every $\varrho(\sigma)_i$ in order to define a piecewise linear function on $\varrho(\sigma)$ as follows: We set $\varphi_0 := 0$. Then φ_1 is determined on σ_1. In order to get $\omega_\varphi(\sigma_1) = \mathrm{lcm}(\Gamma_{C,h}) \cdot \omega_{\Gamma_Y}(\sigma_1)$ we have to set

$$\varphi_1(v_1) := \chi_\sigma \cdot \omega_{\Gamma_Y}(\sigma_1) \text{ with } \chi_\sigma := \frac{\mathrm{lcm}(\Gamma_{C,h})}{\omega_{\Gamma_{C,h}}(\varrho(\sigma))}$$

and φ_1 is entirely determined then. Now φ_2 is determined on σ_2. In order to get $\omega_\varphi(\sigma_2) = \mathrm{lcm}(\Gamma_{C,h}) \cdot \omega_{\Gamma_Y}(\sigma_2)$ we have to set $\varphi_2(v_2) := \chi_\sigma \cdot \omega_{\Gamma_Y}(\sigma_2) + \varphi_1(v_2)$ and φ_2 is entirely determined. Going on this way we set $\varphi_i(v_i) := \chi_\sigma \cdot \omega_{\Gamma_Y}(\sigma_i) + \varphi_{i-1}(v_i)$ for all i and obtain the function $\varphi_{\varrho(\sigma)}$. Using the same procedure we can construct functions $\varphi_{\varrho(\sigma)}$ for all $\sigma \in Y^{(m-1)}$ and it remains to check that all our pieces $\varphi_{\varrho(\sigma)}$, $\sigma \in X^{(m-1)}$ glue together to obtain a globally defined rational function φ on C. Therefore, for all $i = 1, \ldots, r$ we choose a basis $a_1^{(i)}, \ldots, a_{m-1}^{(i)}$ of $\Lambda_{\sigma_i} \subseteq \mathbb{Z}^m$ and a representative $v_i \in \mathbb{Z}^m$ of the primitive normal vector of $\varrho(\sigma)_i$ relative to σ_i. We set

$$\left|\det\left(\begin{array}{c|c|c|c|c} a_1^{(i)} & \cdots & a_{m-1}^{(i)} & -e_m \end{array} \right)\right| =: \alpha_i \in \mathbb{Z}_{>0},$$

where e_j denotes the j-th unit vector in $\mathbb{R}^m \cong V'_{\varrho(\sigma)}$. Using our assumption $D \in \ker(p_*)$ we can conclude that

$$0 = \omega_Y(\sigma) = \sum_{i=1}^{r} \omega_{\Gamma_Y}(\sigma_i) \cdot \alpha_i. \qquad (*)$$

As $a_1^{(i)}, \ldots, a_{m-1}^{(i)}, -e_m$ is a basis of $\mathbb{R}^m$ we can express v_i by a linear combination

$$v_i = c_1^{(i)} a_1^{(i)} + \ldots + c_{m-1}^{(i)} a_{m-1}^{(i)} - c_m^{(i)} e_m.$$

It is easily checked that $c_m^{(i)} = \frac{1}{\alpha_i}$. We can use the above linear combination to calculate $\varphi_i(-e_m)$:

$$\begin{aligned} \varphi_1(v_1) &= \chi_\sigma \cdot \omega_{\Gamma_Y}(\sigma_1) \\ &= c_1^{(1)} \varphi_1(a_1^{(1)}) + \ldots + c_{m-1}^{(1)} \varphi_1(a_{m-1}^{(1)}) + \frac{1}{\alpha_1} \varphi_1(-e_m) \\ &= \frac{1}{\alpha_1} \varphi_1(-e_m) \end{aligned}$$

implies $\varphi_1(-e_m) = \chi_\sigma \cdot \omega_{\Gamma_Y}(\sigma_1)\alpha_1$. Similarly,

$$\begin{aligned} \varphi_2(v_2) &= \chi_\sigma \cdot \omega_{\Gamma_Y}(\sigma_2) + \varphi_1(v_2) \\ &= \chi_\sigma \cdot \omega_{\Gamma_Y}(\sigma_2) + c_1^{(2)} \varphi_1(a_1^{(2)}) + \ldots + c_{m-1}^{(2)} \varphi_1(a_{m-1}^{(2)}) + \frac{1}{\alpha_2} \varphi_1(-e_m) \\ &= c_1^{(2)} \varphi_2(a_1^{(2)}) + \ldots + c_{m-1}^{(2)} \varphi_2(a_{m-1}^{(2)}) + \frac{1}{\alpha_2} \varphi_2(-e_m) \end{aligned}$$

implies $\chi_\sigma \cdot \omega_{\Gamma_Y}(\sigma_2) + \frac{1}{\alpha_2}\varphi_1(-e_m) = \frac{1}{\alpha_2}\varphi_2(-e_m)$ and hence

$$\varphi_2(-e_m) = \chi_\sigma \cdot (\omega_{\Gamma_Y}(\sigma_2)\alpha_2 + \omega_{\Gamma_Y}(\sigma_1)\alpha_1).$$

Inductively we obtain

$$\varphi_i(-e_m) = \chi_\sigma \cdot \sum_{j=1}^{i} \omega_{\Gamma_Y}(\sigma_j)\alpha_j \qquad (**)$$

and hence using $(*)$:

$$\varphi_r(-e_m) = \chi_\sigma \cdot \sum_{j=1}^{r} \omega_{\Gamma_Y}(\sigma_j)\alpha_j = 0.$$

Thus all patches $\varphi_{\varrho(\sigma)}$, $\sigma \in X^{(m-1)}$ fit together on the ray $\{0_{V'}\} + (\{0_V\} \times \mathbb{R}_{\leq 0})$ and we are done in the case $m = 2$.
Now let $m \geq 3$. To see that the patches $\varphi_{\varrho(\sigma)}$ glue together everywhere we have to study the consequences of the balancing condition of Γ_Y: Let $\tau \in \Gamma_{X,h}^{(m-1)}$ be a common face of the cones $\varrho(\sigma^{(1)}), \ldots, \varrho(\sigma^{(s)})$. For all i we may choose a representative v_i of the primitive normal vector of $\varrho(\sigma^{(i)})$ relative to τ such that the equation

$$\sum_{i=1}^{s} \omega_{\Gamma_{X,h}}(\varrho(\sigma^{(i)})) \cdot v_i = 0 \in \Lambda'$$

holds (cf. remark 1.1.9). Like above, for $i = 1, \ldots, s$ let $\sigma_1^{(i)}, \ldots, \sigma_{r_i}^{(i)}$ be all facets of Γ_Y contained in $\varrho(\sigma^{(i)})$ on which p is injective and which meet in a common face $\tau_1 \subseteq \tau$ of dimension $m-2$ (see figure 4.4). Then let $v_j^{(i)}$ be a representative of the primitive

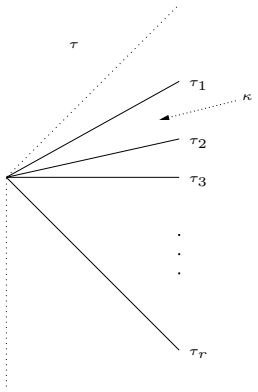


Figure 4.4: The common face τ of $\varrho(\sigma^{(1)}), \ldots, \varrho(\sigma^{(s)})$.

normal vector of $\sigma_j^{(i)}$ relative to τ_1. The balancing condition of Γ_Y implies that

$$\sum_{i=1}^{s}\sum_{j=1}^{r_i} \omega_{\Gamma_Y}(\sigma_j^{(i)}) \cdot [v_j^{(i)}] = 0 \in \Lambda'/\Lambda'_\tau. \qquad (***)$$

Choosing a basis $u_1, \ldots, u_{m-2}$ of Λ'_{τ_1} and extending it by a vector u_{m-1} to a basis of Λ'_τ we can express the vectors $v_j^{(i)}$ by $\mathbb{Z}$-linear combinations

$$v_j^{(i)} = c_1^{(i)}u_1 + \ldots + c_{m-1}^{(i)}u_{m-1} + t_j^{(i)}v_i.$$

Note that the orientations of v_i and $v_j^{(i)}$ imply that $t_j^{(i)} > 0$. Equation $(***)$ then implies

$$\sum_{i=1}^{s}\sum_{j=1}^{r_i} \omega_{\Gamma_Y}(\sigma_j^{(i)}) \cdot t_j^{(i)}[v_i] = 0 \in \Lambda'/\Lambda'_\tau.$$

As $\Gamma_{C,h}$ is locally irreducible we can conclude that there exists some $\lambda \in \mathbb{Q}$ such that

$$\sum_{j=1}^{r_i} \omega_{\Gamma_Y}(\sigma_j^{(i)}) \cdot t_j^{(i)} = \lambda \cdot \omega_{\Gamma_{X,h}}(\varrho(\sigma^{(i)}))$$

for all i. Now we fix $i \in \{1, \ldots, s\}$ and choose a basis $b_1, \ldots, b_{m-2}, b_{m-1} := v_i, b_m := (0_{\Lambda'}, 1)$ of $\Lambda'_{\varrho(\sigma^{(i)})} \cong \mathbb{Z}^m$ like above. Note that in general we cannot simply take again the vectors $u_1, \ldots, u_{m-2}$ we have chosen above because in general $u_1, \ldots, u_{m-2}, b_m$ is not a basis of Λ'_τ. Nevertheless, on the other hand, we can express the vectors $v_j^{(i)}$ by an $\mathbb{R}$-linear combination

$$v_j^{(i)} = d_1^{(i)}u_1 + \ldots + d_{m-2}^{(i)}u_{m-2} + t_j^{(i)}v_i + d_m^{(i)}b_m$$

and can hence calculate using (∗∗):

$$\begin{aligned}
&(\varphi_{\varrho(\sigma^{(i)})})_\kappa(-e_m)\\
&= \chi_\sigma^{(i)} \cdot \sum_{j=1}^{r_i} \omega_{\Gamma_Y}(\sigma_j^{(i)}) \cdot \left|\det\left(\ \tilde{u}_1 \,\middle|\, \cdots \,\middle|\, \tilde{u}_{m-2} \,\middle|\, \tilde{v}_j^{(i)} \,\middle|\, -e_m\ \right)\right|\\
&= \chi_\sigma^{(i)} \cdot \sum_{j=1}^{r_i} \omega_{\Gamma_Y}(\sigma_j^{(i)}) t_j^{(i)} \cdot \underbrace{\left|\det\left(\ \tilde{u}_1 \,\middle|\, \cdots \,\middle|\, \tilde{u}_{m-2} \,\middle|\, e_{m-1} \,\middle|\, -e_m\ \right)\right|}_{=:B}\\
&= \chi_\sigma^{(i)} \cdot \sum_{j=1}^{r_i} \omega_{\Gamma_Y}(\sigma_j^{(i)}) t_j^{(i)} \cdot B
\end{aligned}$$

(here $\tilde{x}$ denotes the representation of x in the basis $b_1, \ldots, b_m$). Note that B is independent of i. Thus we obtain:

$$(\varphi_{\varrho(\sigma^{(i)})})_\kappa(-e_m) = \chi_{\sigma^{(i)}} \cdot \sum_{j=1}^{r_i} \omega_{\Gamma_Y}(\sigma_j^{(i)}) t_j^{(i)} B = \lambda B \cdot \operatorname{lcm}(\Gamma_{C,h})$$

for all i. This proves that the pieces $\varphi_{\varrho(\sigma)}$ of our map φ agree on common faces of codimension one. Moreover, the argument shows that if one of the cones $\varrho(\sigma^{(i)})$ does not contain any facet $\sigma_j^{(i)}$ with non-zero weight on which p is injective, then $\varphi_{\varrho(\sigma^{(i)})}|_\tau = 0$ for all i.

It remains to check that our pieces of $\varphi_{\varrho(\sigma)}$ glue together on cones of higher codimension as well: Therefore let $\tau < \varrho(\sigma_1), \varrho(\sigma_2)$ be a common face of $\varrho(\sigma_1)$ and $\varrho(\sigma_2)$, $\sigma_1, \sigma_2 \in X^{(m-1)}$. If $\sigma_1, \sigma_2 \notin Y^{(m-1)}$ then $\varphi_{\varrho(\sigma_1)} = 0 = \varphi_{\varrho(\sigma_2)}$ and we are done. Hence let $\sigma_1 \in Y^{(m-1)}$. We have to distinguish between two cases: If all facets $\varrho(\sigma)$ with $\varrho(\sigma) > \tau$ contain facets σ_j with non-zero weights on which p is injective then $\varphi_{\varrho(\sigma_1)}|_\tau = \varphi_{\varrho(\sigma_2)}|_\tau$ by applying our above argument several times. Otherwise $\varphi_{\varrho(\sigma_1)}|_\tau = 0 = \varphi_{\varrho(\sigma_2)}|_\tau$ anyway.

Hence φ is in fact a rational function on C. The faces of $\varphi \cdot \Gamma_{X,h}$ with non-zero weights on which p is injective are by construction exactly the faces of $\operatorname{lcm}(\Gamma_{C,h}) \cdot \Gamma_Y$ on which p is injective and these faces occur with correct weights. As both $\varphi \cdot \Gamma_{X,h}$ and $\operatorname{lcm}(\Gamma_{C,h}) \cdot \Gamma_Y$ fulfill the balancing condition it follows that $[\varphi \cdot \Gamma_{X,h}] = [\operatorname{lcm}(\Gamma_{C,h}) \cdot \Gamma_Y] = \operatorname{lcm}(\Gamma_{C,h}) \cdot D$. Hence $\varphi_D := [\varphi] \in \operatorname{Div}(\Gamma_{C,h})$ is our wanted Cartier divisor with $\varphi_D \cdot \Gamma_{C,h} = \operatorname{lcm}(\Gamma_{C,h}) \cdot D$. □

Using this lemma we are able to prove the first part of the promised 1:1-correspondence between Weil divisors and Cartier divisors:

Theorem 4.2.5
Let $C \in Z_m^{\mathrm{aff}}(V)$ be a cycle such that $\operatorname{Div}(C) \xrightarrow{\sim} Z_{m-1}^{\mathrm{aff}}(C)$, let $h \in \mathcal{K}^*(C)$ be a rational function and let $D \in Z_{m-1}^{\mathrm{aff}}(\Gamma_{C,h})$. If $m \leq 2$ or $\Gamma_{C,h}$ is locally irreducible then there exists a Cartier divisor $\varphi_D \in \operatorname{Div}(\Gamma_{C,h})$ with $\varphi_D \cdot \Gamma_{C,h} = \operatorname{lcm}(\Gamma_{C,h}) \cdot D$.

Proof. $D \in Z_{m-1}^{\mathrm{aff}}(\Gamma_{C,h})$ implies that $p_*D \in Z_{m-1}^{\mathrm{aff}}(C)$. By assumption there exists a Cartier divisor $\varphi'_D \in \operatorname{Div}(C)$ with $\varphi'_D \cdot C = p_*D$. By proposition 4.1.6 we know that

$$p_*D = \varphi'_D \cdot C = p_*(p^*\varphi'_D \cdot \gamma(C)) = p_*(p^*\varphi'_D \cdot \Gamma_{C,h}).$$

Hence $D - p^*\varphi'_D \cdot \Gamma_{C,h} \in \ker(p_*)$ and by lemma 4.2.4 there exists a Cartier divisor $\varphi''_D \in \mathrm{Div}(\Gamma_{C,h})$ with $\varphi''_D \cdot \Gamma_{C,h} = \mathrm{lcm}(\Gamma_{C,h}) \cdot (D - p^*\varphi'_D \cdot \Gamma_{C,h})$. Thus we obtain

$$(\varphi''_D + \mathrm{lcm}(\Gamma_{C,h}) \cdot p^*\varphi'_D) \cdot \Gamma_{C,h} = \mathrm{lcm}(\Gamma_{C,h}) \cdot D.$$

□

Now we prove the missing part of the 1:1-correspondence. Note that in contrast to the first part we need no special assumptions here.

Theorem 4.2.6
Let $C \in Z^{\mathrm{aff}}_m(V)$ be a cycle such that $\mathrm{Div}(C) \xrightarrow{\sim} Z^{\mathrm{aff}}_{m-1}(C)$, let $h \in \mathcal{K}^*(C)$ be a rational function and let $\Phi \in \mathrm{Div}(\Gamma_{C,h})$ with $\Phi \cdot \Gamma_{C,h} = 0$. Then $\Phi = 0 \in \mathrm{Div}(\Gamma_{C,h})$.

Proof. Let (X, ω_X) be a reduced representative of C and let $v_1, \ldots, v_k \in |X|$ be a basis of $\langle |X| \rangle \subseteq V$. For every cone $\sigma \in X^{(m)}$ and all $\tau < \sigma$, $\tau \in X^{(m-1)}$ we can choose representatives $v_{\sigma,\tau}$ of the primitive normal vector of σ relative to τ such that

$$\sum_{\sigma > \tau} \omega_X(\sigma) v_{\sigma,\tau} = 0$$

by remark 1.1.9. Moreover, for every cone $\tau \in X$ we choose a basis $x^\tau_1, \ldots, x^\tau_{d_\tau}$ of Λ_τ. For $\sigma \in X^{(m)}$ with $v^\sigma_1, \ldots, v^\sigma_{t_\sigma} \in \sigma$, $v^\sigma_j \in \{v_1, \ldots, v_k\}$, we have linear relations

$$\begin{aligned} f_1(\{v_{\sigma,\tau} | \tau < \sigma, \tau \in X^{(m-1)}\} \cup \{v^\sigma_1, \ldots, v^\sigma_{t_\sigma}\} \cup \{x^\tau_j | \tau \leq \sigma\}) &= 0, \\ &\vdots \\ f_{s_\sigma}(\{v_{\sigma,\tau} | \tau < \sigma, \tau \in X^{(m-1)}\} \cup \{v^\sigma_1, \ldots, v^\sigma_{t_\sigma}\} \cup \{x^\tau_j | \tau \leq \sigma\}) &= 0. \end{aligned}$$

Our assumption $\mathrm{Div}(C) \xrightarrow{\sim} Z^{\mathrm{aff}}_{m-1}(C)$ then implies that the following system of equations in the variables $\psi_\sigma(v_{\sigma,\tau})$, $\psi_\sigma(x^\tau_j)$ and $\psi_\sigma(v^\sigma_j)$ has exactly the solution 0:

$$\begin{aligned} \forall \tau \in X^{(m-1)} : &\ \textstyle\sum_{\sigma > \tau} \omega_X(\sigma) \psi_\sigma(v_{\sigma,\tau}) = 0, \\ \forall \sigma_1, \sigma_2 \in X^{(m)} : &\ \psi_{\sigma_1}(x^\tau_j) = \psi_{\sigma_2}(x^\tau_j)\ \forall \tau < \sigma_1, \sigma_2, \forall j, \\ \forall \sigma \in X^{(m)} : &\ f_i(\{\psi_\sigma(v_{\sigma,\tau}) | \tau < \sigma, \tau \in X^{(m-1)}\} \cup \{\psi_\sigma(v^\sigma_1), \ldots, \psi_\sigma(v^\sigma_{t_\sigma})\} \cup \\ &\ \{\psi_\sigma(x^\tau_j) | \tau \leq \sigma\}) = 0\ \forall i, \\ \forall \sigma \in X^{(m)} : &\ \psi_\sigma(v^\sigma_j) = 0\ \forall j. \end{aligned}$$

Now we set $\tilde{v}_i := (v_i, h_\sigma(v_i))$ for all i, we set $\tilde{x}^\tau_j := (x^\tau_j, h_\tau(x^\tau_j))$ for all τ and j and $\tilde{v}_{\sigma,\tau} := (v_{\sigma,\tau}, h_\sigma(v_{\sigma,\tau}))$ for all $\tau \in X^{(m-1)}$, $\sigma \in X^{(m)}$ with $\tau < \sigma$. Moreover we set $\tilde{\tau} := (\mathrm{id} \times h_\tau)(\tau)$ and $\varrho(\tau) := \tilde{\tau} + (\{0_{V'}\} \times \mathbb{R}_{\leq 0})$. Now we choose a representative $\varphi \in \mathcal{K}^*(X)$ of Φ with $\varphi((0_V, 0)) = 0$, $\varphi(\tilde{v}_i) = 0$ for all i and additionally $\varphi((0_V, -1)) = 0$. Note that $\Phi \cdot \Gamma_{C,h} = 0$ implies that φ is linear on every face of $\Gamma_{X,h}$. Then we consider the following system of equations:

$$\begin{aligned} \forall \tau \in X^{(m-1)} : &\ \omega_{\Gamma_{X,h}}(\varrho(\tau)) \varphi_{\varrho(\tau)}((0_V, -1)) + \textstyle\sum_{\sigma > \tau} \omega_X(\sigma) \varphi_{\tilde{\sigma}}(\tilde{v}_{\sigma,\tau}) = 0, \\ \forall \sigma_1, \sigma_2 \in X^{(m)} : &\ \varphi_{\tilde{\sigma}_1}(\tilde{x}^\tau_j) = \varphi_{\tilde{\sigma}_2}(\tilde{x}^\tau_j)\ \forall \tau < \sigma_1, \sigma_2, \forall j, \\ \forall \sigma \in X^{(m)} : &\ f_i(\{\varphi_{\tilde{\sigma}}(\tilde{v}_{\sigma,\tau}) | \tau < \sigma, \tau \in X^{(m-1)}\} \cup \{\varphi_{\tilde{\sigma}}(\tilde{v}^\sigma_1), \ldots, \varphi_{\tilde{\sigma}}(\tilde{v}^\sigma_{t_\sigma})\} \cup \\ &\ \{\varphi_{\tilde{\sigma}}(\tilde{x}^\tau_j) | \tau \leq \sigma\}) = 0\ \forall i, \\ \forall \sigma \in X^{(m)} : &\ \varphi_{\tilde{\sigma}}(\tilde{v}^\sigma_j) = 0\ \forall j, \\ \forall \tau \in X \setminus X^{(m)} : &\ \varphi_{\varrho(\tau)}((0_V, -1)) = 0. \end{aligned}$$

As this system is fulfilled by the values of our given function φ and since we can simplify this system to the one above we have the unique solution $\varphi_{\tilde{\sigma}}(\tilde{v}_{\sigma,\tau}) = \varphi_{\tilde{\sigma}}(\tilde{x}_j^{\tau}) = \varphi_{\tilde{\sigma}}(\tilde{v}_j^{\sigma}) = 0$ for all σ, τ and j. Hence $\Phi = 0 \in \mathrm{Div}(\Gamma_{C,h})$. □

Let $C \in Z_m^{\mathrm{aff}}(V)$ with $\mathrm{Div}(C) \cong Z_{m-1}^{\mathrm{aff}}(C)$ like above. The preceding theorem enables us to define an intersection product of two cycles in a modification $\Gamma_{C,h}$ of C if one of them can be expressed by a Cartier divisor:

Definition 4.2.7
Let $C \in Z_m^{\mathrm{aff}}(V)$ be a cycle such that $\mathrm{Div}(C) \xrightarrow{\sim} Z_{m-1}^{\mathrm{aff}}(C)$ and let $\Gamma_{C,h}$ be a modification of C. Let $D \in Z_k^{\mathrm{aff}}(\Gamma_{C,h})$ be any cycle and let $E \in Z_{m-1}^{\mathrm{aff}}(\Gamma_{C,h})$ be a Weil divisor such that there exists $\varphi_E \in \mathrm{Div}(\Gamma_{C,h})$ with $\varphi_E \cdot \Gamma_{C,h} = E$. Then we define:

$$E \cdot D := \varphi_E \cdot D.$$

Example 4.2.8
The *moduli space $\mathcal{M}_{0,n,\mathrm{trop}}$ of n-marked abstract tropical curves* embedded into the real vector space $\mathbb{R}^{\binom{n}{2}-n}$ as defined in [GKM07, section 3] arises from $\mathbb{R}^{n-3}$ by a finite series of modifications

$$\mathbb{R}^{n-3} = C_0 \rightsquigarrow C_1 \rightsquigarrow C_2 \rightsquigarrow \cdots \rightsquigarrow C_{\binom{n}{2}-2n+3} = \mathcal{M}_{0,n,\mathrm{trop}}$$

such that every C_i is locally irreducible and $\mathrm{lcm}(C_i) = 1$ for all i (a proof for this fact can be found in [H07, pages 44ff]). Hence we can conclude by theorems 4.2.5 and 4.2.6 that $\mathrm{Div}(\mathcal{M}_{0,n,\mathrm{trop}}) \cong Z_{n-4}^{\mathrm{aff}}(\mathcal{M}_{0,n,\mathrm{trop}})$ and we can define an intersection product of arbitrary cycles with Weil divisors as given in definition 4.2.7.

Example 4.2.9
The tropical linear space L_k^n introduced in definition 3.1.1 arises from $\mathbb{R}^k$ by a finite series of modifications

$$\mathbb{R}^k \rightsquigarrow L_k^{k+1} \rightsquigarrow L_k^{k+2} \rightsquigarrow \cdots \rightsquigarrow L_k^n$$

via maps $\max\{0, x_1, \ldots, x_k\}, \ldots, \max\{0, x_1, \ldots, x_{n-1}\}$. Moreover, every L_k^{k+i} is locally irreducible and $\mathrm{lcm}(L_k^{k+i}) = 1$ for all i. Using the same maps this also holds for tropical fans $\widetilde{F}$ associated to open subsets of L_k^n as in corollary 3.1.14. Using proposition 4.1.7 we can extend this statement to all tropical fans $\widetilde{F} = \widetilde{F_1} \times \ldots \times \widetilde{F_r}$, where all $\widetilde{F_i}$ are tropical fans associated to open subsets of L_k^n as in corollary 3.1.14. Hence we can define an intersection product of arbitrary cycles with Weil divisors on these spaces as introduced in definition 4.2.7. Moreover, we can deduce the following corollary.

Corollary 4.2.10
Let $C \in Z_n$ be a smooth tropical variety (cf. definition 3.2.1). Then the map

$$\mathrm{Div}(C) \longrightarrow Z_{n-1}(C) : \varphi \mapsto \varphi \cdot C$$

is an isomorphism.

Proof. Let $(((X, |X|), \omega_X), \{\Phi_\sigma\})$ be a representative of C as in definition 3.2.1 and let $D \in Z_{n-1}(C)$ be a cycle of codimension one. For all $\sigma \in X$ we can regard $D \cap S_\sigma$ as an open tropical cycle in $\widetilde{F_\sigma}$ via the map Φ_σ. Let $\widetilde{D \cap S_\sigma}$ be any tropical cycle in $\widetilde{F_\sigma}$ restricting to $D \cap S_\sigma$. By example 4.2.9 and theorem 4.2.5 there is a Cartier divisor $\varphi_\sigma \in \mathrm{Div}(\widetilde{F_\sigma})$ with $\varphi_\sigma \cdot \widetilde{F_\sigma} = \widetilde{D \cap S_\sigma}$. As all these Cartier divisors φ_σ are unique by theorem 4.2.6 they agree on overlaps $S_\sigma \cap S_{\sigma'}$ and hence define a global Cartier divisor $\varphi \in \mathrm{Div}(C)$. Moreover, φ is locally unique and thus globally unique. Hence the claim follows. □

5 Chern classes of tropical vector bundles

In this chapter we introduce tropical vector bundles, morphisms and rational sections of these bundles and define the pull-back of a tropical vector bundle and of a rational section along a morphism. Most of the definitions presented here for tropical vector bundles will be contained in [T09] for the case of line bundles. Afterwards we use the bounded rational sections of a tropical vector bundle to define the Chern classes of this bundle and prove some basic properties of Chern classes. Finally we give a complete classification of all vector bundles on an elliptic curve up to isomorphisms.

5.1 Tropical vector bundles

In this section we will introduce our basic objects such as tropical vector bundles, morphisms of tropical vector bundles and rational sections.

Definition 5.1.1 (Tropical matrices)
A tropical matrix is an ordinary matrix with entries in the tropical semi-ring

$$(\mathbb{T} = \mathbb{R} \cup \{-\infty\}, \oplus, \odot),$$

where $a \oplus b = \max\{a, b\}$ and $a \odot b = a + b$. We denote by $\mathrm{Mat}(m \times n, \mathbb{T})$ the set of tropical $m \times n$ matrices. Let $A \in \mathrm{Mat}(m \times n, \mathbb{T})$ and $B \in \mathrm{Mat}(n \times p, \mathbb{T})$. We can form a tropical matrix product $A \odot B := (c_{ij}) \in \mathrm{Mat}(m \times p, \mathbb{T})$ where $c_{ij} = \bigoplus_{k=1}^{m} a_{ik} \odot b_{kj}$. Moreover, let $G(r \times s) \subseteq \mathrm{Mat}(r \times s, \mathbb{T})$ be the subset of tropical matrices with at most one finite entry in every row. Let $G(r)$ be the subset of $G(r \times r)$ containing all tropical matrices with exactly one finite entry in every row and every column.

Remark 5.1.2
Note that a matrix $A \in G(r \times s)$ does, in general, not induce a map $f_A : \mathbb{R}^s \to \mathbb{R}^r : x \mapsto A \odot x$ as the vector $A \odot x$ may contain entries that are $-\infty$. To obtain a map $f_A : \mathbb{R}^s \to \mathbb{R}^r$ anyway we use the following definition: Let $x \in \mathbb{R}^s$ and $A \odot x = (y_1, \ldots, y_r) \in \mathbb{T}^r$ with $y_i = -\infty$ for $i \in I$ and $y_i \in \mathbb{R}$ for $i \notin I$. Then we define $f_A(x) := (\widetilde{y_1}, \ldots, \widetilde{y_r}) \in \mathbb{R}^r$ with $\widetilde{y_i} := 0$ for $i \in I$ and $\widetilde{y_i} := y_i$ for $i \notin I$.

Notation 5.1.3
For an element σ of the symmetric group S_r we denote by A_σ the tropical matrix

$A_\sigma = (a_{ij}) \in \mathrm{Mat}(r \times r, \mathbb{T})$ given by

$$a_{ij} := \begin{cases} 0, & \text{if } j = \sigma(i) \\ -\infty, & \text{else.} \end{cases}$$

Moreover, for $a_1, \ldots, a_r \in \mathbb{R}$ we denote by $D(a_1, \ldots, a_r)$ the tropical diagonal matrix $D(a_1, \ldots, a_r) = (d_{ij}) \in \mathrm{Mat}(r \times r, \mathbb{T})$ given by

$$d_{ij} := \begin{cases} a_i, & \text{if } i = j \\ -\infty, & \text{else.} \end{cases}$$

Note that every element $M \in G(r)$ can be written as $M = A_\sigma \odot D(a_1, \ldots, a_r)$ for some $\sigma \in S_r$ and some numbers $a_1, \ldots, a_r \in \mathbb{R}$. Moreover, $G(r)$ together with tropical matrix multiplication is a group with neutral element $E := D(0, \ldots, 0)$.

Lemma 5.1.4
$G(r)$ is precisely the set of invertible tropical matrices, i.e.

$$G(r) = \{A \in \mathrm{Mat}(r \times r, \mathbb{T}) | \exists A' \in \mathrm{Mat}(r \times r, \mathbb{T}) : A \odot A' = A' \odot A = E\}.$$

Proof. The inclusion

$$G(r) \subseteq \{A \in \mathrm{Mat}(r \times r, \mathbb{T}) | \exists A' \in \mathrm{Mat}(r \times r, \mathbb{T}) : A \odot A' = A' \odot A = E\}$$

is obvious. Thus, let $A, A' \in \mathrm{Mat}(r \times r, \mathbb{T})$ be given such that $A \odot A' = A' \odot A = E$. Assume that $A = (a_{ij})$ contains more than one finite entry in a row or column. For simplicity of notation we assume that $a_{11}, a_{12} \neq -\infty$. As $A \odot A' = E$ we can conclude that the first two rows of A' look as follows:

$$A' = \left(\begin{array}{cccc} \alpha & -\infty & \ldots & -\infty \\ \beta & -\infty & \ldots & -\infty \\ \hline & & * & \end{array} \right) \text{ for some } \alpha, \beta \in \mathbb{R}.$$

As moreover $A' \odot A = E$ holds, we can conclude from the second line of A' and the first column of A that

$$a_{11} + \beta = -\infty,$$

which is a contradiction to $a_{11}, \beta \in \mathbb{R}$. □

We have all requirements now to state our main definition:

Definition 5.1.5 (Tropical vector bundles)
Let X be a tropical cycle. A *tropical vector bundle* over X of rank r is a tropical cycle F together with a morphism $\pi : F \to X$ and a finite open covering $\{U_1, \ldots, U_s\}$ of X as well as a homeomorphism $\Phi_i : \pi^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbb{R}^r$ for every $i \in \{1, \ldots, s\}$ such that

(a) for all i we obtain a commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\Phi_i} & U_i \times \mathbb{R}^r \\ & \searrow^{\pi} & \downarrow{\scriptstyle p_1} \\ & & U_i \end{array}$$

where $p_1 : U_i \times \mathbb{R}^r \to U_i$ is the projection to the first factor,

(b) for all i, j the composition $p_j^{(i)} \circ \Phi_i : \pi^{-1}(U_i) \to \mathbb{R}$ is a regular invertible function, where $p_j^{(i)} : U_i \times \mathbb{R}^r \to \mathbb{R} : (x, (a_1, \ldots, a_r)) \mapsto a_j$,

(c) for every $i, j \in \{1, \ldots, s\}$ there exists a *transition map* $M_{ij} : U_i \cap U_j \to G(r)$ such that

$$\Phi_j \circ \Phi_i^{-1} : (U_i \cap U_j) \times \mathbb{R}^r \to (U_i \cap U_j) \times \mathbb{R}^r$$

is given by $(x, a) \mapsto (x, M_{ij}(x) \odot a)$ and the entries of M_{ij} are regular invertible functions on $U_i \cap U_j$ or constantly $-\infty$,

(d) there exist representatives F_0 of F and X_0 of X such that $F_0 = \{\pi^{-1}(\tau) | \tau \in X_0\}$ and $\omega_{F_0}(\pi^{-1}(\tau)) = \omega_{X_0}(\tau)$ for all maximal polyhedra $\tau \in X_0$.

An open set U_i together with the map $\Phi_i : \pi^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbb{R}^r$ is called a *local trivialization* of F. Tropical vector bundles of rank one are called *tropical line bundles*.

Remark 5.1.6
Let $V_1, \ldots, V_t$ be any open covering of X. Then the covering $\{U_i \cap V_j\}$ together with the restricted homeomorphisms $\Phi_i|_{\pi^{-1}(U_i \cap V_j)}$ and transition maps $M_{ij}|_{(U_i \cap V_k) \cap (U_j \cap V_l)}$ fulfills all requirements of definition 5.1.5, too, and hence defines again a vector bundle. As the open covering, the homeomorphisms and the transition maps are part of the data of definition 5.1.5 this new bundle is (according to our definition) different from our initial one even though they are "the same" in some sense. Hence, in the following we will identify vector bundles that arise by such a construction one from the other:

Definition 5.1.7
Let $\pi : F \to X$ together with open covering $U_1, \ldots, U_s$, homeomorphisms Φ_i and transition maps M_{ij} and $\pi : F \to X$ together with open covering $V_1, \ldots, V_t$, homeomorphisms Ψ_i and transition maps N_{ij} be two tropical vector bundles according to definition 5.1.5. We will identify these vector bundles if the vector bundles $\pi : F \to X$ with open covering $\{U_i \cap V_j\}$ and restricted homeomorphisms $\Phi_i|_{\pi^{-1}(U_i \cap V_j)}$ respectively $\Psi_j|_{\pi^{-1}(U_i \cap V_j)}$ and transition maps $M_{ij}|_{(U_i \cap V_k) \cap (U_j \cap V_l)}$ respectively $N_{kl}|_{(U_i \cap V_k) \cap (U_j \cap V_l)}$ are equal.

Remark 5.1.8
Let $\pi_1 : F_1 \to X$ and $\pi_2 : F_2 \to X$ be two vector bundles on X. By definition 5.1.7 we can always assume that F_1 and F_2 satisfy definition 5.1.5 with the same open covering.

Remark 5.1.9
Let $\pi : F \to X$ be a vector bundle with open covering $U_1, \dots, U_s$ and transition maps M_{ij} as in definition 5.1.5. On the common intersection $U_i \cap U_j \cap U_k$ we obviously have $M_{ij}(x) = M_{kj}(x) \odot M_{ik}(x)$. This last equation is called *cocycle condition*. Conversely, given an open covering $U_1, \dots, U_s$ of X and maps $M_{ij} : U_i \cap U_j \to G(r)$ such that the entries of $M_{ij}(x)$ are regular invertible functions on $U_i \cap U_j$ or constantly $-\infty$ and the cocycle condition $M_{ij}(x) = M_{kj}(x) \odot M_{ik}(x)$ holds on $U_i \cap U_j \cap U_k$, we can construct a vector bundle $\pi : F \to X$ with this given open covering and transition functions M_{ij}: Take the disjoint union $\coprod_{i=1}^{s}(U_i \times \mathbb{R}^r)$ and identify points $(x, y) \sim (x, M_{ij}(x) \odot a)$ to obtain the topological space $|F|$. We have to equip this space with the structure of a tropical cycle. As this construction is exactly the same as for tropical line bundles, we only sketch it here and refer to [T09] for more details. Let $(((X_0, |X_0|, \{\varphi_\sigma\}), \omega_{X_0}), \{\Phi_\sigma\})$ be a representative of X. We define $F_0 := \{\pi^{-1}(\sigma) | \sigma \in X_0\}$ and $\omega_{F_0}(\pi^{-1}(\sigma)) := \omega_{X_0}(\sigma)$ for all maximal polyhedra $\sigma \in X_0$. Our next step is to construct the polyhedral charts $\widetilde{\varphi}_{\pi^{-1}(\sigma)}$ for F_0: Let $\sigma \in X_0$ be given and let $U_{i_1}, \dots, U_{i_t}$ be all open sets with non-empty intersection with σ. Moreover, let $\{V_i | i \in I\}$ be the set of all connected components of all $\sigma \cap U_{i_k}$. Every such set V_i comes from a set $U_{j(i)}$ of the given open covering. Hence, for every pair $k, l \in I$ we have a restricted transition map $N_{kl} := M_{j(k),j(l)}|_{V_k \cap V_l}$. This implies that for all $k, l \in I$ the entries of $N_{kl} \circ \Phi_\sigma^{-1}$ are (globally) integer affine linear functions on $V_k \cap V_l$. As σ is simply connected, for every such entry $h \in \mathcal{O}^*(V_k \cap V_l)$ of N_{kl} there exists a unique continuation $\widetilde{h} \in \mathcal{O}^*(\sigma)$. Hence we can extend all transition maps $N_{kl} : V_k \cap V_l \to G(r)$ to maps $N'_{kl} : \sigma \to G(r)$. Now we choose for every $i \in I$ a point $P_i \in V_i$ and for all pairs $k, l \in I$ a path $\gamma_{kl} : [0, 1] \to \sigma$ from P_k to P_l. Let $k, l \in I$ be given. As the image of γ_{kl} is compact there exists a finite covering $V_{\mu_1}, \dots, V_{\mu_c}$ of $\gamma_{kl}([0, 1])$. For $x \in V_l$ we set

$$S(\gamma_{kl})(x) := (N'_{\mu_1,\mu_2}(x))^{-1} \odot \dots \odot (N'_{\mu_{c-1},\mu_c}(x))^{-1} \in G(r).$$

Now fix some $k_0 \in I$. For all $l \in I$ we define maps

$$\widetilde{\varphi}^{(l)}_{\pi^{-1}(\sigma)} : V_l \times \mathbb{R}^r \cong \pi^{-1}(V_l) \to \mathbb{R}^{n_\sigma + r} : (x, a) \mapsto (\varphi_\sigma(x), S(\gamma_{k_0 l})(x) \odot a).$$

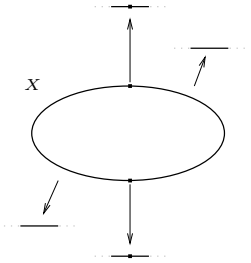
These maps agree on overlaps and hence glue together to an embedding

$$\widetilde{\varphi}_{\pi^{-1}(\sigma)} : \pi^{-1}(\sigma) \to \mathbb{R}^{n_\sigma + r}.$$

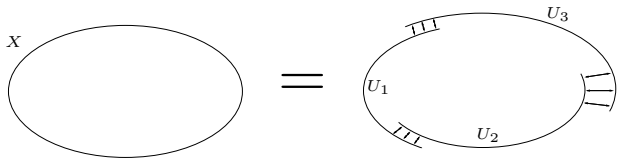
In the same way we can construct the fan charts $\widetilde{\Phi}_{\pi^{-1}(\sigma)}$. Then we define F to be the equivalence class

$$F := \left[(((F_0, |F_0|, \{\widetilde{\varphi}_{\pi^{-1}(\sigma)}\}), \omega_{F_0}), \{\widetilde{\Phi}_{\pi^{-1}(\sigma)}\})\right].$$

Example 5.1.10
Throughout the chapter, the curve $X := X_2$ from example 1.4.5 will serve us as a central example. Recall that X arises by gluing open fans as drawn in the figure:



Moreover, recall from definition 1.4.4 that the transition functions between these open fans composing X are integer affine linear. This implies that the curve X has a well-defined lattice length L. We can cover X by open sets U_1, U_2, U_3 as drawn in the following figure:



The easiest way to construct a (non-trivial) vector bundle of rank r on X is fixing a (non-trivial) transition map $M_{12} : U_1 \cap U_2 \to G(r)$ and defining $M_{23} : U_2 \cap U_3 \to G(r)$, $M_{31} : U_3 \cap U_1 \to G(r)$ to be the trivial maps $x \mapsto E$ for all x. We will see later that in fact every vector bundle of rank r on X arises in this way.

Knowing what tropical vector bundles are, there are a few notions related to this definition we want to introduce now:

Definition 5.1.11 (Direct sums of vector bundles)
Let $\pi_1 : F_1 \to X$ and $\pi_2 : F_2 \to X$ be two vector bundles of rank r and r', respectively, with a common open covering $U_1, \ldots, U_s$ and transition maps $M_{ij}^{(1)}$ and $M_{ij}^{(2)}$, respectively, satisfying definition 5.1.5 (see remark 5.1.8). We define the *direct sum bundle* $\pi : F_1 \oplus F_2 \to X$ to be the vector bundle of rank $r + r'$ we obtain from the gluing data

- $U_1, \ldots, U_s$
- $M_{ij}^{(1)} \times M_{ij}^{(2)} : U_i \cap U_j \to G(r + r') : x \mapsto \begin{pmatrix} M_{ij}^{(1)}(x) & -\infty \\ -\infty & M_{ij}^{(2)}(x) \end{pmatrix}$.

Definition 5.1.12 (Subbundles)
Let $\pi : F \to X$ be a vector bundle with open covering $U_1, \ldots, U_s$ and homeomorphisms Φ_i according to definition 5.1.5. A subcycle $E \in Z_l(F)$ is called a subbundle of rank r' of F if $\pi|_E : E \to X$ is a vector bundle of rank r' such that we have for all $i = 1, \ldots, s$:

$$\Phi_i|_{(\pi|_E)^{-1}(U_i)} : (\pi|_E)^{-1}(U_i) \overset{\cong}{\to} U_i \times \langle e_{j_1}, \ldots, e_{j_{r'}} \rangle_{\mathbb{R}}$$

for some $1 \leq j_1 < \ldots < j_{r'} \leq r$, where the e_j are the standard basis vectors in $\mathbb{R}^r$.

Remark 5.1.13
If $\pi : F \to X$ is a vector bundle of rank r with subbundle E of rank r' like in definition 5.1.12 this implies that there exists another subbundle E' of rank $r - r'$ with

$$\Phi_i|_{(\pi|_{E'})^{-1}(U_i)} : (\pi|_{E'})^{-1}(U_i) \overset{\cong}{\to} U_i \times \langle e_j | j \notin \{j_1, \ldots, j_{r'}\}\rangle_{\mathbb{R}}$$

and hence that $F = E \oplus E'$ holds.

Definition 5.1.14 (Decomposable bundles)
Let $\pi : F \to X$ be a vector bundle of rank r. We say that F is *decomposable* if there exists a subbundle $\pi|_E : E \to X$ of F of rank $1 \leq r' < r$. Otherwise we call F an *indecomposable vector bundle*.

As announced in the very beginning of this section we also want to talk about morphisms and, in particular, isomorphisms of tropical vector bundles:

Definition 5.1.15 (Morphisms of vector bundles)
A morphism of vector bundles $\pi_1 : F_1 \to X$ of rank r and $\pi_2 : F_2 \to X$ of rank r' is a morphism $\Psi : F_1 \to F_2$ of tropical cycles such that

(a) $\pi_1 = \pi_2 \circ \Psi$ and

(b) there exist an open covering $U_1, \ldots, U_s$ according to definition 5.1.5 for both F_1 and F_2 (see remark 5.1.8) and maps $A_i : U_i \to G(r' \times r)$ for all i such that

$$\Phi_i^{F_2} \circ \Psi \circ (\Phi_i^{F_1})^{-1} : U_i \times \mathbb{R}^r \to U_i \times \mathbb{R}^{r'}$$

is given by $(x, a) \mapsto (x, f_{A_i(x)}(a))$ (cf. 5.1.2) and the entries of A_i are regular invertible functions on U_i or constantly $-\infty$.

An isomorphism of tropical vector bundles is a morphism of vector bundles $\Psi : F_1 \to F_2$ such that there exists a morphism of vector bundles $\Psi' : F_2 \to F_1$ with $\Psi' \circ \Psi = \mathrm{id} = \Psi \circ \Psi'$.

Lemma 5.1.16
Let $\pi_1 : F_1 \to X$ and $\pi_2 : F_2 \to X$ be two vector bundles of rank r over X. Then the following are equivalent:

(a) There exists an isomorphism of vector bundles $\Psi : F_1 \to F_2$.

(b) There exist a common open covering $U_1, \ldots, U_s$ of X and transition maps $M_{ij}^{(1)}$ for F_1 and $M_{ij}^{(2)}$ for F_2 satisfying definition 5.1.5 (cf. remark 5.1.8) and maps $E_i : U_i \to G(r)$ for $i = 1, \ldots, s$ such that

- *the entries of E_i are regular invertible functions on U_i or constantly $-\infty$ and*
- *for all i, j holds $E_j(x) \odot M_{ij}^{(1)}(x) = M_{ij}^{(2)}(x) \odot E_i(x)$ for all $x \in U_i \cap U_j$.*

Proof. $(a) \Rightarrow (b)$: We claim that the maps $A_i : U_i \to G(r \times r)$ of definition 5.1.15 are the wanted maps E_i. As Ψ is an isomorphism we can conclude that $A_i(x)$ is an invertible matrix for all $x \in U_i$, i.e. that $A_i : U_i \to G(r)$. Hence it remains to check that $A_j(x) \odot M_{ij}^{(1)}(x) = M_{ij}^{(2)}(x) \odot A_i(x)$ holds for all $x \in U_i \cap U_j$: Let i, j be given. As $\Psi : F_1 \to F_2$ is an isomorphism, the diagram

$$\begin{array}{ccc} (U_i \cap U_j) \times \mathbb{R}^r & \xrightarrow{\Phi_i^{F_2} \circ \Psi \circ (\Phi_i^{F_1})^{-1}} & (U_i \cap U_j) \times \mathbb{R}^r \\ \downarrow{\scriptstyle \Phi_j^{F_1} \circ (\Phi_i^{F_1})^{-1}} & & \downarrow{\scriptstyle \Phi_j^{F_2} \circ (\Phi_i^{F_2})^{-1}} \\ (U_i \cap U_j) \times \mathbb{R}^r & \xrightarrow[\Phi_j^{F_2} \circ \Psi \circ (\Phi_j^{F_1})^{-1}]{} & (U_i \cap U_j) \times \mathbb{R}^r \end{array}$$

commutes. Hence $A_j(x) \odot M_{ij}^{(1)}(x) = M_{ij}^{(2)}(x) \odot A_i(x)$ holds.
$(b) \Rightarrow (a)$: Conversely, let the maps $E_i : U_i \to G(r)$ be given. The equation

$$E_j(x) \odot M_{ij}^{(1)}(x) = M_{ij}^{(2)}(x) \odot E_i(x)$$

for all $x \in U_i \cap U_j$ ensures that the maps

$$U_i \times \mathbb{R}^r \to U_i \times \mathbb{R}^r : (x, a) \mapsto (x, E_i(x) \odot a)$$

on the local trivializations can be glued to a globally defined map $\Psi : |F_1| \to |F_2|$. Moreover, this map is a morphism as π_1, π_2 are morphisms and the maps $p_j^{(i)} \circ \Phi_i^{F_1}$, $p_j^{(i)} \circ \Phi_i^{F_2}$ and the finite entries of E_i are regular invertible functions (cf. definition 5.1.5). The equation $E_j(x) \odot M_{ij}^{(1)}(x) = M_{ij}^{(2)}(x) \odot E_i(x)$ implies that

$$E_j^{-1}(x) \odot M_{ij}^{(2)}(x) = M_{ij}^{(1)}(x) \odot E_i^{-1}(x)$$

holds for all $x \in U_i \cap U_j$, where $E_k^{-1}(x) := (E_k(x))^{-1}$ for all $x \in U_k$. As the finite entries of $E_k^{-1} : U_k \to G(r)$ are again regular invertible functions we can also glue the maps

$$U_i \times \mathbb{R}^r \to U_i \times \mathbb{R}^r : (x, a) \mapsto (x, E_i^{-1}(x) \odot a)$$

on the local trivializations to obtain the inverse morphism $\Psi' : |F_2| \to |F_1|$, which proves that Ψ is an isomorphism. □

The morphisms we have just introduced admit another important operation, namely the pull-back of a vector bundle:

Definition 5.1.17 (Pull-back of vector bundles)
Let $\pi : F \to X$ be a vector bundle of rank r with open covering $U_1, \ldots, U_s$ and transition maps M_{ij} as in definition 5.1.5, and let $f : Y \to X$ be a morphism of tropical cycles. Then the *pull-back bundle* $\pi' : f^*F \to Y$ is the vector bundle we obtain by gluing the patches $f^{-1}(U_1) \times \mathbb{R}^r, \ldots, f^{-1}(U_s) \times \mathbb{R}^r$ along the transition maps $M_{ij} \circ f$. Hence we obtain the commutative diagram

$$\begin{array}{ccc} f^*F & \xrightarrow{f'} & F \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow[f]{} & X \end{array}$$

where f' and π' are locally given by $f' : f^{-1}(U_i) \times \mathbb{R}^r \to U_i \times \mathbb{R}^r : (y, a) \mapsto (f(y), a)$ and $\pi' : f^{-1}(U_i) \times \mathbb{R}^r \to f^{-1}(U_i) : (y, a) \mapsto y$.

To be able to define Chern classes in the second section we need the notion of a rational section of a vector bundle:

Definition 5.1.18 (Rational sections of vector bundles)
Let $\pi : F \to X$ be a vector bundle of rank r. A *rational section* $s : X \to F$ of F is a continuous map $s : |X| \to |F|$ such that

(a) $\pi(s(x)) = x$ for all $x \in |X|$ and

(b) there exist an open covering $U_1, \ldots, U_s$ and homeomorphisms Φ_i satisfying definition 5.1.5 (cf. definition 5.1.7) such that the maps $p_j^{(i)} \circ \Phi_i \circ s : U_i \to \mathbb{R}$ are rational functions on U_i for all i, j,

where $p_j^{(i)} : U_i \times \mathbb{R}^r \to \mathbb{R}$ is given by $(x, (a_1, \ldots, a_r)) \mapsto a_j$. A rational section $s : X \to F$ is called *bounded* if the above maps $p_j^{(i)} \circ \Phi_i \circ s$ are bounded for all i, j.

Remark 5.1.19
Let $\pi : L \to X$ be a line bundle and $s : X \to L$ a rational section. By definition, the map $p^{(i)} \circ \Phi_i \circ s$ is a rational function on U_i for all i. Moreover, on $U_i \cap U_j$ the maps $p^{(i)} \circ \Phi_i \circ s$ and $p^{(j)} \circ \Phi_j \circ s$ differ by a regular invertible function only. Hence s defines a Cartier divisor $\mathcal{D}(s) \in \mathrm{Div}(X)$.

There is a useful statement on these Cartier divisors $\mathcal{D}(s)$ in [T09] that we want to cite here including its proof:

Lemma 5.1.20
Let $\pi : L \to X$ be a line bundle and let $s_1, s_2 : X \to L$ be two bounded rational sections. Then $\mathcal{D}(s_1) - \mathcal{D}(s_2) = h$ for some bounded rational function $h \in \mathcal{K}^*(X)$, i.e. $\mathcal{D}(s_1)$ and $\mathcal{D}(s_2)$ are rationally equivalent.

Proof. Let $U_1, \ldots, U_s$ be an open covering of X with transition maps M_{ij} and homeomorphisms Φ_i according to definition 5.1.5 such that for all i both $s_1^{(i)} := p^{(i)} \circ \Phi_i \circ s_1$ and $s_2^{(i)} := p^{(i)} \circ \Phi_i \circ s_2$ are rational functions on U_i (cf. definition 5.1.18). We define $h_i := s_1^{(i)} - s_2^{(i)} \in \mathcal{K}^*(U_i)$. As we have $s_1^{(i)} - s_1^{(j)} = s_2^{(i)} - s_2^{(j)} = M_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ for all i, j these maps h_i glue together to $h \in \mathcal{K}^*(X)$. Hence we have

$$\begin{aligned} \mathcal{D}(s_1) - \mathcal{D}(s_2) &= [\{(U_i, s_1^{(i)})\}] - [\{(U_i, s_2^{(i)})\}] \\ &= [\{(U_i, s_1^{(i)} - s_2^{(i)})\}] \\ &= [\{(U_i, h_i)\}] \\ &= [\{(|X|, h)\}]. \end{aligned}$$

□

Remark 5.1.21
Lemma 5.1.20 implies that we can associate to any line bundle L admitting a bounded rational section s a Cartier divisor class $\mathcal{D}(F) := [\mathcal{D}(s)]$ that only depends on the bundle L and not on the choice of the rational section s.

Combining both the notion of a morphism of vector bundles and the notion of a rational section we can define the following:

Definition 5.1.22 (Pull-back of rational sections)
Let $\pi : F \to X$ be a vector bundle of rank r and $f : Y \to X$ a morphism of tropical varieties. Moreover, let $s : X \to F$ be a rational section of F with open covering $U_1, \dots, U_s$ and homeomorphisms $\Phi_1, \dots, \Phi_s$ as in definition 5.1.18. Then we can define a rational section $f^*s : Y \to f^*F$ of f^*F, the *pull-back section* of s, as follows: On $f^{-1}(U_i)$ we define

$$f^*s : f^{-1}(U_i) \to f^{-1}(U_i) \times \mathbb{R}^r : y \mapsto (y, (p_i \circ \Phi_i \circ s \circ f)(y)),$$

where $p_i : U_i \times \mathbb{R}^r \to \mathbb{R}^r$ is the projection on the second factor. Note that for $y \in f^{-1}(U_i) \cap f^{-1}(U_j)$ the points $(y, (p_i \circ \Phi_i \circ s \circ f)(y))$ and $(y, (p_j \circ \Phi_j \circ s \circ f)(y))$ are identified in f^*F if and only if $(f(y), (p_i \circ \Phi_i \circ s \circ f)(y))$ and $(f(y), (p_j \circ \Phi_j \circ s \circ f)(y))$ are identified in F. But this is the case as $(f(y), (p_i \circ \Phi_i \circ s \circ f)(y)) = (\Phi_i \circ s)(f(y)) \sim (\Phi_j \circ s)(f(y)) = (f(y), (p_j \circ \Phi_j \circ s \circ f)(y))$. Hence we can glue our locally defined map f^*s to obtain a map $f^*s : Y \to f^*F$.

We finish this section with the following statement on vector bundles on simply connected tropical cycles which will be of use for us later on:

Theorem 5.1.23
Let $\pi : F \to X$ be a vector bundle of rank r on the simply connected tropical cycle X. Then F is a direct sum of line bundles, i.e. there exist line bundles $L_1, \dots, L_r$ on X such that $F = L_1 \oplus \dots \oplus L_r$.

Proof. We show that every vector bundle of rank $r \geq 2$ on X is decomposable. Let $U_1, \dots, U_s$ be an open covering of X and let

$$M_{ij}(x) = D(\varphi_{i,j}^{(1)}, \dots, \varphi_{i,j}^{(r)})(x) \odot A_{\sigma_{ij}}(x) =: D_{ij}(x) \odot A_{\sigma_{ij}}(x), \quad x \in U_i \cap U_j$$

with $\varphi_{i,j}^{(1)}, \dots, \varphi_{i,j}^{(r)} \in \mathcal{O}^*(U_i \cap U_j)$ and $\sigma_{ij}(x) \in S_r$ be transition functions according to definition 5.1.5. We only have to show that it is possible to track the first coordinate of the $\mathbb{R}^r$-factor in $U_1 \times \mathbb{R}^r$ consistently along the transition maps: Let $\gamma : [0,1] \to |X|$ be a closed path starting and ending in $P \in U_1$. Decomposing γ into several paths if necessary, we may assume that γ has no self-intersections, i.e. that $\gamma|_{[0,1)}$ is injective. As $\gamma([0,1])$ is compact we can choose an open covering $V_1, \dots, V_t$ of $\gamma([0,1])$ such that for all j we have $V_j \subseteq U_i$ for some index $i = i(j)$, $P \in V_1 = V_t \subseteq U_1$, all sets V_j and all intersections $V_j \cap V_{j+1}$ are connected and all intersections $V_j \cap V_{j'}$ for non-consecutive indices are empty. For sets V_j and $V_{j'}$ with non-empty intersection we have restricted transition maps $\widetilde{M}_{V_j,V_{j'}}(x) = \widetilde{D}_{V_j,V_{j'}}(x) \odot A_{\sigma_{V_j,V_{j'}}}$ induced by the transition maps between $U_{i(j)} \supseteq V_j$ and $U_{i(j')} \supseteq V_{j'}$. Note that the permutation parts $A_{\sigma_{V_j,V_{j'}}}$ of the transition maps do not depend on x as all intersections $V_j \cap V_{j'}$ are connected and the permutations have to be locally constant. We define $I_\gamma := \sigma_{V_{t-1},V_t} \circ \dots \circ \sigma_{V_1,V_2}(1)$. We have to check that $I_\gamma = 1$ holds. First we show that I_γ does not depend on the choice of the covering $V_1, \dots, V_t$. Hence, let $V_1', \dots, V_{t'}'$ be another covering as above. We may

assume that all intersections $V_j \cap V'_{j'}$ are connected, too. Between any two sets $A, B \in \{V_1, \ldots, V_t, V'_1, \ldots, V'_{t'}\}$ with non-empty intersection we have restricted transition maps $\widetilde{M}_{A,B}(x) = \widetilde{D}_{A,B}(x) \odot A_{\sigma_{A,B}}$ as above. Moreover, let $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_p = 1$ be a decomposition of $[0,1]$ such that for all i we have $\gamma([\alpha_i, \alpha_{i+1}]) \subseteq V_j \cap V'_{j'}$ for some indices j, j'. Let i_0 be the maximal index such that $\gamma([\alpha_{i_0}, \alpha_{i_0+1}]) \subseteq V_a \cap V'_b$ and

$$\sigma_{V_{a-1},V_a} \circ \ldots \circ \sigma_{V_1,V_2} = \sigma_{V'_b,V_a} \circ \sigma_{V'_{b-1},V'_b} \circ \ldots \circ \sigma_{V'_1,V'_2}$$

is still fulfilled. Assume that $i_0 < p-1$. Let $\gamma([\alpha_{i_0+1}, \alpha_{i_0+2}]) \subseteq V_{a'} \cap V'_{b'}$. Hence $\gamma(\alpha_{i_0+1}) \in V_a \cap V'_b \cap V_{a'} \cap V'_{b'}$ and we can conclude using the cocycle condition:

$$\begin{aligned}
\sigma_{V_a,V_{a'}} \circ \sigma_{V_{a-1},V_a} \circ \ldots \circ \sigma_{V_1,V_2} &= \sigma_{V_a,V_{a'}} \circ \sigma_{V'_b,V_a} \circ \sigma_{V'_{b-1},V'_b} \circ \ldots \circ \sigma_{V'_1,V'_2} \\
&= \sigma_{V_a,V_{a'}} \circ \sigma_{V'_{b'},V_a} \circ \sigma_{V'_b,V'_{b'}} \circ \sigma_{V'_{b-1},V'_b} \circ \ldots \circ \sigma_{V'_1,V'_2} \\
&= \sigma_{V'_{b'},V_{a'}} \circ \sigma_{V'_b,V'_{b'}} \circ \sigma_{V'_{b-1},V'_b} \circ \ldots \circ \sigma_{V'_1,V'_2},
\end{aligned}$$

a contradiction to our assumption. Hence $i_0 = p-1$ and we can conclude that I_γ is independent of the chosen covering.
If γ and γ' are paths that pass through exactly the same open sets U_i in the same order, then we can conclude that $I_\gamma = I_{\gamma'}$ holds as exactly the same transition functions are involved. Hence, a continuous deformation of γ does not change I_γ. As X is simply connected we can contract γ to a point. This implies $I_\gamma = I_{\gamma_0}$, where γ_0 is the constant path $\gamma_0(t) = P$ for all t. Thus $I_\gamma = I_{\gamma_0} = 1$. This proves the claim. □

There is a related theorem in [T09] which we want to state here. As we will not need the result in this work, we will omit the proof and refer to [T09] instead.

Theorem 5.1.24
Let $\pi : L \to X$ be a line bundle on the simply connected tropical cycle X. Then L is trivial, i.e. $L \cong X \times \mathbb{R}$ as a vector bundle.

Combing both theorem 5.1.23 and theorem 5.1.24 we can conclude the following:

Corollary 5.1.25
Let $\pi : F \to X$ be a vector bundle of rank r on the simply connected tropical cycle X. Then F is trivial, i.e. $F \cong X \times \mathbb{R}^r$ as a vector bundle.

5.2 Chern classes

In this section we will introduce Chern classes of tropical vector bundles and prove basic properties. To be able to do this we need some preparation:

Definition 5.2.1
Let $\pi : F \to X$ be a vector bundle of rank r and let $s : X \to F$ be a rational section with open covering $U_1, \ldots, U_s$ as in definition 5.1.18. We fix a natural number $1 \leq k \leq r$ and a subcycle $Y \in Z_l(X)$. By definition, $s_{ij} := p_j^{(i)} \circ \Phi_i \circ s : U_i \to \mathbb{R}$ is a rational function on U_i for all i, j. Hence, for all i we can take local intersection products

$$(s^{(k)} \cdot Y) \cap U_i := \sum_{1 \leq j_1 < \ldots < j_k \leq r} s_{ij_1} \cdots s_{ij_k} \cdot (Y \cap U_i).$$

Since $s_{i'j} = s_{i\sigma(j)} + \varphi_j$ on $U_i \cap U_{i'}$ for some $\sigma \in S_r$ and some regular invertible map $\varphi_j \in \mathcal{O}^*(U_i \cap U_{i'})$, the intersection products $(s^{(k)} \cdot Y) \cap U_i$ and $(s^{(k)} \cdot Y) \cap U_{i'}$ coincide on $U_i \cap U_{i'}$ and we can glue them to obtain a global intersection cycle $s^{(k)} \cdot Y \in Z_{l-k}(X)$.

Lemma 5.2.2
Let $\pi : F \to X$ be a vector bundle of rank r, fix $k \in \{1, \dots, r\}$ and let $s : X \to F$ be a rational section. Moreover, let $Y \in Z_l(X)$ be a cycle and let $\varphi \in \mathcal{K}^*(Y)$ be a bounded rational function on Y. Then the following equation holds:

$$s^{(k)} \cdot (\varphi \cdot Y) = \varphi \cdot (s^{(k)} \cdot Y).$$

Proof. The claim follows immediately from the definition of the product $s^{(k)} \cdot Y$. □

Lemma 5.2.3
Let $\pi : F \to X$ and $\pi' : F' \to X$ be two isomorphic vector bundles of rank r with isomorphism $f : F \to F'$. Moreover, fix $k \in \{1, \dots, r\}$, let $s : X \to F$ be a rational section and let $Y \in Z_l(X)$ be a cycle. Then the following equation holds:

$$s^{(k)} \cdot Y = (f \circ s)^{(k)} \cdot Y \in Z_{l-k}(X).$$

Proof. Let $U_1, \dots, U_s$ be an open covering of X satisfying definition 5.1.5 for both F and F' and let $s_{ij} := p_j^{(i)} \circ \Phi_i \circ s : U_i \to \mathbb{R}$ and $(f \circ s)_{ij} := p_j^{(i)} \circ \Phi_i \circ f \circ s : U_i \to \mathbb{R}$ as in definition 5.2.1. By lemma 5.1.16 the isomorphism f can be described on $U_i \times \mathbb{R}^r$ by $(x, a) \mapsto (x, E_i(x) \odot a)$ with $E_i(x) = D(\varphi_1, \dots, \varphi_r) \odot A_\sigma$ for some regular invertible functions $\varphi_1, \dots, \varphi_r \in \mathcal{O}^*(U_i)$ and a permutation $\sigma \in S_r$. Hence $(f \circ s)_{ij} = s_{i\sigma(j)} + \varphi_j$ on U_i and thus

$$\sum_{1 \le j_1 < \dots < j_k \le r} s_{ij_1} \cdots s_{ij_k} \cdot (Y \cap U_i) = \sum_{1 \le j_1 < \dots < j_k \le r} (f \circ s)_{ij_1} \cdots (f \circ s)_{ij_k} \cdot (Y \cap U_i),$$

which proves the claim. □

To be able to prove the next theorem, which will be essential for defining Chern classes, we first need some generalizations of our previous definitions:

Definition 5.2.4 (Infinite tropical cycle)
We define an *infinite tropical polyhedral complex* to be a tropical polyhedral complex according to definition 1.4.4 but we do not require the set of polyhedra X to be finite. In particular, all open fans F_σ have still to be open tropical fans according to definition 1.4.3. Then an *infinite tropical cycle* is an infinite tropical polyhedral complex modulo refinements analogous to definition 1.4.13.

Definition 5.2.5 (Infinite rational functions and infinite Cartier divisors)
Let C be an infinite tropical cycle and let U be an open set in $|C|$. As in definition 1.5.1 an *infinite rational function* on U is a continuous function $\varphi : U \to \mathbb{R}$ such that there exists a representative $(((X, |X|, \{m_\sigma\}_{\sigma \in X}), \omega_X), \{M_\sigma\}_{\sigma \in X})$ of C, which may now be an infinite tropical polyhedral complex, such that for each face $\sigma \in X$ the map $\varphi \circ m_\sigma^{-1}$ is locally integer affine linear (where defined). Analogously it is possible to define *infinite*

regular invertible functions on U.
A *representative of an infinite Cartier divisor* on C is then a set $\{(U_i, \varphi_i) |\ i \in I\}$, where $\{U_i\}$ is an open covering of $|C|$ and φ_i is an infinite rational function on U_i. An *infinite Cartier divisor* on C is then a representative of an infinite Cartier divisor modulo the equivalence relation given in definition 1.5.1.

Remark 5.2.6
Using these basic definitions it is possible to generalize many other concepts to the infinite case. In particular, as our infinite objects are locally finite, it is possible to perform intersection theory as before.

Definition 5.2.7 (Tropical vector bundles on infinite cycles)
Let X be an infinite tropical cycle. A *tropical vector bundle* over X of rank r is an infinite tropical cycle F together with a morphism $\pi : F \to X$ such that properties (a)–(d) given in definition 5.1.5 are fulfilled with the difference that the open covering $\{U_i\}$ of X may now be infinite.

Now we are ready to prove the announced theorem:

Theorem 5.2.8
Let $\pi : F \to X$ be a vector bundle of rank r and $s_1, s_2 : X \to F$ two bounded rational sections. Then $s_1^{(k)} \cdot Y$ and $s_2^{(k)} \cdot Y$ are rationally equivalent, i.e.

$$[s_1^{(k)} \cdot Y] = [s_2^{(k)} \cdot Y] \in A_*(X)$$

holds for all subcycles $Y \in Z_l(X)$.

Proof. Let $p : |\widetilde{X}| \to |X|$ be the universal covering space of $|X|$. We can locally equip $|\widetilde{X}|$ with the tropical structure inherited form X and obtain an infinite tropical cycle $\widetilde{X}$ according to definition 5.2.4. Moreover, pulling back F along p, we obtain a tropical vector bundle p^*F on $\widetilde{X}$ according to definition 5.2.7. As $\widetilde{X}$ is simply connected we can conclude by lemma 5.1.23 that $p^*F = L_1 \oplus \ldots \oplus L_r$ for some infinite tropical line bundles $L_1, \ldots, L_r$ on $\widetilde{X}$. Hence, the bounded rational sections p^*s_1 and p^*s_2 correspond to r infinite tropical Cartier divisors as in definition 5.2.5 each, which we will denote by $\varphi_1, \ldots, \varphi_r$ and $\psi_1, \ldots, \psi_r$, respectively. By lemma 5.1.20 we can conclude that for all i these Cartier divisors differ by bounded infinite rational functions only, i.e. $\varphi_i - \psi_i = h_i$ for some bounded infinite rational function h_i on $\widetilde{X}$. In particular,

$$\left(\sum_{1 \le j_1 < \ldots < j_k \le r} \varphi_{j_1} \cdots \varphi_{j_k} - \sum_{1 \le j_1 < \ldots < j_k \le r} \psi_{j_1} \cdots \psi_{j_k} \right) \cdot \widetilde{X} = \widetilde{h} \cdot \widetilde{\xi}_2 \cdots \widetilde{\xi}_k \cdot \widetilde{X}$$

with a bounded infinite rational function $\widetilde{h}$ and infinite Cartier divisors $\widetilde{\xi}_i$. Then we can define a rational function h, which is then also bounded, and Cartier divisors ξ_i on X as follows: Let $U \subseteq |X|$ and $\widetilde{U} \subseteq |\widetilde{X}|$ be open subsets such that $p|_{\widetilde{U}} : \widetilde{U} \to U$ is bijective with inverse map $p' : U \to \widetilde{U}$. Then we locally define $h|_U := (p')^* \widetilde{h}|_{\widetilde{U}}$ and $\xi_i|_U := (p')^* \widetilde{\xi}_i|_{\widetilde{U}}$. Note that h and ξ_i are well-defined as the Cartier divisors φ_i and ψ_i,

respectively, are the same on every possible set $\widetilde{U} \overset{\cong}{\to} U$. As we locally have

$$(s_1^{(k)} \cdot Y) \cap U = p_* \left(\sum_{1 \leq j_1 < \ldots < j_k \leq r} \varphi_{j_1} \cdots \varphi_{j_k} \cdot (p')_*(Y \cap U) \right)$$

and

$$(s_2^{(k)} \cdot Y) \cap U = p_* \left(\sum_{1 \leq j_1 < \ldots < j_k \leq r} \psi_{j_1} \cdots \psi_{j_k} \cdot (p')_*(Y \cap U) \right)$$

we can conclude that

$$(s_1^{(k)} - s_2^{(k)}) \cdot Y = h \cdot \xi_2 \cdots \xi_k \cdot Y,$$

which proves the claim. □

Now we are ready to give a definition of Chern classes:

Definition 5.2.9 (Chern classes)
Let $\pi : F \to X$ be a vector bundle of rank r admitting bounded rational sections. For $k \in \{1, \ldots, r\}$ we define the k-th Chern class of F to be the endomorphism

$$c_k(F) : A_*(X) \to A_*(X) : [Y] \mapsto [s^{(k)} \cdot Y],$$

where $A_*(X) = \bigoplus_i A_i(X)$ and $s : X \to F$ is any bounded rational section. Note that the map $c_k(F)$ is well-defined by lemma 5.2.2 and independent of the choice of the rational section s by theorem 5.2.8. Moreover, we define $c_0(F) : A_*(X) \to A_*(X)$ to be the identity map and $c_k(F) : A_*(X) \to A_*(X)$ to be the zero map for all $k \notin \{0, \ldots, r\}$. To stress the character of an intersection product of $c_k(F)$ we usually write $c_k(F) \cdot Y$ instead of $c_k(F)(Y)$ for $Y \in A_*(X)$.

Remark 5.2.10
Note that lemma 5.2.3 implies that isomorphic vector bundles have the same Chern classes.

As announced in the beginning we finish this section with proving some basic properties of Chern classes:

Theorem 5.2.11 (Properties of Chern classes)
Let $\pi : F \to X$ and $\pi' : F' \to X$ be vector bundles of rank r and r', respectively, admitting bounded rational sections. Moreover, let $f : \widetilde{X} \to X$ be a morphism of tropical cycles. Then the following holds:

(a) $c_i(F) = 0$ *for all* $i \notin \{0, \ldots, \mathrm{rank}(F)\}$,

(b) $c_i(F) \cdot (c_j(F') \cdot Y) = c_j(F') \cdot (c_i(F) \cdot Y)$ *for all* $Y \in A_*(X)$,

(c) $f_*(c_i(f^*F) \cdot Y) = c_i(F) \cdot f_*(Y)$ *for all* $Y \in A_*(\widetilde{X})$,

(d) $c_i(f^*F) \cdot f^*(Y) = f^*(c_i(F) \cdot Y)$ *for all* $Y \in A_*(X)$ *if* X *and* $\widetilde{X}$ *are smooth varieties,*

(e) $c_k(F \oplus F') = \sum_{i+j=k} c_i(F) \cdot c_j(F')$

(f) $c_1(F) \cdot Y = \mathcal{D}(F) \cdot Y$ *for all* $Y \in A_*(X)$ *if* $r = \operatorname{rank}(F) = 1$*, where* $\mathcal{D}(F)$ *is the Cartier divisor class associated to* F*.*

Proof. Properties (a) and (e) follow immediately from definition 5.2.9, property (b) follows from the fact that the intersection product is commutative and property (f) follows from remark 5.1.21.
(c): The projection formula implies

$$f_*(c_i(f^*F) \cdot Y) = f_*([(f^*s)^{(i)} \cdot Y]) = [s^{(i)} \cdot f_*Y] = c_i(F) \cdot f_*Y,$$

where s is any bounded rational section of F.
(d): Applying theorem 3.3.3 (c) and (f) we obtain

$$c_i(f^*F) \cdot f^*Y = [(f^*s)^{(i)} \cdot f^*Y] = [f^*(s^{(i)} \cdot Y)] = f^*[s^{(i)} \cdot Y] = f^*(c_i(F) \cdot Y),$$

where s is again any bounded rational section of F. □

Remark 5.2.12
In "classical" algebraic geometry even the following, generalized version of property (e) is true: Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of vector bundles, then $c_k(F) = \sum_{i+j=k} c_i(F') \cdot c_j(F'')$. In the tropical world it is not entirely clear what an exact sequence of tropical vector bundles should be. Nevertheless, in some sense the "classical" statement is true in tropical geometry as well: Let $\pi_1 : F_1 \to X$ and $\pi_2 : F_2 \to X$ be tropical vector bundles of rank r_1 and r_2, respectively, and let $U_1, \ldots, U_s$ be an open covering of X such that all requirements of definition 5.1.5 are fulfilled for F_1 and F_2 simultaneously. Moreover, let $f : F_1 \to F_2$ be an injective morphism of tropical vector bundles such that $(\Phi_i^{F_2} \circ f \circ (\Phi_i^{F_1})^{-1})(U_i \times \mathbb{R}^{r_1}) = U_i \times \langle e_{i_1}, \ldots, e_{i_{r_1}} \rangle_{\mathbb{R}}$ for all i, i.e. such that the image of F_1 under f is a subbundle F' of F_2 (cf. definition 5.1.12). Then we can conclude by remark 5.1.13 that F_2 is decomposable into $F_2 = F' \oplus F'' \cong F_1 \oplus F''$ for some other subbundle F'' of F_2. Hence we can conclude by theorem 5.2.11 that $c_k(F_2) = \sum_{i+j=k} c_i(F_1) \cdot c_j(F'')$.

5.3 Vector bundles on an elliptic curve

In this section we will give a complete classification of all vector bundles on an elliptic curve up to isomorphism. One characteristic to distinguish different bundles will be the following:

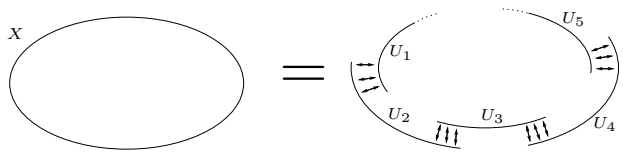
Definition 5.3.1 (Degree of a vector bundle)
Let $X := X_2$ be the curve from example 1.4.5 and let $\pi : F \to X$ be a vector bundle of rank r. We define the *degree* of F to be the number

$$\deg(F) := \deg(c_1(F) \cdot X).$$

As already advertised in example 5.1.10 vector bundles on the elliptic curve X can be described by a single transition function. We will prove this fact in the following lemma:

Lemma 5.3.2
Again, let $X := X_2$ be the curve from example 1.4.5 and let $\pi : F \to X$ be a vector bundle of rank r. Then F is isomorphic to a vector bundle $\pi' : F' \to X$ that admits an open covering $U_1', \dots, U_s'$ and transition maps M_{ij}' such that at most one transition map is non-trivial.

Proof. Let $U_1, \dots, U_s$ be the open covering with transition maps M_{ij} for F according to definition 5.1.5. We may assume that all sets U_i are connected and that for all i, j the intersections $U_i \cap U_j$ are connected as well. Moreover, we may assume that the sets U_i are numbered consecutively as shown in the figure. For simplicity of notation we will consider our indices modulo s.



We can write every map $M_{i,i+1}$, $i = 1, \dots, s$, as

$$M_{i,i+1}(x) = D(\varphi_{i,i+1}^{(1)}, \dots, \varphi_{i,i+1}^{(r)})(x) \odot A_{\sigma_{i,i+1}} =: D_i(x) \odot P_i$$

for some regular invertible functions $\varphi_{i,i+1}^{(k)} \in \mathcal{O}^*(U_i \cap U_{i+1})$ and permutations $\sigma_{i,i+1} \in S_r$. We will show that we can replace successively all the transition maps $M_{i,i+1}$ but one by the constant map $M_{i,i+1}' : U_i \cap U_{i+1} \to G(r) : x \mapsto E$ and the resulting vector bundle F' is isomorphic to F: Choose $j_0 \in \{2, \dots, s\}$. Note that if we are given a regular invertible function $\varphi \in \mathcal{O}^*(U_i \cap U_j)$ there is a unique regular invertible function $\widetilde{\varphi} \in \mathcal{O}^*(U_i)$ such that $\widetilde{\varphi}|_{U_i \cap U_j} = \varphi$. As they are regular invertible functions, too, we can extend in exactly the same way the finite entries of the matrix D_{j_0} along the chain $U_{j_0-1}, U_{j_0-2}, \dots, U_{i+1}$ to any set U_{i+1} for $i \in \{2, \dots, j_0 - 1\}$. By abuse of notation we will denote this continuation of D_{j_0} as well by D_{j_0}. Now, we take $U_i' := U_i$ for all $i = 1, \dots, s$ and

$$M_{i,i+1}'(x) := \begin{cases} P_{j_0} \odot D_{j_0}(x) \odot M_{i,i+1}(x) \odot D_{j_0}(x)^{-1} \odot P_{j_0}^{-1}, & \text{if } i \in \{2, \dots, j_0 - 1\} \\ M_{i,i+1}(x), & \text{if } i \in \{j_0 + 1, \dots, s\}. \end{cases}$$

Moreover, we set $M_{12}'(x) := P_{j_0} \odot D_{j_0}(x) \odot D_1(x) \odot P_1$ and $M_{j_0,j_0+1}'(x) := E$. To check that the vector bundle F' we obtain from this gluing data is isomorphic to F we apply lemma 5.1.16: We set

$$E_i(x) := \begin{cases} D_{j_0}(x) \odot P_{j_0}, & \text{if } i \in \{2, \dots, j_0\} \\ E, & \text{else,} \end{cases}$$

and get

$$\begin{aligned} (D_{j_0} \odot P_{j_0}) \odot (D_1 \odot P_1) &= (D_{j_0} \odot P_{j_0} \odot D_1 \odot P_1) \odot E \\ (D_{j_0} \odot P_{j_0}) \odot (D_2 \odot P_2) &= (D_{j_0} \odot P_{j_0} \odot D_2 \odot P_2 \odot D_{j_0}^{-1} \odot P_{j_0}^{-1}) \odot (D_{j_0} \odot P_{j_0}) \\ &\vdots \quad \vdots \\ E \odot (D_{j_0} \odot P_{j_0}) &= E \odot (D_{j_0} \odot P_{j_0}). \end{aligned}$$

This finishes our proof. □

To classify all vector bundles on our elliptic curve X we give now a non-redundant parametrization of all indecomposable vector bundles on X. Arbitrary vector bundles are then just direct sums of these building blocks.

Theorem 5.3.3 (Vector bundles on elliptic curves)
Let $X := X_2$ be the curve from example 1.4.5. Then the set of indecomposable vector bundles of rank r and degree d is in natural bijection with $\gcd(r,d) \cdot X$, i.e. with points of the curve X stretched to $\gcd(r,d)$ times the original length.

Proof. Let $\pi : F \to X$ be an indecomposable vector bundle of rank r with open covering $U_1, \ldots, U_s$ and transition maps M_{ij} according to definition 5.1.5. Again, we may assume that all sets U_i are connected, that for all i, j the intersections $U_i \cap U_j$ are connected as well and that the sets U_i are numbered consecutively. Moreover, by lemma 5.3.2 we may assume that M_{12} is the only non-trivial transition map. Let $M_{12}(x) = D(\varphi_1, \ldots, \varphi_r)(x) \odot A_\sigma =: D(x) \odot A_\sigma$ for some regular invertible functions $\varphi_1, \ldots, \varphi_r \in \mathcal{O}^*(U_1 \cap U_2)$ and a permutation $\sigma \in S_r$. As F is indecomposable σ must by a single cycle. Hence there exists $\varrho \in S_r$ such that $\varrho\sigma\varrho^{-1} = (12 \ldots r)$. We will apply lemma 5.1.16 to show that we can replace $M_{12}(x)$ by $M'_{12}(x) := A_\varrho \odot D(x) \odot A_{\varrho^{-1}} \odot A_{(12\ldots r)}$ without changing the isomorphism class of F: We set $E_i(x) := A_\varrho$ for all x and all i and obtain

$$\begin{aligned} A_\varrho \odot (D(x) \odot A_\sigma) &= (A_\varrho \odot D(x) \odot A_{\varrho^{-1}} \odot A_{(12\ldots r)}) \odot A_\varrho \\ A_\varrho \odot E &= E \odot A_\varrho \\ \vdots &\quad \vdots \\ A_\varrho \odot E &= E \odot A_\varrho. \end{aligned}$$

Hence we may assume that $\sigma = (12 \ldots r)$. Our next step is to apply lemma 5.1.16 to show that we may replace $D(x) = D(\varphi_1, \ldots, \varphi_r)$ by $D'(x) = D(\varphi', 0, \ldots, 0)$ for some $\varphi' \in \mathcal{O}^*(U_1 \cap U_2)$ without changing the isomorphism class of F. For $i = 1, \ldots, r$ let α_i be the slope of φ_i and let L be the (lattice) length of our curve X. For $i = 2, \ldots, r$ we set $\delta_i := \sum_{j=i}^{r} (j - i + 1) \cdot \alpha_j$. Moreover, we define $\varphi' := \varphi_1 + \ldots + \varphi_r - \delta_2 L$. Note that if we are given a regular invertible function $\psi \in \mathcal{O}^*(U_i \cap U_j)$ there is a unique regular invertible function $\widetilde{\psi} \in \mathcal{O}^*(U_i)$ such that $\widetilde{\varphi}|_{U_i \cap U_j} = \varphi$. Hence we can extend our regular invertible functions $\varphi_1, \ldots, \varphi_r$ along the chain $U_2, U_3, \ldots, U_s, U_1$ to any of the sets $U_1, \ldots, U_s$. Note that on $U_1 \cap U_2$ the extension of φ_i to U_2 and the extension of φ_i to U_1 differ exactly by $\alpha_i L$. We use these continuations to define the maps E_i:

$$E_i(x) := D(\widetilde{\varphi_2} + \ldots + \widetilde{\varphi_r} - \delta_2 L, \widetilde{\varphi_3} + \ldots + \widetilde{\varphi_r} - \delta_3 L, \ldots, \widetilde{\varphi_r} - \delta_r L, 0),$$

where for entries of E_i the map $\widetilde{\varphi_j}$ denotes the continuation of φ_j to U_i. Hence we obtain on $U_1 \cap U_2$:

$$\begin{aligned} & E_2 \odot M_{12} \\ = \; & D(\widetilde{\varphi_2} + \ldots + \widetilde{\varphi_r} - \delta_2 L, \ldots, \widetilde{\varphi_r} - \delta_r L, 0) \odot (D(\varphi_1, \ldots, \varphi_r) \odot A_\sigma) \\ = \; & D(\varphi_2 + \ldots + \varphi_r - \delta_2 L, \ldots, \varphi_r - \delta_r L, 0) \odot (D(\varphi_1, \ldots, \varphi_r) \odot A_\sigma) \\ = \; & D(\varphi_1 + \ldots + \varphi_r - \delta_2 L, \varphi_2 + \ldots + \varphi_r - \delta_3 L, \ldots, \varphi_{r-1} + \varphi_r - \delta_r L, \varphi_r) \odot A_\sigma \end{aligned}$$

and

$$\begin{aligned} & M'_{12} \odot E_1 \\ = \; & (D(\varphi_1 + \ldots + \varphi_r - \delta_2 L, 0, \ldots, 0) \odot A_\sigma) \odot D(\widetilde{\varphi_2} + \ldots + \widetilde{\varphi_r} - \delta_2 L, \ldots, \widetilde{\varphi_r} - \delta_r L, 0) \\ = \; & (D(\varphi_1 + \ldots + \varphi_r - \delta_2 L, 0, \ldots, 0) \odot A_\sigma) \odot D(\varphi_2 + \ldots + \varphi_r - \delta_3 L, \ldots, \varphi_r - \delta_{r-1} L, 0) \\ = \; & D(\varphi_1 + \ldots + \varphi_r - \delta_2 L, \varphi_2 + \ldots + \varphi_r - \delta_3 L, \ldots, \varphi_{r-1} + \varphi_r - \delta_r L, \varphi_r) \odot A_\sigma. \end{aligned}$$

The other conditions are trivially fulfilled as $E_i|_{U_i \cap U_{i+1}} = E_{i+1}|_{U_i \cap U_{i+1}}$ for all $i \neq 1$. Hence we may assume that $M_{12}(x) = D(x) \odot A_\sigma = D(\varphi', 0, \ldots, 0)(x) \odot A_{(12\ldots r)}$. As F is a vector bundle of degree d the affine linear map φ' must have slope $-d$. Thus, the transition map M_{12} is determined by the isomorphism class of F up to translations of φ'. To prove the claim it remains to show that two vector bundles F and F' as above with transition maps $M_{12}(x) = D(\varphi, 0, \ldots, 0)(x) \odot A_{(12\ldots r)}$ and $M'_{12}(x) = D(\varphi + cL, 0, \ldots, 0)(x) \odot A_{(12\ldots r)}$ are isomorphic if and only if c is an integer multiple of $\gcd(r, d)$: By lemma 5.1.16 F and F' are isomorphic if and only if for all $i = 1, \ldots, s$ there exists a map $E_i : U_i \to G(r)$ such that for all i the equation $E_{i+1}(x) \odot M_{i,i+1}(x) = M'_{i,i+1}(x) \odot E_i(x)$ holds for all $x \in U_i \cap U_{i+1}$. As $M_{i,i+1}$ is trivial for all $i \neq 1$ these equations imply $E_i|_{U_i \cap U_{i+1}} = E_{i+1}|_{U_i \cap U_{i+1}}$ for all $i \neq 1$. Hence F and F' are isomorphic if and only if there exist a permutation $\tau \in S_r$ and regular invertible functions $\psi_1, \ldots, \psi_r \in \mathcal{O}^*(U_1 \cap U_2)$ with continuations $\widetilde{\psi_1}, \ldots, \widetilde{\psi_r}$ to all sets $U_1, \ldots, U_s$ along the chain $U_2, U_3, \ldots, U_s, U_1$ such that

$$(D(\widetilde{\psi_1}, \ldots, \widetilde{\psi_r}) \odot A_\tau) \odot (D(\varphi, 0, \ldots, 0) \odot A_\sigma) = (D(\varphi + cL, 0, \ldots, 0) \odot A_\sigma) \odot (D(\widetilde{\psi_1}, \ldots, \widetilde{\psi_r}) \odot A_\tau)$$

holds on $U_1 \cap U_2$. In particular, the last equation implies $A_\tau \odot A_\sigma = A_\sigma \odot A_\tau$ and hence $\tau = \sigma^k$ for some $k \in \mathbb{Z}$. Thus F and F' are isomorphic if and only if there exist $k \in \mathbb{Z}$ and $\psi_1, \ldots, \psi_r$ as above such that

$$D(\widetilde{\psi_1}, \ldots, \widetilde{\psi_k}, \widetilde{\psi_{k+1}} + \varphi, \widetilde{\psi_{k+2}}, \ldots, \widetilde{\psi_r}) \odot A_{\sigma^{k+1}} = D(\varphi + cL + \widetilde{\psi_r}, \widetilde{\psi_1}, \ldots, \widetilde{\psi_{r-1}}) \odot A_{\sigma^{k+1}}.$$

Let α_i be the slope of ψ_i. Then on $U_1 \cap U_2$ the continuation of ψ_i to U_2 and the continuation of ψ_i to U_1 differ exactly by $\alpha_i L$. Hence we obtain the system of equations

$$\begin{array}{lcl} \psi_1 & = & \varphi + cL + \psi_r + \alpha_r L \\ \psi_2 & = & \psi_1 + \alpha_1 L \\ \vdots & & \vdots \\ \psi_k & = & \psi_{k-1} + \alpha_{k-1} L \\ \psi_{k+1} + \varphi & = & \psi_k + \alpha_k L \\ \psi_{k+2} & = & \psi_{k+1} + \alpha_{k+1} L \\ \vdots & & \vdots \\ \psi_r & = & \psi_{r-1} + \alpha_{r-1} L. \end{array}$$

In particular, we can conclude that $\alpha_1 = \ldots = \alpha_k$ and $\alpha_{k+1} = \ldots = \alpha_r$. Hence F and F' are isomorphic if and only if there exist $\alpha_1, \alpha_r, k \in \mathbb{Z}$ such that

$$-c = (r - k) \cdot \alpha_r + k \cdot \alpha_1 \text{ and } \alpha_1 = -d + \alpha_r,$$

or equivalently if and only if there exist $\alpha_r, k \in \mathbb{Z}$ with

$$-c = r\alpha_r - k \cdot d.$$

This finishes the proof. □

Remark 5.3.4
Note that the claim of theorem 5.3.3 coincides with the equivalent result in "classical" algebraic geometry (see [A57, theorem 7]).

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Appendix: Pictures of tropical surfaces

Tropical varieties arise in a natural way as images of algebraic varieties under valuation maps. Let us make this precise:

Definition (Tropicalization of algebraic varieties)
Let K be the field of *Puiseux series with complex coefficients*, i.e.

$$K := \left\{ \sum_{q\in\mathbb{R}} a_q t^q \;\middle|\; \begin{array}{l} a_q \in \mathbb{C} \text{ and } \{q \in \mathbb{R} | a_q \neq 0\} \subseteq \mathbb{R} \text{ is bounded} \\ \text{below and has no accumulation points} \end{array} \right\}$$

with the usual addition and multiplication of power series. This field K is algebraically closed and admits a non-archimedean valuation

$$\text{val} : K^* \longrightarrow \mathbb{R} : \sum_{q\in\mathbb{R}} a_q t^q \mapsto \min\{q \in \mathbb{R} | a_q \neq 0\} \in \mathbb{R}.$$

We use this valuation to define the map

$$\text{Val} : K^n \to \mathbb{T}^n : (x_1, \ldots, x_n) \mapsto (-\text{val}(x_1), \ldots, -\text{val}(x_n)).$$

Let $X \subseteq K^n$ be an algebraic variety. Then the set $\text{Trop}(X) := \text{Val}(X \cap (K^*)^n) \subseteq \mathbb{R}^n$ is called the *tropicalization* of X.

Remark
The tropicalization $\text{Trop}(X)$ of an algebraic variety X carries a natural structure of a tropical variety in $\mathbb{R}^n$. More details on this fact can be found, for example, in [S05].

A special case of this process is the case where X is a hypersurface, i.e. the zero locus of a single polynomial over K. In this situation it is much easier to describe the tropicalization of X.

Definition (Tropicalization of polynomials)
Let $f = \sum_{i_1,\ldots,i_n} a_{i_1\ldots i_n} z_1^{i_1} \cdots z_n^{i_n} \in K[z_1, \ldots, z_n]$ be a polynomial over the field K. Then we call

$$\text{Trop}(f) := \max_{i_1,\ldots,i_n} \{-\text{val}(a_{i_1\ldots i_n}) + i_1 x_1 + \ldots + i_n x_n\}$$

the *tropicalization* of f. Obviously, the function $\text{Trop}(f) : \mathbb{R}^n \to \mathbb{R}$ is piecewise integer linear. We denote by $V(\text{Trop}(f))$ the *corner locus* of $\text{Trop}(f)$, i.e. the set of all points $x \in \mathbb{R}^n$ where $\text{Trop}(f)$ is not differentiable.

With these definitions we get the following statement:

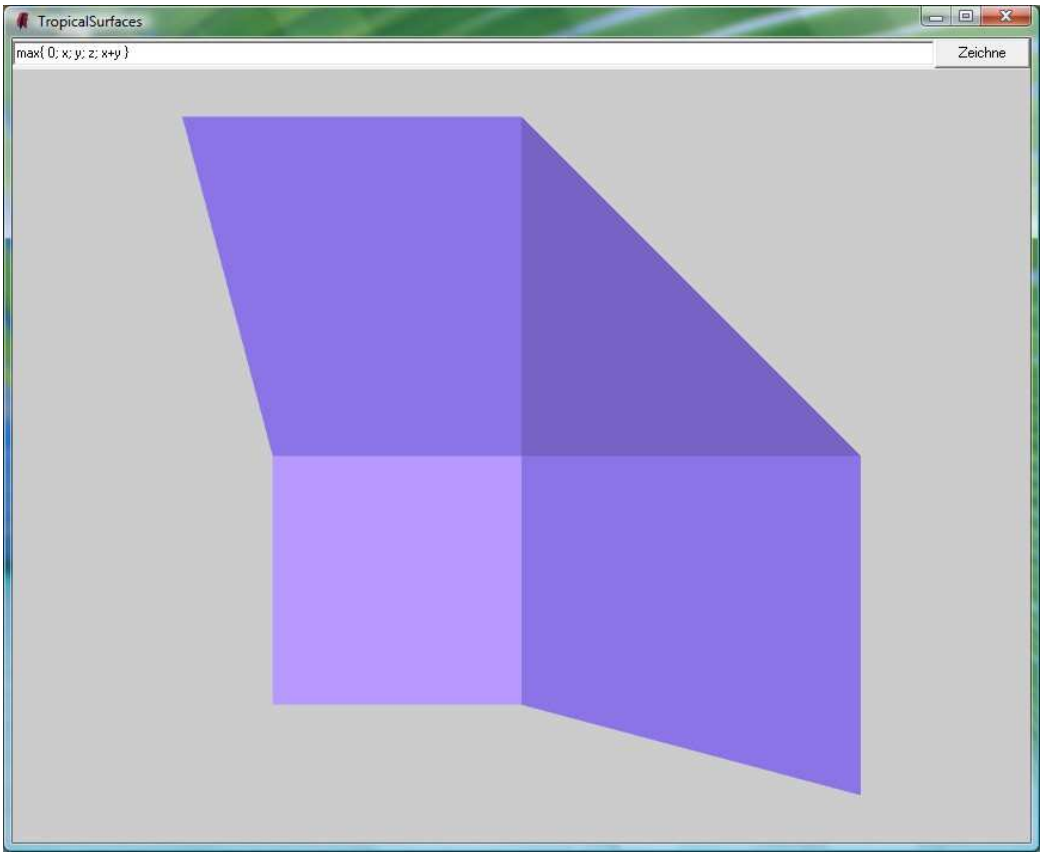
Proposition
Let $f \in K[z_1, \ldots, z_n]$ be a polynomial over K and $V(f)$ its zero locus. Then the following equation holds:

$$\mathrm{Trop}(V(f)) = V(\mathrm{Trop}(f)).$$

Proof. A proof of this fact can be found in [EKL04]. □

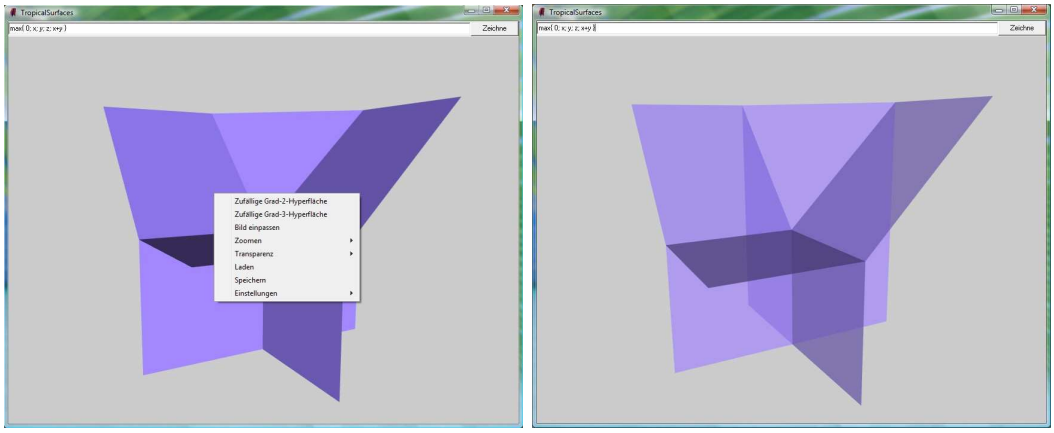
In fact, every tropical hypersurface, i.e. every tropical variety of codimension one in $\mathbb{R}^n$, arises as a tropicalization of a hypersurface $V(f) \subseteq K^n$ (see for example [M05]) and every tropical cycle of codimension one in $\mathbb{R}^n$ is a difference of tropical hypersurfaces. As the corner locus of a tropical polynomial is relatively easy to calculate, tropical varieties of the form $V(\mathrm{Trop}(f))$ are an important source of examples in tropical geometry. For some purposes the easiest possible case of curves in $\mathbb{R}^2$ cannot provide sufficient examples and one has to deal with hypersurfaces in 3-space. This was the reason for creating a computer program to calculate the corner locus of a tropical polynomial in 3 variables: TropicalSurfaces. The program is written in Delphi and binary versions as well as the source code are available on `http://www.mathematik.uni-kl.de/~allermann/software.html`.

The usage of TropicalSurfaces is easy: Type a tropical polynomial in the text box and press the button labeled with "Zeichne".

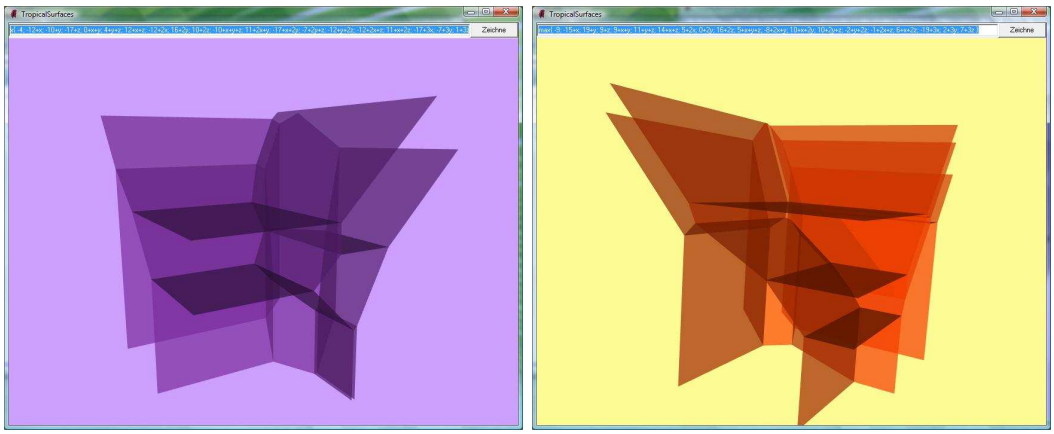


A picture of the corner locus of the given polynomial is drawn. Click with the left mouse button into the drawing area and move the cursor to rotate the picture in 3-space. If you press the right mouse button within the picture, a context menu appears. Here you can

- zoom in and out,
- change transparency settings,
- save the displayed tropical surface as a binary or text file,
- load tropical surfaces from binary or text files,
- export the displayed picture as a pixel graphic, POV-Ray file or scalable vector graphic,
- change other settings, e.g. the color scheme and the clipping area.



Moreover, you can create random polynomials of degrees 2 and 3 via the context menu and display the corresponding tropical surfaces.



Wissenschaftlicher Werdegang

25.07.1981	geboren in Mainz
2001	Abitur am Gymnasium am Römerkastell, Alzey
seit 10/2002	Studium der Mathematik an der TU Kaiserslautern
09/2006	Diplom in Mathematik, TU Kaiserslautern
seit 10/2006	Doktorand bei Prof. Dr. Andreas Gathmann, TU Kaiserslautern

Curriculum Vitae

07/25/1981	born in Mainz, Germany
2001	Abitur at the Gymnasium am Römerkastell, Alzey, Germany
since 10/2002	Study of Mathematics at the TU Kaiserslautern, Germany
09/2006	Diplom in Mathematics, TU Kaiserslautern
since 10/2006	Ph.D. studies with Prof. Dr. Andreas Gathmann, TU Kaiserslautern