

THE NUMBERS OF TROPICAL PLANE CURVES THROUGH POINTS IN GENERAL POSITION

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ABSTRACT. We show that the number of tropical curves of given genus and degree through some given general points in the plane does not depend on the position of the points. In the case when the degree of the curves contains only primitive integral vectors this statement has been known for a while now, but the only known proof was indirect with the help of Mikhalkin's Correspondence Theorem that translates this question into the well-known fact that the numbers of *complex* curves in a toric surface through some given points do not depend on the position of the points. This paper presents a direct proof entirely within tropical geometry that is in addition applicable to arbitrary degree of the curves.

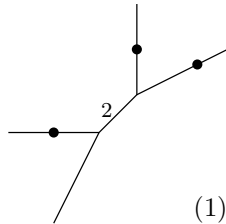
1. INTRODUCTION

Tropical geometry recently attracted a lot of attention. One reason for this is the possibility to relate *complex* enumerative geometry to the (hopefully simpler) *tropical* enumerative geometry. For example, Mikhalkin has proven the so-called "Correspondence Theorem" which asserts that the numbers of complex curves (of given genus and homology class) in toric surfaces through some given points are equal to the numbers of certain plane tropical curves through the same number of given points [Mi1]. Furthermore, Mikhalkin gave a nice purely combinatorial way of computing these numbers of tropical curves. Nishinou and Siebert were able to prove the same for rational curves in (higher-dimensional) complete toric varieties [NS].

Of course the numbers of complex curves in a toric surface through some given general points do not depend on the position of the points. It is therefore a corollary of the Correspondence Theorem that the corresponding numbers of plane tropical curves through some given general points cannot depend on the position of the points either (see remark 4.9 for details). From a purely tropical point of view this statement is far from being obvious however. It is the main result of this paper to prove this statement within the framework of tropical geometry. In addition, our result will be more general than what is known so far since not all numbers of plane tropical curves have been related to numbers of complex curves yet: for some numbers that should correspond to relative Gromov-Witten invariants (i.e. curves with fixed local multiplicities to a given divisor) there is no Correspondence Theorem yet (see remark 4.10 for details).

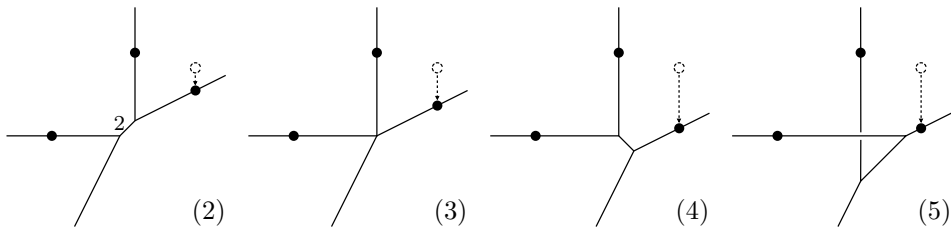
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Let us briefly describe the tropical set-up. A plane tropical curve can roughly be thought of as a weighted graph in the real plane whose edges are (possibly unbounded) line segments of rational slope (for a precise definition see section 2). These graphs must satisfy the so-called *balancing condition*: at each vertex the weighted sum of the primitive integral vectors along the edges starting from this vertex must be zero. For example, the following picture shows a tropical curve through 3 given points in the plane:



(the balancing condition at e.g. the left vertex is $2 \cdot (1,1) + (-1,0) + (-1,-2) = (0,0)$). The *degree* of a tropical curve is the data of the slopes of the unbounded edges together with their weights; we have to fix this for enumerative questions. It is easily seen that the above tropical curve (1) cannot be deformed (with its degree, i.e. the slopes of its unbounded edges fixed) to another tropical curve that still passes through the three given points.

Let us now move one of the three given points, say the rightmost one down. This has the effect of making the internal edge shorter as in (2) below until it finally disappears and the graph acquires a vertex of valence 4 as in (3). But if we move the point further down something strange happens: our tropical curve now deforms into *two different* curves (4) and (5) that pass through the given points. So a naive count would give the result that the number of tropical curves through the given points *does depend* on the position of the points, as we have deformed one curve (1) into two curves (4) and (5).



The solution to this problem is that tropical curves have to be counted with multiplicities (as it is in fact required by the Correspondence Theorem). In contrast to the complex case these multiplicities can be greater than 1 even for points in general position. In the above example they turn out to be 4 for (1), 3 for (4), and 1 for (5), so that the total weighted sum before and after the deformation is still the same (for the precise definition of the multiplicities see definition 4.6).

In fact, this example shows already the general idea of our proof. In the space \mathbb{R}^{2n} of n points in the plane we consider the subspace of configurations at which the topology of the tropical curves passing through them changes, as e.g. in (3) above. This is a space of real codimension 1 in \mathbb{R}^{2n} (in fact it is a union of convex polyhedra

of dimension $2n - 1$). Whenever we pass such a “wall” in \mathbb{R}^{2n} we have to show that the weighted sum of tropical curves through the points before and after crossing the wall is the same. Once we have done this we know that we can go from any configuration of points to any other along a path in \mathbb{R}^{2n} , and that the corresponding (weighted) count of tropical curves through these points will remain constant along the path (in particular when crossing the walls). In the whole analysis we can of course neglect the “boundaries of the walls” since they have codimension 2 in \mathbb{R}^{2n} and can be avoided by the path. This amounts to only consider configurations of points in sufficiently general position.

This paper is organized as follows. In section 2 we give a rigorous definition of tropical curves with marked points. We then build up suitable moduli spaces of such marked curves in section 3. Section 4 studies the “evaluation map” that maps an n -marked tropical curve to its configuration of points in \mathbb{R}^{2n} . The main work of our paper, i.e. the “wall crossing analysis” described above, is contained in the proof of our main theorem 4.8.

Finally we should say clearly that our paper gives a rather ad hoc solution to the problem described above. From a theoretical point of view it would certainly be more desirable to solve the problem in the same way as in complex geometry: develop a tropical version of intersection theory and define the numbers of curves through given points as intersection products on suitable “moduli spaces of tropical stable maps”. Different collections of points should then just correspond to “rationally equivalent cycles” in \mathbb{R}^{2n} , from which it should follow by general principles that the resulting intersection products are the same. We hope that the ideas of our paper will be useful to set up such a theory.

2. PLANE TROPICAL CURVES

Let us start by defining tropical curves. Our definition differs slightly from the ones given in [Mi1] and [NS]. The reason for this is that [Mi1] and [NS] consider mostly “generic tropical curves”, i.e. tropical curves that pass through given points (or subspaces) in “general position”. In our paper however the points will be in slightly less general position, leading to “degenerate” tropical curves. As a consequence, we have to be more careful in the definition of (possibly degenerate) tropical curves.

As in the classical case of complex geometry there are two ways to describe curves in an ambient space: either as certain one-dimensional *subspaces* or as abstract curves together with a *map* to the ambient space. While these two notions describe the same objects for generic curves, they differ e.g. in the cases where the curves have multiple components (resp. the map is not generically injective). Both viewpoints have their advantages, and in fact our definition of tropical curves in this paper is based on a mixture of these two ideas. As a consequence the resulting moduli spaces will be quite well suited for our computations, but unfortunately rather unsatisfactory from a purely theoretic point of view.

Tropical curves are based on graphs. The graphs we will need are “graphs with multiple and unbounded edges allowed”. To make this precise and set up the notation we will give a rigorous definition.

Definition 2.1

A *graph* is a tuple $\Gamma = (\Gamma^0, \Gamma', \partial, j)$ where

- Γ^0 and Γ' are finite sets (whose elements are called *vertices* and *flags*, respectively);
- $\partial : \Gamma' \rightarrow \Gamma^0$ is a map (called the *boundary map*);
- $j : \Gamma' \rightarrow \Gamma'$ is a map with $j \circ j = \text{id}$ (called the *glueing map*).

By the condition $j \circ j = \text{id}$ the relation

$$F_1 \sim F_2 :\iff F_1 = F_2 \text{ or } F_1 = j(F_2)$$

for $F_1, F_2 \in \Gamma'$ is an equivalence relation on the Γ' whose equivalence classes all consist of one or two elements. The equivalence class $[F]$ of a flag $F \in \Gamma'$ is called an *edge* of Γ . It is called an *unbounded edge* or *end* if $[F] = \{F\}$ has only one element, and an *internal edge* otherwise. We denote the sets of edges, unbounded edges, and internal edges by Γ^1 , Γ_∞^1 , and Γ_0^1 , respectively. For a vertex $V \in \Gamma^0$ we denote by $\text{val } V := \#\{F \in \Gamma'; \partial F = V\} \in \mathbb{N}$ the *valence* of V .

The *topological model* $|\Gamma|$ of a graph Γ is obtained by starting with a disjoint union of a point $*_V$ for every vertex $V \in \Gamma^0$ and a semi-open interval $[0, 1)_F$ for every flag $F \in \Gamma'$, and glueing

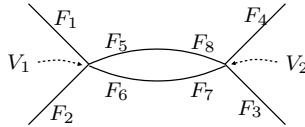
- the point $0 \in [0, 1)_F$ to the point $*_{\partial F}$ for every flag $F \in \Gamma'$;
- the intervals $[0, 1)_{F_1}$ and $[0, 1)_{F_2}$ on the common open subset $(0, 1)$ along the map $t \mapsto 1 - t$ for every pair of flags F_1, F_2 with $F_1 \sim F_2$ and $F_1 \neq F_2$.

A graph is called *connected* if its topological model is. The *genus* of a connected graph Γ is defined to be $g(\Gamma) = \dim H^1(|\Gamma|, \mathbb{R})$.

In the topological model every vertex $V \in \Gamma^0$ corresponds to a point $*_V \in |\Gamma|$, every flag $F \in \Gamma'$ corresponds to a semi-open interval $[0, 1) \subset |\Gamma|$, and every edge $E \in \Gamma^1$ (without its endpoints) corresponds to an open interval $(0, 1) \subset |\Gamma|$. By abuse of notation we will denote these points and intervals simply by $V \in |\Gamma|$, $F \subset |\Gamma|$, and $E \subset |\Gamma|$, respectively.

Example 2.2

The following picture shows (the topological model of) a graph Γ of genus 1 with 2 vertices $\Gamma^0 = \{V_1, V_2\}$ (of valence 4 each) and 8 flags $\Gamma' = \{F_1, \dots, F_8\}$:



The graph has 6 edges: four unbounded edges $E_i = \{F_i\}$ for $i = 1, \dots, 4$, and two internal edges $E_5 = \{F_5, F_8\}$, $E_6 = \{F_6, F_7\}$.

We are now ready to give the definition of a (plane) tropical curve.

Definition 2.3

A *tropical curve* is a triple $C = (\Gamma, \omega, h)$, where

- Γ is a connected graph;

- (b) $\omega : \Gamma^1 \rightarrow \mathbb{N}_{>0}$ is a map (called the *weight function*);
- (c) $h : |\Gamma| \rightarrow \mathbb{R}^2$ is a continuous proper map that embeds every flag $F \subset |\Gamma|$ into a (unique) affine line in \mathbb{R}^2 with rational slope. Moreover, if we denote by $u(F) \in \mathbb{Z}^2$ the primitive integral vector that starts at $h(\partial F)$ and points in the direction of $h(F)$ we set $v(F) := \omega([F]) \cdot u(F)$ and require that at every vertex $V \in \Gamma^0$
- the vectors $\{v(F); F \in \Gamma^1 \text{ with } \partial F = V\}$ span \mathbb{R}^2 as a vector space; and
 - the *balancing condition*

$$\sum_{F \in \Gamma^1: \partial F = V} v(F) = 0$$

holds.

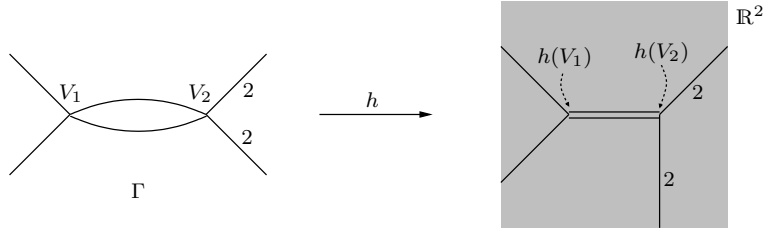
The genus $g(C)$ of a tropical curve $C = (\Gamma, \omega, h)$ is defined to be the genus of its underlying graph Γ . Two tropical curves (Γ, ω, h) and $(\tilde{\Gamma}, \tilde{\omega}, \tilde{h})$ are called isomorphic if there is a homeomorphism $\varphi : \Gamma \rightarrow \tilde{\Gamma}$ such that $\tilde{h} \circ \varphi = h$ and $\omega(E) = \tilde{\omega}(\varphi(E))$ for all edges $E \in \Gamma^1$.

Remark 2.4

The condition in definition 2.3 (c) that h is proper means precisely that the unbounded edges of Γ map to unbounded rays in \mathbb{R}^2 . Note that the requirement that the flags around every vertex span \mathbb{R}^2 imply together with the balancing condition that every vertex has valence at least 3. If two flags around a vertex are mapped to the same line in \mathbb{R}^2 then this vertex must have valence at least 4.

Example 2.5

The following picture shows an example of a tropical curve $C = (\Gamma, \omega, h)$ of genus 1 based on the graph Γ of example 2.2. We have labeled the edges $E \in \Gamma^1$ by their weight $\omega(E)$ and left out this label if the weight is 1.



In this example the two internal edges are mapped to the same image line. Note that the images of the flags around both vertices span \mathbb{R}^2 as a vector space as required by definition 2.3 (c). Although h embeds every flag in \mathbb{R}^2 it is not a global embedding. The balancing conditions at the two vertices are

$$(-1, 1) + (-1, -1) + (1, 0) + (1, 0) = (0, 0) \text{ at } V_1$$

$$\text{and } (-1, 0) + (-1, 0) + 2 \cdot (0, -1) + 2 \cdot (1, 1) = (0, 0) \text{ at } V_2.$$

In the rest of this paper when we draw tropical curves we will for simplicity only draw the image $h(\Gamma)$, with the edges labeled by their weight.

An important notion is that of the degree of a tropical curve:

Definition 2.6

Let G be the free abelian semigroup generated by $\mathbb{Z}^2 \setminus \{(0, 0)\}$. We denote the addition in G by the symbol \oplus to distinguish it from the addition in \mathbb{Z}^2 . For an element $\Delta = u_1 \oplus \cdots \oplus u_n$ that is a sum of n vectors $u_1, \dots, u_n \in \mathbb{Z}^2$ we set $\#\Delta := n$.

If $C = (\Gamma, \omega, h)$ is a tropical curve we define the *degree* of C to be

$$\deg C := \bigoplus_{\{F\} \in \Gamma_\infty^1} v(F)$$

where $v(F)$ is as in definition 2.3 (c).

Example 2.7

The tropical curve C of example 2.5 has degree

$$\deg C = (-1, 1) \oplus (-1, -1) \oplus (0, -2) \oplus (2, 2).$$

Note that for every tropical curve $C = (\Gamma, \omega, h)$ we have $\#\deg C = \#\Gamma_\infty^1$ by definition. Moreover, if $\deg C = v_1 \oplus \cdots \oplus v_n$ with $v_1, \dots, v_n \in \mathbb{Z}^2$ we must have $v_1 + \cdots + v_n = 0$ by the balancing condition.

To study tropical curves through given points in the plane we now have to consider curves with marked points on them:

Definition 2.8

For $n \in \mathbb{N}$ we say that an *n -marked tropical curve* is a tuple (C, x_1, \dots, x_n) where

- (a) $C = (\Gamma, \omega, h)$ is a tropical curve;
- (b) $x_1, \dots, x_n \in |\Gamma|$ are points with the following property: for every $i = 1, \dots, n$ there is a flag $F_i \subset |\Gamma|$ with $x_i \in F_i$ such that $|\Gamma| \setminus ([F_1] \cup \cdots \cup [F_n])$ contains no loops and no connected components with more than one unbounded end. (Here as usual $[F]$ denotes the (open) edge corresponding to a flag F . Note that some of the points x_1, \dots, x_n may be vertices, and it is allowed that some of them coincide.)

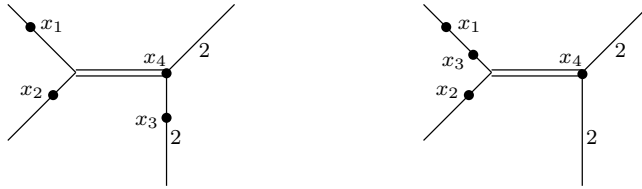
An isomorphism $(C, x_1, \dots, x_n) \rightarrow (\tilde{C}, \tilde{x}_1, \dots, \tilde{x}_n)$ is an isomorphism $C \rightarrow \tilde{C}$ of tropical curves taking x_i to \tilde{x}_i for all $i = 1, \dots, n$.

Remark 2.9

Let $C = (\Gamma, \omega, h)$ be a tropical curve, and let $x_1, \dots, x_n \in C$ be points none of which lies on a vertex of Γ . Then in part (b) of definition 2.8 the edges $[F_i]$ for the marked points x_i are uniquely defined: they are simply the edges on which the marked points lie. So in this case (C, x_1, \dots, x_n) is a marked tropical curve if and only if $|\Gamma| \setminus \{x_1, \dots, x_n\}$ contains no loops and no connected components with more than one unbounded end.

Example 2.10

In the picture below the left choice of 4 marked points makes the tropical curve of example 2.5 into a marked tropical curve, whereas the right one does not.



To see that condition (b) of definition 2.8 is satisfied for the curve on the left but not for the one on the right note first that (analogously to remark 2.9) the edge $[F_i]$ corresponding to the point x_i is uniquely determined for $i = 1, 2, 3$ in both cases: it is simply the edge on which x_i lies. For the flag F_4 however we have four choices, namely the four flags starting at the vertex x_4 . For the curve on the left condition (b) of definition 2.8 is satisfied if we pick for F_4 one of the flags pointing to the left. For the curve on the right however any choice of flag F_4 would lead to a connected component of the space $|\Gamma| \setminus ([F_1] \cup \dots \cup [F_4])$ that has either two unbounded ends (if we pick a flag pointing to the left) or a loop (if we pick one of the other flags).

Remark 2.11

It is easy to see that there is always a certain lower bound on the number n of marked points that we need to make a given tropical curve of genus g and degree Δ into a marked tropical curve. Recall that we have to make $|\Gamma|$ into a space that has no loops and no connected component with more than one unbounded end by removing one (open) edge for every marked point. This means that we need at least g marked points to break all loops in the graph and at least $\#\Delta - 1$ more marked points to remove or separate all the unbounded ends. So we must always have the inequality $n \geq \#\Delta + g - 1$ for an n -marked tropical curve of genus g and degree Δ .

Remark 2.12

The reason for condition (b) in definition 2.8 is that it ensures for sufficiently generic tropical curves that there are no deformations of the curve if we fix the images $h(x_i)$ of the marked points in \mathbb{R}^2 (see proposition 4.2 for a precise statement). For example, in example 2.10 the curve on the left cannot be deformed with fixed images of the marked points, whereas in the curve on the right the vertex on which x_4 lies can be moved to the right (thereby increasing the length of the “double edge”, moving the two edges of weight 2 horizontally, and letting x_4 move onto one of the middle edges).

For more special tropical curves however it is unlikely that definition 2.8 does the “right thing”. So if one wants to build up a general theory of marked tropical curves then definition 2.8 will probably not be the “correct” one.

3. MODULI SPACES OF TROPICAL CURVES

We are now ready to define the moduli spaces of marked tropical curves that will be our main object of study. For the rest of this paper we fix the following set-up: we choose a non-negative integer g and a degree $\Delta \in G$, and we set $n := \#\Delta + g - 1$. This number n (which has already appeared in remark 2.11) will turn out to be the required number of points through which we get a finite non-zero number of

tropical curves of genus g and degree Δ . We will therefore only consider tropical curves with precisely this number n of marked points.

Definition 3.1

The *moduli space* $\overline{\mathcal{M}}_{g,\Delta}$ of marked tropical curves of genus g and degree Δ is defined to be the set of all n -marked tropical curves (C, x_1, \dots, x_n) of degree Δ modulo isomorphisms such that $n = \#\Delta + g - 1$, $g(C) \leq g$, and at least $g - g(C)$ of the marked points lie on vertices.

Remark 3.2

We have to allow curves of lower genus in our definition to ensure that the moduli spaces are “closed under degenerations” (see proposition 3.12). For an example consider again the left curve of example 2.10. If we shrink the length of the double edge to zero then the resulting curve will be of genus 0 (since we do not allow edges of length zero). Note also that by definition 2.8 (b) there must be a marked point on every loop of a tropical curve. So if we reduce the genus of a curve by a degeneration process that shrinks k loops to a point as above then this will necessarily result in k marked points lying on vertices after the degeneration. This is the reason for the condition on the marked points in definition 3.1.

Let us now study the structure of these moduli spaces. The first thing we do is to sort the elements of $\overline{\mathcal{M}}_{g,\Delta}$ according to their “combinatorial type”:

Definition 3.3

Let $(C, x_1, \dots, x_n) \in \overline{\mathcal{M}}_{g,\Delta}$ be a marked tropical curve, where $C = (\Gamma, \omega, h)$. For every $i = 1, \dots, n$ we denote by $s(i) \in \Gamma^0 \amalg \Gamma^1$ the unique stratum (vertex or edge) of Γ on which x_i lies. The *combinatorial type* of (C, x_1, \dots, x_n) is the data $\alpha = (\Gamma, \omega, u, s)$ (where $u : \Gamma' \rightarrow \mathbb{Z}^2 \setminus \{(0, 0)\}$ was introduced in definition 2.3 (c)), i.e. it is given by the weighted graph, the direction of every edge of the graph in \mathbb{R}^2 , and the information on which edges (or vertices) the marked points lie. The *codimension* of such a combinatorial type α is defined to be

$$\begin{aligned} \text{codim } \alpha := & \sum_{V \in \Gamma^0} (\text{val } V - 3) \\ & + (g - g(C)) \\ & + \#\{i = 1, \dots, n; s(i) \in \Gamma^0\}. \end{aligned}$$

We denote by $\mathcal{M}_{g,\Delta}^\alpha$ the subset of $\overline{\mathcal{M}}_{g,\Delta}$ that corresponds to marked tropical curves of combinatorial type α . (We will see in proposition 3.9 and example 3.10 that $\text{codim } \alpha$ is closely related, but not always equal to the codimension of $\mathcal{M}_{g,\Delta}^\alpha$ in $\overline{\mathcal{M}}_{g,\Delta}$.)

Remark 3.4

Note that the codimension of a combinatorial type as defined above is visibly the sum of three non-negative integers. In particular it is always a non-negative integer itself, and it is zero if and only if the curves of this type have only vertices of valence 3, have genus *equal* to g , and have no marked points on vertices. If this is the case then we see moreover by the argument of remark 2.11 that n is the *minimal* number of marked points needed to satisfy the condition of remark 2.9.

This means that then $|\Gamma| \setminus \{x_1, \dots, x_n\}$ has exactly $\#\Delta$ connected components, and that each of these components contains exactly one unbounded end.

Remark 3.5

For future computations we will need another description of the codimension of a combinatorial type α . To derive it, note that the genus of a graph Γ is given by $g(C) = \#\Gamma_0^1 - \#\Gamma^0 + 1$. Moreover, counting the vertices of the graph leads to the equation $\#\Gamma_\infty^1 + 2\#\Gamma_0^1 = \sum_{V \in \Gamma^0} \text{val } V$. Combining these two equations and using the relations $\#\Gamma_\infty^1 = \#\Delta$ and $n = \#\Delta + g - 1$ we arrive at the result

$$\text{codim } \alpha = 2n - 2 + 2g(C) - \#\Gamma_0^1 - \#\{i; s(i) \in \Gamma^1\}.$$

Remark 3.6

Let α be a combinatorial type occurring in $\overline{\mathcal{M}}_{g,\Delta}$ that corresponds to tropical curves of genus strictly less than g . Then there must be at least one marked point on a vertex by definition 3.1. In particular we must then have $\text{codim } \alpha \geq 2$ by definition 3.3.

Proposition 3.7

There are only finitely many combinatorial types occurring in a given moduli space $\overline{\mathcal{M}}_{g,\Delta}$.

Proof:

This follows essentially from [NS] proposition 2.1. There it is shown that the number of possibilities for (Γ, ω, u) is finite for a given degree Δ and genus of the curves. Furthermore, the genus of the curves in $\overline{\mathcal{M}}_{g,\Delta}$ is bounded by g , and there are only finitely many possibilities for s . \square

We will now study the spaces $\mathcal{M}_{g,\Delta}^\alpha$ for fixed combinatorial types separately. Of course the idea of the codimension of a combinatorial type α is that it should correspond to the codimension of its moduli space $\mathcal{M}_{g,\Delta}^\alpha$ in the whole space $\overline{\mathcal{M}}_{g,\Delta}$. Unfortunately this is not true in general (see example 3.10). It is true however in codimensions up to 2 (which will be sufficient for our purposes) *except* for one special case:

Definition 3.8

We say that a combinatorial type $\alpha = (\Gamma, \omega, u, s)$ is *exceptional* if $\text{codim } \alpha = 2$ and Γ has two vertices of valence 4 that are joined by two edges. In other words we can say that α is exceptional if and only if it contains the picture of example 2.2 as a “subgraph”, all other vertices are of valence 3, the genus of the graph is maximal, and no marked points lie on vertices.

Proposition 3.9

For every combinatorial type α occurring in $\overline{\mathcal{M}}_{g,\Delta}$ the space $\mathcal{M}_{g,\Delta}^\alpha$ is naturally an (unbounded) open convex polyhedron in a real affine space, i.e. a subset of a real affine space given by finitely many linear strict inequalities. For its dimension we have

$$\dim \mathcal{M}_{g,\Delta}^\alpha \begin{cases} = 2n & \text{if } \text{codim } \alpha = 0; \\ = 2n - 1 & \text{if } \text{codim } \alpha = 1 \text{ or } \alpha \text{ is exceptional}; \\ \leq 2n - 2 & \text{otherwise.} \end{cases}$$

Proof:

The proof of this proposition is based on the ideas of [Mi1] proposition 2.23. Our result is similar to [Shu] lemma 2.2 but differs in that we consider parametrized and not embedded tropical curves (so that we cannot apply Shustin's technique of Newton polyhedra).

Fixing the combinatorial type of a tropical curve $C = (\Gamma, \omega, h)$ simply means that we fix the weighted graph, the slopes of the images of all edges in \mathbb{R}^2 , and the edges (resp. vertices) on which the marked points lie. In contrast, the combinatorial type does not fix the position of the curve in the plane, the lengths of the images of the internal edges, and the position of the marked points (that do not lie on vertices) on their respective edges. So $\mathcal{M}_{g,\Delta}^\alpha$ can be thought of as a subset of the real affine space A whose coordinates are

- (a) the position $h(V) \in \mathbb{R}^2$ of a fixed "root vertex" $V \in \Gamma^0$;
- (b) the lengths of the images $h(E) \subset \mathbb{R}^2$ of the internal edges $E \in \Gamma_0^1$;
- (c) for every marked point x_i lying on an edge $s(i) = E_i \in \Gamma^1$ its distance in \mathbb{R}^2 from a neighboring vertex, i.e. the number $|h(x_i) - h(\partial F_i)|$ for a fixed flag $F_i \in \Gamma'$ with $F_i \in E_i$.

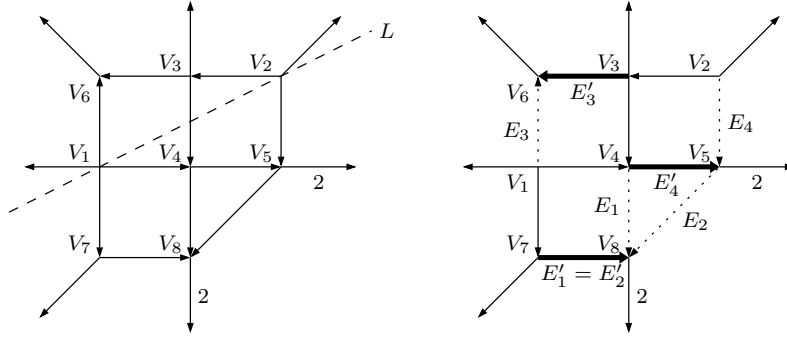
Note that a different choice of root vertex in (a) or flag in (c) would simply correspond to an affine isomorphism. Therefore $\mathcal{M}_{g,\Delta}^\alpha$ is naturally a subset of a real affine space A . Moreover, by remark 3.5 the dimension of A is

$$\dim A = 2 + \#\Gamma_0^1 + \#\{i; s(i) \in \Gamma^1\} = 2n + 2g(C) - \text{codim } \alpha.$$

Note however that the affine coordinates described above can only be chosen independently if $g(C) = 0$. Otherwise, each loop in the graph Γ leads to two linear equations on the lengths of its internal edges describing the condition that the image of this loop closes up in the plane \mathbb{R}^2 . As it suffices to consider these conditions for a chosen set of generators of $H^1(\Gamma, \mathbb{Z})$ we arrive at a total of $2g(C)$ linear conditions. So we would expect $\mathcal{M}_{g,\Delta}^\alpha$ to have dimension $\dim A - 2g(C) = 2n - \text{codim } \alpha$. This is in general only a lower bound however since the $2g(C)$ conditions above need not be independent (see e.g. example 3.10). The main work in the proof of our proposition is now to give a good estimate on how many of these conditions are independent.

To do so, pick a fixed tropical curve in $\mathcal{M}_{g,\Delta}^\alpha$ and a vertex $V_1 \in \Gamma^0$ of maximal valence. Let $L \subset \mathbb{R}^2$ be a fixed line through its image point $h(V_1)$. (In some cases we will specify later which line is to be picked here. For the moment all our constructions work for an arbitrary line L .)

Order the vertices of Γ starting with V_1 so that the distance of their image points from L is increasing (if for some vertices this distance is equal we order them arbitrarily). Orient the edges so that they point from the lower to the higher vertex. The unbounded edges are always oriented so that they point from its vertex to infinity. Note that the balancing condition of definition 2.3 (c) implies that every vertex has at least one adjacent edge pointing in a direction of strictly increasing distance from L . So every vertex must have at least one adjacent edge which is oriented away from it. (This is in fact the only property of the chosen order and orientations that we will need.) An example is shown in the picture below on the left:



We will now distinguish recursively $2g(C)$ edges $E_1, \dots, E_{g(C)}, E'_1, \dots, E'_{g(C)}$ as follows. For $i = 1, \dots, g(C)$ we let E_i be an (internal) edge contained in a loop of $\Gamma \setminus \{E_1, \dots, E_{i-1}\}$ such that the vertex that this edge points to is the highest possible (in the chosen ordering). Then $T := \Gamma \setminus \{E_1, \dots, E_{g(C)}\}$ is a maximal tree in Γ . In particular, for all $i = 1, \dots, g(C)$ the edge E_i closes a unique loop Γ_i in $T \cup \{E_i\} \subset \Gamma$. We let E'_i be the unique (internal) edge of T that is contained in Γ_i and adjacent to the vertex that E_i points to. As an example, in the picture above on the right (where $g(C) = 4$) we have drawn the edges E_1, \dots, E_4 as dotted arrows and E'_1, \dots, E'_4 in bold. The maximal tree T consists exactly of all solid lines. Note that by construction the edges $E_1, \dots, E_{g(C)}$ are all distinct and different from the $E'_1, \dots, E'_{g(C)}$. It may happen however that not all edges $E'_1, \dots, E'_{g(C)}$ are distinct. Note that by construction E_i and E'_i always point to the same vertex, namely to the highest vertex contained in the loop Γ_i .

We will now define a set of *conditional edges* by starting with the $2g(C)$ edges $E_1, \dots, E_{g(C)}, E'_1, \dots, E'_{g(C)}$ and removing some of these edges by applying the following rules at each vertex V :

- (i) if there is at least one edge E'_i pointing to V that is not parallel to its corresponding edge E_i then we keep the edge E'_i with this property such that i is maximal and remove all other edges $E'_1, \dots, E'_{g(C)}$ that point to V ;
- (ii) if there is no such edge then we remove all edges $E'_1, \dots, E'_{g(C)}$ that point to V .

Note that all edges $E_1, \dots, E_{g(C)}$ will end up to be conditional edges, and that all conditional edges will be distinct. In the example above we end up with the 7 conditional edges $E_1, E_2, E_3, E_4, E'_2, E'_3, E'_4$.

We claim that for any (marked) tropical curve in $\mathcal{M}_{g,\Delta}^\alpha$ the lengths of its conditional edges are determined uniquely in terms of the lengths of all other edges. To see this apply the following procedure recursively for $i = g(C), \dots, 1$: assume that we know already the lengths of all edges $E_{i+1}, \dots, E_{g(C)}, E'_{i+1}, \dots, E'_{g(C)}$ as well as of all unconditional edges. Then by construction the only edges in the loop Γ_i whose lengths are not yet known can be E_i and E'_i (if V is the vertex that E_i and E'_i point to then all other edges in Γ_i must point to smaller vertices than V whereas all edges E_j and E'_j with $j < i$ point to vertices greater than or equal to V). If E_i and E'_i are not parallel then the condition that Γ_i closes up in \mathbb{R}^2 determines both their lengths uniquely. Otherwise E'_i is an unconditional edge by (ii), and E_i is again determined uniquely by the condition that Γ_i closes up.

It follows that the dimension of $\mathcal{M}_{g,\Delta}^\alpha$ is at most equal to $\dim A$ minus the number of conditional edges. So let us determine how many conditional edges there are. Note that when going from the $2g(C)$ edges $E_1, \dots, E_{g(C)}, E'_1, \dots, E'_{g(C)}$ to the conditional edges we removed at most $\text{val } V - 3$ edges at each vertex V : in case (i) above there is at least one edge pointing away from V and one pair $\{E_i, E'_i\}$ that we do not remove. In case (ii), if we remove any edge E'_i at all there is at least one edge pointing away from V and one other edge E_i that we do not remove. But these cannot be all edges adjacent to V since then the flags adjacent to V would not span \mathbb{R}^2 in contrast to definition 2.3 (c). So there must be at least one other edge that is not removed, leading again to a total of at least three edges at V that are not removed.

Keeping in mind that we do not remove any edge at the vertex V_1 at all (since no edge points towards it) it follows that the number of conditional edges is at least

$$\begin{aligned} & 2g(C) - \sum_{V \neq V_1} (\text{val } V - 3) \\ & = 2g(C) - \text{codim } \alpha + (\text{val } V_1 - 3) + (g - g(C)) + \#\{i; s(i) \in \Gamma^0\} \end{aligned}$$

so that an upper bound for the dimension of $\mathcal{M}_{g,\Delta}^\alpha$ is

$$2n - (\text{val } V_1 - 3) - (g - g(C)) - \#\{i; s(i) \in \Gamma^0\}. \quad (*)$$

We now consider several cases, stopping at the first one that applies to α :

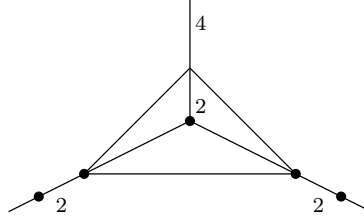
- If $\text{codim } \alpha \leq 1$ then V_1 is the only vertex that can possibly have valence greater than 3. So the number (*) is simply $2n - \text{codim } \alpha$, and it follows that $\dim \mathcal{M}_{g,\Delta}^\alpha = 2n - \text{codim } \alpha$ (as we know already that this number is also a lower bound).
- If the number (*) is at most $2n - 2$ then the dimension statement of the proposition follows immediately.
- If there are two vertices of valence 4 that are not joined by more than one edge then we choose L above to be a line through these two vertices. We can then label these two vertices as V_1 and V_2 . It follows that there is at most one edge pointing to V_2 , i.e. that there are no edges removed at V_2 either in the above procedure. Hence we can subtract $\text{val } V_2 - 3$ from the number (*). So again we conclude that $\dim \mathcal{M}_{g,\Delta}^\alpha \leq 2n - 2$.
- The only case left is that we have $g = g(C)$, no marked points on vertices, only vertices of valence 3 and 4, and all vertices of valence 4 joined by at least 2 edges between each pair of them. This is only possible if there are exactly 2 vertices of valence 4, and if these vertices are joined by exactly 2 edges. In other words, α is exceptional. In this case the dimension is obviously the same as for the combinatorial type where the double edge is replaced by one edge of added weight. As this new type has codimension 1 it follows that $\dim \mathcal{M}_{g,\Delta}^\alpha = 2n - 1$ in this case.

We have therefore proven the dimension statement of the proposition. So to finish the proof it suffices to notice that in the linear subspace of A determined by the $2g(C)$ (not necessarily independent) loop conditions the subset $\mathcal{M}_{g,\Delta}^\alpha$ is simply the open convex polyhedron given by the strict inequalities that all lengths of the internal edges (b) are positive, and that all marked points on edges (c) lie

in the interior of their respective edges. The polyhedron is unbounded since the coordinates of the root vertex (a) are not restricted. \square

Example 3.10

The following example shows that we do *not* have $\dim \mathcal{M}_{g,\Delta}^\alpha = 2n - \text{codim } \alpha$ in general. We consider marked tropical curves of genus $g = 3$ and degree $\Delta = (-4, -2) \oplus (4, -2) \oplus (0, 4)$ that have $n = \#\Delta + g - 1 = 5$ marked points and are of the following combinatorial type α (with an arbitrary choice of labeling of the marked points):



We have $\text{codim } \alpha = 6$ since there are three 4-valent vertices in the graph and three marked points on vertices. We would therefore expect $\dim \mathcal{M}_{g,\Delta}^\alpha$ to be $2n - \text{codim } \alpha = 10 - 6 = 4$. We have $\dim \mathcal{M}_{g,\Delta}^\alpha = 5$ however: there are 2 dimensions for moving the marked points on edges, 2 dimensions for translations of the curve in the plane, and 1 more dimension for rescaling the whole curve.

Definition 3.11

Let α be a combinatorial type occurring in a moduli space $\overline{\mathcal{M}}_{g,\Delta}$. By proposition 3.9 the space $\mathcal{M}_{g,\Delta}^\alpha$ of curves of this given type is naturally an open subset of a real affine space A_α . We denote by $\overline{\mathcal{M}}_{g,\Delta}^\alpha$ the closure of $\mathcal{M}_{g,\Delta}^\alpha$ in A_α .

Proposition 3.12

Let α be a combinatorial type occurring in a moduli space $\overline{\mathcal{M}}_{g,\Delta}$. Then every point in $\overline{\mathcal{M}}_{g,\Delta}^\alpha$ can naturally be thought of as a marked tropical curve in $\overline{\mathcal{M}}_{g,\Delta}$. The corresponding map $i_\alpha : \overline{\mathcal{M}}_{g,\Delta}^\alpha \rightarrow \overline{\mathcal{M}}_{g,\Delta}$ maps the boundary $\partial \mathcal{M}_{g,\Delta}^\alpha$ to the union of the strata $\mathcal{M}_{g,\Delta}^{\alpha'}$ such that

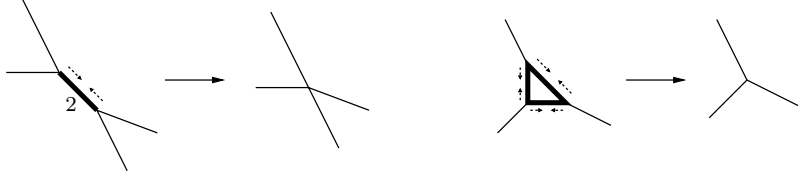
- (a) the number of internal edges plus the number of marked points lying on edges is smaller for α' than for α ;
- (b) $\text{codim } \alpha' > \min(1, \text{codim } \alpha)$.

Moreover, the restriction of i_α to any inverse image of such a stratum $\mathcal{M}_{g,\Delta}^{\alpha'}$ is an affine map.

Proof:

Recall that by the proof of proposition 3.9 the boundary $\partial \mathcal{M}_{g,\Delta}^\alpha$ is given by tuples (C, x_1, \dots, x_n) with $C = (\Gamma, \omega, h)$ such that some marked points x_i with $s(i) \in \Gamma^1$ lie on the boundary of their respective edges (i.e. on a vertex) and/or some interior edges are mapped to a line of length 0 in \mathbb{R}^2 (i.e. to a point). In the former case this simply changes the combinatorial type (C, x_1, \dots, x_n) so that the points x_i are now required to map to a vertex instead of to an edge, i.e. so that $s(i) \in \Gamma^0$. In the

latter case we will simply change the graph by removing all edges E such that $h(E)$ is a point and glueing the vertices in ∂E to one vertex as in the following picture:



In the picture we have drawn in bold the edges whose lengths tend to zero and are finally mapped to a point and removed.

It is easy to see that the conditions of definition 2.3 (c) are still satisfied for the new graph. Hence the result is a tropical curve. Together with the marked points it is in fact an n -marked tropical curve (i.e. it satisfies condition (b) of definition 2.8) since with the notation of definition 2.8 (b) none of the processes above can open up a loop in $|\Gamma| \setminus ([F_1] \cup \dots \cup [F_n])$ or connect two unbounded edges in $|\Gamma| \setminus ([F_1] \cup \dots \cup [F_n])$ that have not been connected before. This shows that the points in the boundary $\partial \mathcal{M}_{g,\Delta}^\alpha$ can naturally be thought of as marked tropical curves in $\overline{\mathcal{M}}_{g,\Delta}$ themselves.

Let us now analyze which combinatorial types can occur on the boundary, i.e. in the image $i_\alpha(\partial \mathcal{M}_{g,\Delta}^\alpha)$. It is clear by the construction above that condition (a) of the proposition must be satisfied. To show (b) we distinguish two cases: if a combinatorial type α' occurring in the boundary has genus strictly less than g then $\text{codim } \alpha' > 1$ by remark 3.6. On the other hand, if α' (and hence also α) corresponds to curves of genus g then we have $\text{codim } \alpha' > \text{codim } \alpha$ by (a) and the formula of remark 3.5. This proves (b).

Finally, it is clear that the restriction of i_α to the inverse image of any stratum $\mathcal{M}_{g,\Delta}^{\alpha'}$ is an affine map since the affine structure on any stratum is given by the position of the curve in the plane, the lengths of the internal edges, and the position of the marked points on edges. \square

Remark 3.13

One would expect that in general the boundary of a stratum $\mathcal{M}_{g,\Delta}^\alpha$ corresponds only to marked curves of higher codimensions. This is not true however: the curve in example 3.10 can be degenerated by shrinking the triangles to zero size, arriving at a tropical curve of genus 0 with three edges and one vertex, and three of the marked points lying on the vertex. This new combinatorial type in the boundary has codimension 6, which is the same codimension as the one of the curves that we started with. Proposition 3.12 (b) ensures however that the codimension of the curves in the boundary is always bigger if we start with a stratum of codimension 0 or 1.

Remark 3.14

By propositions 3.7 and 3.12 we can think of the moduli space $\overline{\mathcal{M}}_{g,\Delta}$ as being obtained by starting with finitely many unbounded closed convex polyhedra $\overline{\mathcal{M}}_{g,\Delta}^\alpha$ and then glueing them together by attaching each boundary $\partial \mathcal{M}_{g,\Delta}^\alpha$ with affine maps to some polyhedra $\overline{\mathcal{M}}_{g,\Delta}^{\alpha'}$ such that the number of internal edges plus the number of marked points lying on edges is smaller for α' than for α . In particular,

this makes $\overline{\mathcal{M}}_{g,\Delta}$ into a topological space with a natural stratification such that each stratum is an unbounded open convex polyhedron.

One would expect that the moduli space $\overline{\mathcal{M}}_{g,\Delta}$ (or more probably a similar space with a slightly different definition of marked tropical curves) can in fact be given the structure of a tropical variety itself. However, the theory of abstract tropical varieties is still very much in its beginnings (see e.g. [Mi2]) so that we will not use this language here.

4. ENUMERATIVE GEOMETRY OF TROPICAL CURVES

We are now ready to count tropical curves through some given points in the plane. As in the previous section we fix a genus $g \in \mathbb{N}$ and degree $\Delta \in G$ and consider tropical curves with $n := \#\Delta + g - 1$ marked points.

Definition 4.1

We define the *evaluation map* to be

$$\begin{aligned} \pi : \quad \quad \quad \overline{\mathcal{M}}_{g,\Delta} &\rightarrow \mathbb{R}^{2n} \\ ((\Gamma, \omega, h), x_1, \dots, x_n) &\mapsto (h(x_1), \dots, h(x_n)). \end{aligned}$$

Proposition 4.2

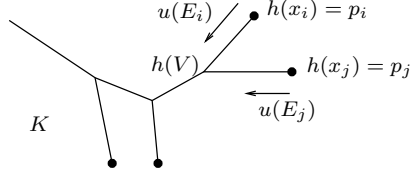
The evaluation map $\pi : \overline{\mathcal{M}}_{g,\Delta} \rightarrow \mathbb{R}^{2n}$ is affine on each closed stratum $\overline{\mathcal{M}}_{g,\Delta}^\alpha$ (and hence in particular continuous). Moreover, it is injective on each stratum $\mathcal{M}_{g,\Delta}^\alpha$ with $\text{codim } \alpha = 0$.

Proof:

For a fixed combinatorial type it is clear that the positions $h(x_i) \in \mathbb{R}^2$ of all marked points are linear functions in the affine coordinates constructed in the proof of proposition 3.9. Hence π is affine on each closed stratum $\overline{\mathcal{M}}_{g,\Delta}^\alpha$.

Now let α be a combinatorial type with $\text{codim } \alpha = 0$, and let $p_1, \dots, p_n \in \mathbb{R}^2$. We are going to show that π is injective on $\mathcal{M}_{g,\Delta}^\alpha$, i.e. that there is at most one marked tropical curve $(C = (\Gamma, \omega, h), x_1, \dots, x_n)$ of type α with $h(x_i) = p_i$ for all i . Note that $\text{codim } \alpha = 0$ implies in particular that all vertices of Γ have valence 3.

As $\text{codim } \alpha = 0$ none of the marked points can lie on a vertex. Let $E_i = s(i) \in \Gamma^1$ be the edge on which x_i lies. By remark 3.4 $|\Gamma \setminus \{x_1, \dots, x_n\}|$ has $\#\Delta$ connected components each of which has genus 0 and contains exactly one unbounded end. Let K be one of these connected components. We can pick two distinct points x_i and x_j in K such that the corresponding edges E_i and E_j are adjacent to the same vertex V of K . Then both the direction and a starting point of the line segments $h(E_i)$ and $h(E_j)$ are fixed by u and the points p_i and p_j . Note that the directions of the two line segments cannot be the same by remark 2.4 since $\text{val } V = 3$. Hence these data determine $h(V)$ (and thus the lengths of E_i and E_j) uniquely as the intersection point $h(E_i) \cap h(E_j)$. (It may happen that no such intersection exists, in which case there is no marked tropical curve of the given type through the points p_i .)



We can now remove the two edges E_i and E_j and the marked points x_i and x_j from K and replace them by the one marked point V (whose image point $h(V)$ is also fixed). Applying the same arguments as above again we conclude by induction that $h(K)$ is uniquely defined. Of course, this holds for all connected components of $|\Gamma| \setminus \{x_1, \dots, x_n\}$, and therefore there is at most one possibility (up to isomorphism) for the map h . \square

Definition 4.3

We set

$$\mathcal{P} = \mathbb{R}^{2n} \setminus \bigcup_{\alpha} \pi(\mathcal{M}_{g,\Delta}^{\alpha})$$

where the union is taken over all combinatorial types α occurring in $\overline{\mathcal{M}}_{g,\Delta}$ such that

- π is not injective on $\mathcal{M}_{g,\Delta}^{\alpha}$; or
- α is non-exceptional and $\text{codim } \alpha \geq 2$.

We say that n points $p_1, \dots, p_n \in \mathbb{R}^2$ are *in general position* if $(p_1, \dots, p_n) \in \mathcal{P} \subset \mathbb{R}^{2n}$.

Our goal will be to show that the number of tropical curves (counted with a suitable multiplicity) through n given points $p_1, \dots, p_n \in \mathbb{R}^2$ in general position does not depend on the choice of points.

Remark 4.4

Note that our definition of points being in general position is a lot weaker than the definitions of [Mi1] and [NS]. In [Mi1] and [NS] points in general position require (in our language) that there are no curves of a combinatorial type of positive codimension through these points, whereas we exclude only curves of codimension at least 2. As a consequence, our space of points in general position will be *connected* in \mathbb{R}^{2n} :

Lemma 4.5

- (a) *The locus $\mathcal{P} \subset \mathbb{R}^{2n}$ of points in general position is connected.*
- (b) *The map $\pi : \overline{\mathcal{M}}_{g,\Delta} \rightarrow \mathbb{R}^{2n}$ has finite fibers over $\mathcal{P} \subset \mathbb{R}^{2n}$.*

Proof:

- (a) By definition the complement of \mathcal{P} is a union of two types of sets:
 - subspaces $\pi(\mathcal{M}_{g,\Delta}^{\alpha}) \subset \mathbb{R}^{2n}$ such that π is not injective on $\mathcal{M}_{g,\Delta}^{\alpha}$. As π is an affine map this means that the fiber dimension of π is positive. Moreover, by proposition 4.2 we must have $\text{codim } \alpha \geq 1$, and thus the dimension of $\mathcal{M}_{g,\Delta}^{\alpha}$ is at most $2n - 1$ by proposition 3.9. Therefore the codimension of the image $\pi(\mathcal{M}_{g,\Delta}^{\alpha})$ is at least 2 in \mathbb{R}^{2n} .

- subspaces $\pi(\mathcal{M}_{g,\Delta}^\alpha) \subset \mathbb{R}^{2n}$ such that α is non-exceptional and of codimension at least 2. Then $\mathcal{M}_{g,\Delta}^\alpha$ has dimension at most $2n - 2$ by proposition 3.9. Therefore its image has again codimension at least 2 in \mathbb{R}^{2n} .

It follows that \mathcal{P} is the complement of a subset of codimension at least 2 in \mathbb{R}^{2n} and thus connected.

- (b) This follows immediately since by definition π is injective on each of the (finitely many) strata $\mathcal{M}_{g,\Delta}^\alpha$ in the preimage of \mathcal{P} .

□

To be able to count tropical curves through points in general position we finally need one more ingredient: we need to know the multiplicities with which to count the curves. We will only construct these multiplicities for curves that can actually occur through points in general position.

Definition 4.6

Let (C, x_1, \dots, x_n) with $C = (\Gamma, \omega, h)$ be a tropical curve in $\overline{\mathcal{M}}_{g,\Delta}$. Assume that its combinatorial type α is either exceptional or has codimension at most 1. Note that then all vertices of Γ are of valence 3 or 4.

For a 3-valent vertex $V \in \Gamma^0$ with adjacent flags $F_1, F_2, F_3 \in \Gamma'$ we set

$$\text{mult}_C V := |\det(v(F_1), v(F_2))| \in \mathbb{N}_{>0}$$

(note that by the balancing condition $v(F_1) + v(F_2) + v(F_3) = 0$ this definition does not depend on how the three flags are numbered). For a 4-valent vertex $V \in \Gamma^0$ with adjacent flags $F_1, F_2, F_3, F_4 \in \Gamma'$ we set

$$\text{mult}_C V := \max |\det(v(F_i), v(F_j)) \cdot \det(v(F_k), v(F_l))| \in \mathbb{N}_{>0}$$

where the maximum is taken over all i, j, k, l such that $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Finally, we define the *multiplicity* $\text{mult } C$ of the curve C to be the product of the multiplicities of all its vertices.

Using this multiplicity we can now define the function

$$N_{g,\Delta} : \mathcal{P} \rightarrow \mathbb{N} \\ (p_1, \dots, p_n) \mapsto \sum \text{mult } C$$

where the sum is taken over all $(C, x_1, \dots, x_n) \in \pi^{-1}(p_1, \dots, p_n)$. Note that this is a finite sum by lemma 4.5 (b), and that the multiplicities $\text{mult } C$ are defined since all curves in the preimage $\pi^{-1}(\mathcal{P})$ are by definition 4.3 of a combinatorial type that is exceptional or of codimension at most 1.

Remark 4.7

In contrast to the case of counting complex curves this multiplicity with which we have to count curves through points in general position will *not* always be 1 (not even with the stronger definition of general position of [Mi1] and [NS]). It is always a positive integer though.

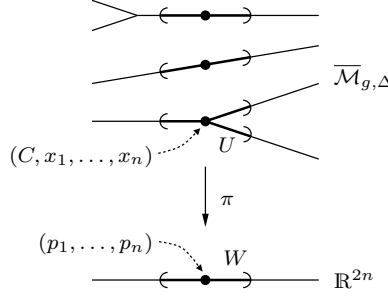
We are now ready to prove our main theorem:

Theorem 4.8

The function $N_{g,\Delta} : \mathcal{P} \rightarrow \mathbb{N}$ is constant. In other words, the number of tropical curves of genus g and degree Δ through n points in general position does not depend on the position of the points.

Proof:

As \mathcal{P} is connected by lemma 4.5 (a) it suffices to show that $N_{g,\Delta}$ is locally constant. So let $W \subset \mathbb{R}^{2n}$ be a small open subset of a point $(p_1, \dots, p_n) \in \mathcal{P}$. Recall that $\pi^{-1}(p_1, \dots, p_n)$ is a finite set by lemma 4.5 (b). Hence as π is continuous and affine on each of the finitely many closed strata $\overline{\mathcal{M}}_{g,\Delta}^\alpha$ we can pick W small enough so that the inverse image $\pi^{-1}(W)$ is a finite disjoint union of open subsets of $\overline{\mathcal{M}}_{g,\Delta}$ each of which contains exactly one point in the inverse image $\pi^{-1}(p_1, \dots, p_n)$. Of course it suffices to show that $N_{g,\Delta}$ is constant when restricted to any of these open subsets.



So let $(C, x_1, \dots, x_n) \in \pi^{-1}(p_1, \dots, p_n)$ be a marked tropical curve, and let $U \subset \pi^{-1}(W)$ be a small open neighborhood of this curve as above. If α is the combinatorial type of (C, x_1, \dots, x_n) then α is exceptional or $\text{codim } \alpha \leq 1$ by definition 4.3.

If $\text{codim } \alpha = 0$ then (C, x_1, \dots, x_n) cannot be in the boundary of any other stratum $\mathcal{M}_{g,\Delta}^{\alpha'}$ by proposition 3.12 (b). Hence (after possibly shrinking W and U) α is the only combinatorial type occurring in U . As $\dim \mathcal{M}_{g,\Delta}^\alpha = 2n$ by proposition 3.9 and π is affine and injective on this stratum by definition 4.3 we conclude that $\pi|_U : U \rightarrow W$ is an isomorphism. So in this case $N_{g,\Delta}$ is trivially constant when restricted to U . In the picture above this is the case for the upper two connected components of $\pi^{-1}(W)$.

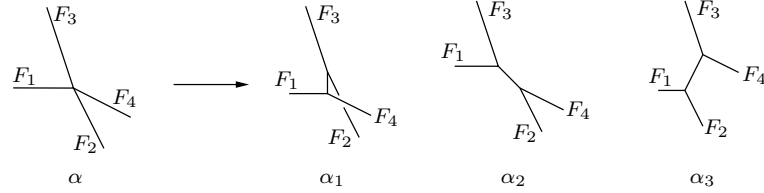
We can therefore assume from now on that α is exceptional or $\text{codim } \alpha = 1$. Then $\dim \mathcal{M}_{g,\Delta}^\alpha = 2n - 1$ by proposition 3.9, and hence the image $\pi(U \cap \mathcal{M}_{g,\Delta}^\alpha)$ is of codimension 1 in W by definition 4.3. We can therefore think of the image of this stratum as a “wall” that divides W into two halves. We have to show that the restriction of $N_{g,\Delta}$ to U stays constant when crossing this wall. So we have to analyze which combinatorial types occur in U on both sides of the wall. After possibly shrinking W and U these are of course just the types that contain $\mathcal{M}_{g,\Delta}^\alpha$ in their boundary. In the picture above this is the case for the bottom connected component of $\pi^{-1}(W)$ (where we have one combinatorial type on the left and two combinatorial types on the right side of the wall).

As α is exceptional or $\text{codim } \alpha = 1$ there are by definition 3.3 four cases to check:

- (a) α is non-exceptional, and the graph Γ has one vertex V of valence 4:

As this is the most interesting case (that we have already mentioned in the introduction) we will discuss it in detail. In fact this is the only case in which the number of tropical curves through the given points would not be constant if we did not count them with their correct multiplicities.

Denote by F_1, \dots, F_4 the four flags with $\partial F_i = V$ for $i = 1, \dots, 4$, and let $v_i := v(F_i)$ (using definition 2.3 (c)). We will assume for simplicity that no two of these vectors are parallel and leave the other (much easier) limit cases to the reader. Then there are exactly three different types $\alpha_1, \alpha_2, \alpha_3$ that have α in their boundary. Each of them replaces the 4-valent vertex by two 3-valent ones that are separated by a new edge. The remaining two edges of each of the two new vertices are given by the four flags F_1, \dots, F_4 in any possible way: we obtain type α_i for $i = 1, 2, 3$ when F_i and F_4 come together at one vertex and the other two flags at the other vertex. In each case the weight and direction of the new edge is determined uniquely by the balancing condition of definition 2.3 (c).



(If some of the vectors v_i are parallel then not all of these types occur.)

Now let $(p'_1, \dots, p'_n) \in W \setminus \pi(\mathcal{M}_{g,\Delta}^\alpha)$ be a collection of points that is not on the wall, and let $(C', x'_1, \dots, x'_n) \in \pi^{-1}(p'_1, \dots, p'_n)$. We have to determine which of the combinatorial types $\alpha_1, \alpha_2, \alpha_3$ are possible for (C', x'_1, \dots, x'_n) . Note that in any case the new edge of C' cannot contain a marked point. By remark 3.4 the connected component of $|\Gamma'| \setminus \{x'_1, \dots, x'_n\}$ that contains the new edge has no loops and exactly one unbounded end. Therefore the new edge is connected to an unbounded end via exactly one of the four flags F_1, \dots, F_4 . By symmetry we may assume that this flag is F_4 . Then by the argument of the proof of proposition 4.2 for each possible combinatorial type the lines in \mathbb{R}^2 on which F_1, F_2 , and F_3 lie are fixed by the points p'_1, \dots, p'_n , whereas the line for F_4 is not.

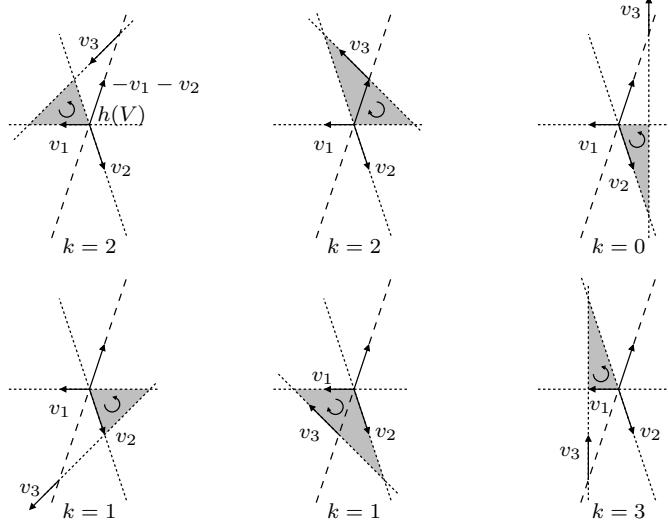
Let us now determine if the collection of points (p'_1, \dots, p'_n) admits a tropical curve of combinatorial type α_3 through them. Let $V = \partial F_1 = \partial F_2$ be the common vertex of F_1 and F_2 . As the result is obviously invariant under a relabeling of flags $F_1 \leftrightarrow F_2$ we may assume without loss of generality that

$$\det(v_1, v_2) > 0, \quad (1)$$

i.e. that the (oriented) angle between the vectors $v_1 = v(F_1)$ and $v_2 = v(F_2)$ is less than π . The new internal edge of the curve starts at $h(V)$ and points in the direction $-v_1 - v_2$. So a curve of type α_3 through the given points exists if and only if the line on which F_3 lies intersects the ray with direction $-v_1 - v_2$ starting at V . As this condition is invariant under a change of direction $v_3 \leftrightarrow -v_3$ we may also assume without loss of generality that

$$\det(v_3, v_1 + v_2) > 0. \quad (2)$$

Let us consider the triangle T that is cut out by the three lines on which the three flags F_1 , F_2 , and F_3 lie. The following six cases can occur:



In the top three cases there is a tropical curve of type α_3 through the marked points (with v_4 determined by the balancing condition), whereas in the bottom three cases there is no such curve. The cases in the left, middle, and right column differ by the slope of v_3 compared to the slopes of v_1 and v_2 .

In any case the lines on which F_1 , F_2 , and F_3 lie, taken in this order, define an orientation on the triangle T that we have also indicated in the picture above. Of course this also defines an orientation on the edges of T . Let $k \in \{0, 1, 2, 3\}$ be the number of vectors v_1 , v_2 , v_3 that point against this orientation on their respective edge, and set

$$\mu := (-1)^k \cdot \prod_{\substack{V' \in \Gamma^0 \\ V' \neq V}} \text{mult}_C V' \in \mathbb{Z}.$$

By looking at all the cases above we see that there is a tropical curve of type α_3 through the given points p'_1, \dots, p'_n if and only if $\mu > 0$ (note that the product over $\text{mult}_C V'$ in the definition of μ is always positive by definition). To be precise it may also happen that the line on which F_3 lies is parallel to one of the other two lines. In this case the three lines do not determine a unique triangle but rather two “unbounded” triangles. One can check immediately that then both unbounded triangles give the same value of μ , and that again there is a tropical curve of type α_3 through the given points if and only if $\mu > 0$.

Note that the number μ changes sign if we exchange the labeling of F_1 and F_2 or if we replace v_3 by $-v_3$. So we can remove our assumptions (1) and (2) above and conclude that in any case there is a curve of type α_3 through the given points if and only if the number

$$\mu_3 := \mu \cdot \det(v_1, v_2) \cdot \det(v_3, v_1 + v_2)$$

is positive. In this case the curve is then unique by proposition 4.2, and by definition its multiplicity is just μ_3 .

As the number μ is invariant under cyclic permutations $F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow F_1$ it now follows that for $i = 1, 2, 3$ there is a curve of type α_i through the given points if and only if $\mu_i > 0$, where

$$\begin{aligned}\mu_1 &:= \mu \cdot \det(v_2, v_3) \cdot \det(v_1, v_2 + v_3), \quad \text{and} \\ \mu_2 &:= \mu \cdot \det(v_3, v_1) \cdot \det(v_2, v_3 + v_1),\end{aligned}$$

and that in this case it counts with multiplicity μ_i .

Finally, an elementary calculation shows that $\mu_1 + \mu_2 + \mu_3 = 0$. From this it follows immediately that the number of curves in $\pi^{-1}(p'_1, \dots, p'_n) \cap U$, counted with their respective multiplicities, is

$$\sum_{i: \mu_i > 0} \mu_i = \max\{|\mu_1|, |\mu_2|, |\mu_3|\},$$

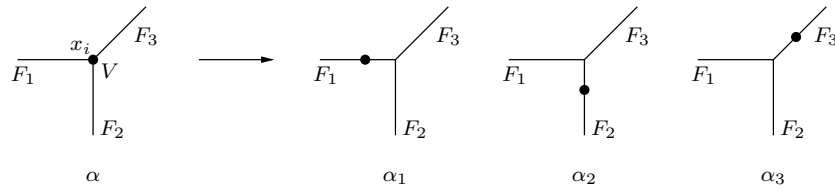
which is by definition the multiplicity of the one curve in $\pi^{-1}(p_1, \dots, p_n)$ with a 4-valent vertex.

- (b) α is non-exceptional, and the genus of Γ is $g(C) = g - 1$:

This is a contradiction to $\text{codim } \alpha = 1$ by remark 3.6. Hence this case cannot occur.

- (c) α is non-exceptional, and there is a marked point x_i on a vertex V of Γ (i.e. $s(i) \in \Gamma^0$ for one i):

The idea here is the same as in case (a), but the analysis is a lot simpler. The point x_i must lie on a 3-valent vertex V whose adjacent flags we denote by F_1 , F_2 , and F_3 . There are at most three combinatorial types that have α in their boundary, namely the types α_k for $k = 1, 2, 3$ where the marked point x'_i lies on the edge $[F_k]$ and the remaining data of the curve stay the same:



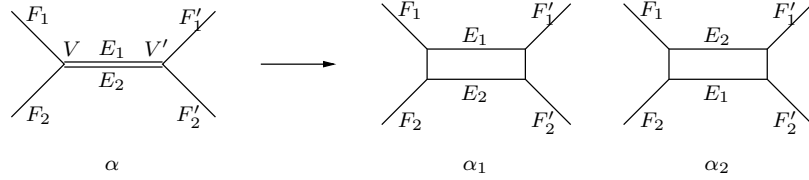
By definition 2.8 (b) at least one of these cases will be allowed, i.e. lead to a marked tropical curve (namely if x_i is moved onto the interior of the flag chosen in this definition). After possibly relabeling the flags we can assume that type α_1 is possible. By remark 3.4 one can then reach an unbounded end of $|\Gamma| \setminus \{x'_1, \dots, x'_n\}$ in α_1 from both sides of the point x'_i along exactly one path. So one can reach such an unbounded end via F_1 (starting to the left of the marked point), and (after possibly relabeling the flags $F_2 \leftrightarrow F_3$) via F_2 starting from V , but not via F_3 starting from V .

First of all this means that type α_3 is impossible, since this would connect the two unbounded ends behind F_1 and F_2 in contradiction to definition 2.8 (b). Moreover, we see that (analogously to case (a) above) for both

α_1 and α_2 the line on which F_3 lies is fixed by the other marked points, whereas the lines on which F_1 and F_2 lie are not. It is now immediate that we always have exactly one of the cases α_1 and α_2 , depending on whether the point p'_i lies on the one or the other side of the line on which F_3 lies. The multiplicity is obviously the same in both cases as it does not depend on the position of the marked points.

(d) α is exceptional:

Note first that the two edges E_1 and E_2 joining the two 4-valent vertices V and V' are distinguishable since by definition 2.8 (b) at least one of them must have a marked point on it. At both vertices V and V' the balancing condition implies that the two other flags must point to different sides of the line on which E_1 and E_2 lie. We denote by F_1 and F'_1 (resp. F_2 and F'_2) the flags pointing to the one (resp. the other) side. Then there are exactly two combinatorial types α_1, α_2 that contain α in their boundary:



Obviously, the curves of types α , α_1 , and α_2 all have the same multiplicity. Applying the analysis of (a) to both vertices V and V' we see that the types α_1 and α_2 always occur on different sides of the wall, whereas α occurs on the wall itself. Hence it follows that restriction of $N_{g,\Delta}$ to U is constant. \square

Remark 4.9

Let us finally recap what we have shown and how our results relate to the existing literature.

Let X be a (complex) smooth projective toric surface defined by a fan Σ , and let G' be the sub-semigroup of G (as introduced in definition 2.6) generated by all primitive integral vectors along the rays of Σ . It is then well-known that every element $\Delta = u_1 \oplus \dots \oplus u_k \in G'$ with $u_1 + \dots + u_k = 0$ corresponds to a homology class of complex curves in X . Pick a genus $g \geq 0$, and set $n = \#\Delta + g - 1$. Moreover, choose n points $q_1, \dots, q_n \in X$ in general position that lie in the torus $(\mathbb{C}^*)^2 \subset X$. Define the map $\text{Log} : (\mathbb{C}^*)^2 \rightarrow \mathbb{R}^2$ by $\text{Log}(z_1, z_2) := (\log |z_1|, \log |z_2|)$ and set $p_i := \text{Log}(q_i)$ for $i = 1, \dots, n$. It is then the main result of [Mil] (the so-called ‘‘Correspondence Theorem’’) that the number of *complex* curves in X of class Δ and genus g through the points q_1, \dots, q_n is equal to the number of *tropical* plane curves of degree Δ and genus g through the points p_1, \dots, p_n (counted with their multiplicities as constructed in definition 4.6). In particular, since the number of such complex curves does not depend on the points q_1, \dots, q_n , it follows as a corollary that the number of such tropical curves does not depend on the position of the points p_1, \dots, p_n either. In this paper we have given a proof of this last statement within tropical geometry, i.e. without referring to the (highly non-trivial) Correspondence Theorem.

For example, if we choose for Δ the degree that contains d times the vectors $(-1, 0)$, $(0, -1)$, and $(1, 1)$ each then tropical curves of degree Δ correspond to complex curves in \mathbb{P}^2 of degree d .

Remark 4.10

Our results are in fact more general than what we have just described in remark 4.9. Namely, our main theorem 4.8 is also applicable in the following two cases in which there is no analogue of the “Correspondence Theorem” yet:

- (a) if the degree Δ does not only contain primitive integral vectors, i.e. if there are unbounded ends with weights greater than 1;
- (b) if some of the unbounded ends are “fixed”, i.e. if not only their slope is fixed but also the line in \mathbb{R}^2 on which they lie. We have not included this set-up explicitly in our definitions since this would have made the notations too complicated. The necessary modifications in the constructions and proofs are very straightforward however if one thinks of a fixed unbounded end as an unbounded end with a marked point “at infinity” on it. For example, every fixed unbounded end reduces the required number n of marked points by 1, and in definition 2.8 (b) we have to replace “no connected component with more than one unbounded end” by “no connected component with more than one unbounded end that is not fixed”.

Intuitively, case (a) corresponds to curves with fixed multiplicities to the corresponding toric divisors (e.g. tropical curves of degree $(-2, 0) \oplus (0, -1) \oplus (0, -1) \oplus (1, 1) \oplus (1, 1)$ correspond to complex conics in \mathbb{P}^2 tangent to the line determined by the vector $(-1, 0)$). Case (b) corresponds to fixed intersection points of the complex curves with these toric divisors. Combining both cases we should in some cases be able to construct a tropical analogue of relative Gromov-Witten invariants, i.e. of complex curves with fixed multiplicities to a given divisor in maybe fixed points of this divisor. We will discuss this in detail in a forthcoming paper [GM].

REFERENCES

- [GM] Andreas Gathmann and Hannah Markwig, *The Caporaso-Harris formula and plane relative Gromov-Witten invariants in tropical geometry*, preprint math.AG/0504392.
- [Mi1] Grigory Mikhalkin, *Enumerative tropical geometry in \mathbb{R}^2* , J. Amer. Math. Soc. **18** (2005), 313–377, preprint math.AG/0312530.
- [Mi2] Grigory Mikhalkin, *Tropical geometry*, <http://www.math.toronto.edu/~mikha/book.ps>.
- [NS] Takeo Nishinou and Bernd Siebert, *Toric degenerations of toric varieties and tropical curves*, preprint math.AG/0409060.
- [Shu] Eugenii Shustin, *A tropical approach to enumerative geometry*, Algebra Anal. **17** (2005) no. 2, 170–214, preprint math.AG/0211278.

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