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Enumerative geometry of rational and elliptic tropical curves in $\mathbb{R}^m$

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Preface

In tropical geometry, algebraic varieties are replaced by combinatorial objects in the hope that the combinatorial objects still carry relevant information about the algebraic counterparts while being easier to study. Ideally, one can formulate a question both on the algebraic and on the tropical side, answer it on the tropical side and transfer the answer back to the algebraic setup. The transformation process is called "tropicalization" and there are various ways to describe it, see e.g. the introductory article [**Gat06**]. The combinatorial objects on the tropical side are pure-dimensional polyhedral complexes, which fulfill a "balancing condition" at each polyhedron of codimension one.

The technique of transforming algebraic questions into tropical ones has already been used successfully, especially in enumerative geometry of plane curves. A starting point was the correspondence theorem by Grigory Mikhalkin [**Mik05**], which states that the number of projective curves of degree d and genus g passing $3d+g-1$ points in $\mathbb{P}^2$ is equal to the number of plane tropical curves of degree d and genus g passing the same number of points in $\mathbb{R}^2$, counted with multiplicity. It is not only known that these numbers do not depend on the position of the points, they can also be calculated. On the algebraic side, in [**CH98**] a recursive formula is established that determines these numbers in any genus. The same and similar ideas have been applied on the tropical side [**GM07b**] [**BM08**], yielding the same recursive formula. There have been various other results in plane enumerative tropical geometry, e.g. [**IKS08, BGM12, GMS12, BBM11**].

Algebraic enumerative geometry. The main strategy to count geometric objects fulfilling a collection of conditions is to construct a moduli space which parameterizes the objects. The set of objects which fulfill a given condition is then a subspace of the moduli space. By intersecting the arising subspaces, we obtain the points in the moduli space that correspond to the objects that fulfill all desired conditions at the same time. Since we want to count objects fulfilling given conditions, the conditions are chosen in way that the number of these objects is finite, i.e. the intersection of the subspaces is zero-dimensional. The main tool to study enumerative questions is therefore intersection theory on moduli spaces.

In algebraic geometry, main objects of interest are the moduli spaces $\bar{\mathcal{M}}_{g,n}(\mathbb{P}^m, d)$ of stable maps $(C, x_1, \ldots, x_n, f)$ of degree d with n marked points in genus g, see e.g. [**BM96**]. Such a stable map $(C, x_1, \ldots, x_n, f)$ consists of a curve C of genus g with n marked points $x_1, \ldots, x_n$ and of a map $f : (C, x_1, \ldots, x_n) \to \mathbb{P}^m$ that fulfills that the push-forward $f_*[C]$ has degree d and that f has finitely many automorphisms. One considers stable maps to $\mathbb{P}^m$ instead of algebraic curves of degree d embedded in $\mathbb{P}^m$ because their moduli space behaves in a better way.

The moduli space $\bar{\mathcal{M}}_{g,n}(\mathbb{P}^m, d)$ is a Deligne-Mumford stack and has an expected virtual dimension. Its actual dimension, however, can be larger and needs not to be constant. The space $\bar{\mathcal{M}}_{g,n}(\mathbb{P}^m, d)$ is considered a compactification of the subspace $\mathcal{M}_{g,n}(\mathbb{P}^m, d)$ which consists of stable maps whose underlying curves are smooth. However, $\mathcal{M}_{g,n}(\mathbb{P}^m, d)$ is in general not dense in $\bar{\mathcal{M}}_{g,n}(\mathbb{P}^m, d)$. In the case that the underlying curves are rational the dimensions of the two spaces coincide.

For each marked point x_i with $i \in [n]$ there exists an evaluation map

$$\begin{aligned} \mathrm{ev}_i : \bar{\mathcal{M}}_{g,n}(\mathbb{P}^m, d) &\to \mathbb{P}^m \\ (C, x_1, \ldots, x_n, f) &\mapsto f(x_i), \end{aligned}$$

which evaluates the map f at the point x_i. The maps ev_i are morphisms.

If we want to count curves of degree d in $\mathbb{P}^m$ which fulfill a general collection $\mathcal{L} = (L_i)_{i\in[n]}$ of incidence conditions in $\mathbb{P}^m$, i.e. which intersect all subvarieties $L_1, \ldots, L_n$ of $\mathbb{P}^m$, there exist two approaches using moduli spaces to tackle this problem.

In [**Vak00**], the following approach is used for counting rational and elliptic curves in $\mathbb{P}^m$ (i.e. the genus g of the curves is zero or one). One considers the closure $\mathcal{X}^1$ of the set of smooth stable maps $(C, x_1, \ldots, x_n, f) \in \mathcal{M}_{g,n}(\mathbb{P}^m, d)$ which fulfill all conditions $L_1, \ldots, L_n$, i.e. it holds $\mathrm{ev}_i(x_i) \in L_i$ for all $i \in [n]$. (It is moreover demanded in the elliptic case that no component of arithmetic genus one is contracted to a point.) If the dimension of $\mathcal{X}^1$ is zero, the enumerative number $N^1_{\mathrm{cplx}}(d, g, \mathcal{L})$ of curves of degree d and genus g in $\mathbb{P}^m$ is defined as the degree $\deg(\mathcal{X}_1)$ of $\mathcal{X}^1$. The number does not change if we replace $L_1, \ldots, L_n$ by rationally equivalent varieties.

Another approach using so-called Gromov-Witten-invariants is the following, see e.g. [**BM96**]. There can be constructed a naturally defined virtual fundamental class

$$[\bar{\mathcal{M}}_{g,n}(\mathbb{P}^m, d)]^{\mathrm{vir}} \in A_*(\bar{\mathcal{M}}_{g,n}(\mathbb{P}^m, d))$$

in the chow group of the moduli space. This virtual class has the expected virtual dimension of $\bar{\mathcal{M}}_{g,n}(\mathbb{P}^m, d)$. The pull-back of the subvarieties L_i along the evaluation maps ev_i can be interpreted as the geometric condition that the curves pass the subvarieties L_i, and hence we consider the intersection product

$$\mathcal{X}^2 = \mathrm{ev}_1^*[L_1] \cdot \cdots \cdot \mathrm{ev}_n^*[L_n] \cdot [\bar{\mathcal{M}}_{g,n}(\mathbb{P}^m, d)]^{\mathrm{vir}}.$$

If $\mathcal{X}^2$ is zero-dimensional, the virtual number $N^1_{\mathrm{cplx}}(d, g, \mathcal{L}) = \deg(\mathcal{X}^2)$ does not change if we replace $L_1, \ldots, L_n$ by rationally equivalent varieties.

In the case of rational curves, the virtual fundamental class is equal to the usual fundamental class and the two invariants coincide, i.e. it holds $N^1_{\mathrm{cplx}}(d, g, \mathcal{L}) = N^2_{\mathrm{cplx}}(d, g, \mathcal{L})$. In general, this is not true.

Let us turn to the question how and if these enumerative invariants can be calculated.

As already mentioned above, in the case of plane curves (i.e. curves in $\mathbb{P}^2$) a recursive formula is established in [**CH98**] that allows us to calculate the enumerative numbers $N^1_{\mathrm{cplx}}(d, g, \mathcal{L})$ in any genus. Even more, with the formula also relative invariants can be calculated: The term relative means that we impose, in addition to the incidence conditions $L_1, \ldots, L_n$, tangency conditions to subvarieties $T_1, \ldots, T_j$ of a hyperplane $H \subset \mathbb{P}^m$.

The idea used in [**CH98**] is the following: The incidence and tangency conditions to the hyperplane H, which is in this case a line, are specialized (i.e. the varieties are translated) in a way that the curves C which fulfill all incidence and tangency conditions split up into components $C_0, \ldots, C_l$ such that C_0 lies in the hyperplane H and such that the other components $C_1, \ldots, C_l$ are not (completely) contained in H. Instead of counting the curves C of degree d and genus g, the decomposed curves $C_0, \ldots, C_l$ (fulfilling derived incidence and tangency conditions) are counted. The data of the decomposed curves is simpler than that of the original curves and a recursion appears.

The same idea of specializing the incidence and tangency conditions is used in [**Vak00**] to calculate the enumerative numbers $N^1_{\mathrm{cplx}}(d, g, \mathcal{L})$ in the case of rational and elliptic curves in any dimension.

There are no known formulae which allow to compute the enumerative numbers $N^1_{\mathrm{cplx}}(d, g, \mathcal{L})$ in the case that the genus is greater than one and that the dimension of the ambient space is strictly greater than two.

In the case of virtual invariants, a constructive formula has been proven in [**Gat03**] with which the virtual invariants can be calculated in any degree, genus and dimension, using relations between them.

Tropical enumerative geometry. Ideas that are similar to the explained techniques in algebraic enumerative geometry have also been used successfully in tropical geometry, and various algebraic results mentioned above have been proven also on the tropical side.

A tropical curve is a metric graph with unbounded leaves that fulfills a balancing condition at every vertex. The tropical analogue of an n-marked stable map to $\mathbb{P}^m$ of degree d and genus

g is a parametrized n-marked tropical curve (C, h) of degree d and genus g in $\mathbb{R}^m$. With the algebraic setup in mind, we can think of (C, h) as a triple $(C, x_1, \ldots, x_n, h)$ where C is a smooth (abstract) tropical curve of genus g, $x_1, \ldots, x_n$ are marked points on C and $h : C \to \mathbb{R}^m$ is a tropical morphism. The curve has degree d if $h(C)$ has precisely d leaves in each of the standard directions $-e_1, \ldots, -e_m, -e_0 = \sum_{i=1}^m e_i \in \mathbb{R}^m$.

The translation of the enumerative algebraic question dealt with above to tropical geometry is the following: Given a collection $\mathcal{L}$ of incidence conditions $L_1, \ldots, L_m$ in $\mathbb{R}^m$, which are tropical L_m^m-directional varieties (i.e. their recession fan has standard directions and is therefore some L_k^m), how many parameterized n-marked tropical curves (C, h) of degree d and genus g, counted with multiplicity $\operatorname{mult}(C, h, \mathcal{L})$, fulfill all conditions $L_1, \ldots, L_n$? We denote this number by $N_{\text{trop}}(d, g, \mathcal{L})$.

Why do we count with multiplicity and what does $\operatorname{mult}(C, h, \mathcal{L})$ stand for? Remember that the main motivation for dealing with tropical geometry is its relation to algebraic geometry. We want to answer algebraic questions on the tropical side. So assume that the collection $\mathcal{L}'$ of incidence conditions $L_1', \ldots, L_n'$ in $\mathbb{P}^m$ tropicalizes to the tropical varieties $L_1, \ldots, L_n$ in $\mathbb{R}^m$. Then the multiplicity $\operatorname{mult}(C, h, \mathcal{L})$ is defined as the number of stable maps $(C, x_1, \ldots, x_n, f)$ that fulfill all algebraic incidence conditions $L_1', \ldots, L_1'$ and such that f_*C tropicalizes to $h(C)$. In particular, if $\operatorname{mult}(C, h, \mathcal{L})$ is greater than zero, (C, h) fulfills all tropical incidence conditions L_i (which are the tropicalizations of the algebraic varieties L_i').

It follows from this definition of the multiplicities that the algebraic invariant $N^1_{\text{cplx}}(d, g, \mathcal{L}')$ is equal to the number $N_{\text{trop}}(d, g, \mathcal{L})$. Therefore, also the tropical number $N_{\text{trop}}(d, g, \mathcal{L})$ does not depend on the position of the varieties $L_1, \ldots, L_n$ in $\mathbb{R}^m$.

In the case of plane tropical curves and in the case of rational tropical curves in any dimension, there exists correspondence theorems which state a formula for calculating the multiplicities $\operatorname{mult}(C, h, \mathcal{L})$, see [**Mik05, NS06, Tyo12**]. For elliptic curves in $\mathbb{R}^m$ with $m > 2$, there exist only partial correspondence theorems dealing with the case that the edges in the loop of the curve span the ambient space $\mathbb{R}^m$, see [**Tyo12, Nis09**]. Moreover, there are additional (but not complete) results in [**Spe07, Kat10**] dealing with the question which elliptic tropical curves in $\mathbb{R}^m$ are realizable.

In the case of rational curves, the multiplicities $\operatorname{mult}(C, h, \mathcal{L})$ do not only have an interpretation on the algebraic but also purely on the tropical side. The tropical moduli space $\mathcal{M}_{0,n}(d, \mathbb{R}^m)$ of parameterized n-marked rational tropical curves of degree d in $\mathbb{R}^m$ is a well-studied tropical variety, i.e. a pure-dimensional polyhedral complex which fulfills a balancing condition at every cell of codimension one. As on the algebraic side, there exist evaluation maps ev_i, which are morphisms, that evaluate a curve (C, h) at the marked point x_i, i.e. $\operatorname{ev}_i(C, h) = h(x_i)$. Using tropical intersection theory as developed in [**Sha12, AR10, FR12**], we can define the intersection product $\prod_{i=1}^n \operatorname{ev}_i^* L_i \cdot \mathcal{M}_{0,n}(d, \mathbb{R}^m)$. Using the correspondence theorems, it has been shown in [**Rau05**] that the multiplicity $\operatorname{mult}(C, h, \mathcal{L})$ is equal to the weight of the point (C, h) in this intersection product. It hence holds in the rational case that the invariant $N_{\text{trop}}(d, 0, \mathcal{L})$ is equal to the degree of the intersection product

$$\prod_{i=1}^n \operatorname{ev}_i^* L_i \cdot \mathcal{M}_{0,n}(d, \mathbb{R}^m).$$

Note the similarity to the algebraic definition of the virtual invariants $N^2_{\text{cplx}}(d, 0, \mathcal{L}')$.

Tropical moduli spaces of abstract n-marked tropical curves (i.e. curves without a map to $\mathbb{R}^m$) have been defined in any genus, e.g. [**Cap12, Cha12**]. However, for genus greater than zero these moduli spaces do not have a structure as tropical varieties but only as topological spaces or stacky fans (think of the latter as fans modulo a group action on the cones). Moduli spaces $\mathcal{M}'_{g,n}(d, \mathbb{R}^m)$ of parameterized n-marked tropical curves of degree d and genus g have been studied in [**KM09**] in the case of plane elliptic curves and in [**Her09**] in any genus and any dimension. These moduli spaces have a structure as a weighted polyhedral complex and as a tropical local orbit space, respectively. We can think of tropical local orbit spaces as abstract tropical varieties modulo a group action on the cells of the underlying polyhedral complex. However, for $m > 2$ these moduli spaces are, at least in the elliptic case, not appropriate for tackling enumerative questions. On the one hand, not all elliptic curve which are known to be realizable are contained in the local orbit space $\mathcal{M}'_{1,n}(\Delta, \mathbb{R}^m)$ defined in [**Her09**]. (All curves $(C, h) \in \mathcal{M}'_{1,n}(\Delta, \mathbb{R}^m)$ fulfill that the direction vectors of edges in and at the loop of $h(C)$ span $\mathbb{R}^m$. However, all 3-valent well-spaced elliptic curves in $\mathbb{R}^m$ are realizable - and well-spacedness does not restrict the dimension of the space spanned by the edges in and at the loop, see [**Spe07**]). On the other hand, it can be checked in small examples of curves in $\mathbb{R}^3$ that the intersection product

$$\prod_{i=1}^{n} \mathrm{ev}_i^*(L_i) \cdot \mathcal{M}'_{1,n}(\Delta, \mathbb{R}^m)$$

is in general not independent of the position of the varieties $L_1, \ldots, L_n$ in $\mathbb{R}^m$.

Results of this thesis. We study enumerative tropical questions in $\mathbb{R}^m$ for $m \geq 2$. Our approach is purely tropical and based on tropical intersection theory as developed in [**Sha12, AR10, FR12**].

In chapter 2 we deal with rational curves in $\mathbb{R}^m$. Using tropical intersection theory on the moduli spaces of rational parametrized curves, we establish with purely tropical means a recursive formula that allows to determine the degree of

$$\prod_{i=1}^{n} \mathrm{ev}_i^* L_i \cdot \mathcal{M}_{0,n}(d, \mathbb{R}^m),$$

where $L_1, \ldots, L_n$ is a collection of L^m_m-directional tropical varieties, i.e. the varieties have standard directions. As in the algebraic case, as a side product also relative invariants can be computed. The formula is the tropical analogue of the recursive formula proven in [**Vak00**] with which one can calculate the enumerative number of algebraic rational curves of degree d in $\mathbb{P}^m$ passing a configuration of subvarieties of $\mathbb{P}^m$.

As mentioned above, there already existed an indirect proof of this recursive formula using the detour over algebraic curves via correspondence theorems. Our approach is a tropical version of [**Vak00**] merged with ideas used in the plane tropical case [**GM07b**]. A related result about the recursive structure of the multiplicity of a rational tropical curve is stated in [**BM07**].

In chapter 3 we study elliptic curves.

- We develop a combinatorial notion of a well-spaced elliptic curve in $\mathbb{R}^m$, which is based on the known necessary and sufficient conditions on the realizability of elliptic curves in $\mathbb{R}^m$. In particular, for $m > 2$, a well-spaced elliptic curve fulfills all known necessary realizability conditions and all elliptic curves which are known to be realizable are also well-spaced.
- We construct a pure-dimensional weighted polyhedral complex $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ parameterizing I-marked well-spaced curves of degree Δ in $\mathbb{R}^m$, where I is an index set labeling the marked points of a curve.
- We define an open and dense weighted subcomplex $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)^{\mathrm{reg}}$ of $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ containing only elliptic curves which have an "honest" loop, i.e. the first Betti number of the support of the curve is one. The weighted polyhedral complex $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)^{\mathrm{reg}}$ is an abstract (open) tropical variety, non-regular curves without honest loop are missing and $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)^{\mathrm{reg}}$ hence has "holes".
- We prove an enumerative statement: Let $L_1, \ldots, L_n$ be translated tropical fans in $\mathbb{R}^m$ that are complete intersections, i.e. they can be cut out by rational functions on $\mathbb{R}^m$. Then the degree of the intersection product

$$\prod_{i=1}^{n} \mathrm{ev}_i^* L_i \cdot \mathcal{M}_{1,n}(\Delta, \mathbb{R}^m)^{\mathrm{reg}}$$

  is independent of the position of the translated fans L_i in $\mathbb{R}^m$ as long as their position is general. The degree of this intersection product can be interpreted as the the number of well-spaced elliptic curves in $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)^{\mathrm{reg}}$ passing a general configuration of tropical fans in $\mathbb{R}^m$, counted with an intersection-theoretic multiplicity. It is not known of this tropical invariant whether it is equal to the corresponding enumerative or virtual algebraic invariant.

At the end of the thesis an index of the notations is provided. The figure on page 4 has been produced with the help of the polymake application a-tint written by Simon Hampe using jReality.

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CHAPTER 1

Preliminaries

In this chapter, we recall some definitions and results concerning polyhedral complexes and tropical varities, intersection theory of tropical varieties as developed in [**Sha12, AR10, FR12**] and moduli spaces of rational tropical curves, e.g. [**Mik07**] [**GKM09**]. The developed notions and the stated results will be needed in the subsequent chapters.

1.1. Polyhedral complexes and tropical varieties

If Λ is a lattice, we denote the dual lattice by $\Lambda^\vee$. We regard $\mathbb{R}^m$ as a vector space containing the lattice $\mathbb{Z}^m$ with the usual embedding. We denote the standard unit vectors by $e_1, \dots, e_m$ and define $e_0 := -\sum_{i\in[m]} e_i$.

Definition and notaion 1.1.1 (Saturation and index of a lattice)
Let Λ be a lattice and let $\Lambda_\mathbb{R} := \Lambda \otimes_\mathbb{Z} \mathbb{R}$ be the corresponding real vector space containing Λ. Given a sublattice $\Lambda' \subset \Lambda$, we call

$$\operatorname{sat}(\Lambda') := \Lambda \cap (\Lambda' \otimes_\mathbb{Z} \mathbb{R})$$

the saturation of Λ'.

A morphism of lattices $f : \Lambda_1 \to \Lambda_2$ corresponds to a linear map of the corresponding real vector spaces $f : (\Lambda_1)_\mathbb{R} \to (\Lambda_2)_\mathbb{R}$ which maps Λ_1 to Λ_2. By abuse of notation we denote the two maps identically. We define the index of f by

$$\operatorname{ind} f := [\operatorname{sat}(\operatorname{im} f) : \operatorname{im} f] \in \mathbb{N}.$$

Given a sublattice Λ' of a lattice Λ we define the index of Λ' as the index of the inclusion map $i : \Lambda' \hookrightarrow \Lambda$. For elements $y_1, \dots, y_s \in \Lambda$ we define $\operatorname{ind}(y_1, \dots, y_s)$ as the index of the lattice spanned by $y_1, \dots, y_s$.

Definition 1.1.2 ((General) cone, polyhedron)
A general polyhedron in $\Lambda_\mathbb{R}$ is a subset $\emptyset \neq \tau \subset \Lambda_\mathbb{R}$ that can be described by finitely many affine linear integral equalities and inequalities, i.e. a set of the form

$$\tau = \{x \in \Lambda_\mathbb{R} | f_i(x) = a_i \text{ for all } i \in I, f_j(x) \geq b_j \text{ for all } j \in J, f_k(x) > c_k \text{ for all } k \in K\}$$

for some finite index sets I, J, K, affine linear forms $f_i, f_j, f_k \in \Lambda^\vee$ and real numbers a_i, b_j, c_k.

We denote by $\mathrm{W}(\tau)$ the smallest linear subspace of $\Lambda_\mathbb{R}$ containing all $x - y$ for all $x, y \in \tau$ and by $\Lambda(\tau)$ the lattice $\mathrm{W}(\tau) \cap \Lambda$. We define the dimension of τ to be the dimension of $\mathrm{W}(\tau)$.

A general polyhedron τ is called a polyhedron if it is closed, i.e. the affine linear forms f_i, f_j, f_k can be chosen in a way that no strict inequalities occur.

A (general) polyhedron is called a (general) cone if all a_i, b_j and c_k can be chosen to be zero.

Definition 1.1.3 ((Abstract) polyhedral complex)
An abstract polyhedral complex $\mathcal{X} = (X, Y, \{\varphi_\sigma\})$ is a topological space Y together with a finite set X of closed subsets of Y and embedding maps $\varphi_\sigma : \sigma \to (\Lambda(\sigma))_\mathbb{R} = \Lambda(\sigma) \otimes_\mathbb{Z} \mathbb{R}$ for every $\sigma \in X$ (i.e. φ_σ is continuous and injective), called polyhedral charts, such that

a) for all $\sigma, \sigma' \in X$ with $\sigma \cap \sigma' \neq \emptyset$ there exist $n \in \mathbb{N}$ and $\sigma_1, \dots, \sigma_n \in X$ with $\sigma \cap \sigma' = \cup_{i=1}^n \sigma_i$,
b) every image $\varphi_\sigma(\sigma)$, $\sigma \in X$, is a general polyhedron in $(\Lambda(\sigma))_\mathbb{R}$ such that $\mathrm{W}(\varphi_\sigma(\sigma)) = (\Lambda(\sigma))_\mathbb{R}$, i.e. $\mathrm{W}(\varphi_\sigma(\sigma))$ (which is defined in 1.1.2) is not contained in a proper subspace of $(\Lambda(\sigma))_\mathbb{R}$,

c) for every pair $\sigma, \sigma' \in X$ with $\sigma \subset \sigma'$ the maps $\varphi_\sigma \circ \varphi_{\sigma'}^{-1}$ and $\varphi_{\sigma'} \circ \varphi_\sigma^{-1}$ are affine $\mathbb{Z}$-linear where defined,
d) $Y = \dot{\bigcup}_{\sigma \in X} \varphi_\sigma^{-1}(\varphi_\sigma(\sigma)^\circ)$, where $\varphi_\sigma(\sigma)^\circ$ denotes the interior of $\varphi_\sigma(\sigma)$ in $\mathrm{W}(\varphi_\sigma(\sigma))$.

X is called the polyhedral structure of $\mathcal{X}$ and is denoted by $\mathrm{pol}(\mathcal{X})$, Y is the support $\mathrm{supp}(\mathcal{X})$ of $\mathcal{X}$. The elements of X are called polyhedra or cells of $\mathcal{X}$ and $\sigma \in X$ is called a face of $\sigma' \in X$ if $\sigma \subset \sigma'$. An inclusion-maximal polyhedron is called a facet. The relative interior of $\sigma \in X$ is defined as $\sigma^\circ = \varphi_\sigma^{-1}(\varphi_\sigma(\sigma)^\circ)$. The dimension of a polyhedron $\sigma \in X$ is defined as the dimension of $\varphi_\sigma(\sigma)$ and the dimension of the polyhedral complex as the maximal dimension of its polyhedra. We will denote the set of all k-dimensional polyhedra of $\mathcal{X}$ by $X^{(k)}$, elements are called k-cells. A polyhedral complex is called pure-dimensional if all facets have the same dimension.

For all $\sigma \in X$, we define $\mathrm{W}(\sigma) = \mathrm{W}(\varphi_\sigma(\sigma))$ and $\Lambda(\sigma) = \Lambda(\varphi_\sigma(\sigma))$.

A polyhedral complex X in $\Lambda_\mathbb{R}$ is an abstract polyhedral complex $(X, Y, \{\varphi_\sigma\})$ with $Y \subset \Lambda_\mathbb{R}$ such that all polyhedral charts $\varphi_\sigma : \sigma \to \Lambda(\sigma) \subset \Lambda_\mathbb{R}$ are translations by an element of $\Lambda_\mathbb{R}$ (where the target space is restricted to $\Lambda(\sigma)$) and such that all $\sigma \in X$ are closed in $\Lambda_\mathbb{R}$. We call it an open polyhedral complex if there exists $\sigma \in X$ that is not closed in $\Lambda_\mathbb{R}$.

We denote $(X, Y, \{\varphi_\sigma\})$ by X only because Y is given by $\bigcup_{\sigma \in X} \sigma \subset \Lambda_\mathbb{R}$ and because the polyhedral charts φ_σ are translations such that $\varphi_\sigma(\sigma) \subset \mathrm{W}(\sigma)$.

Definition 1.1.4 ((Open) subcomplex, refinement)
Let $\mathcal{X} = (X, \mathrm{supp}(\mathcal{X}), \{\varphi_\sigma\})$, $\mathcal{Y} = (Y, \mathrm{supp}(\mathcal{Y}), \{\psi_\tau\})$ be two abstract polyhedral complexes. We say that $\mathcal{X}$ is a subcomplex of $\mathcal{Y}$ if

a) for every $\sigma \in X$ there exists $\tau \in Y$ such that σ is a closed subset of τ,
b) for every pair $\sigma \in X$, $\tau \in Y$ with $\sigma \subset \tau$ the maps $\psi_\sigma \circ \psi_\tau^{-1}$ and $\psi_\tau \circ \varphi_\sigma^{-1}$ are integer affine linear where defined.

In particular $|X|$ is a closed subset of $|Y|$. If there exist $\sigma \in X$ and $\tau \in Y$ such that the inclusion $\sigma \subset \tau$ is not closed, we call $\mathcal{X}$ an open subcomplex of $\mathcal{Y}$. We call $\mathcal{X}$ a refinement of $\mathcal{Y}$ if $\mathcal{X}$ is a subcomplex of $\mathcal{Y}$ that fulfills $|X| = |Y|$.

Definition 1.1.5 ($U_\mathcal{X}(\sigma)$)
Let $\mathcal{X}$ be an abstract polyhedral complex and $\sigma \in X$. The polyhedral complex $U_\mathcal{X}(\sigma)$ is defined as the open subcomplex of $\mathcal{X}$ whose support is

$$\mathrm{supp}(U_\mathcal{X}(\sigma)) = \bigcup_{\tau \in X, \sigma^\circ \subset \tau} \tau^\circ$$

and whose polyhedral structure is given by $\{\tau \cap \mathrm{supp}(U_\mathcal{X})(\sigma) | \tau \in \mathrm{pol}(\mathcal{X})\}$.

We just write $U(\sigma)$ instead of $U_\mathcal{X}(\sigma)$ if no confusion can occur. A subset $U \subset \mathrm{supp}(X)$ is called polyhedral set if there exist $\sigma_1, \ldots, \sigma_k \in \mathrm{pol}(\mathcal{X})$ such that $U = \bigcup_{i \in [k]} \mathrm{supp}(U_\mathcal{X}(\sigma_i))$. In particular, if $\sigma \in \mathrm{pol}(\mathcal{X})$ and $\sigma^\circ \subset U$, it follows $\tau^\circ \subset U$ for all $\tau \in \mathrm{pol}(\mathcal{X})$ with $\sigma \subset \tau$.

Definition 1.1.6 ((Open) fan, normal vector)
A fan in $\Lambda_\mathbb{R}$ is a polyhedral complex in $\Lambda_\mathbb{R}$ that is a refinement of a polyhedral complex in $\Lambda_\mathbb{R}$ whose polyhedra are all (closed) cones in $\Lambda_\mathbb{R}$. An open fan F in $\Lambda_\mathbb{R}$ is an open polyhedral complex in $\Lambda_\mathbb{R}$ such that there exists a fan G in $\Lambda_\mathbb{R}$ that fulfills that F is a refinement of $U_G(\{0\})$. Note that every fan is an open fan.

Let F be an open fan and let $\sigma, \tau \in F$ with $\tau \subset \sigma$ and $\dim \tau = \dim \sigma - 1$. Then there exists $f \in \Lambda(\sigma)^\vee$ with $f|_{\mathrm{W}(\tau)} = 0_{\mathrm{W}(\tau)}$ and $f(p) > 0$ for all $p \in \sigma \setminus \tau$. The primitive integer vector

$$u_{\sigma/\tau} \in \mathrm{W}(\sigma)/\,\mathrm{W}(\tau)$$

which fulfills $f(u) > 0$ and $\langle \Lambda(\tau), \{u\} \rangle = \Lambda(\sigma)$ (which is the span of $\Lambda(\tau)$ and u in $\Lambda_\mathbb{R}$) for all $u \in u_{\sigma/\tau}$ is called normal vector of σ with respect to τ. It is unique in $\mathrm{W}(\sigma)/\,\mathrm{W}(\tau)$.

Remark 1.1.7
Every abstract polyhedral complex has a refinement such that the intersection of two polyhedra is either empty or a single polyhedron:

Let us consider polyhedra in $\mathbb{R}^m$. A polyhedron $\tau \subset \mathbb{R}^m$ has a natural structure of a polyhedral complex $\mathrm{pol}(\tau)$ in $\mathbb{R}^m$. We define a refinement: Choose a rational point $p \in \tau^\circ$ (i.e. a point such that the convex hull of p and $\sigma \in \mathrm{pol}(\tau)$ is a rational polyhedron for all $\sigma \in \mathrm{pol}(\tau)$) and for all $\sigma \in \mathrm{pol}(\tau) \setminus \{\tau\}$ define $\sigma_p \subset \mathbb{R}^m$ as the convex hull of p and σ. Set

$$\mathrm{pol}(\tau(p)) := \{\sigma_p | \sigma \in \mathrm{pol}(\tau) \setminus \{\tau\}\} \cup \{\{p\}\} \cup \{q \in \mathrm{pol}(\tau)^{(0)}\}.$$

Then, $\mathrm{pol}(\tau(p))$ is a refinement of $\mathrm{pol}(\tau)$ that has the following property: For all maximal $\sigma \in \mathrm{pol}(\tau(p))$ there exists precisely one $\sigma' \in \mathrm{pol}(\tau)$ of codimension one which fulfills $\sigma' \subset \sigma$.

Using this idea for refining an abstract polyhedral complex $\mathcal{X}$, starting with the cells of dimension one in $\mathrm{pol}(\mathcal{X})$ and proceeding to the facets, we get a refinement $\mathcal{X}'$ of $\mathcal{X}$ for which the following is true: For every $\sigma_1, \sigma_2 \in \mathrm{pol}(\mathcal{X}')$ with $\sigma_1 \cap \sigma_2 \neq \emptyset$, there exists $\sigma \in \mathrm{pol}(\mathcal{X}')$ with $\tau_1 \cap \tau_2 = \tau$.

Definition 1.1.8 (Weighted (reduced) polyhedral complex, refinement)
A d-dimensional weighted (abstract) polyhedral complex is a pure-dimensional (abstract) polyhedral complex $\mathcal{X}$ of dimension $d \in \mathbb{N}$ together with a weight function $\omega_{\mathcal{X}} : \mathrm{pol}(\mathcal{X})^{(d)} \to \mathbb{Z}$. A weighted open fan is defined similarly. If no confusion can occur, we denote it by $\mathcal{X}$ only.

Set $X^* = \{\sigma \in \mathrm{pol}(\mathcal{X}) | |\exists \tau \in \mathrm{pol}(\mathcal{X})^{(d)} : \sigma \subset \tau, \omega(\tau) \neq 0\}$, $|X^*| = \bigcup_{\sigma \in X^*} \sigma$ and $\mathcal{X}^* = (X^*, |X^*|)$. Then $(\mathcal{X}^*, \omega|_{(X^*)^{(d)}})$ is a weighted (abstract) polyhedral complex, too. We call $(\mathcal{X}, \omega_{\mathcal{X}})$ reduced if it is equal to $(\mathcal{X}^*, \omega|_{(X^*)^{(d)}})$.

$(\mathcal{Y}, \omega_Y)$ is called refinement of the weighted polyhedral complex $(\mathcal{X}, \omega_X)$ if $\mathcal{Y}^*$ is a refinement of $\mathcal{X}^*$ and if for all facets $\sigma \in \mathrm{pol}(\mathcal{Y})$ and $\tau \in \mathrm{pol}(\mathcal{X})$ with $\sigma \subset \tau$, it holds $\omega(\sigma) = \omega(\tau)$.

We define the polyhedral structure and the support of $(\mathcal{X}, \omega_{\mathcal{X}})$ as the polyhedral structure and the support of $\mathcal{X}$.

Definition 1.1.9 (Tropical fan)
An (open) tropical fan $\mathcal{F} = (F, \omega)$ of dimension $d \in \mathbb{N}$ in W is a weighted (open) d-dimensional fan that fulfills the following balancing condition for all $\tau \in F^{(d-1)}$:

$$\sum_{\sigma : \tau \subset \sigma \in F^{(d)}} \omega(\sigma) \cdot u_{\sigma/\tau} = 0 \in \mathrm{W} / \mathrm{W}(\tau)$$

Example 1.1.10 (The tropical linear spaces L_k^m)
Tropical fans that will appear frequently are the fans (L_k^m, ω) in $\mathbb{R}^m$: Set

$$L_k^m = \{\{0\}, \mathrm{Conv}(\mathbb{R}_{\geq 0} \cdot e_{i_1}, \ldots, \mathbb{R}_{\geq 0} \cdot e_{i_k}, 0) | i_j \in \{0, \ldots, m\} \, \forall \, j \in [k]\}$$

and let the weight function ω be one on every facet.

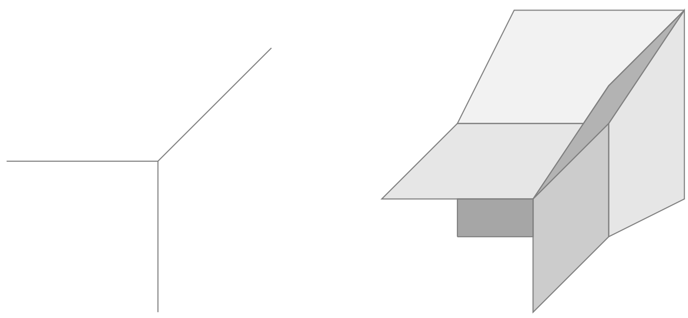


FIGURE 1. The tropical linear spaces L_1^2 and L_2^3.

Definition 1.1.11 ((General) morphism of (open) tropical fans)
Let $\mathcal{F}_1, \mathcal{F}_2$ be (open) tropical fans in $(\Lambda_1)_{\mathbb{R}}$ and $(\Lambda_2)_{\mathbb{R}}$. A morphism $f : \mathcal{F}_1 \to \mathcal{F}_2$ is a map $\mathrm{supp}(\mathcal{F}_1) \to \mathrm{supp}(\mathcal{F}_2)$ that is the restriction of an affine $\mathbb{Z}$-linear map $\mathrm{W}_1 \to \mathrm{W}_2$, i.e. the composition of a translation by a real vector and a $\mathbb{Z}$-linear map, such that for all $\sigma \in \mathcal{F}_1$ there exists

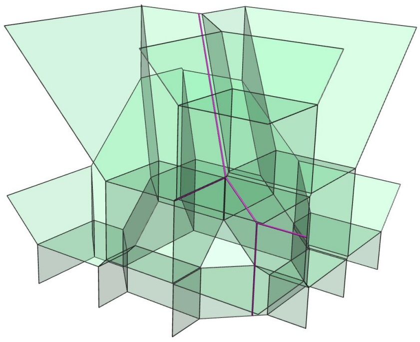


FIGURE 2. Example of a tropical surface (green) in $\mathbb{R}^3$ and a subvariety (pink) that is a curve.

$\tau \in \mathcal{F}_2$ with a closed inclusion $f(\sigma) \subset \tau$. If we drop the condition that the inclusion $f(\sigma) \subset \tau$ is closed, we call f a general morphism.

An isomorphism of tropical fans is a bijective morphism $f : \mathcal{F}_1 \to \mathcal{F}_2$ whose inverse is a morphism, too, and that respects the weight functions of $\mathcal{F}_1$ and $\mathcal{F}_2$, i.e. $\omega_{\mathcal{F}_1}(\sigma) = \omega_{\mathcal{F}_2}(\tau)$ for all facets σ of $\mathcal{F}_1$ and τ of $\mathcal{F}_2$ with $f(\sigma) \subset \tau$.

If $U_1 \subset \text{supp}(\mathcal{F}_1)$, $U_2 \subset \text{supp}(\mathcal{F}_2)$ are polyhedral sets and $g : U_1 \to U_2$ is a map that is on each connected component the restriction of a (iso-)morphism of tropical fans then we call g tropical (iso-)morphism, too.

Remark 1.1.12
There exist refinements of $\mathcal{G}_1$ and $\mathcal{G}_2$ of $\mathcal{F}_1$ and $\mathcal{F}_2$ such that $f(\sigma) \in \mathcal{G}_2$ for all $\sigma \in \mathcal{G}_1$.

Definition 1.1.13 ((Abstract, open) tropical variety, polyhedral neighborhood)
An abstract tropical variety X is a topological space $|X|$ together with a finite atlas $\{\varphi_i : U_i \to \text{supp}(F_i)\}$ where $\{U_i\}$ is an open cover of $\text{supp}(X)$ and where all φ_i are homeomorphisms into the support of an open tropical fan F_i, which are called fan charts, such that the transition maps $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are tropical isomorphisms for all i, j. (In particular $\varphi_i(U_i \cap U_j)$ and $\varphi_j(U_i \cap U_j)$ are polyhedral sets in F_i and F_j, respectively.)

X^* is the non-zero part of X, namely the abstract tropical variety with underlying topological space

$$\text{supp}(X^*) = \{x \in |X| | \exists i : \varphi_i(x) \text{ is contained in a facet of } F_i \text{ with a non-zero weight}\} \subset |X|$$

and atlas $\{\phi_i|_{U_i \cap \text{supp}(X^*)}\}$.

We define the support $\text{supp}(X)$ of the tropical variety X as the non-zero part $\text{supp}(X^*)$ of the underlying topological space $|X|$.

A tropical variety is an abstract tropical variety such that the underlying topological space is a closed subset of a real vector space $\Lambda_{\mathbb{R}}$ and such that all fan charts are translations by an element of $\Lambda_{\mathbb{R}}$. If the underlying topological space is not closed in $\Lambda_{\mathbb{R}}$, we call it an open tropical variety.

We identify abstract tropical varieties $X = (|X|, \{\varphi_i^1 : U_i^1 \to F_i^1\})$ and $X' = (|X'|, \{\varphi_j^2 : U_j^2 \to F_j^2\})$ that fulfill $\text{supp}(X^*) = \text{supp}((X')^*)$ if their atlases are compatible, i.e. if $\varphi_i^1 \circ (\varphi_j^2)^{-1}$ are tropical isomorphisms where defined.

Construction and definition 1.1.14 (Polyhedral structure of an abstract tropical variety, polyhedral neighborhood and set)
Let $X = (\text{supp}(X), \{\varphi_i : U_i \to \text{supp}(F_i)\})$ be a representative of an abstract tropical variety. Set $\text{pol}(X) = \bigcup_i \{\overline{\varphi_i^{-1}(\sigma)} | \sigma \in F_i\}$. Then $\text{pol}(X)$ together with polyhedral charts induced by φ_i is an abstract polyhedral complex $\mathcal{X}$. This is true because the F_i are open tropical fans and because of the following: For all $\sigma \in F_i$ with $\varphi_i^{-1}(\sigma^\circ) \cap U_j \neq \emptyset$ it holds $\varphi_j \circ \varphi_i^{-1}(\sigma^\circ) = \tau^\circ$ for some $\tau \in F_j$, see

definitions 1.1.11 and 1.1.13. As weight function $\omega : \mathrm{pol}(X) \to \mathbb{Z}$ we choose the one that is induced by the weight functions of the open tropical fans F_i (which is well-defined since isomorphisms of tropical fans respect the weight functions).

If we choose a different representative $\mathcal{X}'$ of the same tropical variety, $\mathcal{X}$ and $\mathcal{X}'$ have a common refinement.

An abstract polyhedral complex $\mathcal{X}$ that arises in this way from the tropical variety X is called a polyhedral structure on X.

Let $p \in \mathrm{supp}(X)$. We call a neighborhood $U \subset \mathrm{supp}(X)$ a polyhedral neighborhood of p in X if there exists a polyhedral structure $\mathcal{X}$ on X containing $\{p\}$ such that $U = U_{\mathcal{X}}(\{p\})$, see 1.1.5 for the notation. We call $U \subset \mathrm{supp}(X)$ a polyhedral set in X if there exist points $p_1, \ldots, p_k \in U$ with polyhedral neighborhoods $U_1, \ldots, U_k \subset \mathrm{supp}(X)$ such that $U = \bigcup_{i=1}^k U_i$.

Remark 1.1.15
Let X be an abstract tropical variety with polyhedral structure $\mathcal{X}$. For $\sigma \in \mathrm{pol}(\mathcal{X})$, the polyhedral neighborhood $U_{\mathcal{X}}(\sigma)$ of the polyhedron σ inherits from X a structure as tropical variety.

Example 1.1.16
There exist different abstract tropical varieties which have the same underlying topological space. Here is an example:

We equip $\mathbb{R}^2$ with the lattice spanned by the standard unit vectors and let the topological space $|X| := \mathbb{R}^2 \setminus \{(0,0)\}$ be given by the plane without the origin. Let X be the tropical variety that has $|X|$ as underlying topological space, let the polyhedral structure be induced by the rays $\mathbb{R}_{<0} \cdot e_1$, $\mathbb{R}_{<0} \cdot e_2$, $\mathbb{R}_{>0} \cdot e_1$ and $\mathbb{R}_{>0} \cdot e_2$ and let all fan charts be the identity maps. We define all weights to be one.

Next, we consider the topological space

$$|Y| := \{x \in \mathbb{R}^2 | x_1 \geq 0, x_2 > 0, x_1 - 3x_2 \geq 0\}/\sim,$$

where $\sim$ is given as follows:

$$y \sim y' \Leftrightarrow y, y' \in \{x \in \mathbb{R}^2 | x_2 > 0, x_1 = 0\} \cup \{x \in \mathbb{R}^2 | x_2 > 0, x_1 = 3x_2\} \text{ and } y_2 = y_2'$$

Note that $[(0,0)] \notin |Y|$. The topological space $|Y|$ is illustrated in figure 3 and the illustrated sets σ_i, τ_i induce an abstract polyhedral complex $\mathcal{Y}$ with support $|Y|$, $i \in [3]$ (where σ_i are the rays $\{[((i-1)x_2, x_2)] | x_2 > 0\} \subset |Y|$). The set $\{U(\sigma_i) | i \in 1,2,3\}$ is an open cover of $|Y|$, see notation 1.1.5.

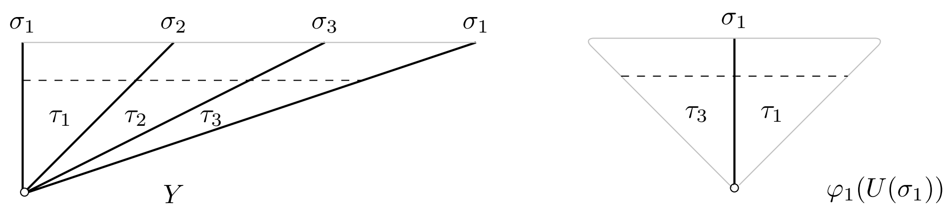


FIGURE 3. Left: The polyhedral structure of Y defined in example 1.1.16. Right: The image of $U(\sigma_1)$ under the fan chart φ_{σ_1}, which maps the intersection of $U(\sigma_1)$ and the dashed line to the dashed line.

Except φ_1, let the fan charts $\varphi_i : U(\sigma_i) \to \mathbb{R}^2$ be given by the identity map. The fan chart φ_1 is given by $[(x_1, x_2)] \mapsto (x_1 - 3x_2, x_2)$ on τ_3°, by $[(x_1, x_2)] \mapsto (x_1, x_2)$ on τ_1° and by $[(0, x_2)] \mapsto (0, x_2)$ on σ_1° (see figure 3). Let again all weights be one and denote this tropical variety by Y.

The topological spaces $|X|$ and $|Y|$ are homeomorphic. However, the tropical varieties X and Y do not agree. For example, in Y there exists a loop which is a straight line in the lattice (the dashed line in figure 3), which is not possible in X.

Definition 1.1.17 (Subvariety)
Let $X = \{|X|, \{\phi_i\}\}$ be an abstract tropical variety. We call an abstract tropical variety $C = \{|C|, \{\psi_j\}\}$ subvariety of X if $|C|$ is a closed subset of $|X|$ and if $\phi_i \circ \psi_j^{-1}$ and $\psi_j \circ \phi_i^{-1}$ are tropical morphisms where defined. We call C an open subvariety of X, if C fulfills the weaker condition (compared to $|C| \subset |X|$ closed) that there exist polyhedral structures $\mathcal{C}$ on C and $\mathcal{X}$ on X such that $\mathcal{C}$ is an open subcomplex of $\mathcal{X}$.

We denote the set of subvarieties of X by $Z(X)$ and the set of d-dimensional subvarieties of X by $Z_d(X)$.

Example 1.1.18
The open unit interval with weight one is an open subvariety of $\mathbb{R}$ with weight one, but not a subvariety.

Lemma 1.1.19 ([**AR10**]: 5.15)
Let X be an abstract tropical variety. Given two closed subvarieties X_1, X_2 of dimension d, one can define their sum $X_1 + X_2$, which is again a subvariety of X (see e.g. 5.14 in [**AR10**]). $Z_d(X)$ together with this operation "+" forms an abelian group.

Remark 1.1.20
The statement of the previous lemma is not true for open subvarieties. Consider for example the open intervals $(0, 1)$ and $(0, 2)$ with weight one as open subvarieties of $\mathbb{R}$ with weight one. Trying to define the sum $(0, 1) + (0, 2)$, it seems reasonable to choose $(0, 1) \cup (0, 2) = (0, 2)$ as underlying topological space. Moreover, it seems reasonable to take $\{(0, 1], \{1\}, [1, 2)\}$ as polyhedral structure and define the weights as $\omega((0, 1]) = 2$ and $\omega([1, 2)) = 1$. However, this open polyhedral complex is not balanced at $\{1\}$. In order to remedy this problem, we would have to allow polyhedral structures with superposed polyhedra, e.g. $\{(0, 1), (0, 1], \{1\}, [1, 2)\}$. This is possible but not used in this thesis. Therefore, we do not develop this concept here.

Definition 1.1.21 (Degree)
Let X be a zero-dimensional tropical variety. Then we define its degree by

$$\deg(X) := \langle X \rangle := \sum_{p \in \operatorname{supp}(X)} \omega(p).$$

1.2. Intersection theory on tropical varieties

We recall some main definitions and results in tropical intersection theory from [**Rau09**], [**All09**] and [**FR12**]. Additionally, we prove some corollaries that will be needed later on.

Intersection with rational functions.

Definition 1.2.1 (Rational function)
Let X be an abstract tropical variety and let $U \subset \operatorname{supp}(X)$ be an (open) polyhedral set. A rational function on U is a continuous map $\varphi : U \to \mathbb{R}$ such that there exists a polyhedral structure $\mathcal{X}$ on X with polyhedral charts φ_σ for which $\varphi \circ \varphi_\sigma^{-1}$ is affine $\mathbb{Z}$-linear for all $\sigma \in \mathcal{X}$. We call such a polyhedral structure on X a φ-polyhedral structure and a representative that induces such a structure a φ-representative. Note that the restriction of a rational function to a polyhedral subset $U' \subset U$ is a rational function on U'.

Let the polyhedral complex $\mathcal{X}$ be a polyhedral structure on X. Then, for all $\sigma \in \operatorname{pol}(\mathcal{X})$, there exists a unique $\lambda \in \Lambda(\sigma)^\vee$ and $c \in \mathbb{R}$ such that $\varphi \circ \varphi_\sigma^{-1}| = (\lambda + c)|_\sigma$. We set $\varphi_\sigma := \lambda$.

Construction 1.2.2 (Intersection with rational functions)
Let X be an abstract tropical variety of dimension $d \in \mathbb{N}$, C a subvariety of X and $\varphi : U \to \mathbb{R}$ a rational function on a polyhedral set $U \supset \operatorname{supp}(C)$. Choose a φ-representative $(\operatorname{supp}(C), \{\varphi_i : U_i \to \operatorname{supp}(F_i)\})$ with corresponding polyhedral structure $\mathcal{C}$ and denote by $\sigma_i \in \operatorname{pol}(\mathcal{C})$ the polyhedron with $U_i = U_{\mathcal{C}}(\sigma_i)$. We define the intersection product $\varphi \cdot C$ as the abstract tropical variety $(\operatorname{supp}(\mathcal{C}^{(d-1)}), \{\varphi_i'\})$ where $\varphi_i' = \varphi_i|_{U_i \cap \operatorname{supp}(\mathcal{C}^{(d-1)})}$ and the weight function is given by

$$\begin{aligned} \omega_{\varphi \cdot C} : \mathcal{C}^{(d-1)} &\to \mathbb{Z} \\ \tau &\mapsto \sum_{\substack{\sigma \in \mathcal{C}^{(d)} \\ \tau \subset \sigma}} \varphi_\sigma\Big(\omega(\sigma) v_{\sigma/\tau}\Big) - \varphi_\tau\Big(\sum_{\substack{\sigma \in \mathcal{C}^{(d)} \\ \tau \subset \sigma}} \omega(\sigma) v_{\sigma/\tau}\Big), \end{aligned}$$

where $v_{\sigma/\tau}$ are arbitrary representatives of the normal vectors $u_{\sigma/\tau}$. (Note that $\sum \omega(\sigma) v_{\sigma/\tau} \in \Lambda_\tau$ due to the balancing condition.)

Example 1.2.3
The tropical linear spaces L_k^m arise as the intersection product

$$(\max\{0, x_1, \ldots, x_m\})^{m-k} \cdot \mathbb{R}^m.$$

Proposition 1.2.4 ([**Rau09**]: 1.2.13, [**AR10**]: 6.4 and 6.7)
Let X be an abstract tropical variety and φ a rational function on X. Then $\varphi \cdot X$ is a subvariety of X of codimension one. It holds $\varphi \cdot (\psi \cdot X) = \psi \cdot (\varphi \cdot X)$.

Remark 1.2.5
Let X be a tropical fan in $\mathbb{R}^m$ and let $\varphi_1, \ldots, \varphi_r : \mathbb{R}^m \to \mathbb{R}$ be integer linear functions. For $i \in [r]$ define $\psi_i : \mathbb{R}^m \to \mathbb{R}$, $x \mapsto \max\{\varphi_i(x), 0\}$. Then, for any $\sigma \in \operatorname{pol}(\mathcal{X})$, $\prod_{i \in [r]} \psi_i \cdot U(\sigma)$ is the translation of a tropical fan in $\mathbb{R}^m$.

Remark 1.2.6 (On the support and the weights in an intersection product)
Let $r \in \mathbb{N}$ and let X be a tropical fan in $\mathbb{R}^m$. Let $\varphi_i : X \to \mathbb{R}^m$ be rational functions and $a_i \in \mathbb{R}$, $i \in [r]$, and let $\mathcal{X}$ be a φ_i-representative for all $i \in [r]$. Then, by the definition of the intersection product, the reduced part of the support of

$$\prod_{i \in [r]} \max\{\varphi_i, a_i\} \cdot X$$

is contained in the union of the $(\dim X - r)$-dimensional polyhedra $\sigma \in \mathcal{X}$ which fulfill that $\max\{\varphi_i, a_i\}|_{U(\sigma)}$ is not affine linear for all $i \in [r]$. (If $\max\{\varphi_j, a_j\}|_{U(\sigma)}$ were affine linear, the weight of σ in $\max\{\varphi_j, a_j\} \cdot U(\sigma)$ and hence in $\prod_{i \in [r]} \max\{\varphi_i, a_i\} \cdot X$ would be zero.)

Definition 1.2.7 (Star, e.g. [**Rau09**]: 1.2.3)
Let X be a tropical variety in a real vector space W and let $\mathcal{X}$ be a polyhedral structure on X. Let $\sigma \in \text{pol}(\mathcal{X})$ and denote by $q : \mathrm{W} \to \mathrm{W} / \mathrm{W}(\sigma)$ the quotient map. For $\tau \in \text{pol}(\mathcal{X})$ with $\sigma \subset \tau$ denote by $\overline{\tau}$ the closed cone in $\mathrm{W} / \mathrm{W}(\sigma)$ that is spanned by the image of $\tau - \sigma$ under q. We define $\text{Star}_{\mathcal{X}}(\sigma)$ as the tropical fan

$$\text{Star}_{\mathcal{X}}(\sigma) = \{\overline{\tau} | \sigma \subset \tau \in \text{pol}(\mathcal{X})\}$$

where the weights are given by $\omega(\overline{\tau}) = \omega(\tau)$. We call $\text{Star}_{\mathcal{X}}(\sigma)$ the star of $\mathcal{X}$ at σ and we denote the associated tropical fans by $\text{Star}_X(\sigma)$.

For a point $p \in X$ we define $\text{Star}_X(p)$ as the tropical variety associated to $\text{Star}_{\mathcal{X}}(\{p\})$ for some polyhedral structure $\mathcal{X}$ of X with $\{p\} \in \text{pol}(\mathcal{X})$.

Proposition 1.2.8 (Locality of the intersection product, [**Rau09**]: 1.2.12)
Let X be a tropical variety in W and let $\varphi_1, \ldots, \varphi_l$ be rational functions on X. Let $\sigma \in \text{pol}(\mathcal{X})$ for some φ_i-representative $\mathcal{X}$, $i \in [l]$. Then the intersection product can be computed locally on $\text{Star}_X(\sigma)$, i.e.

a) $\text{Star}_{\varphi_1 \cdots \varphi_l \cdot X}(\sigma) = (\varphi_1)_\sigma \cdots (\varphi_l)_\sigma \cdot \text{Star}_X(\sigma)$
b) If $l = \text{codim}(\sigma)$, then

$$\omega_{\varphi_1 \cdots \varphi_l \cdot X}(\sigma) = \omega_{(\varphi_1)_\sigma \cdots (\varphi_l)_\sigma \cdot \text{Star}_X(\sigma)}(\{0\}).$$

Lemma 1.2.9 ([**Rau09**]: 1.2.9)
Let $h_1, \ldots, h_l$ be integer linear functions on W with $l \leq \dim \mathrm{W} = r$ and define the rational functions $\varphi_i = \max\{h_i, 0\}$ on W. Let $H : \mathrm{W} \to \mathbb{R}^l$ be the linear function that is given by $x \mapsto (h_1(x), \ldots, h_l(x))$ and assume that H has full rank. Then $\varphi_1 \cdots \varphi_l \cdot \mathrm{W}$ is equal to $\ker(H)$ with weight $\text{ind}\, H = |\mathbb{Z}^l / H(\Lambda)|$.

Definition 1.2.10 (Morphism of tropical varieties)
Let X, Y be abstract tropical varieties. A tropical morphism $f : X \to Y$ is a map

$$f : \text{supp}(X) \to \text{supp}(Y)$$

such that there exist representatives $\mathcal{X} = (|X|, \{\varphi_i : U_i^X \to \mathrm{W}_i^X\})$, $\mathcal{Y} = (|Y|, \{\psi_j : U_j^Y \to \mathrm{W}_j^Y\})$ such that all $\psi_j \circ f \circ \varphi_i^{-1}$ are morphisms of open tropical fans, see 1.1.11. If the maps $\psi_j \circ f \circ \varphi_i^{-1}$ are general morphisms of open tropical fans, we call f a general tropical morphism. An isomorphism of tropical varieties is a bijective morphism whose inverse is a tropical morphism, too, and which respects the weight functions on X and Y.

Remark 1.2.11
The restriction of a morphism of tropical varieties $f : X \to Y$ to a subvariety is again a morphism. This is not true for the restriction to an open subvariety: In this case the image $f(\sigma)$ of a polyhedron $\sigma \in \mathcal{X}$, $\mathcal{X}$ a polyhedral structure on X, need not be closed in $\text{supp}(Y)$. The restriction of a general tropical morphism to an open subvariety is again a general tropical morphism.

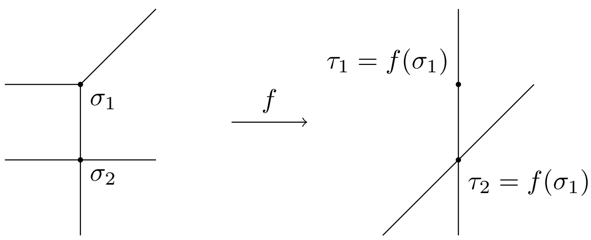


FIGURE 4. Look at the tropical varieties X (left) and Y (right) above in $\mathbb{R}^2$ whose weights are all one. The map which is given on $U(\sigma_1)$ by $(x_1, x_2) \mapsto (0, x_2)$ and on $U(\sigma_2)$ by $(x_1, x_2) \mapsto (x_1, x_1 + x_2)$ is a morphism which is not globally given by an affine $\mathbb{Z}$-linear map.

Definition 1.2.12 (Pull-back, e.g. [**Rau09**]: 1.3.3)
Let $f : X \to Y$ be a general morphism of abstract tropical varieties and let φ be a rational function on Y. We call the rational function $f^*\varphi$ on X the pull-back of φ along f.

Construction 1.2.13 (Push-forward)
Let $f : X \to Y$ be a morphism of (abstract) tropical varieties. Choose representatives $\mathcal{X}$ of X and $\mathcal{Y}$ such that $f(\sigma) \in \mathrm{pol}(\mathcal{Y})$ for all $\sigma \in \mathrm{pol}(\mathcal{X})$. Define

$$\mathrm{pol}(f_*\mathcal{X}) := \{f(\sigma) | \sigma \in \mathcal{X} \text{ is contained in a facet on which } f \text{ is injective}\}$$

with weight function

$$\omega_{f_*\mathcal{X}}(\tau) = \sum_{\substack{\sigma\in\mathcal{X}:\\ f(\sigma)=\tau}} \omega_{\mathcal{X}}(\tau) \cdot |\Lambda(\tau) : f_\sigma(\Lambda(\sigma))|$$

where $f_\sigma : \mathrm{W}(\sigma) \to \mathrm{W}(\tau)$ is the linear map induced by the restriction of f to σ. Put $\mathrm{supp}(f_*\mathcal{X}) = \bigcup_{\sigma\in\mathrm{pol}(f_*\mathcal{X})} \sigma$.

Lemma 1.2.14 ([**AR10**]: 7.4)
Use the notation from the previous construction. Then the abstract polyhedral complex $f_*\mathcal{X}$ is a polyhedral structure on a subvariety of Y. We denote this subvariety by f_*X.

Lemma 1.2.15 ([**Rau09**]: 1.3.8 and 1.3.9)
Let $f : X \to Y$, $f' : X' \to Y'$ and $g : Y \to Z$ be morphisms of abstract tropical varieties. Then it holds for all subvarieties W of X and W' of X'

a) $(f \times f')_*(W \times W') = f_*(W) \times f'_*(W')$ and
b) $(g \circ f)_*W = g_*(f_*W)$.

Theorem 1.2.16 (Projection formula, [**AR10**]: 7.7)
Let $f : X \to Y$ be a morphism of (abstract) tropical varieties and let φ be a rational function on Y. Then it holds for all $C \in Z(X)$ that

$$\varphi \cdot (f_*C) = f_*(f^*\varphi \cdot C).$$

Intersecting tropical varieties. We will describe properties of an intersection product of tropical varieties from [**FR12**] that is compatible with the intersection of a tropical variety with a rational function. This is not possible for all varieties, only on those that locally look like matroidal fans.

Definition 1.2.17 (Matroid variety, matroidal fan)
Let $M = (E, \mathcal{B})$ be a loop-free matroid with ground set $E = [m]$ of rank d. To a chain of flats $\mathcal{F} = (\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_{p-1} \subsetneq F_p = E)$ we associate the cone

$$\langle\mathcal{F}\rangle := \left\{ \sum_{i=1}^{p} \lambda_i \cdot v_{F_i} : \lambda_1, \dots, \lambda_{p-1} \geq 0, \lambda_p \in \mathbb{R} \right\},$$

where $v_F = -\sum_{i\in F} e_i$ for $F \subset E$. Define $\mathrm{pol}(\mathcal{B}(M))$ as the set of cones in $\mathbb{R}^m$ that correspond to a chain of flats of M and define the weight function ω to be one on every facet. Then $\mathcal{B}(M) = (\mathrm{pol}(\mathcal{B}(M), \omega)$ is a tropical fan in $\mathbb{R}^m$ with lineality space (at least) $L = \mathbb{R} \cdot e_0$. Denote by $B(M)$ the corresponding tropical variety and by $B(M)/L$ the tropical variety with lineality space L modded out. A tropical variety that is of the form $B(M)$ or $B(M)/L$ for some matroid M is called matroid variety. $\mathcal{B}(M)$ and $\mathcal{B}(M)/L$ are called matroidal fans.

Remark 1.2.18
Another definition of the support of $\mathcal{B}(M)$ is the following. For $p \in \mathbb{R}^m$ define the p-weight of a basis $B \subset M$ as $\sum_{i\in B} p_i$. Let M_p be the matroid that is given by the p-minimal bases of M. Using this notation, the support of $\mathcal{B}(M)$ is given by the points $p \in \mathbb{R}^m$ such that the matroid M_p is still loop-free.

Example 1.2.19
The tropical linear spaces L_k^m are matroid varieties that correspond to the uniform matroids $U_{k+1,m+1}$ (that have rank $k+1$, $m+1$ elements and all subsets of the ground set with $k+1$ elements are bases) modulo lineality space.

Lemma 1.2.20 ([**FS05**]: 4.2)
Let X be a matroidal fan and let $\sigma \in X$ be a cone. Then the star $\mathrm{Star}_X(\sigma)$ is a matroidal fan, too.

Definition 1.2.21 (Smooth variety, [**FR12**]: 6.1)
A smooth variety is an abstract tropical variety X that has a representative $(\mathrm{supp}(X), \{\varphi_i : U_i \to \mathrm{supp}(F_i)\})$ such that all F_i are open matroidal fans.

Let C, D be subvarieties of a smooth variety X. In [**FR12**], an intersection product $C \cdot D \in Z(X)$ on X is constructed. It has the following properties:

Lemma 1.2.22 (Properties of the intersection product, [**FR12**]: 6.4)
For all subcycles C, D, E of a smooth variety X and all rational functions $\varphi_1, \dots, \varphi_s$ on X the following holds:

a) If X is a matroid variety and if C and D are fans, then $C \cdot D$ is a fan, too.
b) $\mathrm{supp}(C \cdot D) \subset \mathrm{supp}(C) \cap \mathrm{supp}(D)$.
c) $(\varphi_1 \cdot C) \cdot D = \varphi_1 \cdot (C \cdot D)$.
d) $C \cdot X = C$.
e) $C \cdot D = D \cdot C$.
f) $(C \cdot D) \cdot E = C \cdot (D \cdot E)$.
g) $(C + D) \cdot E = C \cdot E + D \cdot E$.
h) If $C = \varphi_1 \cdots \varphi_s \cdot X$, then $C \cdot D = \varphi_1 \cdots \varphi_s \cdot D$.
i) $(A_1 \times A_2) \cdot (B_1 \times B_2) = (A_1 \cdot B_2) \times (A_2 \cdot B_2)$ if A_1, B_1 and A_2, B_2 are subvarieties of two smooth varieties X_1 and X_2, respectively.

Remark 1.2.23 ([**FR12**]: 6.3 and 6.4)
The intersection product can be calculated locally on $\mathrm{Star}_X(p)$: Let C, D be subvarieties of a smooth variety X and let $p \in \mathrm{supp}(X)$. Then the following holds on $\mathrm{Star}_X((p)$:

$$\mathrm{Star}_{C \cdot D}(p) = \mathrm{Star}_C(p) \cdot \mathrm{Star}_D(p)$$

Definition 1.2.24 (Pull-back of varieties, [**FR12**]: 8.1)
Let $f : X \to Y$ be a morphism of smooth tropical varieties and let C be a subvariety of Y. Then the pull-back of C along f is defined as

$$f^*C \coloneqq \pi_*(\Gamma_f \cdot (X \times C)),$$

where $\pi : X \times Y \to X$ is the projection to the first factor and Γ_f is the graph of f, i.e. $\Gamma_f \coloneqq {\gamma_f}_*(X)$ with $\gamma_f : X \to X \times Y$, $x \mapsto (x, f(x))$.

Lemma 1.2.25 (Properties of the pull-back, projection formula [**FR12**]: 8.2 and 8.3)
Let $f : X \to Y$, $g : Y \to Z$ be morphisms of smooth tropical varieties. Moreover, let C and C' be subcycles of Y, D of X and E of Z. Then the following holds:

a) $f^*(C + C') = f^*C + f^*C'$.
b) $\mathrm{supp}(f^*C) \subset f^{-1}(\mathrm{supp}(C))$.
c) The projection formula $C \cdot f_*D = f_*(f^*C \cdot D)$ is valid.
d) $f^*(C \cdot C') = f^*(C) \cdot f^*(C')$.
e) $(g \circ f)^*E = f^*(g^*E)$.
f) $f^*C = f^*\varphi_1 \cdots f^*\varphi_k \cdot X$ if $\varphi_1, \dots, \varphi_k$ are rational functions on Y with $C = \varphi_1 \cdots \varphi_k \cdot Y$.

Lemma 1.2.26
Let X be a smooth variety and let A, B be subvarieties of X. Denote by d the diagonal morphism $d : X \to X \times X, x \mapsto (x, x)$. Then it holds $d^*(A \times B) = A \cdot B$.

PROOF. Let $\pi : X \times X \to X$ be the projection onto the first factor. Since $d_* X$ is the diagonal Δ_X in $X \times X$, we get using the projection formula

$$\begin{aligned} A \cdot B &= \pi_*(\Delta_M \cdot (A \times B)) = \pi_*((A \times B) \cdot d_* X) \\ &= \pi_*(d_*(d^*(A \times B) \cdot X)) = (\pi \circ d)_*(d^*(A \times B) \cdot X) \\ &= \mathrm{id}_*(d^*(A \times B)) = d^*(A \times B) \end{aligned}$$

□

Lemma 1.2.27
Let $M, M_1, \ldots, M_n$ be smooth tropical varieties and for $i \in [n]$ let $f_i : M \to M_i$ be morphisms. For subvarieties A_i of M_i it holds

$$(f_1, \ldots, f_n)^*(\prod_{i=1}^n A_i) \cdot M = \prod_{i=1}^n f_i^* A_i \cdot M.$$

PROOF. It suffices to show the claim for $n = 2$. Let $d : M \to M \times M$, $x \mapsto (x, x)$ be again the diagonal morphism. We get $(f_1, f_2) = (f_1 \times f_2) \circ d$, and it follows with lemmata 1.2.25, 1.2.26 and 1.2.15

$$\begin{aligned} (f_1, f_2)^*(A_1 \times A_2) \cdot M &= ((f_1 \times f_2) \circ d)^*(A_1 \times A_2) \cdot M \\ &= d^*(f_1 \times f_2)^*(A_1 \times A_2) \cdot M \\ &= d^*(f_1^*(A_1) \times f_2^*(A_2)) \cdot M \\ &= f_1^*(A_1) \cdot f_2^*(A_2) \cdot M \end{aligned}$$

□

Lemma 1.2.28 (Pull-back of subvarieties of matroid varieties)
Let $f : F \to X$ be a general tropical morphism where X is a matroid variety and F an arbitrary tropical fan. Let $\varphi_1, \ldots, \varphi_k$ and $\psi_1, \ldots, \psi_k$ be rational functions on X with

$$C = \varphi_1 \cdots \varphi_k \cdot M = \psi_1 \cdots \psi_k \cdot M.$$

Assume that there exists a matroid variety Y such that F is a subvariety of Y and a morphism $\tilde{f} : Y \to X$ with $\tilde{f}|_F = f$. Then it holds

$$f^*\varphi_1 \cdots f^*\varphi_k \cdot F = f^*\psi_1 \cdots f^*\psi_k \cdot F \text{ and } \operatorname{supp}(f^*\varphi_1 \cdots f^*\varphi_k \cdot F) \subset f^{-1}(\operatorname{supp}(C)).$$

PROOF.

$$\begin{aligned} f^*\varphi_1 \cdots f^*\varphi_k \cdot F &= \tilde{f}^*\psi_1 \cdots \tilde{f}^*\psi_k \cdot F \cdot Y \\ &= \tilde{f}^* C \cdot F \cdot Y \end{aligned}$$

The second statement follows from lemma 1.2.25. □

Corollary 1.2.29
Let $f : X \to \mathbb{R}^m$ be a general tropical morphism and let $C = \varphi_1 \cdots \varphi_k \cdot \mathbb{R}^m = \psi_1 \cdots \psi_k \cdot \mathbb{R}^m$. Then the following is true:

$$f^*\varphi_1 \cdots f^*\varphi_k \cdot X = f^*\psi_1 \cdots f^*\psi_k \cdot X \text{ and } \operatorname{supp}(f^*\varphi_1 \cdots f^*\varphi_k \cdot F) \subset f^{-1}(\operatorname{supp}(C))$$

PROOF. Since intersection products can be calculated locally on stars, we only have to consider the case that X is a tropical fan in $\mathbb{R}^l$ and f the restriction of a $\mathbb{Z}$-linear map. We can hence extend f to a morphism $\tilde{f} : \mathbb{R}^l \to \mathbb{R}^m$ with $\tilde{f}|_X = f$. Therefore the prerequisites of the previous lemma are fulfilled and the claim follows. □

Definition 1.2.30
Let $f : X \to \mathbb{R}^m$ be a general morphism and let $C = \varphi_1 \cdots \varphi_k \cdot \mathbb{R}^m$. Using the previous lemma, the following definition is well-defined:

$$f^* C := f^*\varphi_1 \cdots f^*\varphi_k \cdot X$$

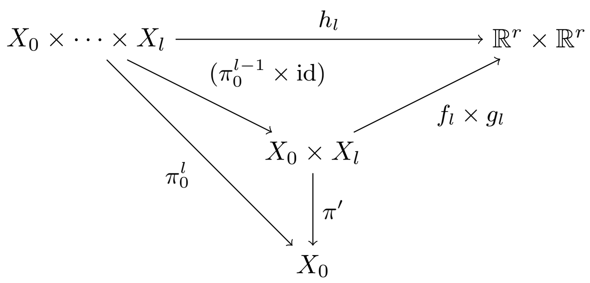


FIGURE 5. The maps h_l and π'.

Lemma 1.2.31 ([**Rau09**], 2.3.10)
Let Σ be a complete unimodular fan in $\mathbb{R}^m$ and let $B_1, \ldots, B_m$ be a basis of the group $Z_*(\Sigma)$ of Σ-directional varieties in $\mathbb{R}^m$. Let $X \sim \sum_i \lambda_i B_i$, $Y \sim \sum_j \mu_j B_j$ be two Σ-directional cycles in $\mathbb{R}^m$ with complementary dimension. Then

$$\deg(Z \cdot (X \times Y)) = \deg(X \cdot Y) = \sum_{i,j} \deg(X \cdot B_i)\beta_{ij} \deg(Y \cdot B_j),$$

where $(\beta_{ij})_{ij}$ denotes the inverse of $\alpha := (\deg(B_i \cdot B_j))_{ij}$ and $\deg(\cdot)$ is set to zero if the dimension of the argument is non-zero.

Lemma 1.2.32
Let $l \in \mathbb{N}$ and let $X_0, \ldots, X_l$ be subvarieties of a matroidal fan and denote by Z the diagonal in $\mathbb{R}^m \times \mathbb{R}^m$. Denote by

$$\pi_k^{j+1} : X_0 \times \cdots \times X_j \to X_k$$

the projection on the k-th factor, $0 \leq k \leq j \leq l$. Moreover, for $k \in [l]$ let $f_k : X_0 \to \mathbb{R}^m$, $g_k : X_k \to \mathbb{R}^m$ be morphisms and define

$$h_k := (f_k \times g_k) \circ (\pi_0^l \times \pi_k^l) : X_0 \times \cdots \times X_l \to \mathbb{R}^m \times \mathbb{R}^m.$$

Then the following formula is valid:

$$(\pi_0^l)_* \left(\left(\prod_{k \in [l]} h_k^*(Z) \right) \cdot (X_0 \times \cdots \times X_l) \right) = \prod_{k \in [l]} f_k^*((g_k)_* X_k) \cdot X_0.$$

PROOF. According to 1.3.27 the pull-back $f_k^*((g_k)_* X_k)$ and the intersection product

$$\prod_{k \in [l]} f_k^*((g_k)_* X_k) \cdot X_0$$

are defined.

For $l = 0$ the claim is obviously true. Now, let $l \in \mathbb{N}$ and define

$$h'_k := (f_k \times g_k) \circ (\pi_0^l \times \pi_k^l) : X_0 \times \cdots \times X_{l-1} \to \mathbb{R}^r \times \mathbb{R}^r$$

for $k \in [l-1]$ and

$$h'_l := f_l \times g_l : X_0 \times X_l \to \mathbb{R}^m \times \mathbb{R}^m.$$

Note that $h_l = h'_l \circ (\pi_0^l \times \mathrm{id})$ and let $\pi' : X_0 \times X_l \to X_0$ be the projection on the first factor. By induction we conclude

$$\begin{aligned}
& (\pi_0^{l+1})_* \left(\left(\prod_{k\in[l]} h_k^*(Z) \right) (X_0 \times \cdots \times X_l) \right) \\
= \quad & (\pi')_*(\pi_0^l \times \mathrm{id})_* \left((h_l)^*(Z) \left(\left(\prod_{k\in[l-1]} (h'_k)^*(Z) \right) \left(\prod_{k\in[l-1]} X_k \right) \times X_l \right) \right) \\
\overset{\text{p.f.}}{=} \quad & (\pi')_* \left((h'_l)^*(Z) \cdot (\pi_0^l \times \mathrm{id})_* \left(\left(\prod_{k\in[l-1]} (h'_k)^*(Z) \right) \left(\prod_{k\in[l-1]} X_k \right) \times X_l \right) \right) \\
\overset{\text{induction}}{=} \quad & (\pi')_* \left((h'_l)^*(Z) \cdot \left(\left(\prod_{i\in[l-1]} f_i^*((g_i)_* X_i) \right) \times X_l \right) \right) \\
\overset{\text{induction}}{=} \quad & \prod_{i\in[l]} f_i^*((g_i)_* X_i) \cdot X_0.
\end{aligned}$$

□

Rational equivalence.

Definition 1.2.33 (Rational equivalence, R_X)
Let Z be a subvariety of the abstract tropical variety X. Define

$$R_Z \coloneqq \{\varphi \in \mathcal{O}(Z) | \varphi(\sigma) \text{ compact } \forall\, \sigma \in \mathrm{pol}(Z),\, \sigma \subset \mathrm{supp}(Z^*)\}.$$

We call Z rationally equivalent to zero if there exist a morphism $f : Z' \to Z$ and $\varphi \in \mathcal{O}(Z')$ with

$$f_*(\varphi \cdot Z') = Z.$$

We call two subvarieties Z_1, Z_2 of X that have the same dimension rationally equivalent if $Z_1 - Z_2 \in Z(X)$ is rationally equivalent to zero.

Remark 1.2.34
This definition of rational equivalence differs slightly from the one given in [**AR08**]. There R_Z is defined as the set of bounded rational functions on Z. However, if we only consider tropical varieties in W, whose polyhedra are closed subsets of W, the two definitions coincide. With the definition from [**AR08**], all points in the open unit interval (an open tropical variety with weight one for example) would be rationally equivalent to zero, which makes this definition inappropriate for enumerative purposes. This is the reason why we changed the definition.

Remark 1.2.35
Note that the restriction $\varphi|_Y$ of $\varphi \in R_X$ to a subvariety Y of X is an element of R_Y. This is true since we demand in the definition of a subvariety that $\mathrm{supp}(Y)$ is a closed subset of $\mathrm{supp}(X)$.

Since intersection products are calculated locally, the following lemma is a consequence of [**AR08**] Lemma 2, [**FR12**] 9.2 and [**All09**] 1.7.6.

Lemma 1.2.36
Let $f : X' \to X$ be a morphism of abstract tropical varieties. Assume that C' and C are subvariety of X' and X, respectively, that are rationally equivalent to zero. Let φ be a rational function on X. Then the following holds:

a) $\varphi \cdot C$ is rationally equivalent to zero.
b) Let X be a matroid variety and let D be another subvariety of X. Then $C \cdot D \in Z(X)$ is also rationally equivalent to zero.
c) $f^*\varphi \in R_{X'}$ if $\varphi \in R_X$.
d) $f_*(C')$ is rationally equivalent to zero.
e) f^*C is rationally equivalent to zero.

f) Assume that C is zero-dimensional. Then $\deg(C) = 0$. In particular, two zero-dimensional varieties which are rationally equivalent have the same degree.

Lemma 1.2.37 (Translations of tropical varieties are rationally equivalent, [**AR08**]: Lemma 3)
Let C be a tropical variety in $\Lambda_{\mathbb{R}}$ and let $C(v)$ denote the translation of C by a vector $v \in \Lambda_{\mathbb{R}}$. Then $C(v)$ is rationally equivalent to C.

Definition 1.2.38 (Recession cone $\mathrm{Re}(\sigma)$)
Let $(\mathcal{X}, \omega_{\mathcal{X}})$ be a weighted polyhedral structure on a d-dimensional tropical variety X in $\Lambda_{\mathbb{R}}$ and let $\sigma \in X$. Define the recession cone $\mathrm{Re}(\sigma) \subset \mathbb{R}^m$ of σ as the cone that consists of all $v \in \mathbb{R}^m$ such that there exists $p \in \sigma$ with $p + \mathbb{R}_{\geq 0} \cdot v \subset \sigma$. If $\mathrm{Re}(\sigma)$ is d-dimensional, we define its weights as

$$\omega(\mathrm{Re}(\sigma)) = \sum_{\substack{\sigma' \in \mathcal{X}: \\ \mathrm{Re}(\sigma) \subset \mathrm{Re}(\sigma')}} \omega_{\mathcal{X}}(\sigma')$$

.

Lemma 1.2.39 ([**AR08**]: Theorem 7)
Let X be a tropical variety. Then there exists a representative $\mathcal{X}$ of X such that $\{\mathrm{Re}(\sigma) | \sigma \in \mathrm{pol}(\mathcal{X})\}$ is a tropical fan, denoted by $\mathrm{Re}(X)$, called the recession fan of X, and does not depend on the chosen representative.

Theorem 1.2.40 ([**AR08**]: Theorem 10)
Let X be a tropical variety in a real vector space $\Lambda_{\mathbb{R}}$. Then $\mathrm{Re}(X)$ is rationally equivalent to X. $\mathrm{Re}(X)$ is the only tropical variety which X is rationally equivalent to and whose polyhedral structures are tropical fans.

Remark 1.2.41 (Recession fan of $\prod_{i \in [r]} \max\{\varphi_i, a_i\} \cdot X$)
Let X be a tropical fan in $\Lambda_{\mathbb{R}}$, let $\varphi_i : \Lambda_{\mathbb{R}} \to \mathbb{R}$ be linear, $i \in [r]$ and let $a \in \mathbb{R}^r$. Then the recession fan of $\prod_{i \in [r]} \max\{\varphi_i, a_i\} \cdot X$ is

$$\prod_{i \in [r]} \max\{\varphi_i, 0\} \cdot X.$$

This is true because $\prod_{i \in [r]} \max\{\varphi_i, a_i\} \cdot X$ and $\prod_{i \in [r]} \max\{\varphi_i, 0\} \cdot X$ are rationally equivalent and because $\prod_{i \in [r]} \max\{\varphi_i, 0\} \cdot X$ is a tropical fan.

1.3. Tropical curves and moduli spaces of rational tropical curves

We recall some general facts about tropical curves and moduli spaces of rational tropical curves, see for example [**GKM09**] and [**Rau09**].

Smooth abstract curves. The linear spaces L_1^m will be the local models of smooth curves.

Definition 1.3.1 (Smooth curves, vertex, edge, leaf, flag)
A (smooth) abstract curve C is a one-dimensional connected closed tropical variety which is locally isomorphic to L_1^r for suitable $r \in \mathbb{N}$. The genus of C is defined as the first Betti number of $\text{supp}(C)$.

We denote by $\mathbf{V}(C)$ the set of points in $\text{supp}(C)$ that are contained in at least three 1-cells of every representative of C. The elements of $\mathbf{V}(C)$ are called vertices of C. The number of 1-cells containing $v \in \mathbf{V}(C)$ is called valence of v. We call a connected subset $E \subset \text{supp}(C)$ an edge of C if the following property is fulfilled: If $E \subsetneq C$ has a boundary point and if there exists a vertex $v \in \mathbf{V}(C)$ with $v \in E$, then v is a boundary point of E. By $\mathbf{E}(C)$ we denote the set of edges of C. If it is clear which curve we refer to, we write $\mathbf{V}$ and $\mathbf{E}$ instead of $\mathbf{V}(C)$ and $\mathbf{E}(C)$. Edges which have only one boundary point are called leaves.

A pair (p, E) consisting of $p \in \text{supp}(C)$ and an edge $E \in \mathbf{E}(C)$ such that p is a boundary point of E is called a flag of C. A pair (p, E') such that E' is a connected subset (which is not a point) of an edge $E \in \mathbf{E}(C)$, such that p' is a boundary point of E and such that the other boundary point of E (if it exists) is a vertex is called a flag segment of C. We denote the set of flag segments of C by $\mathbf{FS}(C)$ and the set of flags of C by $\mathbf{F}(C)$.

Let I be a finite set with $\#I \geq 3$. An (I, G)-marked abstract curve is a triple (C, I, G) where C is a smooth abstract curve with $\#I$ leaves that we label by the elements of I and where $G : \text{supp}(C) \to \mathbb{N}$ is a map that fulfills $\#\{p \in \text{supp}(C) | G(p) > 0\} < \infty$. If a leaf is labeled by $i \in I$, we denote it by x_i. We denote (C, I, G) also by (C, I) or by C and call it I-marked abstract curve if no confusion can occur.

For all $p \in \text{supp}(C)$, the genus of p is defined as $G(p) \in \mathbb{N}$. If g' is the genus of C, the genus g of (C, I, G) is defined as

$$g := g' + \sum_{p \in \text{supp}(C)} G(p).$$

Let $L \subset \text{supp}(C)$ be the set of points that are contained in a circuit of C. We define the loop of C by

$$C_L := L \cup \{p \in \text{supp}(C) | G(p) > 0\} \subset \text{supp}(C).$$

An (I, G)-marked curve is called rational if its genus is zero and elliptic if its genus (as (I, G)-marked curve) is one. (If (C, I, G) is an elliptic curve, either C has genus one and C has only points of genus zero or C is rational and $\text{supp}(C)$ contains exactly one point with a non-zero genus that is one.)

Remark 1.3.2
Note that if an abstract tropical curve C is not L_1^1 and if all circuits of $\text{supp}(C)$ contain more than one edge, there exists a polyhedral structure $\mathcal{C}$ on C such that $\mathbf{V}(C) = \text{pol}(\mathcal{C})^{(0)}$ and $\mathbf{E}(C) = \text{pol}(\mathcal{C})^{(1)}$.

Definition 1.3.3 (Length of an edge)
Let C be an (I, G)-marked curve and let $E \in \mathbf{E}(C)$. Choose a polyhedral structure $\mathcal{C}$ on C in which E is a union of pairwise disjoint 1-cells $E_1, \ldots, E_n \subset \text{supp}(C)$. Then polyhedral charts $\sigma_i : E_i \to \mathbb{R}$ are part of the data of $\mathcal{C}$, $i \in [n]$. The images $\sigma_i(E_i)$ are intervals $I_i \subset \mathbb{R}$. We define the length of E as the sum of the lengths of the intervals I_i, $i \in [n]$, and denote it by $\text{Length}(E)$. (This is well-defined because the composition of polyhedral charts and tropical isomorphisms are affine $\mathbb{Z}$-linear and invertible, see 1.1.3, 1.1.11 and 1.2.10.)

Construction 1.3.4 (Combinatorial morphism, abstract combinatorial type)
Let (C, I, G_C), (D, I, G_D) be abstract marked curves. A combinatorial morphism $f : (C, I, G_C) \to (D, I, G_D)$ is a homeomorphism $f : \text{supp}(C) \to \text{supp}(D)$

a) that respects the labeling of the leaves (i.e. a leaf of C with label $i \in I$ is mapped to the leaf of D that has the same label i) and
b) that respects the genera of the points (i.e. the genus of each $p \in \mathrm{supp}(C)$ is equal to the genus of $f(p)$).

The abstract combinatorial type Γ_C of C is defined as the set of I-marked curves D such that there exists an abstract combinatorial morphism $f : C \to D$. In this case we write $C \sim D$. It is easy to check that $\sim$ is an equivalence relation on the set of I-marked curves.

Remark 1.3.5
Let C, D be I-marked curves. Note that an abstract combinatorial morphism $f : C \to D$ maps edges onto edges and vertices onto vertices.

Construction 1.3.6 ($E^C_{I_1}$, edge induces partition of I)
Let E be a bounded edge of an I-marked curve C that is outside the loop. Then $\mathrm{supp}(C) \setminus E^\circ$ has two connected components. They induce a partition $I = I_1 \dot\cup I_2$ and we denote the edge by $E^C_{I_1} = E^C_{I_2}$.

Construction 1.3.7 (Edge contractions)
Let $E \in \mathbf{E}(C)$ for an (I, G)-marked curve C and assume that E is no leaf. The topological space $\mathrm{supp}(C)/E$ (i.e. we contract the edge E) inherits from C both the structure of a polyhedral complex and of an I-marked curve, that we denote by C/E, in the following way:

If $p, q \in E$ are the boundary points of E that have valence $\mathrm{val}(p)$ and $\mathrm{val}(q)$, locally around $E \in \mathrm{supp}(C)/E$, $\mathrm{supp}(C)/E$ is then isomorphic to $L^{\mathrm{val}(p)+\mathrm{val}(q)-2}$ if $p \neq q$ and to $L^{\mathrm{val}(p)-2}$ if $p = q$. The labeling of the leaves of C/E is chosen such that it is respected by the projection map $\pi : \mathrm{supp}(C) \to \mathrm{supp}(C)/E$. Moreover, π respects the edge lengths. The genus of $p \in \mathrm{supp}(C)/E$ is given by the sum over the genera of the points in $\pi^{-1}\{p\}$ plus the first Betti number of $\pi^{-1}\{p\}$. See figure 6 for two examples.

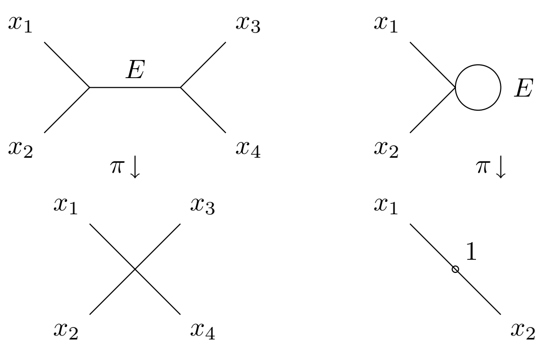


Figure 6. Example of the contraction of an edge E. On the left, all points have genus zero. On the right, in the upper curve all points have genus zero, too. Then the edge E is contracted and turned into a point of genus one in C/E below.

Remark 1.3.8
Note that the contraction of different edges of an I-marked curve C can lead to the same curve, see figure 7 below for an example.

Definition 1.3.9 (Specialization, specialization of a combinatorial type)
Let C be an I-marked curve. An I-marked curve D is called a specialization of C if D is obtained from C by a sequence of edge contractions. We write $D \leq C$. A combinatorial type Γ_D is called a specialization of a combinatorial type Γ_C if there exist $D \in \Gamma_D$ and $C \in \Gamma_C$ such that $D \leq C$. In this case we write $\Gamma_D \leq \Gamma_C$.

Lemma 1.3.10
$\leq$ is a partial order on the set of combinatorial types of I-marked abstract curves.

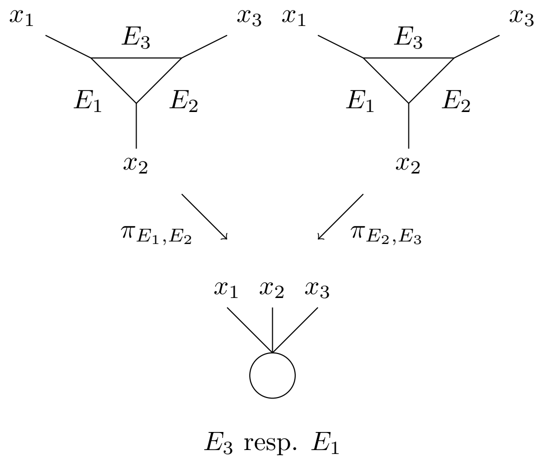


FIGURE 7. On the left, the edges E_1 and E_2 are contracted by π_{E_1,E_2}. On the right, the edges E_2 and E_3 are contracted by π_{E_2,E_3}. The arising curve is the same if E_1 and E_3 have the same length.

PROOF. $\leq$ is reflexive because the identity map id : $\text{supp}(C) \to \text{supp}(C)$ is a combinatorial morphism for all I-marked curves C. It is transitive because concatenations of combinatorial morphisms are again combinatorial morphisms. It remains to show antisymmetry: Assume that Γ_C, Γ_D are combinatorial types which fulfill $\Gamma_C \leq \Gamma_D$ and $\Gamma_C \leq \Gamma_D$. That implies that there exist curves $C \in \Gamma_C$, $D \in \Gamma_D$ and specializations D' of D, C' of C with $C \sim D'$ and $D \sim C'$. D' and C' are obtained from D and C by a sequence of edge contractions. Since the number of edges is finite, we conclude that in both cases no edges are contracted, $C \sim D$ and $\Gamma_C = \Gamma_D$, see also the diagram below. $\square$

$$\begin{array}{ccc} \text{supp}(C') & \overset{\cong}{\longrightarrow} & \text{supp}(D) \\ \uparrow \pi_{C'} & & \pi_{D'} \downarrow \\ \text{supp}(C) & \overset{\cong}{\longleftarrow} & \text{supp}(D') \end{array}$$

FIGURE 8. $\pi_{C'}$ and $\pi_{D'}$ are the projection maps that correspond to the sequence of edge contractions in the previous lemma.

Lemma 1.3.11
Let C, D be (I, G_1), (I, G_2)-marked curves of genus one and let $f_1, f_2 : C \to D$ be combinatorial morphisms. If there exists an edge $E \in \mathbf{E}(C)$ with $f_1(E) \neq f_2(E)$ then $f_1(E) \cup f_2(E) = D_L$, i.e. $f_1(E)$ and $f_2(E)$ are the only edges in the loop of D. In particular, $f_1(v) = f_2(v)$ for all vertices $v \in \mathbf{V}(C)$.

PROOF. If E is an edge outside the loop of C, $f_1(E)$ and $f_2(E)$ are edges outside the loop of D. The set $\text{supp}(C) \setminus E^\circ$ has two connected components that give a partition of $I = I_1 \dot\cup I_2$. Since combinatorial morphisms respect the labeling of the leaves, $\text{supp}(D) \setminus f_1(E^\circ)$ and $\text{supp}(D) \setminus f_2(E^\circ)$ must induce the same partition $I = I_1 \dot\cup I_2$, which implies $f_1(E) = f_2(E)$.

If E is an edge inside the loop of C, $f_1(E)$ and $f_2(E)$ are also edges inside the loop of D. If $f_1(E) \neq f_2(E)$, the loop of C and D contains at least two edges. So let $E \neq E' \in \mathbf{E}(C)$ be another edge in the loop of C. The set $\text{supp}(C) \setminus (E^\circ \cup (E')^\circ)$ has two connected components that give a partition of $I = I_1 \dot\cup I_2$. Again, this partition is respected by combinatorial morphisms and

$\operatorname{supp}(D)\setminus(f_1(E)^\circ\cup f_1(E')^\circ)$ and $\operatorname{supp}(D)\setminus(f_1(E)^\circ\cup f_1(E')^\circ)$ induce the same partition $I = I_1\dot{\cup}I_2$. This implies that $\{f_1(E), f_1(E')\} = \{f_2(E), f_2(E')\}$ for all edges $E' \neq E$ that are inside the loop of C, i.e. $f_2(E') = f_1(E)$ for all edges $E' \neq E$ that are inside the loop. Hence, $f(E)$ and $f_1(E')$ are the only edges in the loop of D. □

Definition 1.3.12 (Morphism of (I, G)-marked curves)
A morphism of marked curves $(C, I, G_C), (D, I, G_D)$ is a morphism $f : C \to D$ that respects the labeling of the leaves (i.e. a leaf with label $i \in I$ of C is mapped to the leaf with label i of D) and the genera of the points (i.e. the genus of $f(p)$ equals the genus of p for all $p \in \operatorname{supp}(C)$).

Corollary 1.3.13
Let C be an (I, G)-marked curve. If C has genus one and if there exists an automorphism $f : C \to C$ that is not the identity, then one of the following statements is true:

a) The loop of C consist of only one edge.
b) The loop of C consists of precisely two edges E_1, E_2 and $f(E_1) = E_2$, $f(E_2) = E_1$.

If C is rational, C has no non-trivial automorphisms.

In order to deal with enumerative questions concerning tropical curves we will study a space that parametrizes I-marked rational curves. To construct such a space we make use of tree metrics. The following definitions and statements can be found for example in [**GKM09**]:

A metric on the set $I = \{i_1, \dots, i_n\}$ with $\#I = n$ can be identified with a point in $\mathbb{R}^{\binom{n}{2}}$ such that the coordinate $\{i, j\}$ describes the distance between $i \in I$ and $j \in I$. The space of metrics coming from I-marked metric trees defines a fan F in $\mathbb{R}^{\binom{n}{2}}$ whose cones are in bijection with combinatorial types of trees. The lineality space of this fan is given by metrics which come from trees without inner edges where all leaves are incident to the single n-valent vertex, so-called star metrics. This space of star metrics can be described by the image of

$$\begin{aligned} \Phi_I : \mathbb{R}^n &\to \mathbb{R}^{\binom{n}{2}} \\ (a_{i_1}, \dots, a_{i_n}) &\mapsto (a_{i_k} + a_{i_l})_{k,l}. \end{aligned}$$

Dividing out the image $\operatorname{im}(\Phi_n)$ from the fan F, which parametrizes tree metrics, we obtain a tropical fan:

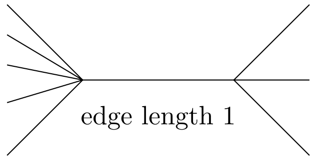
Definition 1.3.14 (The moduli space of abstract curves)
Let I be a set with $\#I = n$. The moduli space of rational I-marked abstract curves $\mathcal{M}_{0,I}$ is the fan in $\mathbb{R}^{\binom{n}{2}}/\operatorname{im}(\Phi_n)$ that parametrizes metric trees with bounded edges of positive length and unbounded edges of infinite length. The lattice is generated by the metric trees which have precisely one bounded edge. For a precise definition, see [**GKM09**], section 3.

In [**Rau09**], proposition 2.1.21, it is shown that the space $\mathcal{M}_{0,I}$ not only parametrizes metric trees but also abstract rational I-marked curves. Therefore, we often speak of rational abstract curves instead of metric trees without different meaning.

The cones of $\mathcal{M}_{0,I}$ are in bijection with combinatorial types of I-marked rational curves. The dimension of a cone equals the number of bounded edges in the corresponding combinatorial type. A point in the interior of a facet corresponds to a 3-valent metric tree with $(n-3)$ bounded edges. Hence, the dimension of $\mathcal{M}_{0,I}$ is $(n-3)$. All facets are equipped with weight 1. Using these weights, $\mathcal{M}_{0,I}$ satisfies the balancing condition and is a tropical fan.

We also denote the associated tropical variety by $\mathcal{M}_{0,I}$.

Definition 1.3.15 ($v_J \in \mathcal{M}_{0,I}$)
The rays of $\mathcal{M}_{0,I}$ with $\#I = n$ are generated by the metric trees $v_J = v_{I\setminus J} \in \mathbb{R}^{\binom{n}{2}}/\Phi_n(\mathbb{R}^n)$, $J \subset I$, with $(v_J)_{i,j} = 0$ if $i, j \in J$ or $i, j \notin J$ and $(v_J)_{i,j} = 1$ else. Hence, the partitions $\{J, I\setminus J\}$ of I with $I \neq J \neq \emptyset$ give a global labeling of the edges of curves in $\mathcal{M}_{0,I}$. Here is an illustration:



Lemma 1.3.16
Consider $\mathcal{M}_{0,I}$ and let $J \subset I$ with $\#J > 1$ and $\#(I \setminus J) > 1$. Define $a \in \mathbb{R}^{\#J}$ with $a_k = \#J - 2$ if $k \in J$ and $a_k = 0$ if $k \notin J$. Then it holds

$$\sum_{\substack{A \subset J, \\ \#A = 2}} v_A = v_J + \Phi_I(a).$$

Proof. Let $k, l \in J$ with $k \neq l$. There exist precisely $\#J - 2$ pairwise different $A \subset J$ with $\#A = 2$, $k \in A$ and $l \notin A$. Hence, the distance of k and l in $\sum v_A$, where the sum runs over all $A \subset J$ with $\#A = 2$, is $2 \cdot (\#J - 2)$. Moreover, for $k \in J$ and $l \notin J$, there exist precisely $\#J - 1$ different $A \subset J$ with $\#A = 2$ and $k \in A$. Thus, the distance of $k \in J$ and any $l \notin J$ in $\sum v_A$ is $\#J - 1$. Finally, the distance of $k, l \notin J$ in $\sum v_A$ is zero. Combining these statements we get that $\sum v_A$ is defined by the metric tree (see picture below)

- with one bounded edge of length one
- such that all leaves $k \in J$ are incident to the same vertex and all have length $\#J - 1$ and
- such that all leaves $l \notin J$ are incident to the other vertex and all have length zero.

This proves the claim.

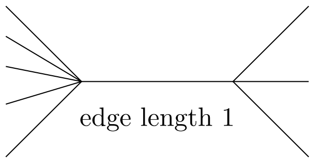


Figure 9. Metric tree defining $\sum v_A$.

□

Parametrized curves.

Definition 1.3.17 (Parametrized curve of degree Δ, marked point, genus, regular)

a) A labeled degree is a map $v : \Delta \to \mathbb{Z}^m \setminus \{0\}$ such that Δ is a finite set and such that the image vectors of the map sum up to zero, i.e. $\sum_{i \in \Delta} v(i) = 0 \in \mathbb{Z}^m$. We denote the degree by Δ if no confusion can occur.
b) Let C be an abstract curve, $p \in \text{supp}(C)$, $\sigma_p : U_C(p) \to \mathbb{R}^{\text{val}(p)-1}$ a fan chart at p and let $f : C \to \mathbb{R}^m$ be a morphism. We define the multiplicity $\text{mult}_f(p)$ of f at p as the index of the map of lattices that is induced by $f \circ \sigma_p^{-1}$. ($\text{mult}_f(p)$ does not depend on the chosen fan chart since the concatenation of σ_p with the inverse of a different fan chart τ_p^{-1} is $\mathbb{Z}$-linear invertible.)
c) Let Δ, I be finite sets with $\Delta \cap I = \emptyset$, $\#I \geq 1$ and $\#(\Delta \cup I) > 3$. Let $(C, \Delta \dot{\cup} I, G)$ be a marked curve and let $h : C \to \mathbb{R}^m$ be a morphism. We call $((C, \Delta \dot{\cup} I, G), h)$ a parametrized (I, G)-marked curve of degree $v : \Delta \to \mathbb{Z}^m \setminus \{0\}$ if

- for every $j \in \Delta$ the image $h(x_j) \subset \mathbb{R}^m$ of the leaf x_j is an unbounded ray that has direction $v(j) \in \mathbb{Z}^m \setminus \{0\}$,
- for every $i \in I$ the leaf x_i is mapped to a point and
- the multiplicity of h at $p \in \text{supp}(C)$ is greater than the genus of p (this condition is necessery in chapter 3 to make the moduli space of well-spaced curves pure-dimensional).

A leaf that is mapped to a point by h is called a marked point of $((C, \Delta\dot{\cup}I, G), h)$. We just write (C, h) instead of $(C, \Delta\dot{\cup}I, G, h)$ if no confusion can occur.

d) A morphism $(C, h) \to (D, g)$ is a morphism of the abstract curves $f : C \to D$ that fulfills $h = g \circ f$.

e) The genus of a parametrized I-marked curve $((C, \Delta\dot{\cup}I, G), h)$ of degree Δ is defined as the genus of the underlying $(\Delta\dot{\cup}I)$-marked curve. If $G(p) = 0$ for all $p \in \text{supp}(C)$, we call (C, h) regular; if there exists $p \in \text{supp}(C)$ with $G(p) > 0$, we call (C, h) non-regular.

f) If (C, h) is an (I, G)-marked curve of degree $v(\Delta)$ with $I = [n]$, we also call it (n, G)-marked.

Construction 1.3.18 ((Weighted) direction vector, $\omega(E)$, $\omega(p, E)$)
Let (C, h) be an I-marked parametrized curve. The direction vector of a flag segment $(p, E) \in \mathbf{FS}(C)$ is the integer primitive vector $\mathrm{v}_{(C,h)}(p, E) \in \mathbb{Z}^m \setminus \{0\}$ such that

$$h(E) \subset h(p) + \mathrm{v}(p, E) \cdot \mathbb{R}_{\geq 0}.$$

$\mathrm{v}_{(C,h)}$ is a function on the set of flags of (C, h). We write v_C or v instead of $\mathrm{v}_{(C,h)}$ if no confusion can occur. Let $\varphi_E : E \to \mathbb{R}$ be a cone chart in a polyhedral structure on C. Then we define

$$\omega_{(C,h)}(E) = \omega(E)$$

as the index of the map of lattices corresponding to $h \cdot {\varphi_E}^{-1}$, and we set $\omega(p, E) = \omega(E)$. We call

$$\mathrm{v}^{\omega}_{(C,h)}(p, E) = \omega_{(C,h)}(E) \cdot \mathrm{v}_{(C,h)}(p, E)$$

the weighted direction vector of $(p, E) \in \mathbf{FS}(C)$ and denote it by $\mathrm{v}^{\omega}(p, E)$ if no confusion can occur.

Definition 1.3.19 (Combinatorial type $\alpha = (\Gamma_\alpha, \mathrm{v}_\alpha)$, $\mathbf{E}(\alpha)$, $\mathbf{V}(\alpha)$, $\mathbf{F}(\alpha)$, $\mathrm{P}(\alpha)$)
Let Γ be the combinatorial type of an abstract (I, G)-marked curve C and let $h : C \to \mathbb{R}^m$ be a morphism. For $E \in \mathbf{E}(C)$ we define

$$[E] \coloneqq \{f(E) | f : C \to D \text{ is a combinatorial morphism of marked curves}\}.$$

Define $[v]$ for $v \in \mathbf{V}(C)$ accordingly and set $\mathbf{V}(\Gamma) = \{[v] | v \in \mathbf{V}(C)\}$, $\mathbf{E}(\Gamma) = \{[E] | E \in \mathbf{E}(C)\}$ and $\mathbf{F}(\Gamma) \coloneqq \{([p], [E]) | (p, E) \in \mathbf{F}(C)\}$.

Let $\bar{\mathrm{v}} : \mathbf{F}(\Gamma) \to \mathbb{R}^m$ be a function. Then we call $(\Gamma, \bar{\mathrm{v}})$ a combinatorial type of parametrized (I, G)-marked curves. The combinatorial type of (C, h) is defined as $\alpha = (\Gamma, \mathrm{v}^\omega)$ with $\mathrm{v}^\omega([(p, E)]) = \omega(E) \cdot \mathrm{v}_{(C,h)}(p, E)$ for all $(p, E) \in \mathbf{F}(C)$. Note that (C, h) and (D, g) have the same combinatorial type if and only if there exists a combinatorial morphism $f : C \to D$ that fulfills

$$\omega_{(C,h)}(E) \cdot \mathrm{v}_{(C,h)}(p, E) = \omega_{(D,g)}(D) \cdot \mathrm{v}_{(D,g)}(f(p), f(E))$$

for all $(p, E) \in \mathbf{FS}(C)$.

We define $\mathbf{V}(\alpha)$, $\mathbf{E}(\alpha)$ and $\mathbf{F}(\alpha)$ as the corresponding sets for Γ, and we just write $\mathbf{V}$, $\mathbf{E}$ and $\mathbf{F}$ if no confusion can occur.

We define $\mathrm{P}(\alpha)$ as the set of curves that have combinatorial type α.

Remark 1.3.20
Let C be an elliptic marked curve which has only two edges $E_1, E_2 \subset C_L$ in the loop that have the same weight. Then there exists an automorphism $f : C \to C$ with $f(E_1) = E_2$ and hence only one edge $[E_1] = [E_2]$ in the loop of the combinatorial type Γ_C of C.

Definition 1.3.21 (Specialization, specialization of a combinatorial type, $\overline{\mathrm{P}(\alpha)}$)
Let (C,h) be an parametrized I-marked curve. A parametrized I-marked curve (D,g) is called a specialization of (C,h) if D is a specialization of C and if the projection map $\pi : C \to D$ fulfills $\mathrm{v}_C(p,E) = \mathrm{v}_D(\pi(p), \pi(E))$ for all flags $(p,E) \in \mathbf{F}(C)$ such that $\pi(E) \in \mathbf{E}(D)$. We write $(D,g) \leq (C,h)$ in this case.

A combinatorial type α is called a specialization of a combinatorial type β if there exist $(D,g) \in \alpha$ and $(C,h) \in \beta$ such that $(D,g) \leq (C,h)$. In this case we write $\alpha \leq \beta$. We denote the set of curves whose combinatorial type is a specialization of α by $\overline{\mathrm{P}(\alpha)}$.

If $\alpha \leq \beta$, there exists a surjective map on the vertices $\pi_\alpha^\beta : \mathbf{V}(\beta) \to \mathbf{V}(\alpha)$ that maps a vertex of β to the corresponding vertex in α.

Lemma 1.3.22
$\leq$ is a partial order on the set of combinatorial types of I-marked parametrized curves of degree Δ. (Hence, we can speak of maximal combinatorial types of parametrized curves.)

Proof. The statement follows from the respective statement about abstract curves, see 1.3.10. □

For dealing with enumerative questions, we are interested in a variety parametrizing rational parametrized curves of a given degree.

Definition 1.3.23 ($\mathcal{M}_{0,I}(\Delta, \mathbb{R}^m)$, $\mathrm{W}(\alpha)$)
We define $\mathcal{M}_{0,I}(\mathbb{R}^m, \Delta)$ to be the space that parametrizes the set of rational I-marked parametrized curves of degree Δ in $\mathbb{R}^m$. The construction of this space as tropical variety can be found in [**GKM09**], section 4. Having fixed one root vertex x_i with label $i \in I$ we identify $\mathcal{M}_{0,I}(\Delta, \mathbb{R}^m)$ with $\mathcal{M}_{0,I\dot\cup\Delta} \times \mathbb{R}^m$, where the first factor parametrizes the I-marked abstract curve C and the second factor contains the coordinates of $h(x_i) \in \mathbb{R}^r$. In remark 1.3.24, we explain why C and $h(x_i)$ are sufficient to encode a parametrized rational curve $(C,h) \in \mathcal{M}_{0,I}(\Delta, \mathbb{R}^m)$.

If α is a combinatorial type in $\mathcal{M}_{0,1}(\Delta, \mathbb{R}^m)$, we denote by $M_0(\alpha)$ the set of curves that have combinatorial type α. For shortening notation, we set

$$\mathrm{W}(\alpha) = \mathrm{W}(\overline{\mathrm{M}_0(\alpha)})$$

(where the latter is defined as the smallest linear space containing the polyhedron $\overline{M_0(\alpha)}$, see 1.1.2) and

$$u_{\beta/\alpha} = u_{\overline{\mathrm{M}_0(\beta)}/\overline{M_0(\alpha)}}$$

(where $\alpha \leq \beta$ is a maximal combinatorial type, $\dim \mathrm{W}(\beta)/\mathrm{W}(\alpha) = 1$ and $u_{\overline{\mathrm{M}_0(\beta)}/\overline{M_0(\alpha)}}$ is the normal vector, see 1.1.6).

Remark 1.3.24
Let $C \in \mathcal{M}_{0,I}$, $P \in \mathbb{R}^m$ and let $v : \Delta \to \mathbb{R}^m$ be a labeled degree. Then there exists exactly one rational curve $(C,h) \in \mathcal{M}_{0,I}(\Delta, \mathbb{R}^m)$ such that the position of its root vertex x_i is given by $h(x_i) = P$. We explain why this is true:

Let $v \in \mathbf{V}$ and assume that $h(v)$ is known and that $v \in E$ for some bounded $E \in \mathbf{E}(C)$. Since C is rational, $\mathrm{supp}(C) \setminus E^\circ$ consists of exactly two connected components $C_1, C_2 \neq \emptyset$. These connected components induce a partition $I_1 \dot\cup I_2 = I$ of the set of labels I. By the balancing condition, we conclude that the direction vector $\mathrm{v}(v,E)$ is given by

$$\mathrm{v}(v,E) = -\sum_{i \in I_1} v(i) = \sum_{j \in I_2} v(j) \in \mathbb{R}^r.$$

There exists $v \neq v' \in E \cap \mathbf{V}(C)$ and it holds

$$h(v') = h(v) + \omega(E) \cdot \mathrm{Length}(E) \cdot \mathrm{v}(v,E).$$

Since $\mathrm{supp}(C)$ is connected, we can reconstruct $h(v)$ forall $v \in \mathbf{V}(C)$ and hence $h : C \to \mathbb{R}$.

Remark 1.3.25 (The curves v_J)
In definition 1.3.15, we defined the curves $v_J \in \mathcal{M}_{0,I}$ for a subset $J \subset I$ which have only one edge. In the case of parametrized curves in $\mathcal{M}_{0,I}(\Delta, \mathbb{R}^m)$, we can similarly define curves v_J^i for $J \subset I \cup \Delta$ and $i \in I$ by choosing the position of a root vertex x_i to be $0 \in \mathbb{R}^m$, i.e. $v_J^i = (v_J, 0) \in \mathcal{M}_{0,I\cup\Delta} \times \mathbb{R}^m = \mathcal{M}_{0,I}(\Delta, \mathbb{R}^m)$. In the situations in which the position of the root vertex does not play a role we leave out the choice of the root vertex $i \in I$ and just write v_J. In particular, lemma 1.3.16 is still valid in the case of parametrized curves.

Definition 1.3.26 (Evaluation maps and their pull-back)
We recall the construction of evaluation maps $\mathcal{M}_{0,I}(\Delta, \mathbb{R}^m) \to \mathbb{R}^m$ and $\mathcal{M}_{0,I}(\Delta, \mathbb{R}^m) \to \mathbb{R}^{m-1}$ made in [**GKM09**], section 4.2, which encode the information where the unbounded ends of a curve $(C, h) \in \mathcal{M}_{0,I}(\Delta, \mathbb{R}^m)$ are mapped by h.

If x_i is a leaf labeled by $i \in I$, it is contracted by h and mapped to a point. Hence, the map

$$\begin{aligned} \mathrm{ev}_i : \mathcal{M}_{0,I}(\Delta, \mathbb{R}^m) &\to \mathbb{R}^m \\ (C, h) &\mapsto h(x_i) \end{aligned}$$

is well-defined. In [**GKM09**], proposition 4.8, it is shown that it is a tropical morphism.

If $v : \Delta \to \mathbb{R}^m$ is a labeled degree and if the end x_i is labeled by $i \in \Delta$, the leaf $x_i \in C$ is not contracted by the map h and we have to use $\mathbb{R}^m / \langle v(i) \rangle$ as codomain in order to obtain a well-defined evaluation map

$$\begin{aligned} \mathrm{ev}_i : \mathcal{M}_{0,I}(\Delta, \mathbb{R}^m) &\to \mathbb{R}^m / \langle v(i) \rangle \cong \mathbb{R}^{m-1} \\ (C, h) &\mapsto [h(P)],\ P \in x_i \text{ arbitrary.} \end{aligned}$$

Using the same argument as in [**GKM09**], proposition 4.8, we see that also in this case ev_i is a morphism.

Lemma 1.3.27 ([**AK06**]: chapter 4, [**FR12**]: 7.2 and 7.3)
The spaces $\mathcal{M}_{0,I}$ and $\mathcal{M}_{0,I}(\Delta, \mathbb{R}^m)$ are matroid varieties. Therefore, we can do intersection theory on them using the results stated in section 1.2.

Remark 1.3.28
There exists a proof [**GZ**] of the following statement:

Let $v : \Delta \to \mathbb{R}^m$ be a L_m^m-directional degree and let C_k be L_m^m-directional varieties in $\mathbb{R}^m$ for $k = 1, \dots, l$, e.g. translates of L_k^m. Set $I = \{0, \dots, n\}$. Then the push-forward

$$(\mathrm{ev}_0)_* \prod_{i=1}^{n} \mathrm{ev}_i^* C_k \cdot \mathcal{M}_{0,I}(\Delta, \mathbb{R}^m)$$

is also L_m^m-directional.

CHAPTER 2

Enumerative geometry of rational tropical curves in $\mathbb{R}^m$

In this chapter, we study rational tropical curves in $\mathbb{R}^m$. Given a family of incidence and tangency conditions, which are tropical varieties in $\mathbb{R}^m$ and $\mathbb{R}^{m-1}$, we establish a recursive formula which allows us to count the number of rational tropical curves of a prescribed degree $j : \Delta \to \mathbb{R}^m$ that fulfill all these conditions, i.e. which intersect these tropical varieties. Each curve is counted with an intersection-theoretic multiplicity calculated on the moduli space of parametrized rational tropical curves of degree Δ in $\mathbb{R}^m$.

Our approach to the stated enumerative question is based on tropical intersection theory. It is geared to the method applied in [**Vak00**] for counting rational algebraic curves of a given degree $d \in \mathbb{N}$ in $\mathbb{P}^m$ fulfilling given incidence and tangency conditions to a hyperplane. Moreover, we generalize ideas from [**GM07b**] applied to prove a tropical Caporaso-Harris-type formula for counting rational tropical curves of degree d and genus g passing a general configuration of $3d+g-1$ points in $\mathbb{R}^2$. Merging both approaches and using tropical intersection theory, we establish a recursive formula for counting rational tropical curves in $\mathbb{R}^m$ that coincides with its algebraic counterpart, see [**Vak00**]. A related result, which describes - using floor diagrams - the recursive structure of the multiplicity of a rational curve fulfilling given incidence and tangency conditions, is stated in [**BM07**] and proven in [**BM11**].

2.1. Setup

We will formulate the enumerative question studied in this chapter precisely. We restrict ourselves to degrees of parameterized tropical curves in $\mathbb{R}^m$, which have only standard directions and fulfill another property stated in the definition below (see 1.3.17 for the definition of a degree). Moreover, we prescribe that the considered incidence and tangency conditions (which are varieties in $\mathbb{R}^m$ and $\mathbb{R}^{m-1}$, respectively) have standard directions, i.e. their recession fan is L_k^m or L_k^{m-1} for some $k \in [m-1]$. The numbers $t_{r,e}$ that appear in the following definition stand for the number of tangency conditions of dimension $e \in [m-1]$ that have to be fulfilled by a leaf of the curve which has direction $-e_1$ and weight $r \in \mathbb{N}$.

Remember that $e_1, \ldots, e_m$ are the standard unit vectors in $\mathbb{R}^m$ and that $e_0 = -\sum_{i=1}^m e_i$.

Definition 2.1.1 ((Generalized) projective degree)
Let $t := (t_{r,e})_{r,e \in \mathbb{N}}$, $t_{r,e} \in \mathbb{N}$, be a vector such that $t_{r,e} = 0$ for all $e > m-1$. Set $d(t) := \sum_{r,e} r t_{r,e}$, i.e. t determines $d(t)$. Let $H_m(t) : \Delta_m(t) \to \mathbb{Z}^m \setminus \{0\}$ be the tropical degree with

$$\Delta_m(t) := \{(r,e,j) | 1 \leq j \leq \sum_e t_{r,e}\} \cup [d(t)] \cup \{2 \cdot d(t)+1, \ldots, (m+1) \cdot d(t)\}$$

and

$$\begin{array}{rcl} H_m(t) : \Delta_m(t) & \to & \mathbb{R}^{m-1} \\ (r,e,j) & \mapsto & -re_1 \\ 1, \ldots, d(t) & \mapsto & -e_0 \\ 2d(t)+1, \ldots, 3d(t) & \mapsto & -e_2 \\ \vdots & & \vdots \\ m \cdot d(t)+1, \ldots, (m+1) \cdot d(t) & \mapsto & -e_m. \end{array}$$

We call a tropical degree such that there exists a vector $t = (t_{r,e})$ which has the above requested properties a generalized projective degree $d(t)$ in $\mathbb{R}^m$.

If $t_{r,e} = 0$ for all $r \neq 1$, we call $\Delta_m(t)$ the projective degree $d(t)$ in $\mathbb{R}^m$.

Remark 2.1.2
A parametrized curve of projective degree $d(t)$ in $\mathbb{R}^m$ has precisely $d(t)$ leaves with weight one in each of the standard directions $-e_0, \ldots, -e_m$. A parametrized curve of generalized projective degree $\Delta_m(t)$ resembles a curve of projective degree $d(t)$. The only difference is that a curve of generalized tropical degree $\Delta_m(t)$ may have leaves in direction $-e_1$ whose weight is larger than 1. However, the sum over the weighted direction vectors of all these leaves still equals $-d(t) \cdot e_1$, due to $\sum t_{r,e} = d(t)$. In the remaining standard directions $-e_0$, $-e_2$, $\ldots$, $-e_m$ there are exactly $d(t)$ leaves with weight 1.

The additional parameter e appears in a label $(r, e, j) \in \Delta_m(t)$ because later on we will demand from curves of degree $\Delta_m(t)$ that the end labeled by (r, e, j), which has weight r, passes the support of a tropical variety $\Gamma^j_{r,e}$ of dimension e.

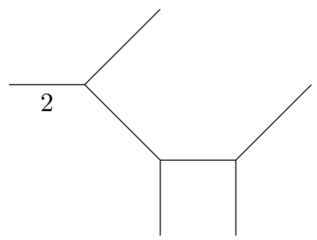


FIGURE 1. $h(C)$ of a curve (C, h) of generalized projective degree 2 in $\mathbb{R}^2$. The weight of the leaf of direction $-e_1$ is two and the weight of the remaining edges is one.

Let $d, m \in \mathbb{N}$, let $i := (i_e)_{e\in\mathbb{N}}$, $i_e \in \mathbb{N}$, be a vector such that $i_e = 0$ for all $e > m$ (i.e. we do not impose incidence conditions of dimension greater than m) and let

$$I := \{(e, j)|e \in \mathbb{N}, 1 \leq j \leq i_e\}.$$

Let $H_m(t) : \Delta_m(t) \to \mathbb{R}^m$ be the generalized projective degree corresponding to the vector $t = (t_{r,e})$.

Denote by ε_e, $e \in \mathbb{N}$, the vector with $(\varepsilon_e)_e = 1$ and $(\varepsilon_e)_f = 0$ for all $e \neq f \in \mathbb{N}$. Define $\varepsilon_{r,e}$, $r, e \in \mathbb{N}$, respectively.

We are going to introduce a subvariety of $\mathcal{M}_I(\Delta_m(t), \mathbb{R}^m)$ parameterizing curves of generalized projective degree d which fulfill given incidence and tangency conditions. The number of incidence conditions of dimension e (which can be fulfilled by an arbitrary point of the curve) will be given by $i_e \in \mathbb{N}$, the number of tangency conditions of dimension e which are fulfilled by the ends in direction $-e_1$ with weight r will be given by $t_{r,e} \in \mathbb{N}$.

Considering $\mathcal{M}_I(\Delta_m(t), \mathbb{R}^m)$, we denote the evaluation map corresponding to the leaf labeled by $(r, e, j) \in \Delta_m(t)$ by

$$\mathrm{ev}^j_{e,r} : \mathcal{M}_I(\Delta_m(t), \mathbb{R}^m) \to \mathbb{R}^m/\langle e_1\rangle = \mathbb{R}^{m-1}.$$

We denote the evaluation map corresponding to the leaf labeled by $(e, j) \in I$ by

$$\mathrm{ev}^j_e : \mathcal{M}_I(\Delta_m(t), \mathbb{R}^m) \to \mathbb{R}^m.$$

Given vectors i and t, let

a) $\Omega := (\Omega^j_e \subset \mathbb{R}^m)_{e,1\leq j\leq i_e}$ be a family of tropical varieties in $\mathbb{R}^m$ such that each Ω^j_e is a translate of L^m_e for all e and all $1 \leq j \leq i_e$, in particular $\dim(\Omega^j_e) = e$, and
b) $\Gamma := (\Gamma^j_{r,e} \subset \mathbb{R}^{m-1})_{e,r,1\leq j\leq t_{r,e}}$ be a family of tropical varieties in $\mathbb{R}^{m-1}$ such that $\Gamma^j_{r,e}$ is a translate of L^{m-1}_e for all e, r and $1 \leq j \leq t_{r,e}$, in particular $\dim(\Gamma^j_{r,e}) = e$.

We call a set Ω that fulfills these properties a set of incidence conditions for i and we call a set Γ that fulfills these conditions a set of tangency conditions for t.

Definition 2.1.3 ($X_m(\mathcal{E})$)
Let i, t be vectors as above, Ω, Γ be sets of incidence and tangency conditions for i and t. We set $\mathcal{E} := (t, \Omega, \Gamma)$.
We define $X_m(\mathcal{E})$ as the subvariety of $\mathcal{M}_I(\Delta_m(t), \mathbb{R}^m)$ which is given by

$$X_m(\mathcal{E}) := \left(\prod_{e,j} (\mathrm{ev}_e^j)^*(\Omega_e^j)\right) \cdot \left(\prod_{r,e,j} (\mathrm{ev}_{r,e}^j)^*(\Gamma_{r,e}^j)\right) \cdot \mathcal{M}_I(\Delta_m(t), \mathbb{R}^m).$$

If $X_m(\mathcal{E})$ is zero-dimensional, we define

$$N_m(\mathcal{E}) := \frac{\deg(X_m(\mathcal{E}))}{(d(t)!)^m}$$

where the factor $(d(t)!)^m$ equals the number of possibilities to label the non-contracted leaves of an I-marked rational curve of degree $\Delta_m(t)$ which do not have direction $-e_1$, i.e. the ends which are not restricted by tangency or incidence conditions. Define $N_m(\mathcal{E})$ to be zero if

$$\dim(X_m(\mathcal{E})) \neq 0.$$

Remark 2.1.4
In order to compute $N_m(\mathcal{E}) = N_m(t, \Omega, \Gamma)$ we can replace the varieties $\Omega_e^j \in \Omega$ and $\Gamma_{r,e}^j \in \Gamma$ by rationally equivalent varieties without changing the number (see [**FR12**], remark 9.2). In particular, we may translate the linear spaces without changing $N_m(\mathcal{E})$ (see [**Rau09**], lemma 1.4.9).
Moreover, all tropical linear spaces whose recession fan is of the form L_e^m are rationally equivalent to L_e^m, see [**AR08**], theorem 7. Although we assume in the following that all elements of Ω and Γ are translates of some L_e^m, the recursive formula we will establish actually allows to count the number of curves which fulfill incidence and tangency conditions that are varieties whose recession fan is of the form L_e^m.
For determining the numbers $N_m(\mathcal{E})$ it is hence sufficient to know the vectors i and t, which determine the tropical degree $\Delta_m(t)$ and the number of linear spaces Ω_e^j and $\Gamma_{r,e}^j$ that have dimension e and that are rationally equivalent to L_e^m. Hence, for shortening and simplifying notation, we denote in the following a set of data given by vectors i, t, which induce sets of incidence and tangency conditions Ω and Γ up to translation, by $\mathcal{F} = (i, t)$. We define

$$N_m(\mathcal{F}) = N_m(i, t) := N_m(t, \Omega, \Gamma),$$

where Ω and Γ are arbitrary sets of tangency and incidence conditions for i and t.

Given $\mathcal{E} = (t, \Omega, \Gamma)$, the dimension of $X_m(\mathcal{E})$ is given by

$$\begin{aligned} & \dim(X_m(t, \Omega, \Gamma)) \\ = \;& \dim(\mathcal{M}_I(\Delta_m(t), \mathbb{R}^m)) - \sum_e (m-e) i_e - \sum_{r,e} (m-1-e) t_{r,e} \\ = \;& (m+1)d + m - 3 - \sum_e (m-1-e) i_e - \sum_{r,e} (m-2+r-e) t_{r,e}. \end{aligned}$$

The cycle $X_m(t, \Omega, \Gamma)$ is zero-dimensional and the numbers $N_m(t, \Omega, \Gamma)$ can only be non-zero if

$$(m+1)d + m - 3 - \sum_e (m-1-e) i_e - \sum_{r,e} (m-2+r-e) t_{r,e} = 0.$$

In order to describe the enumerative relevance of the numbers $N_m(\mathcal{E})$, we state the following lemma.

Lemma 2.1.5 ([**Rau09**], Corollary 2.2.13)
Let X be a polyhedral complex and let $f_k : X \to \mathbb{R}^{m_k}$, $k = 1, \dots, n$, be maps which are affine linear on the cells of X. Moreover, let $Y_1, \dots, Y_k$ be polyhedral complexes in $\mathbb{R}^m$. Then for general translations $Y'_k = Y_k + v_k$, $v_k \in \mathbb{R}^{m_k}$ (i.e. v_k can be chosen from a dense open subset of $\mathbb{R}^{m_k}$), it holds that

a) $Z := \bigcap_k f_k^{-1}(Y'_k)$ is pure-dimensional,
b) the codimension of Z in X equals the sum
$$\operatorname{codim}_X(Z) = \sum_k \operatorname{codim}_{\mathbb{R}^{m_k}}(Y_k),$$
c) the interior of a facet of
$$Z := \bigcap_k f_k^{-1}(Y'_k)$$
is contained in the interior of a facet of X.

Remark 2.1.6 (Enumerative relevance of the numbers $N_m(t, \Omega, \Gamma)$)
Let us interpret the numbers $N_m(t, \Omega, \Gamma)$. Since
$$\operatorname{supp} \operatorname{ev}_k^*(Y) = \operatorname{supp} \left(\mathcal{M}_{0,I \cup \Delta_m(t)} \times Y\right) = \operatorname{ev}_k^{-1}(Y)$$
with respect to the anchor leaf x_k, $k \in I$, for subvarieties Y a of $\mathbb{R}^m$, using the previous lemma 2.1.5 and replacing all Ω_e^j and $\Gamma_{r,e}^j$ by general translations, we conclude that
$$\operatorname{supp} X_m(t, \Omega, \Gamma) = \bigcap_{e,j} (\operatorname{ev}_e^j)^{-1}(\Omega_e^j) \bigcap_{r,e,j} (\operatorname{ev}_{r,e}^j)^{-1}(\Gamma_{r,e}^j).$$
Hence, the points in the interior of a facet of $X_m(t, \Omega, \Gamma)$ correspond to curves
$$(C, h) \in \mathcal{M}_I(\Delta_m(t), \mathbb{R}^m)$$
such that

a) each vertex $v \in \operatorname{supp} C$ is trivalent,
b) $\operatorname{ev}_e^j(C, h) \in \Omega_e^j$ and $\operatorname{ev}_{r,e}^i(C, h) \in \Gamma_{r,e}^i$ for all $e, m, 1 \le j \le i_e$, $1 \le i \le t_{m,e}$,

where the facets are equipped with weights that arise from the intersection products as additional structure. Consequently, if $X_m(t, \Omega, \Gamma)$ is zero-dimensional, $N_m(t, \Omega, \Gamma)$ counts (modulo labeling) the number of curves
$$(C, h) \in \mathcal{M}_{0,I}(\Delta_m(t), \mathbb{R}^m)$$

a) which are trivalent,
b) which intersect all spaces $\Omega_e^j \in \Omega$ and
c) whose leaf labeled by $(r, e, j) \in \Delta_m(t)$ intersects $\Gamma_{r,e}^j \in \Gamma$ (considered as subspaces of $\mathbb{R}^m / \langle e_1 \rangle$),

where each curve is counted with a weight that arises from the intersection product.

Remark 2.1.7
Note the analogy to the algebro-geometric scenario of counting rational curves in projective space. In [**Vak00**], section 2.1, there are defined subschemes $X_m(\mathcal{E})$ of the moduli space of stable maps $\overline{\mathcal{M}}_{0,\sum t_{r,e} + \sum i_e}(\mathbb{P}^m, d)$ which parametrize degree d rational curves in projective space intersecting given linear subspaces Ω_e^j of $\mathbb{P}^m$ and intersecting a hyperplane $H \subset \mathbb{P}^m$ with prescribed multiplicities along linear subspaces $\Gamma_{r,e}^j$ of H. $\overline{\mathcal{M}}_{0,\sum t_{r,e} + \sum i_e}(\mathbb{P}^m, d)$ has the same dimension as $\mathcal{M}_{0,I}(\Delta_m(t), \mathbb{R}^m)$, and also in the classical case the dimension of $X_m(\mathcal{E})$ is given by
$$(m + 1)d + m - 3 - \sum_e (m - 1 - e) i_e - \sum_{r,e} (m - 2 + r - e) t_{r,e}$$
for given vectors i, t.

As in [**GM07b**], section 4, we substitute in our setup the tangency conditions to the hyperplane H encoded by the vector t by the tropical analogue of imposing constraints on the unbounded ends with direction $-e_1$.

2.2. Specialization of the conditions

In order to prove a recursive formula for counting rational curves, we specialize the position of the incidence and tangency conditions. This specialization process causes that the curves in $\operatorname{supp}(X_m(\mathcal{E}))$ split up into "easier" curves, i.e. either the degree or the dimension of the ambient space decreases or the number of incidence conditions decreases, and a recursion appears. The approach in this section is the same as in [**GM07b**].

We assume in this section that $X_m(t, \Omega, \Gamma)$ is zero-dimensional and that Ω, Γ are in general position. Let $\epsilon > 0$ be a small, $T, N > 0$ large real numbers and $0 \leq E \leq r - 2$ such that there exists a variety $\Omega^1_E \in \Omega$ of dimension E in Ω. Denote the zero-dimensional cell of $\Omega^j_e \in \Omega$ by $p^j_e \in \mathbb{R}^m$ ($e \in \mathbb{N}$, $1 \leq j \leq i_e$) and the position of the zero-dimensional cell of $\Gamma^j_{r,e} \in \Gamma$ by $q^j_{r,e} \in \mathbb{R}^{m-1}$ (r, $e \in \mathbb{N}$, $1 \leq j \leq t_{r,e}$). Denote the coordinates of $\mathbb{R}^m$ by $x_1, \ldots, x_m$ and those of $\mathbb{R}^{m-1}$ by $y_2, \ldots, y_m$. We choose the elements of Ω and Γ in each case in general position such that

a) the $x_2, \ldots, x_m$-coordinates of all p^j_e and the $y_2, \ldots, y_m$-coordinates of all $q^j_{r,e}$ lie in the interval $(-\epsilon, \epsilon)$,
b) the x_1-coordinate of all $p^j_e \neq p^1_E$ is in the interval $(-\epsilon, \epsilon)$,
c) the x_1-coordinate of p^1_E is less than $-N$ and greater than $-T$,

i.e. we keep the zero-dimensional cell of all incidence and tangency conditions in a small neighborhood of the origin and we move Ω^1_E very far in direction $-e_1$, see figure 2.

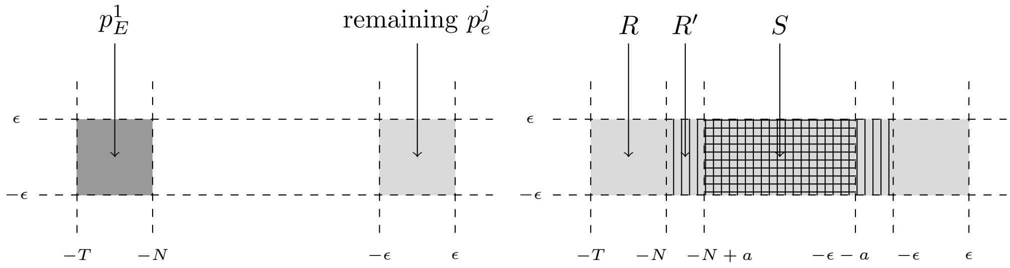


FIGURE 2. Left: All incidence conditions encoded by p^j_e lie in the light gray area except p^1_E, which lies in the dark gray area.
Right: All vertices of a curve $(C, h) \in X_2(\mathcal{E})$ lie in the gray area R. The area with the vertical lines represents R', whereas the area with the horizontal stripes represents S, with $a = 2\epsilon d^2(m+1)$.

Lemma 2.2.1
All vertices of a curve $(C, h) \in \operatorname{supp}(X_m(t, \Omega, \Gamma))$ lie in the set

$$R := \{(x_1, \ldots, x_m) | -T \leq x_1 \leq \epsilon, -\epsilon \leq x_i \leq \epsilon \text{ for all } i = 2, \ldots m\}.$$

PROOF. Assume that there exists a vertex $v \in \operatorname{supp}(C)$ such that the x_2-coordinate of $h(v)$ is smaller than $-\epsilon$. Denote by $v_1, \ldots, v_k$ the vertices of C with minimal x_2-coordinate. We want to show that in this case

$$\bigcap_{e,j} (\mathrm{ev}^j_e)^{-1}(\Omega^j_e) \bigcap_{r,e,j} (\mathrm{ev}^j_{r,e})^{-1}(\Gamma^j_{r,e})$$

is one-dimensional in contradiction to the assumption that Ω and Γ are in general position.

Since the x_2-coordinate of $v_1, \ldots, v_k$ is minimal, there must exist a vertex

$$v \in \{v_1, \ldots, v_k\}$$

one of whose incident edges is pointing downwards, i.e. running along the edge decreases the x_2-coordinate:

If none of these vertices had an incident edge pointing downwards, by the balancing none of these vertices would have an edge pointing upwards. This contradicts the assumption that (C, h) has degree $\Delta_m(t)$. But there are no vertices below $v_1, \ldots, v_k$, hence this edge must be unbounded. Due to the definition of $\Delta_m(t)$ the direction of this edge must be $(0, -1, 0, \ldots, 0)$ and it must have weight 1.

Assume without loss of generality that for some $p \in \mathbb{N}$ all vertices $v_1, \ldots, v_p$ have an incident edge that points downwards and that all $v_{p+1} \ldots, v_k$ have no incident edge that points downwards (and hence none that points upwards). As C is trivalent, it follows from the balancing condition that locally around v_i, $i \in [p]$, the curve (C, h) looks as in figure 3 for some $u_i \in \mathbb{R}^r$ with $(u_i)_2 = 0$.

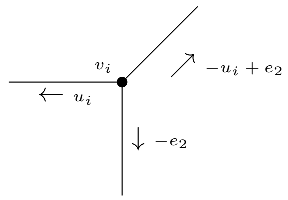


FIGURE 3. Local picture around a vertex $v_i \in \{v_1, \ldots, v_p\}$; $u_i \in \mathbb{R}^r$ is a vector with $(u_i)_2 = 0$.

We will deform the curve (C, h) in the space $\mathcal{M}_{0,I}(\Delta_m(t), \mathbb{R}^m)$: We move all vertices v_i, $i \in [p]$, upwards in direction $-u_i+e_2$ by a small $\delta > 0$ along the incident edge which has direction $-u_i+e_2$, and we move all vertices v_j, $j = p+1, \ldots, k$, upwards in direction e_2 by $\delta > 0$, see figure 4. The length of the edges in direction $-u_i + e_2$, $i \in [p]$, decreases in the local picture and the remaining edges which are incident to $v_1, \ldots, v_s$ move upwards in the x_2-coordinate, changing neither their direction nor the remaining coordinates.

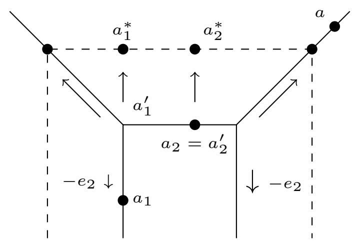


FIGURE 4. Deformation of the curve (C, h) in the local picture.

For small $\delta > 0$ such a deformation does not change the combinatorial type and the deformed curves (C_δ, h_δ) are still trivalent. Moreover, the curves (C_δ, h_δ) differ from (C, h) only locally around the vertices $v_1, \ldots, v_k$. We claim that these curves still satisfy all incidence and tangency conditions, i.e $(C_\delta, h_\delta) \in \operatorname{supp}(X_m(\mathcal{E}))$:

Assume that an incidence condition $\Omega^* \in \Omega$ is fulfilled by an edge E_i incident to v_i, $i \in [s]$ in the point $a_1 \in \operatorname{supp} \Omega^*$, such that the direction vector of (v_i, E_i) has an x_2-coordinate which is zero (e.g. the edge in direction u_i in figure 3). Then $h(C)$ intersects Ω^* in a cell that contains $-e_2 \cdot \mathbb{R}_{\geq 0}$ as ray because the x_2-coordinate of the zero-dimensional cell of Ω^* is larger than the x_2-coordinate of $v_1, \ldots, v_s$. Hence, there exists $\delta > 0$ such that $a_1^* = a_1 + \delta \cdot e_2 \in \operatorname{supp}(\Omega^*)$, (see figure 4 for an example). We conclude that (C_δ, h_δ) fulfills the condition Ω^* for small $\delta > 0$, too, namely in the point a_1^*.

Assume now that $\mathrm{v}(v_i, E_i) = -e_1$. With the argument from above, we conclude that a_1 lies in a cell of Ω^* that has $-e_2 \cdot \mathbb{R}_{\geq 0}$ as ray and $a_1' = h(v_i) \in \operatorname{supp}(\Omega^*)$. Thus, we are in the first case again and it holds that (C_δ, h_δ) fulfills the condition Ω^* for small $\delta > 0$, too.

Assume that an incidence condition $\Omega^* \in \Omega$ is fulfilled in the point $a_2 \in \operatorname{supp}(\Omega^*)$ which lies in the relative interior of an edge E_i incident to v_i, $i \in [s]$, whose direction vector has an x_2-coordinate which is plus one (e.g. the edge in direction $-u_i+e_2$). If we choose $0 < \delta$ smaller than the difference of the x_2-coordinates of a_2 and v_i, it is valid that $a_2 \in h(C_\delta)$. Hence, the deformed curve (C_δ, h_δ) satisfies the incidence condition Ω^*. See figure 4 for an example.

This is a contradiction to the assumption that $X_m(\mathcal{E})$ is zero-dimensional.

If a tangency condition $\Gamma^* \in \Gamma$ is fulfilled by an edge incident to some $v_1, \dots, v_s$, we similarly see that also the curve (C_δ, h_δ) fulfills the condition Γ^* for small $\delta > 0$.

In the same way we see that no vertex can have its x_1-coordinate below $-T$, its x_1-, ..., x_m-coordinate above ϵ nor its x_3-, ..., x_m-coordinate below $-\epsilon$. □

Lemma 2.2.2
Set $a := 2\epsilon d^2(m+1)$. No vertices of a curve $(C,h) \in \operatorname{supp}(X_m(t,\Omega,\Gamma))$ lie in the strip

$$S := \{(x_1, \dots, x_r) | -N + a < x_1 < -\epsilon - a\}.$$

Proof. First note that the previous lemma implies that all edges leaving

$$R' := \{(x_1, \dots, x_r) | -N < x_1 < -\epsilon, -\epsilon \leq x_i \leq \epsilon \text{ for all } i = 2, \dots r\}$$

without passing the hyperplanes $\{x_1 = -N\}$ or $\{x_1 = -\epsilon\}$ must be unbounded and go straight to infinity as there are no vertices outside the area R defined in the last lemma (see figure 2).

Consider a connected component C_0 of $h^{-1}(R')$ and assume that it contains a vertex $v \in C_0$ with $h(v) \in S$. We will show that in this case

$$\bigcap_{e,j} (\mathrm{ev}_e^j)^{-1}(\Omega_e^j) \bigcap_{e,m,j} (\mathrm{ev}_{r,e}^j)^{-1}(\Gamma_{r,e}^j)$$

has dimension one in contradiction to the assumption that Ω, Γ are in general position and $X_m(t,\Omega,\Gamma)$ is zero-dimensional.

Since C and C_0 are connected and balanced and since (C,h) has a generalized projective degree d, $h(C_0)$ intersects the hyperplane $\{x_1 = -N\}$ and we can run from v along C_0 to this hyperplane. We claim that we pass at least one edge with direction $-e_1$ no matter which path we choose:

Assume that there exists a path in C_0 from v to $\{x_1 = -N\}$ which does not contain a flag with direction vector $-e_1$ and let E be an arbitrary edge which is passed on the way from v to $\{x_1 = -N\}$. Let $v_1, v_2 \in E$ be the incident vertices. For degree reasons no entry of the weighted direction vector $u = \omega(E) \cdot \mathrm{v}(v_1, E) = -\omega(E) \cdot \mathrm{v}(v_2, E) \in \mathbb{R}^m$ can have entries u_i, $i \in \{1, \dots, m\}$, with $|u_i| > d$. Hence, the x_1-coordinates of v_1 and v_2 differ by less than $2\epsilon d$. Since the x_1-coordinate of $v \in S$ differs from $-N$ by more than $2\epsilon d^2(m+1)$, we pass more than $d(m+1)$ vertices on the way from v to the hyperplane $\{x_1 = -N\}$. As C is rational, the curve C has more than $d(m+1)$ leaves, which is a contradiction to the assumption that C has a generalized projective degree d in $\mathbb{R}^m$.

If it is possible to run in C_0 from the vertex V to the hyperplane $\{x_1 = -\epsilon\}$, we see by the same argument that we pass at least one edge with direction e_1.

Denote the maximal connected part of C which contains the vertex v and no edge in direction $\pm e_1$ by C_v. We can deform (C,h) to (C_δ, h_δ) without changing the combinatorial type by displacing only C_v in direction $\pm e_1$ by $\delta > 0$ (i.e. we change the length of all edges incident to C_v, which have direction $\pm e_1$, by $\pm\delta$).

These deformed curves (C_δ, h_δ) still fulfill all incidence conditions: Assume that (C,h) fulfills the incidence condition $\Omega^* \in \Omega$ in the point $a \in \operatorname{supp}(\Omega^*) \cap S$. Since the zero-dimensional cells of all incidence conditions except Ω_E^1 have an x_1-coordinate larger than $-\epsilon$ but all points in S have x_1-coordinate smaller than $-\epsilon$, the point a must lie on a facet of Ω^* that contains a ray in direction $-e_1$. Hence, the curve (C_δ, h_δ) still fulfills the condition Ω^*, namely in the point $a \pm \delta e_1 \in \operatorname{supp}(\Omega^*)$. Moreover, it holds $\operatorname{supp}(\Omega_E^1) \cap S = \emptyset$ because $a > \epsilon$. □

Corollary 2.2.3
The intersection $h(C) \cap S$ contains only edges in direction $\pm e_1$.

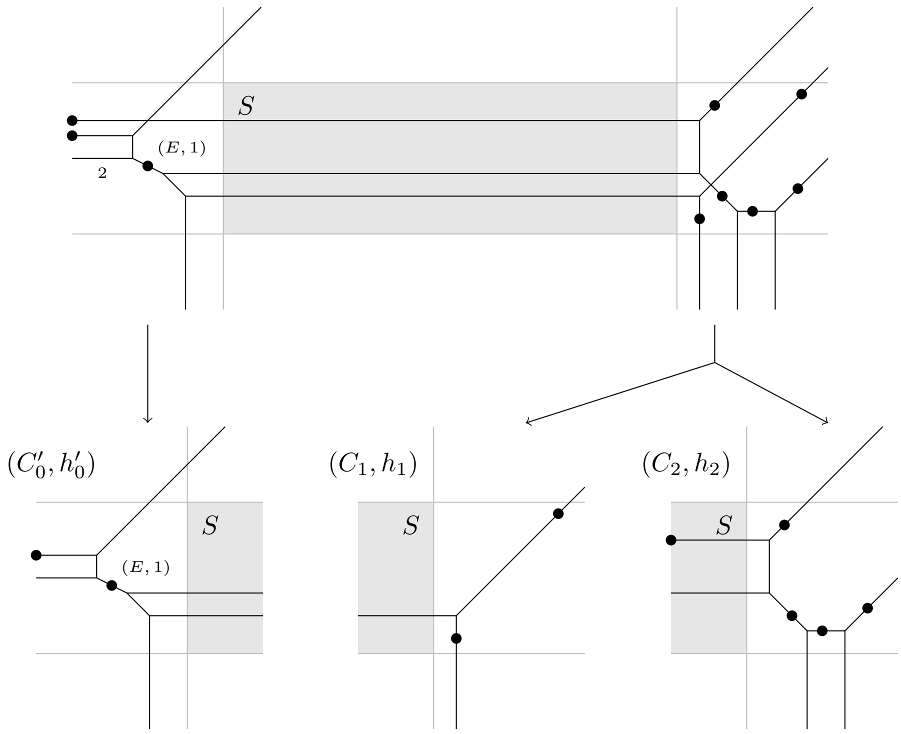


FIGURE 5. A curve (C, h) in some $X_2(\Delta_4, \Omega, \Gamma)$ and the splitting into (C'_0, h'_0), (C_1, h_1) and (C_2, h_2). The black points represent the zero-dimensional incidence and tangency conditions in Ω and Γ. (One-dimensional conditions do not restrict a curve in $\mathbb{R}^2$.)

See figure 5 for an example of a curve in some $X_2(\Delta_4, \Omega, \Gamma)$. By cutting the edges which lie in S and by setting the length of the cut edges to infinity, the curves $(C, h) \in \operatorname{supp} X_m(\mathcal{E})$ decompose into one curve (C'_0, h'_0) which lies "on the left" and curves $(C_1, h_1), \ldots, (C_l, h_l)$, $l \in \mathbb{N}$, which lie "on the right". These decomposed curves are simpler than (C, h) because either the degree or the dimension of the ambient space or the number of incidence conditions has dropped. By counting the number of the decomposed curves in suitable intersection products we will find a recursion which allows to calculate $N_m(\mathcal{E})$. The splitting process will be made more precise in the following section.

Corollary 2.2.4
Let $(C, h) \in X_m(\mathcal{E})$ and assume that $\mathcal{E}$ is general and specialized as described at the beginning of this section. If a point $p = \operatorname{ev}_e^j(C, h)$ with first coordinate $p_1 \leq -\epsilon - a$ fulfills an incidence condition $\Omega_e^j \in \Omega \setminus \{\Omega_E^1\}$, it follows that P is contained in a cell of Ω_e^j which contains the ray $\mathbb{R}_{\leq 0} \cdot e_1$ (at least if we take the coarsest polyhedral structure on Ω_e^j, which is a translate of L_e^m).

2.3. Recursive structure of $\mathcal{M}_{0,I}(\Delta_m(t), \mathbb{R}^m)$

We describe the splitting process of elements of $X_m(\mathcal{E})$, which is possible due to the results of the previous section. The aim is to express the degree of the intersection product $X_m(\mathcal{E})$ on $\mathcal{M}_{0,I}(\Delta_m(t), \mathbb{R}^m)$ via the degree of suitable intersection products on spaces of the form

$$\mathcal{M}_{0,I_0}(\Delta_{m-1}(t'(0)), \mathbb{R}^{m-1}) \times \mathcal{M}_{0,I_1}(\Delta_m(t'(1)), \mathbb{R}^m) \times \cdots \times \mathcal{M}_{0,I_l}(\Delta_m(t'(k)), \mathbb{R}^m),$$

where each of the factors has "easier" data than $\mathcal{M}_{0,I}(\Delta_m(t), \mathbb{R}^m)$.

For a moduli space of rational curves $\mathcal{M}_{0,I}(\Delta, \mathbb{R}^m)$, we denote by $\mathcal{M}_{0,I}(\Delta, \mathbb{R}^m)^\circ$ the subvariety that consists precisely of the trivalent curves and whose weights on the facets are all one.

Assume that the data $\mathcal{E}$ is general and specialized as in the previous section, i.e. one incidence condition has a very small x_1-coordinate and all other coordinates of the incidence and tangency conditions are very close to the origin.

Construction 2.3.1 (Splitting of trivalent curves $(C, h) \in \mathcal{M}_{0,I}(\Delta_m(t), \mathbb{R}^m)^\circ$)
See figure 6 for an example of the following construction. Let

$$(C, h) \in \mathcal{M}_{0,I}(\Delta_m(t), \mathbb{R}^m)^\circ,$$

i.e. C is trivalent, (C, h) lies in the interior of a facet and we do not demand that (C, h) fulfills any incidence or tangency conditions. Denote by C^* the polyhedral complex given by the maximal connected part of (C, h)

a) which contains the leaf $x_{(E,1)}$ labeled by $(E, 1)$ and
b) which fulfills that the path from $x_{(E,1)}$ to any $p \in \operatorname{supp}(C^*)$ contains no flag segment with (primitive) direction vector e_1,

see figure 6 for an example. Let $l \in \mathbb{N}$ be the number of flags of C that are incident to C^* (and all have direction e_1) and denote the corresponding edges by $E_1, \ldots, E_l$. For all $k \in [l]$ denote the weight of the edge E_k by $\omega(E_k) = r^k$. We remove C^* from C and denote the connected polyhedral complex which contains the edge E_i, $i \in [l]$, by C'_i. The complexes $C^*, C'_1, \ldots, C'_l$ induce a partition of the sets Ω, Γ into

$$\Omega = \dot{\bigcup}_{k=0}^{l} \Omega(k) \text{ and } \Gamma = \dot{\bigcup}_{k=0}^{l} \Gamma(k),$$

such that each C'_k, $k = 1, \ldots, l$, contains the marked point labeled by (e, j) and (r, e, j) if and only if $\Gamma^j_{r,e} \in \Gamma(k)$ and $\Omega^j_e \in \Omega(k)$, respectively, and such that C^* contains the marked point labeled by the (e, j) and (r, e, j) if and only if $\Omega^j_e \in \Omega(0)$ and $\Gamma^j_{r,e} \in \Gamma(0)$. Denote the induced partition of the vectors i and t by

$$i = \sum_{k=0}^{l} i(k) \text{ and } t = \sum_{k=0}^{l} t(k).$$

For $k = 0, \ldots, l$, denote the data $(t(k), \Omega(k), \Gamma(k))$ by $\mathcal{E}(k)$.

Consider C'_k, $k \in [l]$, set the length of the edge E_k to infinity and denote the arising curve by C_k. Choose $h_k : C_k \to \mathbb{R}^m$ such that $h_k|_{C'_k} = h|_{C'_k}$.

Let $\pi : \mathbb{R}^m \to \mathbb{R}^{m-1}$, $(x_1, \ldots, x_m) \mapsto (x_2, \ldots, x_m)$, be the projection which forgets the first coordinate. Add to C^* all edges $E_1, \ldots, E_l$, set their length to infinity and denote the arising curve by $C'_0 = C_0$. Let the parametrization $h'_0 : C'_0 \to \mathbb{R}^m$ be induced by h and let the parametrization $h_0 : C_0 \to \mathbb{R}^{m-1}$ be given by $h_0 = \pi \circ h'_0 : C_0 \to \mathbb{R}^{m-1}$.

Since an edge E_k, $k \in [l]$, has direction e_1 and since C is rational, it follows from the balancing condition that, for an appropriate labeling, the curve (C_k, h_k) has degree $\Delta_m(t'(k))$, where

$$t'(k) = t(k) + \varepsilon_{r^k, m-1}.$$

$\varepsilon_{r^k,m-1}$ stands for the leaf coming from the cut edge E_k whose length we set to infinity. $\Delta_m t'(k)$ is a generalized projective degree $d(t'(k)) = d(t(k)) + r^k$.

Set $d_0 = d(t) - \sum_{k=1}^{l}(d(t(k)) + r^k) = d(t) - \sum_{k=1}^{l} d(t'(0))$ and

$$t'(0) = d_0 \cdot \varepsilon_{1,m-2}.$$

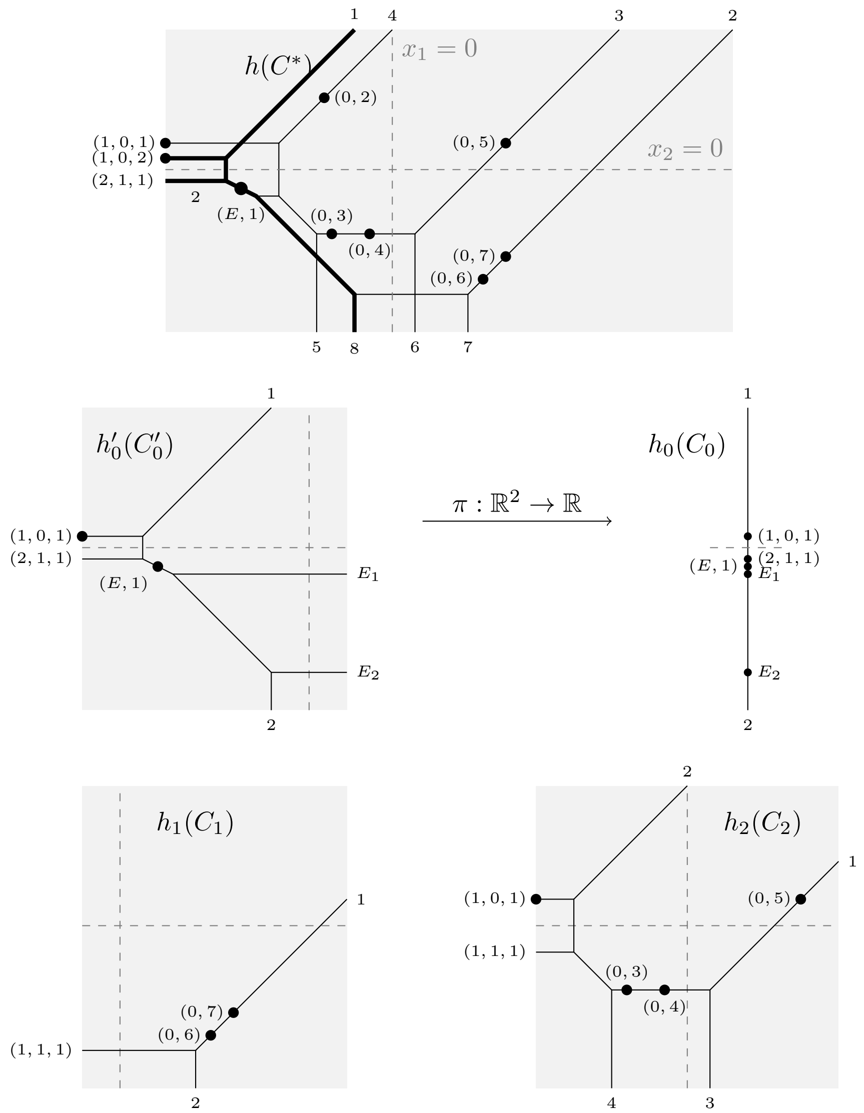


FIGURE 6. A curve $(C, h) \in \mathcal{M}_{0,I}(\Delta_2(t), \mathbb{R}^2)^\circ$ and the splitting into (C_0', h_0'), (C_1, h_1) and (C_2, h_2). t is given by $t_{1,0} = 2$, $t_{2,1} = 1$ and all other $t_{r,e}$ are zero. The thick part represents C^*. Moreover the deformation of $(C_0', h_0' : C_0' \to \mathbb{R}^2)$ into $(C_0, h_0 : C_0 \to \mathbb{R}^1)$. The black points represent the marked points x_i with $i \in I$.

$\Delta_{m-1}(t'(0))$ is a generalized projective degree $d_0 = d(t'(0))$.
Choose the labeling of the marked points on $C_0, \dots, C_l$ in the following way:
For $k \in [l]$, label the leaves in C_k arising from the edge E_k by

$$(r^k, m-1, t(k)_{r^k,m-1} + 1),$$

which means that it is the leaf of weight r^k which has to intersect a variety of dimension $m-1$ with the highest number $t(k)_{r^k,m-1} + 1$. For shortening notation, we denote this leaf by x_{r^k}, too. Label the marked point of C_0, which comes from the edge E_k by E_k (it is hence denoted by x_{E_k}). The labeling of the remaining leaves in C_k, $k = 0, \dots, l$, is chosen

- such that leaves with a higher number in C go to leaves with a higher number in C_k,
- such that (C_k, h_k) has generalized projective degree $\Delta_m(t'(k))$ for $k = 0, \dots, l$ and

- such that a label $(e,j) \in I$ of C goes to a label (e, j_k) with $1 \leq j_k \leq i(k)_e$.

After having chosen the order of the cut edges $E_1, \ldots, E_l$, this labeling is unique. See figure 6 for an example.

Define

$$I_0 := \{(r,e,j)|r,e,j \in \mathbb{N}, 1 \leq j \leq t(0)_{r,e}\} \cup \{(e,j)|e,j \in \mathbb{N}, 1 \leq j \leq i(0)_e\} \cup \{E_k\}_{k\in[l]}$$

and for $k \in [l]$

$$I_k := \{(e,j)|e \in \mathbb{N}, 0 \leq j \leq i(k)_e\}.$$

Let $k \in [l]$. Then I_k stands for the marked points of C_k which come from marked points of C. I_0 has three parts:

a) The marked points labeled by (e,j) of C_0 which come from marked points of C,
b) the marked points labeled by (r,e,j) of C which come from leaves of direction $-e_1$ of C, belong to C_0' and which are contracted by h_0 via the projection to $\mathbb{R}^m/\langle e_1\rangle$ and
c) the marked points labeled by E_k which come from the cut edges E_k linking C_0' to C_k which are contracted by the projection to $\mathbb{R}^{m-1}$.

With this notation, it holds that $(C_k, h_k) \in \mathcal{M}_{0,I_k}(\Delta_{m_k}(t'(k)), \mathbb{R}^{m_k})$ for all $k = 0, \ldots, l$, where $m_0 = m-1$ and $m_k = m$ for all $k \in [l]$. For shortening notation, we set

$$M_0 := \mathcal{M}_{0,I_0}(\Delta_{m-1}(t'(0)), \mathbb{R}^{m-1}) \text{ and } M_k := \mathcal{M}_{0,I_k}(\Delta_m(t'(k)), \mathbb{R}^m) \text{ for all } k \in [l].$$

For $k = 0, \ldots, l$, denote by $\mathrm{ev}_e^{j,(k)} : M_k \to \mathbb{R}^{m_k}$ ($m_0 = m-1$ and $m_k = m$ for $m_k \in [m]$) the evaluation map of the marked point that corresponds to the marked point of $\mathcal{M}_{0,I}(\Delta_m(t), \mathbb{R}^m)$ labeled by (e,j). Define $\mathrm{ev}_{r,e}^{j,(k)} : M_k \to \mathbb{R}^m$ accordingly. $\mathrm{ev}_{r^k} : M_k \to \mathbb{R}^{m-1}$ and $\mathrm{ev}_{E_k} : M_0 \to \mathbb{R}^{m-1}$ are the evaluation maps that correspond to the cut edge E_k ($k \in [l]$).

In particular, the splitting of (C,h) into curves (C_k, h_k), $k = 0, \ldots, l$, induces a partition of the data $\mathcal{E}$ into $\mathcal{E}(k) = (t(k), \Omega(k), \Gamma(k))$.

Remark 2.3.2
The projection of $(C_0, h_0' : C_0 \to \mathbb{R}^m)$ to $(C_0, h_0 : C_0 \to \mathbb{R}^{m-1})$ is necessary because we want to establish a recursive formula which counts tropical curves of generalized projective degree. Since in general the curve (C_0', h_0') does not have a generalized projective degree (there will be unbounded flags with direction e_1 linking C_0' to the curves $C_1, \ldots, C_l$), we cannot count curves of this combinatorial type. However, the curve (C_0, h_0) has a (generalized) projective degree. We will see that we do not lose relevant information in this projection. Since the curves (C_k, h_k), $k \in [l]$, already have a generalized projective degree, no projection is necessary here.

Notation 2.3.3 $(\phi_1, \phi_2 : \mathbb{R}^m \to \mathbb{R})$
Remember that the position of the variety $\Omega_E^1 \in \Omega(0)$ is given by p_E^1, i.e. $\mathrm{supp}(\Omega_E^1) = p_E^1 + \mathrm{supp}(L_E^m)$. Define the rational functions $\phi_1, \phi_2 : \mathbb{R}^m \to \mathbb{R}$ via

$$\phi_1(x_1, \ldots, x_r) = \max\{0, x_1 - (p_E^1)_1, \ldots, x_m - (p_E^1)_m\}$$

and

$$\phi_2(x_1, \ldots, x_r) = \max\{0, x_2 - (p_E^1)_2, \ldots, x_m - (p_E^1)_m\}.$$

Lemma 2.3.4
Let $0 \leq E < m$. Then it holds $\phi_1 \cdot \phi_2^{m-E-1} \cdot \mathbb{R}^m = \Omega_E^1$.

Proof. Without loss of generality, we assume $p_E^1 = 0 \in \mathbb{R}^m$. We prove this statement by induction on $m-E-1$. If $m-E-1 = 0$, i.e. $m-1 = E$, the statement is true. So it remains to calculate

$$\phi_1 \cdot \phi_2^{m-E} = \phi_2 \cdot \phi_1 \cdot \phi_2^{m-E-1} = \phi_2 \cdot L_{E+1}^m.$$

Since ϕ_2 is constant on all facets of L_{E+1}^m (where we use the coarsest polyhedral structure), the support of $\phi_2 \cdot L_{E+1}^m$ is contained in the support of L_E^m (which is the E-skeleton of L_{E+1}^m using the coarsest polyhedral structure). Similar to the calculation of L_E^m, we see that all weights are one. □

Definition 2.3.5 (The varieties M^+, M_k^+ and $M^{(\mathcal{E}(0),\dots,\mathcal{E}(l))}$)
By M^+ we denote the open subvariety of $\mathcal{M}_{0,I}(\Delta_m(t), \mathbb{R}^m)^\circ$ whose support consists of all trivalent curves $(C, h) \in \mathcal{M}_{0,I}(\Delta_m(t), \mathbb{R}^m)$ which fulfill the following condition (which is motivated by the conditions that the curves in the support of $X(\mathcal{E})$ fulfill if $\mathcal{E}$ is specialized as described at the beginning of the previous section):

Assume that $(C, h) \in \mathcal{M}_{0,I}(\Delta, \mathbb{R}^m)$ is trivalent and is split up into $((C_0, h_0), \dots, (C_l, h_l))$. Denote the vertex at the left end of the edge E_k by v_{E_k} and the vertex on the right end by v_{r^k} (where r^k is the multiplicity of the cut edge E_k). Then (C, h) is contained in the support of M^+ if for all $k \in [l]$ the first coordinate of $h(v_{r^k})$ of (C_k, h_k) lies on the right of

$$2\epsilon d^2(m+1) \cdot e_1 + \operatorname{supp}(\phi_1 \cdot \mathbb{R}^m) = p_E^1 + 2\epsilon d^2(m+1) \cdot e_1 + \operatorname{supp} L_E^m.$$

Said differently: If $p \in \operatorname{supp}(\phi_1 \cdot \mathbb{R}^m)$ such that the x_2-, …, x_m-coordinates of p coincide with those of $h_k(v_{r^k})$, then it holds that the difference of the x_1-coordinates of $h_k(v_{r^k})$ and p is greater than $2\varepsilon d^2(m+1)$, i.e. $(h_k(v_{r^k}))_1 - p_1 > 2\varepsilon d^2(m+1)$. Moreover, we demand that the first coordinate of $h(v_{E_k})$ lies on the left of

$$2\epsilon d^2(m+1) \cdot e_1 + \operatorname{supp}(\phi_1 \cdot \ R^m) = p_E^1 + 2\epsilon d^2(m+1) \cdot e_1 + \operatorname{supp} L_E^m.$$

M^+ has the same dimension as $\mathcal{M}_{0,I}(\Delta_m(t), \mathbb{R}^m)$, and we equip all facets with weight one.

We define the varieties M_k^+ accordingly as the subvariety of M_k containing only trivalent curves which fulfill that the vertex v_{r^k} adjacent to the leaf x_{r^k} lies on the right of

$$2\epsilon d^2(m+1) \cdot e_1 + \operatorname{supp}(\phi_1 \cdot \mathbb{R}^m) = p_E^1 + 2\epsilon d^2(m+1) \cdot e_1 + \operatorname{supp} L_E^m.$$

Let $(C, h) \in \mathcal{M}_{0,I}(\Delta_m(t), \mathbb{R}^m)^\circ$, in particular C is trivalent, and let $\mathcal{E}(0), \dots, \mathcal{E}(l)$ be the induced partition of $\mathcal{E}$ as in the construction above (which includes the choice of the labeling of the cut edges). Then we say that (C, h) has splitting type $(\mathcal{E}(0), \dots, \mathcal{E}(l))$. Note that curves which lie in the same facet as (C, h) have the same combinatorial type and hence also splitting type $(\mathcal{E}(0), \dots, \mathcal{E}(l))$ if we choose the labeling of the cut edges consistently.

Denote by

$$M^{(\mathcal{E}(0),\dots,\mathcal{E}(l))}$$

the subvariety of M^+ whose support contains only curves of splitting type

$$(\mathcal{E}(0), \dots, \mathcal{E}(l))$$

and whose weights on the facets are all one.

Remark 2.3.6
A curve $(C, h) \in \mathcal{M}_{0,I}(\Delta_m(t), \mathbb{R}^m)^\circ$ that has splitting type $(\mathcal{E}(0), \dots, \mathcal{E}(l))$ has $l!$ splitting types, namely all

$$(\mathcal{E}(0), \mathcal{E}(\sigma(1)), \cdots, \mathcal{E}(\sigma(l)))$$

with $\sigma \in \mathbb{S}_l$, which correspond to the possible labelings of the l cut edges linking the curve C_0' on the left with l curves C_k' on the right. Therefore it holds

$$M^+ = \sum \frac{1}{l!} M^{(\mathcal{E}(0),\dots,\mathcal{E}(l))},$$

where the sum runs over all (ordered) partitions $(\mathcal{E}(0), \dots, \mathcal{E}(l))$ of $\mathcal{E}$.

Construction 2.3.7 (The variety M_0^+)
Let $(C_0, h_0) \in M_0$. Then we can associate to it a curve $(C_0, h_0' : C_0 \to \mathbb{R}^m)$ that fulfills $h_0 = \pi \circ h_0' : C_0 \to \mathbb{R}^{m-1}$ in the following way - reversing the splitting process:

A leaf of (C_0, h_0') that has direction $-e_2, \dots, -e_m, -e_0 \in \mathbb{R}^{m-1}$ in (C_0, h_0) (the coordinates on $\mathbb{R}^{m-1}$ are $y_2, \dots, y_m$) gets the direction $-e_2, \dots, -e_m, -e_0 \in \mathbb{R}^m$ and weight one. Moreover, the leaves labeled by $(r, e, j) \in \Gamma(0) \subset I_0$ get direction $-e_1 \in \mathbb{R}^m$ and weight r, the leaves labeled by $E_k \in I_0$ get direction $e_1 \in \mathbb{R}^m$ and weight r^k. The contracted leaves labeled by $(e, j) \in I$ are still contracted to a point by h_0'. (C_0, h_0') is a tropical curve since $d(t'(0)) = d(t(0)) - \sum_{k \in [l]} r^k$ (where

$d(t'(0))$ is the projective degree of (C_0, h_0)). The position of the root vertex of (C_0, h_0') is chosen such that $h_0 = \pi \circ h_0'$ and such that

$$h_0'(x_E^1) \in \operatorname{supp}(\phi_1 \cdot \mathbb{R}^m),$$

i.e. the image of the marked point x_E^1 is contained in $\phi_1 \cdot \mathbb{R}^m$. This second condition is possible because $\phi_1 \cdot \mathbb{R}^m$ is a hyperplane. Denote the vertex adjacent to the leaf labeled by E_k by v_{E_k}.
We define M_0^+ as the subvariety of M_0 whose support contains all trivalent curves $(C_0, h_0) \in M_0$ such that (C_0, h_0') has the property that $h_0'(v_{E_k})$ lies on the left of

$$2\epsilon d^2(m+1) \cdot e_1 + \operatorname{supp}(\phi_1 \cdot \mathbb{R}^m),$$

i.e. there exists $p \in \epsilon d^2(m+1) \cdot e_1 + \operatorname{supp}(\phi_1 \cdot \mathbb{R}^m)$ such that the x_2-, …, x_m-coordinates of p coincide with those of $h_0'(v_{E_k})$ and such that the x_1-coordinate of p is greater than that of $h_0'(x_{E_k})$.

For every partition $(\mathcal{E}(0), \ldots, \mathcal{E}(l))$ of $\mathcal{E}$ we are going to construct an open morphism

$$\Psi^{(\mathcal{E}(0),\ldots,\mathcal{E}(l))} : (\mathrm{ev}_E^1)^*\phi_1 \cdot M^{(\mathcal{E}(0),\ldots,\mathcal{E}(l))} \to M_0^+ \times \cdots \times M_l^+.$$

Moreover, we calculate the push-forward of $M^{(\mathcal{E}(0),\ldots,\mathcal{E}(l))}$ along $\Psi^{(\mathcal{E}(0),\ldots,\mathcal{E}(l))}$.

Construction 2.3.8 (The diagonal Z in $\mathbb{R}^{m-1}$, the map $\Psi^{(\mathcal{E}(0),\ldots,\mathcal{E}(l))}$)
Let $Z := \max\{x_1, y_1\} \cdots \max\{x_{m-1}, y_{m-1}\} \cdot (\mathbb{R}^{m-1} \times \mathbb{R}^{m-1})$ denote the diagonal in $\mathbb{R}^{m-1} \times \mathbb{R}^{m-1}$ and, for $k = 1, \ldots, l$, define

$$\begin{aligned} \mathrm{ev}_{k,k} : M_0 \times \cdots \times M_l &\to \mathbb{R}^{m-1} \times \mathbb{R}^{m-1} \\ (C_i, h_i)_{i \in \{0,\ldots,l\}} &\mapsto (\mathrm{ev}_{E_k}(C_0, h_0), \mathrm{ev}_{r^k}(C_k, h_k)). \end{aligned}$$

Denote by $\pi_k^{l+1} : M_0 \times \cdots \times M_l \to M_k$ the projection onto the factor with index $k \in \{0, \ldots, l\}$. Then it holds

$$\mathrm{ev}_{k,k} = (\mathrm{ev}_{E_k} \circ \pi_0^{l+1}) \times (\mathrm{ev}_{r^k} \circ \pi_k^{l+1}).$$

Define the map

$$\begin{aligned} \Psi^{(\mathcal{E}(0),\ldots,\mathcal{E}(l))} : (\mathrm{ev}_E^1)^*(\phi_1) \cdot M^{(\mathcal{E}(0),\ldots,\mathcal{E}(l))} &\to \left(\prod_{k \in [l]} \mathrm{ev}_{k,k}^* Z \right) \cdot \left(M_0^+ \times \cdots \times M_l^+ \right) \\ (C, h) &\mapsto ((C_0, h_0), \ldots, (C_l, h_l)) \end{aligned}$$

where the $(l+1)$-tuple $((C_0, h_0), \ldots, (C_l, h_l))$ arises from splitting up the curve (C, h) as in construction 2.3.1. We show below that this map is well-defined.
We denote $\Psi^{(\mathcal{E}(0),\ldots,\mathcal{E}(l))}$ just by Ψ if no confusion can occur.

Lemma 2.3.9
It holds

$$\Psi_* \left((\mathrm{ev}_E^1)^*(\phi_1) \cdot M^{(\mathcal{E}(0),\ldots,\mathcal{E}(l))} \right) = \frac{(d(t)!)^m \cdot \prod_{k \in [l]} r^k}{\prod_{k=0}^{l} (d(t'(k))!)^m} \left(\prod_{k \in [l]} \mathrm{ev}_{k,k}^*(Z) \right) \cdot \left(M_0^+ \times \cdots \times M_l^+ \right).$$

We will split up the proof into some lemmata.

Remark 2.3.10 (Geometric relevance of $\prod \mathrm{ev}_{k,k}^*(Z)(M_0^+ \times \cdots \times M_l^+)$)
Given a curve $(C, h) \in M^+ \subset \mathcal{M}_{0,I}(\Delta_m(t), \mathbb{R}^m)^\circ$ of splitting type $(\mathcal{E}(0), \ldots, \mathcal{E}(l))$, we are interested in the spaces $M_0^+ \times \cdots \times M_l^+$, whose support contains the $(l+1)$-tuple of split curves $((C_0, h_0), \ldots, (C_l, h_l))$, see construction 2.3.1. This is guaranteed by the condition on the first coordinate of $h(x_{r^k})$ in the construction of M^+ (this point lies on the right of $2\epsilon d^2(m+1) \cdot e_1 + \operatorname{supp}(\phi_1 \cdot \mathbb{R}^m)$. Now we also want to go the other way round: Given a tuple

$$((C_0, h_0), \ldots, (C_l, h_l)) \in M_0^+ \times \cdots \times M_l^+$$

we want to glue the curves together to a curve $(C, h) \in \mathcal{M}_{0,I}(\Delta_m(t), \mathbb{R}^m)$. Obviously, this is not possible for all tuples $((C_0, h_0), \ldots, (C_l, h_l))$ but (at most) for those where the evaluation of a leaf

x_{r^k} of (C_k, h_k) (which comes from the cut edge E_k and which has direction $-e_1$) lies in $h_0(C_0)$ for all $k \in [l]$. Hence, we consider the space

$$\left(\prod_{k\in[l]} \mathrm{ev}^*_{k,k}(Z)\right)(M_0^+ \times \cdots \times M_l^+)$$

in which, by the pull-back of the diagonal, we demand for all $k \in [l]$ that

$$\mathrm{ev}_{E_k}(C_0, h_0) = \mathrm{ev}_{r^k}(C_k, h_k),$$

i.e. we demand that the evaluation of the contracted leaf labeled by E_k of (C_0, h_0) and the evaluation of the non-contracted leaf x_{r^k} of (C_k, h_k) coincide in $\mathbb{R}^{m-1}$. (Remember that the pull-back is contained in the preimage.)

Lemma 2.3.11
It holds that the support of

$$\prod \mathrm{ev}^*_{k,k}(Z)(M_0 \times \cdots \times M_l)$$

is equal to the locus where $\mathrm{ev}_{r^k} \circ \pi_k = \mathrm{ev}_{E_k} \circ \pi_0$ for all $k \in [l]$. Moreover, the weight on each facet is one.

Proof. $M_0 \times \cdots \times M_l$ is isomorphic to

$$(M_{0,I_0\cup\Delta_m(t'(0))} \times \cdots \times M_{0,I_l\cup\Delta_m(t'(l))}) \times (\mathbb{R}^{m-1})^{l+1} \times \mathbb{R}^l$$

where $(\mathbb{R}^{m-1})^{l+1} \times \mathbb{R}^l$ stands for the coordinates of the root vertices of M_k, $k = 0, \ldots, l$. We use the leaf labeled by r^k (coming from the cut edge E_k) as root coordinates for M_k, $k \in [l]$, together with the x_1-coordinate of an arbitrary marked point x_k with $k \in I_k$. As root vertex on M_0 we use alternately the marked points x_{E_k} (also coming from the cut edges E_k). Hence, the evaluation maps ev_{r^k} and ev_{E_k} that appear in $\mathrm{ev}_{k,k}$ are just the identity map on the second factor of the respective moduli space. In order to calculate

$$\prod \mathrm{ev}^*_{k,k}(Z)(M_0 \times \cdots \times M_l),$$

we hence only have to calculate the intersection product

$$\prod_{k\in[l]} (\pi_0^{l+1} \times \pi_k^{l+1})^* Z \cdot (\mathbb{R}^{m-1})^{l+1}$$

where $\pi_j^{l+1} : (\mathbb{R}^{m-1})^{l+1}$ is the projection on the j-the factor. By induction we see that its support is $\{(x, \ldots, x) | x \in \mathbb{R}^{m-1}\} \subset (\mathbb{R}^{m-1})^{l+1}$ and the weights are all one. □

Lemma 2.3.12
The map $\Psi^{(\mathcal{E}(0),\ldots,\mathcal{E}(l))}$ is a surjective morphism.

Proof. Remember that the pull-back is contained in the preimage. Due to the condition that the first coordinates of $h(v_{E_k})$ in M^+ and $h_0'(v_{E_k})$ in M_0^+ lie on the left of $2\varepsilon d^2(m+1)+\mathrm{supp}(\phi_1\cdot\mathbb{R}^m)$ and due to the condition that the first coordinate of $h(v_{r^k})$ in M^+ and $h_k(v_{r^k})$ in M_k^+ lie on the right of $2\varepsilon d^2(m+1) + \mathrm{supp}(\phi_1 \cdot \mathbb{R}^m)$, the map $\Psi^{(\mathcal{E}(0),\ldots,\mathcal{E}(l))}$ is well-defined. (See the construction of the spaces for an explanation of the notation.)
$\Psi^{(\mathcal{E}(0),\ldots,\mathcal{E}(l))}$ is surjective: Given a tuple

$$((C_0, h_0), \ldots, (C_l, h_l)) \in \left(\prod_{k\in[l]} \mathrm{ev}^*_{k,k}(Z)\right) \cdot \left(M_0^+ \times \cdots \times M_l^+\right),$$

we can associate to (C_0, h_0) a curve $(C_0, h_0' : C_0 \to \mathbb{R}^m)$ that fulfills $h_0'(x_E^1) \in \mathrm{supp}(\phi_1 \cdot \mathbb{R}^m)$, see the construction of M_0^+ above. Due to the last lemma, it holds

$$\mathrm{ev}_{E_k}(C_0, h_0) = \mathrm{ev}_{r^k}(C_k, h_k)$$

for all $k \in [l]$. Using the condition on curves $(C_0, h_0) \in \mathrm{supp}(M_0^+)$ and $(C_k, h_k) \in \mathrm{supp}(M_k^+)$, namely that the first coordinate of $h_0'(x_{E_k})$ is less than the first coordinate of $h_k(x_{r^k})$, it follows that we can glue the curves (C_0, h_0') and (C_k, h_k) together - at the leaves x_{E_k} of C_0 and x_{r^k} of C_k - creating a new interior edge of positive length and weight r^k. We choose the labeling of the arising curve $(C,h) \in \operatorname{supp}\left((\mathrm{ev}_E^1)^*(\phi_1)\cdot M^{(\mathcal{E}(0),\dots,\mathcal{E}(l))}\right)$ in a way that the splitting process yields again the tuple $((C_0,h_0),\dots,(C_l,h_l))$. Due to the lemma above, the image of a facet of $(\mathrm{ev}_E^1)^*(\phi_1)\cdot M^{(\mathcal{E}(0),\dots,\mathcal{E}(l))}$ is a facet of $\left(\prod_{k\in[l]} \mathrm{ev}_{k,k}^* Z\right)\cdot\left(M_0^+\times\cdots\times M_l^+\right)$ (if we choose compatible polyhedral structures).

$\Psi^{(\mathcal{E}(0),\dots,\mathcal{E}(l))}$ is a morphism: We consider the edge lengths and the coordinates of one root vertex per moduli space as local coordinates. As root vertex for a curve

$$(C,h)\in \operatorname{supp}((\mathrm{ev}_E^1)^*(\phi_1)\cdot M^{(\mathcal{E}(0),\dots,\mathcal{E}(l))})$$

choose the leaf $(E,1)$ which is part of C_0 (see construction 2.3.1). The edge lengths of C are in $1:1$-correspondence to the edge lengths of the curves $C_0,\dots,C_l$ except for the l edges $E_1,\dots,E_l$ with direction $\pm e_1$ linking the curves $C_1,\dots,C_l$ to the curve C_0' "on the left": In the curves $C_1,\dots,C_l$ the lengths of these edges is set to infinity. Moreover, the $x_2,\dots,x_r$-coordinates of the root vertex of (C,h) correspond to the coordinates of the root vertices of (C_k,h_k), $k=0,\dots,l$. If only the x_1-coordinates of the root vertex of (C,h) varies, $\Psi^{(\mathcal{E}(0),\dots,\mathcal{E}(l))}(C,h)$ remains unchanged. If the edge length of E_k, $k\in[l]$, varies by $a\in\mathbb{R}$, the x_1-coordinate of the root vertex of M_k varies by $r^k\cdot a$, where r^k is the weight of the edge E_k.

□

PROOF OF 2.3.9. $(\mathrm{ev}_E^1)^*(\phi_1)\cdot\mathcal{M}_{0,I}(\Delta_m(t),\mathbb{R}^m)$ is isomorphic to $\mathcal{M}_{I\cup\Delta_m(t)}\times(\phi_1\cdot\mathbb{R}^m)$ using the marked point labeled by $(E,1)$ as root vertex and it holds

$$\operatorname{supp}(M^{(\mathcal{E}(0),\dots,\mathcal{E}(l))})\subset\mathcal{M}_{0,I}(\Delta_m(t),\mathbb{R}^m).$$

Thus the weights in the domain of $\Psi^{\mathcal{E}(0),\dots,\mathcal{E}(l)}$ are all one. Also the weight of a facet of

$$\left(\prod_{k\in[l]}\mathrm{ev}_{k,k}^*(Z)\right)\cdot(M_0^\circ\times\cdots\times M_l^\circ)$$

is one, see lemma 2.3.11.

Since we project the curve (C_0',h_0') on the left to $\mathbb{R}^m/\mathbb{R}\cdot e_1$, changing the x_1-coordinate of $h(x_E^1)$ (with $x_E^1\subset C_0'$), does not change the image of (C,h) under $\Psi^{\mathcal{E}(0),\dots,\mathcal{E}(l)}$. Hence, we see that $\Psi^{\mathcal{E}(0),\dots,\mathcal{E}(l)}$ is injective in a neighborhood of $(C,h)\in\operatorname{supp}((\mathrm{ev}_E^1)^*\phi_1\cdot M^{(\mathcal{E}(0),\dots,\mathcal{E}(l))}$ if and only if the image $h(x_E^1)$ of the marked point x_E^1 of C is contained in a facet of $\phi_1\cdot\mathbb{R}^m$ which does contain $-e_1$ as ray ($\phi_1\cdot\mathbb{R}^m$ is a translate of L_{m-1}^m).

The weight of the facet σ of

$$(\Psi^{(\mathcal{E}(0),\dots,\mathcal{E}(l))})_*(\mathrm{ev}_E^1)^*(\phi_1)\cdot M^{(\mathcal{E}(0),\dots,\mathcal{E}(l))}$$

which contains the tuple of curves $((C_0,h_0),\dots,(C_l,h_l))\in\sigma$ is determined by the index of the linear part of the map $\Psi^{(\mathcal{E}(0),\dots,\mathcal{E}(l))}$ restricted to σ and by the number of preimages of $((C_0,h_0),\dots,(C_l,h_l))$ under $\Psi^{(\mathcal{E}(0),\dots,\mathcal{E}(l))}$ which lie in facets on which the map $\Psi^{(\mathcal{E}(0),\dots,\mathcal{E}(l))}$ is injective. (The weights of the facets do not play a role in this case because they are all one.)

It follows from the proof of surjectivity in the last lemma that the preimages (C,h) of tuples $((C_k,h_k))_{k=0,\dots,l}$ only differ in the labeling of the leaves of C which are labeled by $\{1,\dots,d(t)\}\cup\{2d(t)+1,\dots,(m+1)d(t)\}$, i.e. the leaves which have standard directions apart from $-e_1$. Let us count the number of these labelings:

The splitting process prescribes which ends of C in direction $-e_i$, $i=0,2,\dots,m$ go the the curves $C_0,\dots,C_l$. The labeling of the leaves of C_k respects the order of the corresponding leaves in C, see construction 2.3.1. Hence, there are $\frac{d(t)!}{\prod_{k=0}^{l} d(t'(k))!}$ possibilities to choose a labeling of the $d(t)$ leaves with direction $-e_i$ of C such that the leaves of all C_k get a prescribed labeling. Considering all directions $-e_0, -e_2, \ldots, -e_m$, the number of possibilities to label the leaves of the curve (C,h) without changing $\Psi^{(\mathcal{E}(0),\ldots,\mathcal{E}(l))}(C,h) = ((C_0,h_0),\ldots,(C_l,h_l))$ is given by

$$\frac{(d(t)!)^m}{\prod_{k=0}^{l}(d(t'(k))!)^m}.$$

Finally, we analyze the linear part of $\Psi^{(\mathcal{E}(0),\ldots,\mathcal{E}(l))}$: As already mentioned in the proof of proposition 2.3.12, the map $\Psi^{\mathcal{E}(0),\ldots,\mathcal{E}(l)}$ equals the identity in all coordinates but those which encode the lengths of the edges linking C_0', which lies "on the left", with the curves $C_1,\ldots,C_l$ "on the right". When the length of the edge linking C_0' and C_k varies by $a_k \in \mathbb{R}$, the x_1-coordinate of the root vertex of (C_k,h_k) varies by $r^k \cdot a_k$, where r^k is the weight of the edge linking C_0' to C_k. Hence, the index of $\Psi^{(\mathcal{E}(0),\ldots,\mathcal{E}(l))}$ is given by $\prod_{k\in[l]} r^k$.

Altogether, we conclude that

$$\Psi_*(\mathrm{ev}_E^1)^*(\phi_1)\cdot M^{(\mathcal{E}(0),\ldots,\mathcal{E}(l))} = \frac{(d(t)!)^m \cdot \prod_{k\in[l]} r^k}{\prod_{k=0}^{l}(d(t'(k))!)^m}\left(\prod_{k\in[l]} \mathrm{ev}_{k,k}^*(Z)\right)\left(M_0^+\times\cdots\times M_l^+\right).$$

□

2.4. A recursive formula

Notation 2.4.1
Remember that $\pi : \mathbb{R}^m \to \mathbb{R}^{m-1}$ is the projection on the last $m-1$ coordinates and that $\epsilon > 0$ is a small number such that the zero-dimensional cell of all $\Omega_e^j \in \Omega \setminus \{\Omega_E^1\}$ has x_1-coordinate larger than $-\epsilon$. Denote by $i : \mathbb{R}^{m-1} \to \mathbb{R}$ the inclusion map $(x_1, \ldots, x_{m-1}) \mapsto (0, x_1, \ldots, x_{m-1})$. It holds $\pi \circ i = \mathrm{id}_{\mathbb{R}^{m-1}}$.
The idea behind the following definitions is that we split and deform the incidence and tangency conditions Ω and Γ such that they become incidence and tangency conditions in the split spaces $M_0, \ldots, M_l$. Define

- $\Omega_e^{j,(0)} := \pi_*(\{\max\{x_1, -N\} \cdot \Omega_e^j)$ for $\Omega_e^j \in \Omega(0) \setminus \{\Omega_E^1\}$ (we project the part of Ω_e^j in the region $\{x_1 << 0\}$ to $\mathbb{R}^{m-1}$),
- $\Omega_E^{1,(0)} := (i^*\phi_2)^{m-E-1} \cdot \mathbb{R}^{m-1}$ (we project $\phi_1 \cdot \phi_2^{m-E+1} \cdot \mathbb{R}^m = \Omega_E^1$ except $\phi_1 \cdot \mathbb{R}^m$ to $\mathbb{R}^{m-1}$),
- $\Gamma_{1,m-2}^{j,(0)} = \mathbb{R}^{m-2}$ for $1 \leq j \leq d(t'(0))$ (the leaves of (C_0, h_0) in direction $-e_1$ are unrestricted),
- $\Omega_e^{j,(k)} = \Omega_e^j$ for $\Omega_e^j \in \Omega(k)$, $k \in [l]$,
- $\Gamma_{r,e}^{j,(k)} = \Gamma_{r,e}^j$ for $\Gamma_{r,e}^j \in \Gamma(k)$, $k = 0, \ldots, l$ and
- $\Gamma_{r^k,m-1}^{t(k)_{r^k,m-1}+1,(k)} = \Gamma^{r^k} = \mathbb{R}^{m-1}$ for $k \in [l]$ (the additional leaf in M_k with direction $-e_1$ and weight r^k coming from the cut edge E_k is unrestricted).

Set

- $\Omega(0)' = \{\Omega_e^{j,(0)} | \Omega_e^j \in \Omega(0)\} \cup \{\Gamma_{r,e}^{j,(0)} | \Gamma_{r,e}^j \in \Gamma(0)\}$,
- $\Gamma(0)' = \{\Gamma_{1,m-2}^{j,(0)} | j = 1, \ldots, d(t'(0))\}$,
- $\Omega(k)' = \{\Omega_e^{j,(k)} | \Omega_e^j \in \Omega(k)\}$ for $k = 1, \ldots, l$ and
- $\Gamma(k)' = \{\Gamma_{r,e}^{j,(k)} | \Gamma_{r,e}^j \in \Gamma(k)\} \cup \{\Gamma_{r^k,m-1}^{t(k)_{r^k,m-1}+1,(k)}\}$ and
- $\mathcal{E}(k)' = (t(k)', \Omega(k)', \Gamma(k)')$ for $k = 0, \ldots, l$.

Note that $\Gamma(k)'$ is a set of tangency conditions for the vector $t'(k)$, $k = 0, \ldots, l$, defined in construction 2.3.1.

Proposition 2.4.2
Let $\mathcal{E}$ be general and specialized as described at the beginning of section 2.2. Then the degree of $X_m(\mathcal{E})$ is equal to the degree of

$$\sum \frac{(d(t)!)^m \cdot \left(\prod_{k=1}^l r^k\right)}{l! \cdot \prod_{k=0}^l (d(t'(k))!)^m} \left(\prod_{k \in [l]} (\mathrm{ev}_{k,k})^*(Z)\right) \prod_{k=0}^l X_{m_k}(\mathcal{E}(k)')$$

where the sum runs over all partitions $(\mathcal{E}(0), \ldots, \mathcal{E}(l))$ of $\mathcal{E}$.

Proof. Denote by $\pi_k^{l+1} : M_0 \times \cdots \times M_l$ the projection onto the k-th factor, $k \in \{0, \ldots, l\}$. It holds that

- $\mathrm{ev}_e^{j,(k)} \circ \pi_k^{l+1} \circ \Psi = \mathrm{ev}_e^j$ for all labels (e, j) that go to the curve C_k, $k \in [l]$,
- $\mathrm{ev}_{r,e}^{j,(k)} \circ \pi_k^{l+1} \circ \Psi = \mathrm{ev}_{r,e}^j$ for all labels (r, e, j) that go to the curve C_k, $k = 0, \ldots, l$,
- $\mathrm{ev}_e^{j,(0)} \circ \pi_0^{l+1} \circ \Psi = \pi \circ \mathrm{ev}_e^j$ for all labels (e, j) that got to the curve C_0.

Moreover, for a label $(e, j) \neq (E, 1)$ that goes to C_0 it is true that

$$(\pi \circ \mathrm{ev}_e^j)^* \Omega_e^{j,(0)} = (\pi \circ \mathrm{ev}_e^j)^* \pi_*(\max\{x_1, -N\} \cdot \Omega_e^j) = (\mathrm{ev}_e^j)^* \pi^* \pi_*(\max\{x_1, -N\} \cdot \Omega_e^j) = (\mathrm{ev}_e^j)^* \Omega_e^j,$$

at least in the region $\{x \in \mathbb{R}^m | x_1 << 0\}$ where Ω_e^j is fulfilled by the elements of $\mathrm{supp}(X_m(\mathcal{E}))$, see corollary 2.2.4. Furthermore it holds

$$(\pi \circ \mathrm{ev}_e^j)^* \Omega_E^{1,(0)} = (\mathrm{ev}_e^j)^* \pi^* (i^*\phi_2)^{m-E-1} \cdot \mathbb{R}^{m-1}) = (\mathrm{ev}_e^j)^* (\phi_2^{m-E-1}),$$

because ϕ_2 does not depend on the first coordinate.

Since $\mathcal{E}$ is general and due to 2.2.2, all curves $(C,h) \in \operatorname{supp}(X_m(\mathcal{E}))$ lie in the support of M^+. Moreover, the push-forward preserves the degree of zero-dimensional varieties. Using the projection formula (p.f.), we conclude

$$\begin{aligned}
\deg(X_m(\mathcal{E})) &= \deg\left(\prod_{(e,j)}(\mathrm{ev}_e^j)^*(\Omega_e^j)\prod_{(r,e,j)}(\mathrm{ev}_{r,e}^j)^*(\Gamma_{r,e}^j)\cdot\mathcal{M}_{0,I}(\Delta_m(t),\mathbb{R}^m)\right)\\
&\overset{2.2.2,2.2.1}{=} \deg\left(\prod_{(e,j)}(\mathrm{ev}_e^j)^*(\Omega_e^j)\prod_{(r,e,j)}(\mathrm{ev}_{r,e}^j)^*(\Gamma_{r,e}^j)\cdot M^+\right)\\
&\overset{2.3.6}{=} \sum\frac{1}{l!}\deg\left(\prod_{k=0}^{l}\left(\prod_{\Omega_e^j\in\Omega(k)}\Psi^*(\pi_k^{l+1})^*(\mathrm{ev}_e^{j,(k)})^*\Omega_e^{j,(k)}\prod_{\Gamma_{r,e}^j\in\Gamma(k)}\Psi^*(\pi_k^{l+1})^*(\mathrm{ev}_{r,e}^{j,(k)})^*\Gamma_{r,e}^{j,(k)}\right)\right.\\
&\qquad\left.\cdot(\mathrm{ev}_E^1)^*(\phi_1)\cdot M^{(\mathcal{E}(0),\ldots,\mathcal{E}(l))}\right)\\
&\overset{\text{p.f.}}{=} \sum\frac{1}{l!}\deg\left(\prod_{k=0}^{l}\left(\prod_{\Omega_e^{j,(k)}\in\Omega(k)'}(\pi_k^{l+1})^*(\mathrm{ev}_e^{j,(k)})^*\Omega_e^{j,(k)}\prod_{\Gamma_{r,e}^{j,(k)}\in\Gamma(k)'}(\pi_k^{l+1})^*(\mathrm{ev}_{r,e}^{j,(k)})^*(\Gamma_{r,e}^{j,(k)})\right)\right.\\
&\qquad\left.\cdot\Psi_*\left((\mathrm{ev}_E^1)^*(\phi_1)\cdot M^{(\mathcal{E}(0),\ldots,\mathcal{E}(l))}\right)\right)\\
&\overset{2.3.9}{=} \sum\frac{(d(t)!)^m\prod_{k=1}^{l}r^k}{l!\cdot\prod_{k=0}^{l}(d(t'(k))!)^m}\deg\left[\left(\prod_{k\in[l]}\mathrm{ev}_{k,k}^*(Z)\right)\right.\\
&\qquad\left.\cdot\prod_{k=0}^{l}\left(\prod_{\Omega_e^{j,(k)}\in\Omega(k)'}(\mathrm{ev}_e^{j,(k)})^*\Omega_e^{j,(k)}\prod_{\Gamma_{r,e}^{j,(k)}\in\Gamma(k)'}(\mathrm{ev}_{r,e}^{j,(k)})^*(\Gamma_{r,e}^{j,(k)})\cdot M_k^+\right)\right]
\end{aligned}$$

where the sum runs over all partitions $(\mathcal{E}(0),\ldots,\mathcal{E}(l))$ of $\mathcal{E}$.

With the same argument as in lemma 2.2.1 we see that all tuples of curves $((C_0,h_0),\ldots,(C_l,h_l))$ that lie in the support of the intersection product

$$\left(\prod_{k\in[l]}\mathrm{ev}_{k,k}^*(Z)\right)X_{m-1}(\mathcal{E}(0)')\times X_m(\mathcal{E}(1)')\times\cdots\times X_m(\mathcal{E}(l)')$$

actually lie in the support of $M_0^+\times\cdots\times M_l^+$ - at least if all conditions in $\mathcal{E}$ and also the diagonals Z are general and if $\mathcal{E}$ is specialized as described at the beginning of section 2.2. Hence, the claim follows.

□

Lemma 2.4.3 (Splitting lemma)
With $e_k := \dim(X_m(\mathcal{E}(k)'))$ for all $k \in [l]$ and $m_0 = m-1$ and $m_k = m$ for $k \in [l]$, it is valid that

$$\begin{aligned}
&\deg\left(\left(\prod_{k\in[l]}(\mathrm{ev}_{k,k})^*(Z)\right)\cdot\prod_{k=0}^{l}X_{m_k}(\mathcal{E}(k)')\right)\\
=\ &\deg\left(\left(\prod_{k\in[l]}(\mathrm{ev}_{E_k})^*L_{e_k}^{m-1}\right)\cdot X_{m-1}(\mathcal{E}(0)')\right)\cdot\prod_{k\in[l]}\deg\left((\mathrm{ev}_{r^k})^*L_{m-1-e_k}^{m-1}\cdot X_m(\mathcal{E}(k)')\right),
\end{aligned}$$

where $\mathcal{E}(k)'$ is defined at the beginning of this section.

Proof. We will prove the splitting lemma by induction on l. For $l=0$ the claim is obviously true.

Assume that the lemma is true for $l-1 \in \mathbb{N}$. With the projection formula and since the push-forward preserves the degree of zero-dimensional varieties, it holds that the degree of

$$\left(\prod_{k\in[l]}(\mathrm{ev}_{k,k})^*(Z)\right)\cdot\left(\prod_{k=0}^{l} X_{m_k}(\mathcal{E}(k)')\right)$$

is equal to the degree of

$$Z\cdot\left(\left[\left(\mathrm{ev}_{E_l}\circ\pi_0^l\right)_*\left(\prod_{k\in[l-1]}(\mathrm{ev}'_{k,k})^*(Z)\right)\left(\prod_{k=0}^{l-1} X_{m_k}(\mathcal{E}(k)')\right)\right]\times\left[\left(\mathrm{ev}_{r^l}\right)_* X_m(\mathcal{E}(l)')\right]\right),$$

whose degree is equal to that of

$$\left(\left(\mathrm{ev}_{E_l}\right)_*\left(\pi_0^l\right)_*\left(\prod_{k\in[l-1]}(\mathrm{ev}'_{k,k})^*(Z)\right)\cdot\left(\prod_{k=0}^{l-1} X_{m_k}(\mathcal{E}(k)')\right)\right)\cdot\left(\left(\mathrm{ev}_{r^l}\right)_* X_m(\mathcal{E}(l)')\right).$$

For $k \in [l-1]$ define

$$\mathrm{ev}'_{k,k} := (\mathrm{ev}_{E_k}\times\mathrm{ev}_{r^k})\circ(\pi_0^{l-1}\times\pi_k^{l-1}) : X_{m-1}(\mathcal{E}(0)')\times\prod_{k=1}^{l-1} X_m(\mathcal{E}(k)')\to\mathbb{R}^{m-1}\times\mathbb{R}^{m-1}.$$

Using 1.3.28, it follows that

$$(\mathrm{ev}_{E_l})_*\left(\prod_{k\in[l-1]}(\mathrm{ev}_{E_k})^*(\mathrm{ev}_{r^k})_* X_m(\mathcal{E}(k)')\cdot M_0\right)\ \text{and}\ \left(\mathrm{ev}_{r^l}\right)_* X_m(\mathcal{E}(l)'),$$

which are subvarieties of $\mathbb{R}^m$, have standard directions. Due to lemma 1.2.32, the first term above is equal to

$$(\mathrm{ev}_{E_l})_*\left(\pi_0^l\right)_*\left(\left(\prod_{k\in[l-1]}(\mathrm{ev}'_{k,k})^*(Z)\right)\cdot\left(X_{m_0}(\mathcal{E}(0)')\times\cdots\times X_{m_{l-1}}(\mathcal{E}(l-1)')\right)\right).$$

Hence, the prerequisites of lemma 1.2.31 are fulfilled and it follows

$$\begin{aligned}
&\deg\left(\left(\prod_{k\in[l]}(\mathrm{ev}_{k,k})^*(Z)\right)\cdot\left(\prod_{k=0}^{l} X_{m_k}(\mathcal{E}(k)')\right)\right)\\
=\ &\deg\left(\left(\left(\mathrm{ev}_{E_l}\right)_*\left(\pi_0^l\right)_*\left(\prod_{k\in[l-1]}(\mathrm{ev}'_{k,k})^*(Z)\right)\cdot\left(\prod_{k=0}^{l-1} X_{m_k}(\mathcal{E}(k)')\right)\right)\cdot\left(\left(\mathrm{ev}_{r^l}\right)_* X_m(\mathcal{E}(l)')\right)\right)\\
\overset{1.2.31}{=}\ &\sum_{s,t}\deg\left(L_s^{m-1}\cdot\left(\left(\mathrm{ev}_{E_l}\right)_*\left(\pi_0^l\right)_*\left(\prod_{k\in[l-1]}(\mathrm{ev}'_{k,k})^*(Z)\right)\cdot\left(\prod_{k=0}^{l-1} X_{m_k}(\mathcal{E}(k)')\right)\right)\right)\\
&\cdot\deg\left(L_t^{m-1}\cdot\left(\left(\mathrm{ev}_{r^l}\right)_* X_m(\mathcal{E}(l)')\right)\right)\\
\overset{\text{p.f.}}{=}\ &\sum_{s,t}\deg\left(\left(\prod_{k\in[l-1]}(\mathrm{ev}'_{k,k})^*(Z)\right)\cdot\left[(\mathrm{ev}_{E_l})^* L_s^{m-1}\cdot X_{m-1}(\mathcal{E}(0)')\times\left(\prod_{k\in[l-1]} X_m(\mathcal{E}(k)')\right)\right]\right)\\
&\cdot\deg\left((\mathrm{ev}_{r^l})^* L_{m-1-t}^{m-1}\cdot X_m(\mathcal{E}(l)')\right)\\
\overset{\text{induction}}{=}\ &\deg\left(\left(\prod_{k\in[l]}(\mathrm{ev}_{E_k})^* L_{e_k}^{m-1}\right)\cdot X_{m-1}(\mathcal{E}(0)')\right)\cdot\deg\left((\mathrm{ev}_{r^k})^* L_{m-1-e_k}^{m-1}\cdot X_m(\mathcal{E}(k)')\right),
\end{aligned}$$

where e_k is the dimension of $X_m(\mathcal{E}(k)')$. □

The following notation, which finally allows to state the recursive formula that allows to determine the number of rational tropical curves of generalized projective degree $\Delta_m(t)$ passing a given configuration of tropical varieties with standard directions, is based on the notation used in [**Vak00**], section 2.

Notation 2.4.4 ($\mathcal{F}(m,e)$, $\mathcal{F}(k)$, $\mathcal{F}'(k)$)
Let the data $\mathcal{E} = (t, \Omega, \Gamma)$ be general, $\Omega_E^1 \in \Omega$ be the distinguished element "on the left" as before and let

$$(i(0), t(0)), \ldots, (i(l), t(l))$$

be a partition of $\mathcal{F} = (i,t)$ and set $\mathcal{F}(k) = (i(k), t(k))$ for $k \in [l]$. Set

$$N_m(\mathcal{F}) := N_m(\mathcal{E}).$$

By $\mathcal{F}(r,e)$ we refer to the set of data given by

$$\text{the vector } i - \varepsilon_E \text{ and } t - \varepsilon_{r,e} + \varepsilon_{r,e+E-(m-1)}.$$

If the partition $\mathcal{F}(0), \ldots, \mathcal{F}(l)$ of $\mathcal{F}$ is induced by the partition $(\mathcal{E}(0), \ldots, \mathcal{E}(l))$ of $\mathcal{E}$, remember that we defined $e_k = \dim X_m(\mathcal{E}(k)')$ for $k \in [l]$ and let $i'(k)$ and $t'(k)$ be given by

- $i'(k) := i(k)$ and
- $t'(k) := t(k) + \varepsilon_{r^k, m-1-e_k}$ (which has already been defined before),

where the second summand of $t'(k)$ stands for the additional tangency condition $L^{m-1}_{m-1-e_k}$ imposed on the leaf x_{r^k} of weight r^k in M_k in the last lemma. The remaining entries of $i'(k)$ and $t'(k)$ are induced by $\Omega(k)'$ and $\Gamma(k)'$. For $k = 0$ we set

- $i'(0)_e := i(0)_{e+1} + \#\{e_k | e_k = e\}_{1 \leq k \leq l} + \sum_r t(0)_{r,e}$ for all $e \in \mathbb{N} \setminus \{E, E-1\}$,
- $i'(0)_E := i(0)_{E+1} + 1 + \#\{e_k | e_k = E\}_{1 \leq k \leq l} + \sum_r t(0)_{r,E}$,
- $i'(0)_{E-1} := i(0)_E - 1 + \#\{e_k | e_k = E-1\}_{1 \leq k \leq l} + \sum_r t(0)_{r,E-1}$ and
- $t'(0) := d_0 \varepsilon_{1,m-2}$ (which has already been defined before),

where the terms $\#\{e_k | e_k = e\}$, $e \in [m_-1]$ stands for the additional incidence conditions $L^{m-1}_{e_k}$ of dimension e imposed on the leaf x_{E_k} in M_0 in the last lemma. The remaining entries are induced by $\Omega(0)'$ and $\Gamma(0)'$, see 2.4.1.
For $k = 0, \ldots, l$ set $\mathcal{F}'(k) = (i'(k), t'(k))$.
Define multinomial coefficients with vector arguments via

$$\begin{aligned} \binom{i}{h(0), \ldots, h(l)} &:= \prod_e \binom{i_e}{i(0)_e, \ldots, i(l)_e}, \\ \binom{t}{t(0), \ldots, t(l)} &:= \prod_{r,e} \binom{t_{r,e}}{t(0)_{r,e}, \ldots, t(l)_{r,e}}. \end{aligned}$$

The following formula, which allows to count rational tropical curves of generalized tropical degree $\Delta_m(t)$ which fulfill prescribed incidence and tangency conditions, coincides with the algebro-geometric counterpart in [**Vak00**], theorem 2.20.

Theorem 2.4.5
Let $2 \leq m \in \mathbb{R}^m$. The numbers $N_m(\mathcal{F})$ can be calculated recursively via the formula

$$\begin{aligned} N_m(\mathcal{F}) = \quad & \sum_{r,e} r \cdot t_{r,e} \cdot N_m(\mathcal{F}(r,e)) \\ + \quad & \sum \frac{\prod_{k=1,\ldots,l} r^k}{\operatorname{Aut}(\mathcal{F}(1), \ldots, \mathcal{F}(l)) \cdot d(t'(0))!} \cdot \binom{t}{t(0), \ldots, t(l)} \cdot \binom{i - \varepsilon_E}{i(0) - \varepsilon_E, \ldots, i(l)} \\ & \cdot N_{m-1}(\mathcal{F}'(0)) \cdot \prod_{k=1}^{l} N_m(\mathcal{F}'(k)) \end{aligned}$$

where the second sum runs over all partitions $(\mathcal{F}(0),\dots,\mathcal{F}(l))$ of $\mathcal{F}=(i,t)$ with $d(t'(0))>0$ (see 2.1.1).

The only initial value that is needed for computing all numbers via this formula is that there exists exactly one rational tropical curve (counted with multiplicity) of projective degree 1 which intersects two given general points in $\mathbb{R}^2$ (where one point may be a tangency condition in $\mathbb{R}^2/\mathbb{R}\cdot e_1$).

Proof. We want to calculate recursively

$$N_m(\mathcal{F})=\frac{\deg\left(\prod_{(e,j)}(\mathrm{ev}_e^j)^*\Omega_e^j\cdot\prod_{(m,e,j)}(\mathrm{ev}_{r,e})^*\Gamma_{r,e}^j\cdot\mathcal{M}_{0,I}(\Delta_m(t),\mathbb{R}^m)\right)}{(d(t)!)^m}.$$

Let $\mathcal{F}$ be induced by $\mathcal{E}=(t,\Omega,\Gamma)$, which we assume to be specialized as described at the beginning of section 2.2. With $e_k=\dim(X_{m_k}(\mathcal{E}(k)')$ for $l=0,\dots,l$, $m_0=m-1$ and $m_k=m$ for $k\in[l]$ it follows

$$\begin{aligned}
&N_m(\mathcal{F})=N_m(\mathcal{E})\\
\overset{2.4.2}{=}\ &\sum\frac{\left(\prod_{k=1}^l r^k\right)}{l!\cdot(\prod_{k\in[l]}(d(t'(k))!)^m}\cdot\deg\left(\left(\prod_{k\in[l]}(\mathrm{ev}_{k,k})^*(Z)\right)\left(X_{m-1}(\mathcal{E}(0)')\times\prod_{k=1}^l X_m(\mathcal{E}(k)')\right)\right)\\
\overset{2.4.3}{=}\ &\sum\frac{\left(\prod_{k=1}^l r^k\right)}{l!\cdot(\prod_{k\in[l]}(d(t'(k))!)^m}\cdot\deg\left(\left(\prod_{k\in[l]}(\mathrm{ev}_{E_k})^*L_{e_k}^{m-1}\right)\cdot X_{m-1}(\mathcal{E}(0)')\right)\\
&\cdot\deg\left(\prod_{k\in[l]}((\mathrm{ev}_{r^k})^*L_{m-1-e_k}^{m-1}\cdot X_m(\mathcal{E}(k)'))\right)\\
\overset{2.1.3,\ 2.4.4}{=}\ &\sum\frac{\prod_{k=1,\dots,l}r^k}{\mathrm{Aut}(\mathcal{F}(1),\dots,\mathcal{F}(l))}\cdot N\cdot\frac{N_{m-1}(\mathcal{F}'(0))\cdot\prod_{k=1}^l N_m(\mathcal{F}'(k))}{d(t'(0))!},
\end{aligned}$$

where the sum runs over all partitions all ordered partitions $(\mathcal{E}(0),\dots,\mathcal{E}(l))$ of $\mathcal{E}$ and over all partitions $\mathcal{F}(0),\dots,\mathcal{F}(l)$ of $\mathcal{F}$, respectively, and where

$$N:=\binom{t}{t(0),\dots,t(l)}\cdot\binom{i-\varepsilon_E}{i(0)-\varepsilon_E,\dots,i(l)}.$$

The factor N appears because it is the number of possibilities to distribute the incidence conditions of dimension e (except Ω_E^1) and tangency conditions of dimension e and weight r onto the $l+1$ components of the partition $\mathcal{F}(0),\dots,\mathcal{F}(l)$. We divide by $\mathrm{Aut}(\mathcal{F}(1),\dots,\mathcal{F}(l))$ to make sure that we count each curve in $X_m(\mathcal{E})$ only once.

Take a closer look at the case that the partition $\mathcal{F}(0),\dots,\mathcal{F}(l)$ implies $d(t(0)')=0$, i.e. the curve (C_0,h_0) on the left has projective degree 0 and $h(C_0)$ is a point on an unbounded end of $h(C)$ with direction $-e_1$. Since $\mathcal{E}$ was chosen to be general, all curves that appear in $X_m(\mathcal{E})$ are trivalent. Hence, the partition $(\mathcal{F}(0),\dots,\mathcal{F}(l))$ of $\mathcal{F}$ may only have a non-zero contribution to $N_m(\mathcal{F})$ if the curve (C_0',h_0') fulfills only one incidence and one tangency condition, i.e. $i(0)=\varepsilon_E$, $t(0)=\varepsilon_{r,e}$ for some $(r,e)\in\mathbb{N}^2$ with $t_{r,e}>0$ and $l=1$.

We conclude that

- $\dim(X_{m-1}(\mathcal{E}(0)'))=e+E-(m-1)$,
- $N_{m-1}(\mathcal{F}(0)')=1$,
- $i(1)'=i-\varepsilon_E$,
- $t(1)'=t-\varepsilon_{r,e}+\varepsilon_{r,e+E-(m-1)}$ and hence
- $N_m(\mathcal{F}(1)')=N_m(\mathcal{F}(r,e))$.

The claim follows due to

$$\binom{t}{t(0),t(1)}=t_{r,e}\text{ and }\binom{i-\varepsilon_E}{i(0)-\varepsilon_E,i(1)}=1.$$

□

CHAPTER 3

Enumerative geometry of elliptic tropical curves in $\mathbb{R}^m$

We will give the set $\mathrm{P}_I(\Delta, \mathbb{R}^m)$ of elliptic I-marked parametrized curves of degree $j : \Delta \to \mathbb{R}^m$ a polyhedral structure that reflects the combinatorial types. Afterwards, we will introduce a notion of a well-spaced elliptic curve, which is oriented at the known sufficient and necessary condition on the realizability of elliptic curves. The set $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ of well-spaced elliptic I-marked curves of degree Δ in $\mathbb{R}^m$ is a pure-dimensional subcomplex of $\mathrm{P}_I(\Delta, \mathbb{R}^m)$. We will show that a certain dense open subset $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)^{\mathrm{reg}}$ of $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ consisting only of regular elliptic curves (i.e. they contain only vertices of genus zero and an "honest" loop) is a tropical variety.

Finally, we prove an invariance statement about elliptic curves: The number of well-spaced elliptic curves passing a general configuration of shifted tropical fans in $\mathbb{R}^m$ does not depend on the position of the fans.

3.1. An abstract polyhedral complex parameterizing elliptic tropical curves

Definition 3.1.1 (Parameter space $\mathrm{P}_I(\Delta, \mathbb{R}^m)$ of elliptic curves of degree Δ in $\mathbb{R}^m$)
We define $\mathrm{P}_I(\Delta, \mathbb{R}^m)$ as the set of elliptic parametrized I-marked curves (C, h) of degree Δ in $\mathbb{R}^m$ which fulfill the following condition: If C is elliptic, there exists a flag segment $(p, E) \in \mathbf{FS}(C)$ in the loop of C (i.e. $E \subset C_L$) which fulfills $\mathrm{v}(p, E) \neq 0$. In this case, $h|_E$ is injective. Elliptic means that either C has genus one and only points of genus zero or C is rational and has precisely one point of genus one. If C is elliptic, C_L contains at least two different edges because of the condition that $h|_E$ is injective on at least one bounded edge $E \subset C_L$ in the loop of C. If $I = [n]$, we denote this parameter space also by $\mathrm{P}_n(\Delta, \mathbb{R}^m)$.

The aim of this section is to prove the following proposition. Remember that for a combinatorial type α we denote by $\mathrm{P}(\alpha)$ the set of curves in $\mathrm{P}_I(\Delta, \mathbb{R}^m)$ which have combinatorial type α and by $\overline{\mathrm{P}(\alpha)}$ the set of curves whose combinatorial type is equal to or a specialization of α.

Proposition 3.1.2
$(\{\overline{\mathrm{P}(\alpha)}\}, \mathrm{P}_I(\Delta, \mathbb{R}^m), \{j_\alpha^{l,d}\})$, where α runs over the combinatorial types in $\mathrm{P}_I(\Delta, \mathbb{R}^m)$ and the charts $j_\alpha^{l,d}$ are defined later on in 3.1.9, is an abstract polyhedral complex whose polyhedral structure is given by the combinatorial types. It is in general not pure-dimensional.

By abuse of notation, we denote the abstract polyhedral complex $(\{\overline{\mathrm{P}(\alpha)}\}, \mathrm{P}_I(\Delta, \mathbb{R}^m), \{j_\alpha^{l,d}\})$ by $\mathrm{P}_I(\Delta, \mathbb{R}^m)$, too.

Example 3.1.3 ($\mathrm{P}_I(\Delta, \mathbb{R}^m)$ not pure-dimensional)
Let $\Delta = \{a, b, c, d\}$ and let the degree $j : \Delta \to \mathbb{R}^2$ be given by $j(a) = (-1, 1, -1)$, $j(b) = (2, 1, 0)$, $j(c) = (1, -2, -1)$, $j(d) = (-2, 0, 2)$. The curves $(C_1, h_1), (C_2, h_2)$ in the following figure are elements of $\mathrm{P}_1(\Delta, \mathbb{R}^3)$. Denote their combinatorial types by α_1 and α_2, respectively. Since both curves have only 3-valent vertices of genus zero, their combinatorial types are maximal and they both lie in facets of $\mathrm{P}_1(\Delta, \mathbb{R}^3)$.

The dimension of $\mathrm{P}(\alpha_1)$ is 5: C_1 is determined by the position of the unique marked point in $\mathbb{R}^3$, the length of the unique bounded edge outside the loop and by the length of the path around the loop. However, the dimension of $\mathrm{P}(\alpha_2)$ is 6 because C_2 has an additional bounded edge compared to C_1.

The reason why the dimension of the two maximal cells differ is the following. If the direction vectors a_1, a_2, a_3, a_4 of the edges in the loop span $\mathbb{R}^3$ as in the case of C_1, they must fulfill three conditions to close the loop, one in each coordinate:

$$\sum_{i=1}^{4} \lambda_i a_i = 0 \in \mathbb{R}^3.$$

If the direction vectors b_1, b_2, b_3 of the edges in the loop span a space of dimension two as in the case of C_2, these edges must only fulfill two conditions to close the loop. Hence, C_2 has an additional degree of freedom.

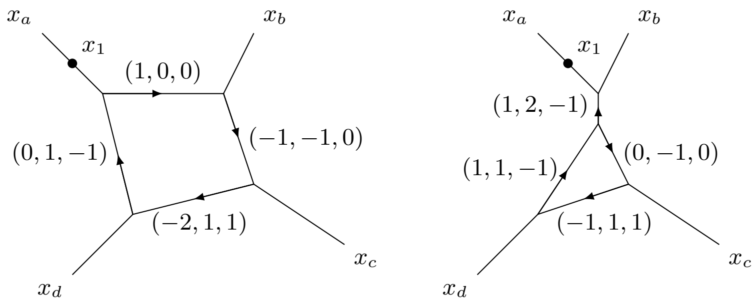


FIGURE 1. The curves (C_1, h_1) (left) and (C_2, h_2) (right) from example 3.1.3. The given vectors are the direction vectors of the bounded edges.

This example can be generalized: Let $(C_1, h_1), (C_2, h_1) \in \mathrm{P}_I(\Delta, \mathbb{R}^m)$ be curves with maximal combinatorial types α_1 and α_2. Let L_1, L_2 be the linear spaces spanned by the direction vectors of flags in loop of C_1 and C_2, then

$$\dim \mathrm{P}(\alpha_1) - \operatorname{codim} L_1 = \dim \mathrm{P}(\alpha_2) - \operatorname{codim} L_2.$$

We present the idea why this statement is true:

Let $\alpha_1 = (\Gamma_1, \mathrm{v}_1)$ and $\alpha_2 = (\Gamma_2, \mathrm{v}_2)$, where Γ_1 and Γ_2 are the abstract combinatorial types. Then Γ_1 and Γ_2 are both elliptic and have the same number of inner edges, namely $\#(I\dot{\cup}\Delta)$, and there are no conditions on the edge lengths. However, the combinatorial types α_1 and α_2 fulfill that the loop is closed, which imposes $\dim L_1$ and $\dim L_2$ conditions on the edge lengths, respectively. It follows $\dim \mathrm{P}(\alpha_1) + \dim L_1 = \dim \mathrm{P}(\alpha_2) + \dim L_2$ and the stated result follows.

Lemma 3.1.4
A curve $(C, h) \in \mathrm{P}_I(\Delta, \mathbb{R}^m)$ is uniquely determined by C, the degree $j : \Delta \to \mathbb{R}^m$, the position of one root vertex $h(x_i)$, $i \in \Delta$, and the non-zero direction vector of a flag in the loop of C (if such a flag exists). In other words: Let $(C, h), (C, g) \in \mathrm{P}_I(\Delta, \mathbb{R}^m)$ and assume that there exists $i \in \Delta$ with $h(x_i) = g(x_i)$. If C is regular, assume moreover that there exists $(p, E) \in \mathbf{F}(C)$ with $E \subset C_L$ bounded and $0 \neq \mathrm{v}_{(C,h)}(p, E) = \mathrm{v}_{(C,g)}(p, E)$. Then it holds $h = g$.

PROOF. If C is non-regular, the first Betti number of $\operatorname{supp}(C)$ is zero and (C, h) is uniquely determined by C and $h(x_i)$, see remark 1.3.24. So let us assume that C is regular. Then there exists a flag (p, E) with $E \subset C_L$ and $d := \mathrm{v}_{(C,h)}(p, E) = \mathrm{v}_{(C,g)}(p, E) \neq 0$. We cut the edge E in the middle, set the length of the two new edges to infinity. We give the arising topological space structures of parametrized I-marked curves (C', h') and (C', g') that are induced by (C, h) and (C, g). (C', h') and (C', g') are rational curves and elements of $\mathcal{M}_{0,I}(\Delta', \mathbb{R}^m)$ with $\Delta' = \Delta\dot{\cup}\{A_1, A_2\}$, where the direction of the leaves x_{A_1} and x_{A_2} labeled by A_1 and A_2 is given by $d_{A_1} = d$ and $d_{A_2} = -d$. Since an element $(D, f) \in \mathcal{M}_{0,I}(\Delta', \mathbb{R}^m)$ is uniquely determined by D and the image of the marked point x_i under f, see above, we conclude $h' = g'$ and therefore $h = g$. $\square$

Example 3.1.5
Here is an example that shows that a degree $j : \Delta \to \mathbb{R}^m$ and an abstract combinatorial type Γ are in general not sufficient to determine a combinatorial type (Γ, v) uniquely. Let the degree

$j : \Delta = \{1,2,3\} \to \mathbb{R}^2$ be given by $j(1) = (1,2)$, $j(2) = (2,-1)$ and $j(3) = (-3,-1)$. In figure 2, $h_1(C_1)$ and $h_2(C_2)$ are shown for two curves $(C_1,h_1),(C_2,h_2) \in \mathrm{P}_\emptyset(\Delta,\mathbb{R}^2)$ which fulfill that C_1 and C_2 have the same abstract combinatorial type.

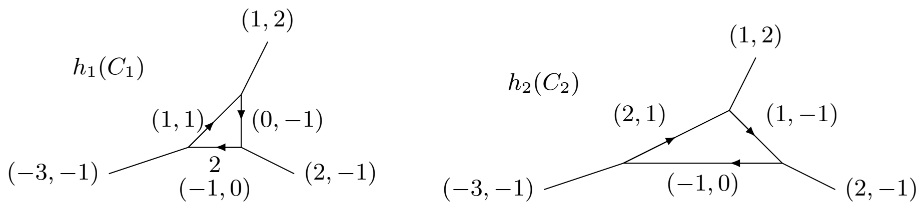


FIGURE 2. The curves C_1 and C_2 have the same abstract combinatorial type and the combinatorial types of (C_1,h_1), (C_2,h_2), which have the same degree, differ.

Let $\beta \le \alpha$ be combinatorial types in $\mathrm{P}_I(\Delta,\mathbb{R}^m)$ and let $[E] \in \mathbf{E}(\beta)$ be an edge. We will show that the edge $[E]$ corresponds to a unique edge $e^\alpha_\beta([E]) \in \mathbf{E}(\alpha)$ of α - at least if α is regular. This correspondence will allow to use the edge lengths as coordinates for the polyhedral charts $j^{l,d}_\alpha$ on $\overline{\mathrm{P}(\alpha)}$ (where $\overline{\mathrm{P}(\alpha)}$ denotes the set of curves whose combinatorial type is finer than α).

Notation 3.1.6
Let α be a combinatorial type in $\mathrm{P}_I(\Delta,\mathbb{R}^m)$ and $(C,h) \in \mathrm{P}(\alpha)$. We define $\tilde{\mathbf{E}}(C) \subset \mathbf{E}(C)$ as the set containing all edges of C except $E_1,E_2 \in \mathbf{E}(C)$ if E_1 and E_2 are the only edges in the loop of C and if they fulfill $\omega(E_1) = \omega(E_2)$ (i.e. $E_1 \cup E_2 = C_L$). $\tilde{\mathbf{E}}(\alpha)$ is defined analogously.

Lemma 3.1.7
Let $\beta \le \alpha$ be combinatorial types of elliptic I-marked curves and let $(C,h),(D,g) \in \mathrm{P}_I(\Delta,\mathbb{R}^m)$ be of type α and β, respectively, such that $(D,g) \le (C,h)$. Denote the projection map by $\pi : \mathrm{supp}(C) \to \mathrm{supp}(D)$. Then the map

$$\begin{array}{rcl} e^\alpha_\beta : \tilde{\mathbf{E}}(\beta) & \hookrightarrow & \mathbf{E}(\alpha) \\ {[E]} & \to & [\overline{\pi^{-1}(E^\circ)}] \end{array}$$

is well-defined, injective and does not depend on the choice of (C,h), (D,g). Moreover, if there are only two edges E_1,E_2 in the loop of D,

$$e^\alpha_\beta([E_1]) := \{[\overline{\pi^{-1}(E_1^\circ)}],[\overline{\pi^{-1}(E_2^\circ)}]\}$$

does not depend on the choice of (C,h) and (D,g).

PROOF. Lemma 1.3.11 states that $f_1(E) = f_2(E)$ if $f_1,f_2 : (C,h) \to (C',h')$ are combinatorial homeomorphisms and if $E \in \tilde{\mathbf{E}}(C)$. In order to prove the lemma, it remains to show that, for specializations $(D_1,g_1),(D_2,g_2) \le (C,h)$ of (C,h) with projection maps $\pi_i : \mathrm{supp}(C) \to \mathrm{supp}(D_i)$, it holds $\pi_1(E) = \pi_2(E)$ for all edges $E \in \tilde{\mathbf{E}}(C)$ which are not contracted by π_1 or π_2.
We prove this statement via the following, where we denote the set of edges that is contracted by π_i by $\mathbf{E}_i$: If $f : D_1 \to D_2$ is a combinatorial morphism, it holds

$$\pi_1^{-1}(E^\circ_{D_1}) = \pi_2^{-1}(f(E^\circ_{D_1}))$$

for all edges $E_{D_1} \in \tilde{\mathbf{E}}(D_1)$ and

$$\pi_1^{-1}(((E^1_{D_1})^\circ) \cup ((E^2_{D_1})^\circ)) = \pi_2^{-1}(f((E^1_{D_1})^\circ) \cup f((E^2_{D_1})^\circ))$$

if $E^1_{D_1}$ and $E^2_{D_1}$ are the only edges in the loop of D_1, i.e. $E^1_{D_1} \cup E^2_{D_1} = (D_1)_L$. Here is the proof:
Let $E \in \mathbf{E}(C) \setminus \mathbf{E}_1$, hence E is not contracted by π_1 and $\pi_1(E) \in \mathbf{E}(D_1)$ is an edge of D_1. Assume first that E is an edge outside the loop of C. Then $\mathrm{supp}(D_1) \setminus \pi_1(E^\circ)$ and $\mathrm{supp}(D_2) \setminus f(\pi_1(E^\circ))$ induce the same partition of $I = I_1 \dot\cup I_2$. Also $\mathrm{supp}(C) \setminus E^\circ = \mathrm{supp}(C) \setminus \pi_1^{-1}(\pi_1(E^\circ))$ and $\mathrm{supp}(C) \setminus \pi_2^{-1}(f(\pi_1(E^\circ)))$ induce the same partition $I = I_1 \dot\cup I_2$. But that means that $E^\circ = \pi_2^{-1}(f(\pi_1(E^\circ)))$ and E is not contracted by π_2, i.e. $E \in \mathbf{E}(C) \setminus \mathbf{E}_2$.

Assume now that $E \in \mathbf{E}(C) \setminus \mathbf{E}_1$ is an edge in the loop of C. Since the loop of a parametrized curve consists of more than one edge, there exists an edge $E_{D_1} \neq \pi_1(E)$ in the loop of D_1. As in the previous case,

$$\operatorname{supp}(D_1) \setminus (\pi_1(E^\circ) \cup E^\circ_{D_1}) \text{ and } \operatorname{supp}(D_2) \setminus (f(\pi_1(E^\circ)) \cup f(E^\circ_{D_1}))$$

induce the same partition of I (because combinatorial morphisms are homeomorphisms that respect the labeling of the leaves), so do therefore

$$\operatorname{supp}(C) \setminus (E^\circ \cup \pi_1^{-1}(E^\circ_{D_1}) \text{ and } \operatorname{supp}(C) \setminus (\pi_2^{-1}(f(\pi_1(E^\circ))) \cup \pi_2^{-1}(f(E^\circ_{D_1}))).$$

It follows that

$$\{E^\circ, \pi_1^{-1}(E^\circ_{D_1})\} = \{\pi_2^{-1}(f(\pi(E^\circ)), \pi_2^{-1}(f(E^\circ_{D_1}))\}$$

for all edges $E_{D_1} \neq \pi(E)$ in the loop of D_1, which implies that $E \in \mathbf{E}(C) \setminus \mathbf{E}_2$. If the loop of D_1 contains more than two edges, it follows $E^\circ = \pi_2^{-1}(f(\pi(E^\circ)))$. □

Definition 3.1.8 (Path)
Let $p, q \in \operatorname{supp}(C)$. A path from p to q with $n \in \mathbb{N}$ flag segments is an n-tuple of flag segments

$$((p_1, E_1), \dots, (p_n, E_n))$$

such that

- $p = p_1$,
- (p_{i+1}, E_i) is a flag segment for all $i \in [n]$ where we set $p_{n+1} \coloneqq q$,
- for any $i, j \in [n]$ with $i \neq j$ the set $E_i \cap E_j$ is either empty or a single point, i.e. the edges of the path overlap only in vertices.

The length of a path $((p_1, E_1), \dots, (p_n, E_n))$ is defined as $\sum_{i=1}^n \operatorname{Length}(E_i)$. Denote the abstract combinatorial type of C by Γ. A path in Γ with n flags is an n-tuple of flags $([F_1], \dots, [F_n])$ with $[F_i] \in \mathbf{F}(\Gamma)$ such that for all $C \in \Gamma$ there exist $F_i^C \in [F_i] \cap \mathbf{F}(C)$ and a path $(F_1^C, \dots, F_n^C)$ in C. A path around the loop of C is a path such that the union of the edges that appear in the path is equal to the loop C_L of C. A path around the loop of Γ is a path such that a corresponding path in C is a path around a loop of C. If $\alpha = (\Gamma, \mathrm{v})$ is a combinatorial type, a path in α is defined as a path in Γ.

We will construct polyhedral charts on $\mathrm{P}_I(\Delta, \mathbb{R}^m)$ on the set $\overline{\mathrm{P}(\alpha)}$ for every combinatorial type α in $\mathrm{P}_I(\Delta, \mathbb{R}^m)$. The coordinates will be given, as in the rational case, by the edge lengths and the position of a root vertex x_i in $\mathbb{R}^m$, $i \in I$. In contrast to the rational case, we have to encode additionally the condition that the loop of the curve is closed, i.e. running around the image of the loop yields the zero vector. This will be done via the kernel of a linear map.

Construction 3.1.9 ($\overline{\mathrm{P}(\alpha)}, j_\alpha^{l,d}$)
Let α be the combinatorial type of a curve in $\mathrm{P}_I(\Delta, \mathbb{R}^m)$. Define $l_\alpha \in \mathbb{N}$ as the number of (bounded) edges inside the loop of α, $b_\alpha \in \mathbb{N}$ as the number of bounded edges outside the loop. Both numbers may be zero.

Let $l : \{[E] \in \mathbf{E}(\alpha) | [E] \text{ bounded}\} \to [b_\alpha + l_\alpha]$ be a bijective map with $l([E]) \leq l_\alpha$ if $[E] \in \mathbf{E}(\alpha)$ is an edge inside the loop of α, i.e. l is a labeling of the bounded edges of α. Choose l in a way such that there exist $[p_i] \in \mathbf{V}(\alpha)$ such that

$$(([p_1], l^{-1}(1)), \dots, ([p_{l_\alpha}], l^{-1}(l_\alpha)))$$

is a path around the loop of α, i.e. l respects the order of the edges in the loop of α. Define $[E_i] \coloneqq l^{-1}(i)$ for $i \in [l_\alpha + b_\alpha]$.

Using the map from the previous lemma, the labeling l induces a labeling of the edges of β for all combinatorial types $\beta \leq \alpha$, where an edge in the loop of β might get two labels if there are precisely two edges in the loop of β. Hence, we get a labeling of the edges of all $(C, h) \in \overline{\mathrm{P}(\alpha)}$.

For $(C, h) \in \overline{\mathrm{P}(\alpha)}$, we define

$$u_{(C,h)} \in \mathbb{R}^{b_\alpha + l_\alpha}$$

as the vector whose i-th coordinate is given by the length of the edge of C with label i if such an edge exists. (Remember that we labeled only bounded edges by l. Otherwise the i-th coordinate is set to zero. Note that if $E \in \mathbf{E}(C)$ has two labels, it is an edge in the loop of C and there is exactly one other edge E' in the loop of C which has the same labels. In this case, E and E' have the same length and weight. Hence, $u_{(C,h)}$ is well-defined.

Choose $d \in I$ and define the map

$$\begin{array}{rcl} j_\alpha^{l,d} : \overline{\mathrm{P}(\alpha)} & \to & \mathbb{R}^{b_\alpha + l_\alpha} \times \mathbb{R}^m \\ (C,h) & \mapsto & (u_{(C,h)}, h(x_d)), \end{array}$$

where x_d is the leaf of C with label $d \in I$ and mapped by h to a point. $j_\alpha^{l,d}$ maps a curve onto its edge lengths and the position of a root vertex x_d.

In order to encode the condition that the loop is closed, we define the map

$$\begin{array}{rcl} A_\alpha^l : \mathbb{R}^{b_\alpha + l_\alpha} & \to & \mathbb{R}^m \\ e_i & \mapsto & \begin{cases} \omega(E_i) \cdot \mathrm{v}([p_i],[E_i]), & \text{if } i \in [l_\alpha], \\ 0, & \text{else,} \end{cases} \end{array}$$

where e_i is the i-th standard unit vector in $\mathbb{R}^{l_\alpha + b_\alpha}$ (i.e. $A_\alpha^l(e_i)$ is the weighted direction vector of the flag $([p_i],[E_i])$ in the loop of α if $i \in [l_\alpha]$) and where $(([p_1],[E_1]),\ldots,([p_{l_\alpha}],[E_{l_\alpha}]))$ is the path around the loop from above in the definition of the labeling $l : \mathbf{E}(\alpha) \to [l_\alpha + b_\alpha]$.

Lemma 3.1.10

Let α be a combinatorial type in $\mathrm{P}_I(\Delta, \mathbb{R}^m)$, define

$$V_\alpha^l := \ker(A_\alpha^l) \cap \mathbb{R}_{\geq 0}^{l_\alpha + b_\alpha} \text{ and } V_{\alpha,+}^l := \ker(A_\alpha^l) \cap \mathbb{R}_{>0}^{l_\alpha + b_\alpha}.$$

Then the map

$$j_\alpha^{l,d} : \overline{\mathrm{P}(\alpha)} \to \mathbb{R}^{l_\alpha + b_\alpha} \times \mathbb{R}^m$$

is injective with

$$j_\alpha^{l,d}(\overline{\mathrm{P}(\alpha)}) = V_\alpha^l \times \mathbb{R}^m \text{ and } j_\alpha^{l,d}(\mathrm{P}(\alpha)) = V_{\alpha,+}^l \times \mathbb{R}^m,$$

where $d \in I$ and $l : \mathbf{E}(\alpha) \to [l_\alpha + b_\alpha]$ is chosen as described in the construction above.

PROOF. Assume that there exist $(D_1, h_1), (D_2, h_2) \in \overline{\mathrm{P}(\alpha)}$ such that

$$j_\alpha^{l,d}(D_1, h_1) = j_\alpha^{l,d}(D_2, h_2).$$

This means that edges in D_2 and D_1 that have the same label also have the same length and that the root vertex x_d has the same image under h_1 and h_2. Denote the combinatorial type of (D_i, h_i) by β_i, $i \in \{1, 2\}$. Since the i-th coordinate of $j_\alpha^{l,d}(D_1, h_1)$ is zero if and only if the i-th coordinate of $j_\alpha^{l,d}(D_2, h_2)$ is zero, in order to construct β_1 and β_2 from α the same subset of edges of α is contracted. Hence, D_1 and D_2 have the same combinatorial type $\beta = \beta_1 = \beta_2$. Due to lemma 3.1.4, we conclude $(D_1, h_1) = (D_2, h_2)$.

Next, we study the image of $j_\alpha^{l,d}$. Since edge lengths are greater than zero and since the loop of an elliptic curve is “closed”, we get

$$j_\alpha^{l,d}(\overline{\mathrm{P}(\alpha)}) \subset V_\alpha^l \times \mathbb{R}^m \text{ and } j_\alpha^{l,d}(\mathrm{P}(\alpha)) \subset V_{\alpha,+}^l \times \mathbb{R}^m.$$

So let us assume that $(u, p) \in V_\alpha^l \times \mathbb{R}^m$ and let us construct a parametrized curve (D, g) with $j_\alpha^{l,d}(D, h) = (u, p)$:

Let α be a combinatorial type with underlying abstract combinatorial type Γ and direction vectors $\mathrm{v} : \mathbf{F}(\alpha) \to \mathbb{R}^m$ and let C be an $I \dot\cup \Delta$-marked curve of combinatorial type Γ whose edge labeled by $i \in [l_\alpha + b_\alpha]$ has length u_i if u_i is non-zero. Denote the vertex incident to the leaf x_d by $v_d \in \mathbf{V}(C)$. Choose the remaining edge lengths arbitrarily. Define D as the $(I \cup \Delta)$-marked curve that comes from C by contracting the edges of C which are labeled by $i \in [l_\alpha + b_\alpha]$ if $u_i = 0$ (see construction 1.3.7), and denote the projection map by $\pi : \mathrm{supp}(C) \to \mathrm{supp}(D)$. Define the morphism $g : D \to \mathbb{R}^m$ by setting $g(x_d) = p$ and $\mathrm{v}_{(D,g)}(\pi(q), \pi(E)) = \mathrm{v}(q, E)$ for all $(q, E) \in \mathbf{F}(C)$ such that E is not contracted by π. Since morphisms are locally affine $\mathbb{Z}$-linear, this data is sufficient to define $g : \mathrm{supp}(D) \to \mathbb{R}^m$. If the first Betti number of supp(C) is zero, there is only one path from $v_d \in \mathrm{supp}(D)$ to $q \in \mathrm{supp}(D)$ and g is well-defined. If the first Betti number of supp(D) is one, there can be two paths from $v_d \in \mathrm{supp}(D)$ to $q \in \mathrm{supp}(D)$. However, $u \in \ker(A^l_\alpha)$ assures that the loop is closed, i.e.

$$0 = \sum \mathrm{Length}(E) \cdot \omega_D(E) \cdot \mathrm{v}(p, E) \in \mathbb{R}^m,$$

where the sum runs over all flags $(p, E) \in \mathbf{F}(C)$ that appear in the path around the loop of C that is used to define A^l_α (which is done in the construction above). Hence, $g(q)$ is well-defined for all $q \in \mathrm{supp}(D)$. Since the contraction of an edge respects the balancing condition and since α is a combinatorial type of tropical curves, it follows that g is a morphism. (D, g) has combinatorial type α if and only if π contracts no edges, i.e. $u_i > 0$ for all $i \in [b_\alpha + l_\alpha]$. □

Remark 3.1.11
Let α_1, α_2 be combinatorial types in $\mathrm{P}_I(\Delta, \mathbb{R}^m)$ with $\overline{\mathrm{P}(\alpha_1)} \cap \overline{\mathrm{P}(\alpha_2)} \neq \emptyset$. Then it is in general not true that there exists a combinatorial type β with $\overline{\mathrm{P}(\alpha_1)} \cap \overline{\mathrm{P}(\alpha_2)} = \overline{\mathrm{P}(\beta)}$. Here is an example:
Let the degree $j : \Delta = [8] \to \mathbb{R}^2$ be given by

$$j(1) := (-1, 2), j(2) := (-1, -2), j(3) := (0, 1), j(4) := (0, 1),$$
$$j(5) := (0, -1), j(6) := (1, 0), j(7) := (0, 1), j(8) := (0, -1).$$

Figure 3 shows two curves $(C_1, h_1), (C_2, h_2) \in \mathrm{P}_{\{a\}}(\Delta, \mathbb{R}^2)$ in the first row. We denote their combinatorial types by α_1 and α_2. The only difference between the two combinatorial types is that the labels x_4 and x_7 are exchanged. In the second row two common specializations (D_1, g_1), $(D_2, g_2) \in \mathrm{P}_{\{a\}}(\Delta, \mathbb{R}^2)$ of (C_1, h_1) and (C_2, h_2) are shown whose combinatorial types differ and are both maximal in $\overline{\mathrm{P}(\alpha_1)} \cap \overline{\mathrm{P}(\alpha_2)}$.
The maximality of the two combinatorial types can be seen by running around the loop of C_1 and C_2 in both directions starting at leaf x_1 and, when we arrive at leaves whose labelings i_1, i_2 differs in the two curves C_1 and C_2, we contract in both curves precisely those edges that lie between the two leaves labeled by i_1 and i_2.

Remark 3.1.12
Let α_1, α_2 be combinatorial types in $\mathrm{P}_I(\Delta, \mathbb{R}^m)$. If $\overline{\mathrm{P}(\alpha_1)} \cap \overline{\mathrm{P}(\alpha_2)} \neq \emptyset$, there exist two combinatorial types β_1, β_2 ($\beta_1 = \beta_2$ is allowed) that fulfill

$$\overline{\mathrm{P}(\alpha_1)} \cap \overline{\mathrm{P}(\alpha_2)} = \overline{\mathrm{P}(\beta_1)} \cup \overline{\mathrm{P}(\beta_2)}.$$

β_1 and β_2 correspond to the two "directions" of the loop in which we can run around it. We do not prove this statement because we do not need it in the following.

Lemma 3.1.13
Let $\beta \le \alpha$ be combinatorial types in $P_I(\Delta, \mathbb{R}^m)$ of curves $(D, g) \le (C, h)$. Let l_α be a labeling of the edges of α, l_β be a labeling of the edges of β with the properties of construction 3.1.9. Moreover let d_α, $d_\beta \in \Delta$. Then the maps $j^{l_\beta, d_\beta}_\beta \circ (j^{l_\alpha, d_\alpha}_\alpha)^{-1}$ and $j^{l_\alpha, d_\alpha}_\alpha \circ (j^{l_\beta, d_\beta}_\beta)^{-1}$ are integer affine linear where defined.

Proof. The coordinates of the maps $j^{l_\alpha, d_\alpha}_\alpha$ and $j^{l_\beta, d_\beta}_\beta$ are the edge lengths and the position of a root vertex. Due to corollary 3.1.7 an edge of β corresponds to a unique edge of α - at least if the loop of β does not contain precisely one edge. However, in this latter case, the lengths of the two corresponding edges in α coincide. Moreover, in $\overline{\mathrm{P}(\alpha)}$ the position of each marked point can be expressed as the position of the leaf x_{d_α} labeled by $d_\alpha \in I$ plus a linear combination of direction vectors of the flags (times the weight of the underlying edge) with edge lengths as coefficients. In particular, the position of the leaf x_{d_β} is a linear function of the edge lengths and of the position of the leaf x_{d_α}, and vice versa. □

Construction 3.1.14 (Topology on $\mathrm{P}_I(\Delta, \mathbb{R}^m)$)
For all combinatorial types α in $\mathrm{P}_I(\Delta, \mathbb{R}^m)$ we endow $\overline{\mathrm{P}(\alpha)}$ with the coarsest topology such that the maps $j^{l,d}_\alpha : \overline{\mathrm{P}(\alpha)} \to \mathbb{R}^{l_\alpha + b_\alpha} \times \mathbb{R}^m$ are continuous. We endow $\mathrm{P}_I(\Delta, \mathbb{R}^m)$ with the topology

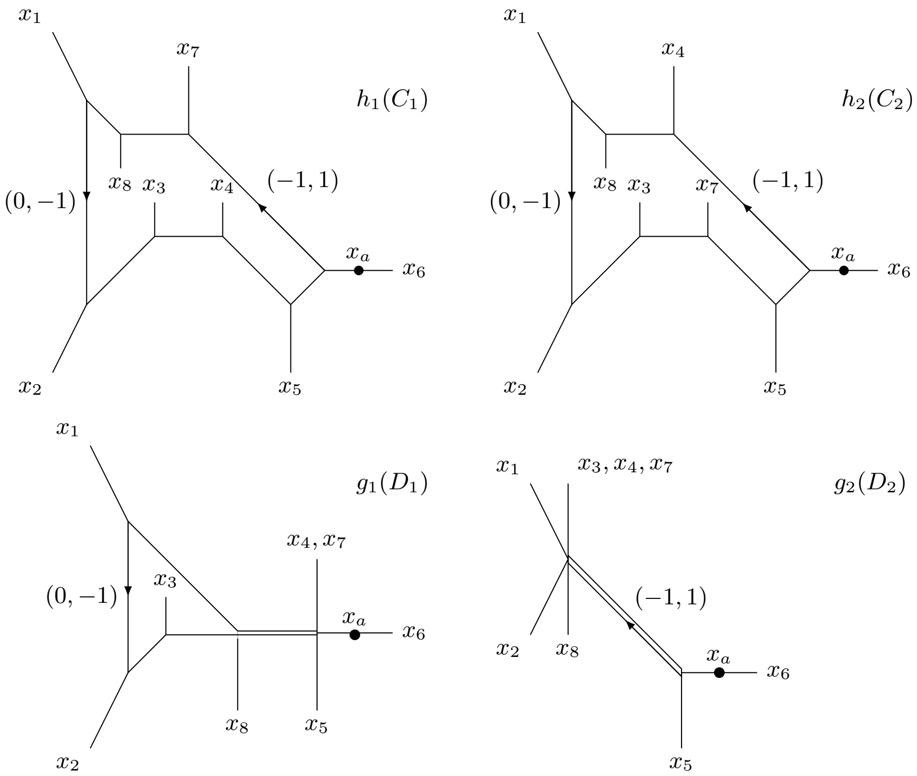


FIGURE 3. The combinatorial types of the curves in the lower row are different but both are maximal common specializations of the combinatorial types of the curves in the first row.

induced by the one on the sets $\overline{\mathrm{P}(\alpha)} \subset \mathrm{P}_I(\Delta, \mathbb{R}^m)$. Then a subset $U \subset \mathrm{P}_I(\Delta, \mathbb{R}^m)$ is open if and only if $j_\alpha^{l,d}(U \cap \overline{\mathrm{P}(\alpha)})$ is open in $\mathrm{im}(j_\alpha^{l,d}) = V_\alpha^l \times \mathbb{R}^m$ for all combinatorial types α, labelings l of the bounded edges of α and $d \in \Delta$. From now on, we consider $\mathrm{P}_I(\Delta, \mathbb{R}^m)$ as a topological space with this topology.

PROOF OF 3.1.2. As topology on $\mathrm{P}_I(\Delta, \mathbb{R}^m)$ we choose the one from the construction above. Then the claim follows from lemma 3.1.10, lemma 3.1.13 and example 3.1.3. $\square$

3.2. The pure-dimensional abstract polyhedral complex $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$

Well-spaced elliptic curves. We will define a polyhedral subcomplex of $\mathrm{P}_I(\Delta, \mathbb{R}^m)$, the subcomplex $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ of well-spaced elliptic curves. The definition of well-spacedness is inspired by the known sufficient and necessary conditions on realizability of elliptic curves in $\mathbb{R}^m$. In order to recall briefly what conditions are known, we need the following notation.

Notation 3.2.1 (Linear spaces related to the loop of a curve, $\mathbf{d}_C(p)$)
Let α be the combinatorial type of an elliptic curve $(C, h) \in \mathrm{P}_I(\Delta, \mathbb{R}^m)$.

a) As (C, h) is elliptic, for each point $p \in \mathrm{supp}(C) \setminus C_L$ there exists a unique vertex $v_p \in C_L$ and a unique path $P_p = ((p_i, E_i))_{i \in [n_p]}$ from p to v_p such that $E_i \not\subset C_L$ for all $i \in [n_p]$, i.e. the path does not pass the loop. The distance $\mathbf{d}_C(p)$ between p and C_L is defined as the length of the path P_p. For all vertices $p \in C_L \cap \mathbf{V}(C)$ in the loop of C, set $\mathbf{d}_C(p) = 0$, and for all points $p \in C_L$ which are no vertices, set $\mathbf{d}_C(p) = -1$, which makes $\mathbf{d}_C(p)$ well-defined for all $p \in \mathrm{supp}(C)$. If no confusion can occur, we just write $\mathbf{d}(p)$.
b) Let $d \in \mathbb{R}$. By $V(C, h)_d$ we denote the linear space that is spanned by the direction vectors of the flag segments $(p, E) \in \mathbf{FS}(C)$ with $\mathbf{d}_C(p) \leq d$, i.e. $V(C, h)_d$ is spanned by the direction vectors of all flag segments whose distance to the loop is at most d. Moreover, we define $V(C, h)_{<d}$ as the linear space that is spanned by the direction vectors of flag segments $(p, E) \in \mathbf{FS}(C)$ with $\mathbf{d}_C(p) < d$. By $L(C, h)_d$, $L(C, h)_{<d}$, we denote the lattices in $V(C, h)_d$ and $V(C, h)_{<d}$ that are spanned by the respective weighted direction vectors $\omega(E) \cdot \mathrm{v}(p, E)$. Note that $V(C, h) := V(C, h)_{<0}$ is the linear space spanned by the direction vectors of the flags in the loop. $V(C, h)_0$ is the linear space spanned by the direction vectors of the flags in and at the loop.
c) We define $V(\alpha)_0$, $V(\alpha)_{<0}$, $V(\alpha)$, $L(\alpha)_0$, $L(\alpha)_{<0}$ as the respective spaces for the combinatorial type α.

If no confusion can occur, we leave out α or (C, h) and just write V_d, $V_{<d}$, L_d and $L_{<d}$.

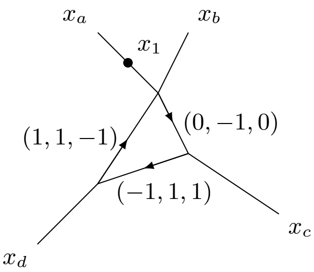


FIGURE 4. The curve C_3 from example 3.2.2. The given vectors are the direction vectors of the bounded edges.

Example 3.2.2 (Nomenclature around the loop of an elliptic curve)
As example of the above definitions, we look at the curves in example 3.1.3. The direction vectors of edges in the loop of C_1 span $\mathbb{R}^3$ and the direction vectors of edges in the loop of C_2 span the 2-dimensional space $V(C_2, h_2)_{<0} = \langle (1, 0, -1), (0, -1, 0) \rangle$. It holds $(1, 2, -1) \in V(C_2, h_2)_{<0}$,

$$V(C_2, h_2)_{<0} = V(C_2, h_2)_0 \text{ and } V(C_1, h_1)_{<0} = V(C_1, h_1)_0 = \mathbb{R}^3.$$

Let us assume that the length of the bounded edge E of C_2 with direction vector $(1, 2, -1)$ that is outside the loop is $d \in \mathbb{R}_{>0}$. We get

$$V(C_2, h_2)_{<d} = V(C_2, h_2)_0 \text{ and } V(C_2, h_2)_d = \mathbb{R}^3.$$

However, if the length of E becomes 0 and (C_2, h_2) degenerates into a curve (C_3, h_3), we get

$$V(C_3, h_3)_{<0} \subsetneq V(C_3)_0 = \mathbb{R}^3.$$

The curve C_3 is shown in figure 4.

Notation 3.2.3 ($\mathbf{V}_H, \mathbf{d}_H, \mathbf{F}_H, \mathbf{V}_d, \mathbf{P}_d, \mathbf{FS}_d$)
Let $(C, h) \in \mathrm{P}_I(\Delta, \mathbb{R}^m)$ and $H \subset \mathbb{R}^m$ a hyperplane. We define

a) the vertices of distance $d \geq 0$ to the loop as $\mathbf{V}_d(C, h) := \{v \in \mathbf{V}(C) | \, \mathbf{d}(v) = d\}$,
b) the points of distance $d \geq 0$ to the loop as $\mathbf{P}_d(C, h) := \{p \in \mathrm{supp}(C) | \, \mathbf{d}(p) = d\}$,
c) the flag segments of distance $d \geq 0$ to the loop as

$$\mathbf{FS}_d(C, h) := \{(p, E) \in \mathbf{FS}(C) | p \in \mathbf{P}_d(C, h), \ \mathbf{d}(q) \geq d \, \forall \, q \in E\},$$

d) the vertices of (C, h) closest to the loop at which a flag runs out of H as

$$\mathbf{V}_H(C, h) := \{v \in \mathrm{supp}(C) | V(C, h)_{<\mathbf{d}(v)} \subset H, \ \exists (v, E) \ \in \mathbf{FS}_{\mathbf{d}(v)}(C) : \mathrm{v}(v, E) \notin H)\},$$

e) the distance $\mathbf{d}_H(C, h)$ of H to the loop of C as the distance of a vertex closest to the loop at which a flag runs out of H, i.e. if $\mathbf{V}_H(C, h) \neq \emptyset$, we define $\mathbf{d}_H(C, h) := \mathbf{d}(v)$ for an arbitrary vertex $v \in \mathbf{V}_H(C, h)$, otherwise we set $\mathbf{d}_H(C, h) = 0$,
f) the flags closest to the loop which run out of H as

$$\mathbf{F}_H(C, h) := \{(p, E) \in \mathbf{F}(C, h) | \, \mathbf{d}(p) = \mathbf{d}_H(C, h), \mathrm{v}(p, E) \notin H\},$$

If it is clear which curve is referred to, we just write $\mathbf{V}_H$, $\mathbf{d}_H$,$\mathbf{F}_H$, $\mathbf{V}_d$, $\mathbf{P}_d$ and $\mathbf{FS}_d$.

If Γ is the abstract combinatorial type of C and if α is the combinatorial type of (C, h), we define $\mathbf{V}_0(\Gamma)$, $\mathbf{V}_0(\alpha)$, $\mathbf{P}_0(\Gamma)$, $\mathbf{P}_0(\alpha)$, $\mathbf{FS}_0(\Gamma)$ and $\mathbf{FS}_0(\alpha)$ analogously and write $\mathbf{V}_0$, $\mathbf{P}_0$ and $\mathbf{FS}_0$ if no confusion can occur.

Remark 3.2.4
Let $(C, h) \in \mathrm{P}_I(\Delta, \mathbb{R}^m)$. Note that $\mathbf{d}_H(C, h) > 0$ if and only if $V(C, h)_0 \subset H$ and $\langle \mathrm{v}(F) | F \in \mathbf{F}(C) \rangle \not\subset H$.

Remark 3.2.5 (On the realizability of elliptic tropical curves)

a) In [**Spe07**], David Speyer gives a sufficient condition for the realizability of an elliptic curve $h : C \to \mathbb{R}^m$ for arbitrary $m \in \mathbb{N}_{\geq 2}$: If

$$\#\mathbf{V}_H \geq 2$$

for all hyperplanes $H \subset \mathbb{R}^m$ with $V(C, h)_{<0} \subset H$ (i.e. there are at least two vertices closest to the loop at which a flag runs out of H), then (C, h) is realizable.
b) In [**Spe07**], David Speyer also proves a partial converse: If (C, h) is 3-valent and realizable in characteristic zero, then $\#\mathbf{V}_H \geq 2$ for all hyperplanes $H \subset \mathbb{R}^m$ with $V(C, h)_{<0} \subset H$.
c) We can reformulate Speyer's results in an alternative way: 3-valent elliptic curves are realizable in characteristic zero if and only if

$$\#\mathbf{F}_H > 2$$

for all hyperplanes $H \subset \mathbb{R}^m$ with $\mathbf{d}_H > 0$. This can be seen as follows:

Assume that $\#\mathbf{V}_H \geq 2$ and that (C, h) is 3-valent. Since (C, h) is 3-valent and balanced it is true that all direction vectors of flags at the loop lie in H if and only if the direction vectors of all flags in the loop lie in H, i.e. $V(C, h) \subset H$ and $V(C, h)_0 \subset H$ are equivalent. Moreover, if $\#\mathbf{V}_H \geq 2$, there exist at least two different vertices $v_1, v_2 \in \mathbf{V}_H$ at which at least one edge runs out of H, i.e. there exist flags $(v_1, E_1), (v_2, E_2) \in \mathbf{F}(C)$ that fulfills $\mathrm{v}(v_1, E_1), \mathrm{v}(v_2, E_2) \notin H$. By definition of $\mathbf{V}_H$, the direction vectors of all flags on the way from the loop to v_1 and v_2 lie in H. By the balancing condition, it follows that there are at least two flags that, seen from loop, lie behind each of the vertices v_1 and v_2 and whose direction vectors are not contained in H, i.e. $\#\mathbf{F}_H \geq 4 > 2$. Now the other direction: If C is 3-valent, at most two edges can run out of H at $v \in \mathbf{V}_H$. Hence, it follows from $\#\mathbf{F}_H > 2$ that there exists at least two vertices at which flags run out of H, which means $\#\mathbf{V}_H \geq 2$.

d) In [**Kat10**], Eric Katz shows a necessary condition on an elliptic curve to be realizable: If (C, h) is realizable and $\#\mathbf{V}_H = 1$ for a hyperplane $H \subset \mathbb{R}^m$ with $V(C,h)_{<0} \subset H$ (i.e. there exists a unique vertex which is closest to the loop at which a flag runs out of H), then the unique vertex $v \in \mathbf{V}_H$ is at least 4-valent.
e) An elliptic curve which fulfills $\#\mathbf{F}_H > 2$ for all hyperplanes $H \subset \mathbb{R}^m$ with $\mathbf{d}_H > 0$ also fulfills Katz' necessary realizability condition: If $\#\mathbf{V}_H = 1$ and $\#\mathbf{F}_H > 2$, there are at least three different edges E_1, E_2, E_3 adjacent to the unique vertex $v \in \mathbf{V}_H$ that fulfill $\mathrm{v}(v, E_i) \notin H$, $i \in [3]$. Hence, the vertex v is at least 4-valent.

We take our reformulation of Speyer's sufficient and necessary condition on the realizability of a 3-valent elliptic curve stated in the remark above as definition of well-spacedness.

Definition 3.2.6 (Well-spacedness, $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$)
We call $(C, h) \in \mathrm{P}_I(\Delta, \mathbb{R}^m)$ well-spaced if

$$\#\mathbf{F}_H > 2$$

for all hyperplanes $H \subset \mathbb{R}^m$ with $\mathbf{d}_H > 0$.
We define the moduli space of elliptic I-marked parametrized curves of degree Δ in $\mathbb{R}^m$ by

$$\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m) := \{(C,h) \in \mathrm{P}_I(\Delta, \mathbb{R}^m) | (C,h) \text{ is well-spaced}\}.$$

If $I = [n]$, we also write $\mathcal{M}_{1,n}(\Delta, \mathbb{R}^m)$.

Remark 3.2.7
By the previous remark, it follows that well-spaced elliptic curves satisfy Eric Katz' necessary realizability condition. Moreover, elliptic curves which are realizable according to David Speyer are well-spaced, and 3-valent curves are well-spaced if and only if they are realizable in characteristic 0.

Remark 3.2.8
Let $(C, h) \in \mathrm{P}_I(\Delta, \mathbb{R}^m)$, $H \subset \mathbb{R}^m$ be a hyperplane with $\mathbf{d}_H > 0$ and $v \in \mathbf{V}_H$, i.e. there exists an edge that is adjacent to the vertex v that fulfills that the direction vector $\mathrm{v}(v, E)$ of the flag (v, E) is not contained in the hyperplane H. Due to the balancing condition, there are at least two different edges $E_1, E_2 \in \mathbf{E}(C)$ adjacent to v which fulfill $\mathrm{v}(v, E_1), \mathrm{v}(v, E_2) \notin H$. It follows that $\#\mathbf{F}_H > 1$. Hence, demanding in the definition of well-spacedness that

$$\#\mathbf{F}_H > 0 \text{ or } \#\mathbf{F}_H > 1$$

would be an empty condition.

Polyhedral structure on $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$. In order to give $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ the structure of an abstract polyhedral complex, we refine the definition of the combinatorial type.

Definition 3.2.9 (Fine combinatorial type of a well-spaced curve, specializations, $\mathrm{M}_1(\Gamma, \mathrm{v}, \leq)$)
A fine combinatorial type is a triple $(\Gamma, \mathrm{v}, \leq)$ where (Γ, v) is a combinatorial type and $\leq$ is a total preorder on the set of vertices of Γ (i.e. $\leq$ is a total, reflexive and transitive) that fulfills the following condition for any vertex $v \in \mathbf{V}(\Gamma)$:

If a path P_v from v to the loop of Γ (see definition 3.2.1) passes a vertex w, the relation $w \leq v$ is fulfilled.

If C is a tropical curve of abstract combinatorial type Γ, the function $\mathbf{d} : \mathbf{V}(C) \to \mathbb{R}_{\geq 0}$, which indicates the distance of a vertex to the loop, defines a total preorder $\leq_{\mathbf{d}}$ on $\mathbf{V}(\Gamma)$ that fulfills the condition from above for all $[v], [w] \in \mathbf{V}(\Gamma)$:

$$[v] \leq_{\mathbf{d}} [w] :\Leftrightarrow \mathbf{d}(v) \leq \mathbf{d}(w).$$

If (Γ, v) is the combinatorial type of $(C, h) \in \mathrm{P}_I(\Delta, \mathbb{R}^m)$, we call $(\Gamma, \mathrm{v}, \leq_{\mathbf{d}})$ the fine combinatorial type of (C, h).

We say that $(\Gamma_1, \mathrm{v}_1, \leq_1)$ is a specialization of $(\Gamma_2, \mathrm{v}_2, \leq_2)$ and write $(\Gamma_1, \mathrm{v}_1, \leq_1) \leq (\Gamma_2, \mathrm{v}_2, \leq_2)$ if

- (Γ_1, v_1) is a specialization of (Γ_2, v_2) and

- $[v] \leq_2 [w]$ implies $\pi([v]) \leq_1 \pi([w])$ for all $[v],[w] \in \mathbf{V}(\Gamma_2)$ where $\pi : \mathbf{V}(\Gamma_2) \to \mathbf{V}(\Gamma_1)$ is the map from definition 1.3.21), i.e. the specialization process respects the total preorder on the vertices.

By $\mathrm{M}_1(\Gamma, \mathrm{v}, \leq)$ we denote the subset of $\mathrm{P}_I(\Delta, \mathbb{R}^m)$ that contains precisely the curves of fine combinatorial type $(\Gamma, \mathrm{v}, \leq)$. By $\overline{\mathrm{M}_1(\Gamma, \mathrm{v}, \leq)}$ we denote the set of curves whose fine combinatorial type is a specialization of $(\Gamma, \mathrm{v}, \leq)$.

The aim of this section is to prove the following theorem.

Theorem 3.2.10
The moduli space $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$, where the degree is given by $j : \Delta \to \mathbb{R}^m$, together with the polyhedral structure given by the fine combinatorial types, is a pure-dimensional subcomplex of $\mathrm{P}_I(\Delta, \mathbb{R}^m)$ of dimension $\#\Delta + \#I + m - \dim\langle j(\Delta)\rangle$.

We split the proof of this statement into several propositions and lemmata.

Proposition 3.2.11
The set $\{\overline{\mathrm{M}_1(\Gamma, \mathrm{v}, \leq)}\}$ of closed subsets of $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$, where $(\Gamma, \mathrm{v}, \leq)$ runs over all fine combinatorial types of curves in $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$, defines a polyhedral structure on $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ as subcomplex of $\mathrm{P}_I(\Delta, \mathbb{R}^m)$.

The key to the proof is the following lemma which implies that

$$\bigcup \{\overline{\mathrm{M}_1(\Gamma, \mathrm{v}, \leq)}\} = \mathcal{M}_{1,I}(\Delta, \mathbb{R}^m),$$

where the union is taken over all fine combinatorial types $(\Gamma, \mathrm{v}, \leq)$ of curves in $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$.

Lemma 3.2.12
Let $(C, h) \in \mathrm{M}_1(\Gamma, \mathrm{v}, \leq)$ be well-spaced. Then a curve $(D, g) \in \overline{\mathrm{M}_1(\Gamma, \mathrm{v}, \leq)}$, whose fine combinatorial type $(\Gamma_D, \mathrm{v}_D, \leq_D)$ is a specialization of $(\Gamma, \mathrm{v}, \leq)$, is also well-spaced.

Proof. Let $(D, g) \in \mathrm{P}_I(\Delta, \mathbb{R}^m)$ such that its fine combinatorial type $(\Gamma_D, \mathrm{v}_D, \leq_D)$ is a specialization of $(\Gamma, \mathrm{v}, \leq)$. Assume that (D, g) is not well-spaced. Then there exists a hyperplane $H \subset \mathbb{R}^m$ with $\mathbf{d}_H(D, g) > 0$ such that $\mathbf{F}_H(D, g) = 2$, in particular $V(D, g)_0 \subset H$. Let $\pi : \mathrm{supp}(C) \to \mathrm{supp}(D)$ be the projection map. Due to

$$V(C, h)_0 \subset V(D, g)_0 \subset H$$

we know that either

$$\#\mathbf{F}_H(C, h) > 2 \text{ or } \mathbf{d}_H(C, h) = 0$$

because (C, h) is well-spaced.

Assume first that $\mathbf{d}_H(C, h) = 0$, which means that either $\langle j(i)|i \in \Delta\rangle \subset H$ or $V(C, h)_0 \not\subset H$, where $j(i)$ is the weighted direction vector of the leaf labeled by $i \in \Delta$. If $\langle j(i)|i \in \Delta\rangle \subset H$, it follows $\mathbf{d}_H(D, g) = 0$. If $V(C, h)_0 \not\subset H$, it follows that

$$V(C, h)_0 \subset V(D, g)_0 \not\subset H$$

and $\mathbf{d}_H(D, g) = 0$. Both statements are a contradiction to $\mathbf{d}_H(D, g) > 0$.

Assume now that $\mathbf{d}_H(C, h) > 0$ and $\#\mathbf{F}_H(C, h) > 2$. Let $(p, E) \in \mathbf{F}_H(C, h)$, i.e. $\mathrm{v}(p, E) \notin H$ and the direction vectors of all edges on the way from the loop to p lie in H. If $\pi(E)$ is an edge of D, then $\pi(p) \in \mathbf{V}_H(D, g)$ because $v \leq p$ implies $\pi(v) \leq_D \pi(p)$ for all vertices $v \in \mathbf{V}(C)$ and because $\mathrm{v}(p, E) = \mathrm{v}(\pi(p), \pi(E))$.

If $\pi(E)$ is a vertex of D, then E is bounded and adjacent to an edge E' with $\mathrm{v}(q, E') \notin H$, due to the balancing condition, where $q \in E$ is the second vertex in E. Either $\pi(E')$ is a point or the edge $\pi(E')$ is adjacent to the vertex $\pi(p)$ of D. Since C is elliptic and since unbounded edges can not be contracted in the specialization process, we arrive recursively at a flag $(p_D, E_D) \in \mathbf{F}(C)$ such that

- the edge E_D is not contracted by π, i.e. $\pi(E_D) \in \mathbf{E}(D)$,
- $\pi(p_D) = \pi(p)$ and

- $\mathrm{v}_{(C,h)}(p_D, E_D) \notin H$.

As $\mathbf{d}(v) \leq \mathbf{d}(w)$ implies $\mathbf{d}(\pi(v)) \leq \mathbf{d}(\pi(w))$ for all vertices $v, w \in \mathbf{V}(C)$, it follows $\pi(p_D) \in \mathbf{V}_H(D, g)$ and hence

$$(\pi(p_D), \pi(E_D)) \in \mathbf{F}_H(D, g).$$

Moreover, $\#\mathbf{F}_H(C, h) \leq \#\mathbf{F}_H(D, g)$: If there exists another flag $(p, E) \neq (p, K) \in \mathbf{F}_H(C, h)$ closest to the loop that runs out of H, it holds $E_D \neq K_D$ because, in the construction of E_D and K_D, we consider only edges outside the loop of C that lie behind the vertex p seen from the loop and the paths from p to p_D and p_K, respectively, do not overlap. We conclude that

$$2 < \#\mathbf{F}_H(C, h) \leq \#\mathbf{F}_H(D, g),$$

which is a contradiction. □

PROOF OF PROPOSITION 3.2.11. Let $(\Gamma, \mathrm{v}, \leq)$ be a fine combinatorial type. Define $\alpha := (\Gamma, \mathrm{v})$. For every $[v] \in \mathbf{V}(\Gamma)$, the concatenation of a polyhedral chart $(j_\alpha^{l,d})^{-1}$ with the map

$$\mathbf{d}_{[v]} : \mathrm{P}(\alpha) \to \mathbb{R}$$

given by $(C, h) \mapsto \mathbf{d}_{(C,h)}(v)$ that maps a vertex onto its distance to the loop is an affine linear map (the coordinates on $(C, h) \in \mathrm{P}(\alpha)$ are the edge lengths and the position of a root vertex in $\mathbb{R}^m$). The set of well-spaced curves of fine combinatorial type $(\Gamma, \mathrm{v}, \leq)$ is cut out from $\mathrm{P}(\Gamma, \mathrm{v})$ by equations of the type

$$(\mathbf{d}_{[v_1]} - \mathbf{d}_{[v_2]}) = 0 \text{ and } (\mathbf{d}_{[v_1]} - \mathbf{d}_{[v_2]}) > 0,$$

where $[v_1], [v_2] \in \mathbf{V}(\Gamma)$. Hence, the set of well-spaced curves of fine combinatorial type $(\Gamma, \mathrm{v}, \leq)$ is a polyhedron in $\mathrm{P}(\Gamma, \mathrm{v})$.

The boundary of $\mathrm{M}_1(\Gamma, \mathrm{v}, \leq)$ consists of curves where some of the inequalities $(\mathbf{d}_{[v_1]} - \mathbf{d}_{[v_2]}) > 0$ become equalities or where edges in the loop are contracted. This happens precisely when the total preorder $\leq$ becomes coarser or the fine combinatorial type specializes, which may imply that edge lengths become zero (i.e. also the combinatorial type specializes). Hence, $\overline{\mathrm{M}_1(\Gamma, \mathrm{v}, \leq)}$ is a closed polyhedron in $\overline{\mathrm{P}(\Gamma, \mathrm{v})}$. Due to the previous lemma it holds moreover that

$$\overline{\mathrm{M}_1(\Gamma, \mathrm{v}, \leq)} \subset \mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$$

if $(\Gamma, \mathrm{v}, \leq)$ is a fine combinatorial type of well-spaced curves. □

The following proposition is a first step on the way to describe the curves of maximal fine combinatorial type in $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ combinatorially.

Proposition 3.2.13
Let $(\alpha, \leq)$ be a maximal fine combinatorial type in $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$. Then $(\alpha, \leq)$ is regular, i.e. α does not contain vertices of genus greater than zero but an "honest" loop.

In order to prove this proposition, we will amongst others parametrize the combinatorial types (Γ, v) that have Γ as underlying abstract combinatorial type. Moreover, we will describe regular resolutions of a non-regular combinatorial type (Γ, v).

Definition 3.2.14 ($\mathbf{F}_p$, $\mathrm{v}(p)$)
Let $(C, h) \in \mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ and let $p \in \operatorname{supp}(C)$ be a point with $\mathbf{d}(p) \geq 0$. Then we define $\mathbf{F}_p(C)$ as the set of flags segments of C that lie directly behind the point p seen from the loop, i.e.

$$\mathbf{F}_p(C) := \{(v, E) \in \mathbf{FS}_{\mathbf{d}(p)}(C) | v = p\}.$$

Moreover, we define $\mathrm{v}_{(C,h)}(p)$ as the sum of the weighted direction vectors of the flags that lie directly behind p, i.e.

$$\mathrm{v}_{(C,h)}(p) := \sum_{(p,E) \in \mathbf{F}_p(C)} \omega(E) \cdot \mathrm{v}_{(C,h)}(p, E).$$

We define $\mathbf{F}_p(\alpha)$ and $\mathrm{v}_\alpha([p])$ analogously for a combinatorial type α. We just write $\mathbf{F}_p$, $\mathrm{v}(p)$ and $\mathrm{v}([p])$ if no confusion can occur.

The next lemma allows to parametrize the combinatorial types (Γ, v) if the abstract combinatorial type Γ is given.

Lemma 3.2.15
Let C be a regular elliptic $I\dot{\cup}\Delta$-marked curve (i.e. C contains an "honest" loop) and let $j : \Delta \to \mathbb{R}^m$ be a degree. Moreover, let

$$((p_1, E_1), \ldots, (p_r, E_r))$$

be a path that runs around the loop of C and let $u \in \mathbb{Z}^m \setminus \{0\}$. Then there exists a parametrized curve $(C, h) \in \mathrm{P}_I(\Delta, \mathbb{R}^m)$ with $\omega(E_r) \cdot \mathrm{v}(p_r, E_r) = u$ if and only if

$$u \in \mathrm{Conv}\left(\sum_{j=1}^{i} \mathrm{v}(p_j) | i \in [r-1]\right)^{\circ},$$

where $^\circ$ denotes the relative interior.

Moreover, there exist $a_i \in \mathbb{R}$, $i = 1, \ldots, r-1$ such that $u = \sum_{i=1}^{r-1} a_i \cdot \mathrm{v}(p_i)$ and it holds $1 > a_1 > \cdots > a_{r-1} > 0$.

Proof. Note that $r \geq 2$ because there exists a flag in the loop of C whose direction vector is non-zero. $\mathrm{v}(p_r, E_r)$ determines the weighted direction vectors a flag (p_i, E_i), $i \in [r]$, in the loop of C to be

$$\omega(E_i)\,\mathrm{v}(p_i, E_i) = \omega(E_r)\,\mathrm{v}(p_r, E_r) - \sum_{j=1}^{i} \mathrm{v}(p_j).$$

Note that it holds $\sum_{i\in[r]} \mathrm{v}(p_j) = 0$ due to the balancing condition. As the loop of $h(C)$ is "closed" and edge lengths are greater than zero, there exists a curve $(C, h) \in \mathrm{P}_I(\Delta, \mathbb{R}^m)$ with $\omega(E_r) \cdot \mathrm{v}(p_r, E_r) = u$ if and only if there exist $0 < \lambda_i \in \mathbb{R}$, $i \in [r]$, which we scale such that $\sum_{i=1}^{r} \lambda_i = 1$ (λ_i is the length of the edge E_i), such that

$$\begin{aligned}
& \sum_{i=1}^{r} \lambda_i \cdot \omega(E_i) \cdot \mathrm{v}(p_i, E_i) = \sum_{i=1}^{r} \lambda_i (u - \sum_{j=1}^{i} \mathrm{v}(p_j)) = 0 \\
\Leftrightarrow \quad & u - \sum_{i=1}^{r-1} \lambda_i (\sum_{j=1}^{i} \mathrm{v}(p_j)) = 0 \\
\Leftrightarrow \quad & u = \sum_{i=1}^{r-1} \lambda_i (\sum_{j=1}^{i} \mathrm{v}(p_j)) \\
\Leftrightarrow \quad & u = \sum_{i=1}^{r-1} (\sum_{j=i}^{r-1} \lambda_j) \cdot \mathrm{v}(p_i)
\end{aligned}$$

The claim follows because $\sum_{i\in[r]} \lambda_i = 1$ and $1 > \lambda_i > 0$ for all $i \in [r]$. □

We will introduce ordered partitions of a set in order to describe resolutions of vertices in the loop of an elliptic curve.

Definition 3.2.16 (Sets of partitions $\mathbf{O}(O_1, \ldots, O_s)$ and $\mathbf{P}(O_1, \ldots, O_s)$)
Let $(O_1, \ldots, O_s)$ be an s-tuple of pairwise disjoint sets. Then we denote by $\mathbf{O}(O_1, \ldots, O_s)$ the set of ordered partitions $(P_1, \ldots, P_r)$ of $\dot{\bigcup}_{i\in s} O_i$ that are finer than $(O_1, \ldots, O_s)$, i.e. there exist $i_1, \ldots, i_s = r \in [r]$ such that $O_j = \dot{\bigcup}_{i=i_{j-1}+1}^{i_j} P_i$ for all $j \in [s]$ (where $i_0 = 0$). By $\mathbf{P}(O_1, \ldots, O_s)$ we denote the set of partitions $\{P_1, \ldots, P_r\}$ of $\dot{\bigcup}_{i\in s} O_i$ that are finer than $\{O_1, \ldots, O_s\}$.

Construction 3.2.17 ($\Gamma_{\mathcal{O}}$)
Let Γ be an abstract combinatorial type of elliptic curves and let $(([v_1], [E_1]), \ldots, ([v_s], [E_s]))$ be a path around the loop of Γ, if Γ is regular. Otherwise, if Γ is non-regular, let $[v_1]$ be the unique

vertex of Γ that has genus 1. $(\mathbf{FS}_{[v_1]}, \ldots, \mathbf{FS}_{[v_s]})$ is an ordered partition of the flags $\mathbf{FS}_0(\Gamma)$ at the loop of Γ. An ordered partition

$$\mathcal{O} = (O_1, \ldots, O_r) \in \mathbf{O}(\mathbf{FS}_{[v_1]}, \ldots, \mathbf{FS}_{[v_s]})$$

that is finer than $(\mathbf{FS}_{[v_1]}, \ldots, \mathbf{FS}_{[v_s]})$ defines a regular combinatorial type $\Gamma_{\mathcal{O}}$ that is a resolution of Γ and that fulfills that, when specializing from $\Gamma_{\mathcal{O}}$ to Γ, only edges in the loop are contracted:
Assume that $C \in \Gamma$ and remove C_L from $\mathrm{supp}(C)$ such that $\mathrm{supp}(C)$ decomposes into $\#\mathbf{FS}_0(C)$ connected components. Add r vertices $v_1, \ldots, v_r$ of genus zero such that precisely all E with $([p], [E]) \in O_i \subset \mathbf{FS}_0(\Gamma)$ are adjacent to v_i, $i \in [r]$. Now we add edges between v_i and v_{i+1} for all $i \in [r-1]$ and between v_1 and v_r. We denote the combinatorial type of this curve by $\Gamma_{\mathcal{O}}$. See figure 5 for an example in the case that Γ is non-regular, i.e. the loop of Γ consists only of one vertex $[v_1]$.
Note that the path around the loop is not unique if there exists a flag in the the loop of Γ (the path can start at different vertices in the loop and starting there can run in two direction around the loop) and that hence different ordered partitions $\mathcal{O}_1, \mathcal{O}_2$ may induce the same combinatorial type $\Gamma_{\mathcal{O}_1} = \Gamma_{\mathcal{O}_2}$. However, for all combinatorial types $\Gamma \leq \Gamma'$ such that in the specialization from Γ' to Γ only edges in the loop are contracted there exists an ordered partition $\mathcal{O}$ that refines $(\mathbf{FS}_{[v_1]}, \ldots, \mathbf{FS}_{[v_s]})$ and that fulfills $\Gamma_{\mathcal{O}} = \Gamma'$.

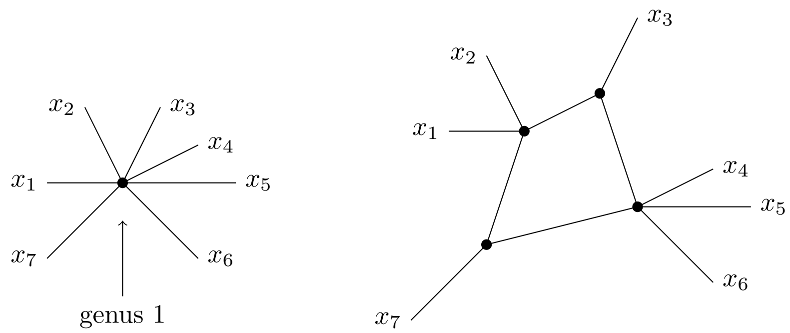


FIGURE 5. On the left: A curve of combinatorial type Γ with one vertex of genus one. On the right: A curve of combinatorial type $\Gamma_{\mathcal{O}}$ where $\mathcal{O} = (\{([p],[x_1]),([p],[x_2])\},\{([p],[x_3])\},\{([p],[x_4]),([p],[x_5]),([p],[x_6])\},\{([p],[x_7])\})$ is an ordered partition finer than $(\mathbf{FS}_0(\Gamma))$.

PROOF OF PROPOSITION 3.2.13. We will show that the fine combinatorial type of a non-regular well-spaced curve $(C, h) \in \mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ is not maximal in $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$. Denote the unique vertex of C which has genus one by $v \in \mathbf{V}(C)$.
First case:
Assume that

$$\dim V(C, h)_0 < \#\mathbf{FS}_0(C) - 1,$$

i.e. the flags that contain the unique vertex $p \in \mathbf{V}(C)$ of genus 1 fulfill at least one relation in addition to balancing at p, and assume $\#\mathbf{FS}_0(C) \geq 4$.
If there exists a vertex of genus zero in C, i.e. $\{v \in \mathbf{V} \,|\, \mathbf{d}(v) > 0\} \neq \emptyset$, define

$$d := \frac{1}{2} \min\{\mathbf{d}(v) | v \in \mathbf{V}(C), \mathbf{d}(v) > 0\},$$

i.e. d is half of the distance of the vertex closest to the loop. If C contains no vertex except the vertex p of genus 1, i.e. $\{v \in V \,|\, \mathbf{d}(v) > 0\} = \emptyset\}$, set $d = \infty$.
The direction vectors $\mathrm{v}(F)$ of flags $F \in \mathbf{FS}_0(C)$ at the loop of C fulfill a relation in addition to the balancing condition at the vertex $p \in \mathbf{V}_0(C)$. Hence, there exists a partition $\{\mathbf{F}_1, \mathbf{F}_2\} \in \mathbf{P}(\mathbf{FS}_0(C))$

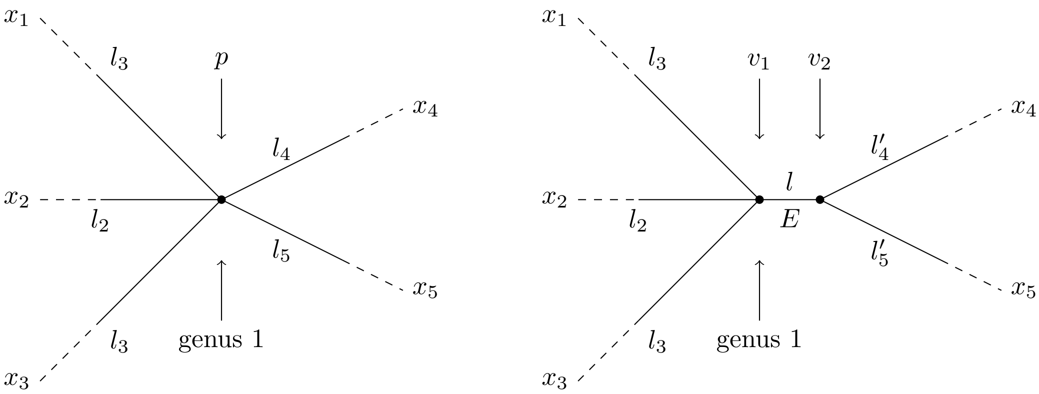


FIGURE 6. $h(C)$ on the left and $h'(C')$ on the right locally around the vertex $h(p)$ of genus one with edge lengths $l = \epsilon$, $l'_4 = l_4 - \epsilon$ and $l'_5 = l_5 - \epsilon$. The partition of $\mathbf{FS}_0(C, h)$ is $\{\{(p, x_1), (p, x_2), (p, x_3)\}, \{(p, x_4), (p, x_5)\}\}$ and $x_4, x_5 \in \langle \mathrm{v}(p, x_i) | i \in [3] \rangle$.

of the flags at the loop of C such that $\langle \mathrm{v}(F) | F \in \mathbf{F}_1 \rangle = V(C, h)_0$ and $\#\mathbf{F}_1, \#\mathbf{F}_2 \geq 2$. (Here we need $\#\mathbf{FS}_0(C) \geq 4$.)

The following construction is illustrated in figure 6. Since C is non-regular, we can add an edge E to C at the vertex p of genus one such that the one vertex $v_1 \in E$ is contained in the flags in $\mathbf{F}_1$ and such that the other vertex $v_2 \in E$ is contained in the flags in $\mathbf{F}_2$. Set the length of the new edge to $0 < \epsilon < d$. Reduce the length of all edges $E \neq E'$ that are adjacent to v_2 by ϵ, which is possible due to the choice of d (as the distance to the loop of the vertex closest to the loop). Assign genus one to v_1 and genus zero to v_2 and denote the arising curve by (C', h') and its combinatorial type by (Γ', v'). Denote the total preorder on the vertices of Γ' that comes from $\mathbf{d}_{(C',h')}$ by $\leq'$. $(\Gamma, \mathrm{v}, \leq)$ is obviously a specialization of $(\Gamma', \mathrm{v}', \leq')$ and the two fine combinatorial types are different.

Moreover, (C', h') is well-spaced: Let $H \subset \mathbb{R}^m$ be a hyperplane with $\mathbf{d}_H(C', h') > 0$. Since $V(C', h')_{d'} = V(C', h')_0$ for all $0 \leq d' < d$, it follows $\mathbf{d}_H(C', h') > \epsilon$ because we chose $\epsilon < d$. Moreover, the curve (C, h) looks as (C', h') in the region where the distance to the loop is greater than or equal to d. Therefore the well-spacedness of (C', h') follows from the well-spacedness of (C, h).

Second case:

We assume that $\dim V(\Gamma)_0 = \#\mathbf{FS}_0(\Gamma) - 1$, i.e. the direction vectors of the flags containing the unique vertex $[p] \in \mathbf{V}_0(\Gamma)$ of genus one fulfill no relation in addition to balancing at $[p]$. Let $\mathbf{FS}_0(\Gamma) = \{[F_1], \ldots, [F_{r+1}]\}$ be the the set of flags that contain the vertex $[p]$ of genus one. For $i \in [r+1]$ set

$$u_i := \omega(E_i) \cdot \mathrm{v}([F_i]).$$

Due to the balancing conditions, it holds $\sum_{i=1}^{r+1} u_i = 0$ and $u_1, \ldots, u_r$ are linearly independent because of $\dim V(\Gamma)_0 = \#\mathbf{FS}_0(\Gamma) - 1$. Set

$$\mathrm{FD}(u_1, \ldots, u_{r+1}) = \{\sum_{i=1}^{r} \lambda_i u_i | 0 \leq \lambda_i < 1\},$$

which is a fundamental domain of the lattice $L(u_1, \ldots, u_r)$ spanned by $u_1, \ldots, u_r$. By definition of $\mathrm{P}_I(\Delta, \mathbb{R}^m)$, the multiplicity of C at $p \in \mathrm{supp}(C)$ is greater than one, which means that $\mathrm{ind}(L(u_1, \ldots, u_r)) > 1$. This implies that $\#(\mathbb{Z}^m \cap \mathrm{FD}(u_1, \ldots, u_{r+1})) > 1$ and there exists a non-zero element

$$u \in \mathbb{Z}^m \cap \mathrm{FD}(u_1, \ldots, u_{r+1}).$$

We claim that there hence exists an ordered partition $\mathcal{O} = (O_1, \dots, O_s) \in \mathbf{O}(\mathbf{FS}_0(\Gamma))$ with $s > 1$ and $[F_{r+1}] \in O_s$ such that

$$u \in \operatorname{Conv}\left(\sum_{j=1}^{i} \left(\sum_{k:[F_k]\in O_j} u_k \right) | i = 1, \dots, s \right)^{\circ} :$$

Let $(O_1, \dots, O_s) \in \mathbf{O}(\mathbf{FS}_0(\Gamma))$ be an ordered partition of the set of flags $\mathbf{FS}_0(\Gamma) = \{[F_i] | i \in [r+1]\}$ at the loop of Γ that fulfills $[F_{r+1}] \in O_s$. Denote by $\lambda(O_1, \dots, O_s) \subset \mathrm{FD}(u_1, \dots, u_{r+1})$ the subset which consists of all

$$v = \sum_{i=1}^{r} \lambda_i u_i$$

that fulfill

- $0 \leq \lambda_1, \dots, \lambda_r < 1$,
- $\lambda_k = \lambda_l$ if there exists $i \in [r]$ with $[F_k], [F_l] \in O_i$ and
- $\lambda_k > \lambda_l$ if there exist $i, j \in [r]$ with $i < j$ and $[F_k] \in O_i$, $[F_l] \in O_j$.

$$\{\lambda(O_1, \dots, O_s) | (O_1, \dots, O_s) \in \mathbf{O}(\{[F]_1, \dots, [F_{r+1}]\}),\ [F_{r+1}] \in O_s\}$$

is by definition a partition of $\mathrm{FD}(u_1, \dots, u_{r+1})$. It holds

$$\operatorname{Conv}\left(\sum_{j=1}^{i} \left(\sum_{k:[F_k]\in O_j} u_k \right) | i = 1, \dots, s \right)^{\circ} = \lambda(O_1, \dots, O_s)$$

and therefore

$$\mathrm{FD}(u_1, \dots, u_{r+1}) = \dot{\bigcup} \operatorname{Conv}\left(\sum_{j=1}^{i} \left(\sum_{k:[F_k]\in O_j} u_k \right) | i = 1, \dots, s \right)^{\circ},$$

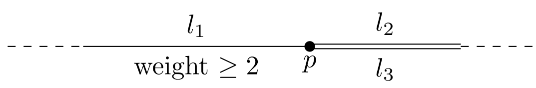
where the disjoint union runs over all ordered partitions $(O_1, \dots, O_s) \in \mathbf{O}(\mathbf{FS}_0(\Gamma))$ with $\mathbf{FS}_0(\Gamma) = \{[F_1], \dots, [F_{r+1}]\}$ which fulfill $[F_{r+1}] \in O_s$.
So let $(O_1, \dots, O_s)$ be the ordered partition such that

$$u \in \operatorname{Conv}\left(\sum_{j=1}^{i} \left(\sum_{k:[F_k]\in O_j} u_k \right) | i = 1, \dots, s \right)^{\circ}.$$

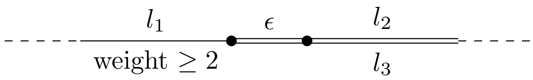
Since $0 \neq u \in \mathbb{Z}^m$, it holds $s > 1$. Due to lemma 3.2.15, there exists a regular curve $(C', h') \in \mathrm{P}_I(\Delta, \mathbb{R}^m)$ which has combinatorial type $(\Gamma_{\mathcal{O}}, \mathrm{v}_{\mathcal{O}})$ such that $u \in \mathbb{R}^m$ is the weighted direction vector of a flag in the loop. Remember that $(\Gamma_{\mathcal{O}}, \mathrm{v}_{\mathcal{O}})$ specializes to (Γ, v). Moreover, (Γ, v) and $(\Gamma_{\mathcal{O}}, \leq_{\mathcal{O}})$ are different because $s > 1$. We define $\leq_{\mathcal{O}}$ on $\mathbf{V}(\Gamma_{\mathcal{O}})$ via

$$[v] \leq_{\mathcal{O}} [w] \Leftrightarrow \pi([v]) \leq \pi([w])$$

where $\pi : \mathbf{V}(\Gamma_{\mathcal{O}}) \to \mathbf{V}(\Gamma)$ is the projection map. Since in the specialization process only edges in the loop are contracted and since (C, h) was chosen to be well-spaced, it follows that also a curve (C', h') of fine combinatorial type $(\Gamma_{\mathcal{O}}, \mathrm{v}_{\mathcal{O}}, \leq_{\mathcal{O}})$ is well-spaced. Hence, $(\Gamma, \mathrm{v}, \leq)$ is not maximal in $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$.
Third (and last) case:
Assume that $\dim V(C, h)_0 < \#\mathbf{FS}_0(C, h) - 1$ and $\#\mathbf{FS}_0(C, h) = 3$, hence $\dim V(C, h)_0 = 1$. ($\#\mathbf{FS}_0(C, h) < 3$ does not occur: That would imply $\dim V(C, h)_0 \leq 0$. Hence, the multiplicity of the vertex $p \in V(C)$ of genus one would be one, which is not allowed.) Then, locally around the vertex $p \in \operatorname{supp}(C)$ of genus one, the image of C under h looks as follows, where the l_i denote the lengths of the edges:



Since the weight of the edge on the left is greater than or equal to 2, we can resolve the vertex p into a loop, see the figure below, and get a well-spaced curve that specializes to (C, h). Hence, the fine combinatorial type of (C, h) is not maximal.



$\square$

Here is a combinatorial description of the well-spaced elliptic curves contained in maximal polyhedra of $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$.

Proposition 3.2.18
A curve $(C, h) \in \mathcal{M}_{1,n}(\Delta, \mathbb{R}^m)$ has a maximal fine combinatorial type if and only if

a) C is regular,
b) $\dim V_0/V_{<0} = \#\mathbf{FS}_0 - \#\mathbf{P}_0$ (i.e. the direction vectors of the flags at the loop fulfill no relation modulo $V_{<0}$ in addition to balancing at the vertices in the loop) and
c) $\dim(V_d/V_{<d}) = \#\mathbf{FS}_d - \#\mathbf{P}_d - 1$ for all $d > 0$ with $\mathbf{V}_d \neq \emptyset$ (i.e. the direction vectors of flag segments with distance $d > 0$ to the loop fulfill precisely one relation modulo $V_{<d}$ in addition to balancing at the vertices with distance d to the loop - at least if there exists a vertex which has distance d to the loop).

In the proof of this proposition we use an alternative description of well-spacedness that is based on the relations that the direction vectors of flag segments with distance d to the loop fulfill in $V_d/V_{<d}$. This description allows to construct a well-spaced resolution if the conditions stated in the proposition above are not fulfilled.

Definition 3.2.19 (d-well-spaced, $V(A)_{<d}$)
Let $(C, h) \in \mathrm{P}_I(\Delta, \mathbb{R}^m)$ and $0 < d \in \mathbb{R}$. (C, h) is called d-well-spaced if

$$\#\mathbf{F}_H > 2$$

for all hyperplanes $H \subset \mathbb{R}^m$ with distance $\mathbf{d}_H = d$ to the loop.
For a set $A \subset \mathbf{FS}_d(C, h)$ of flag segments with distance d to the loop define

$$V(A)_{<d} := \langle V(C, h)_{<d} \cup \{\mathrm{v}(F) | F \in A\} \rangle$$

as the linear space spanned by the direction vectors of flags that are closer to the loop than d and by the direction vectors of the elements of A.

Remark 3.2.20
If there exists no vertex $v \in \mathbf{V}(C)$ with distance $\mathbf{d}(v) = d$ to the loop, there exists no hyperplane $H \subset \mathbb{R}^m$ with distance $\mathbf{d}_H = d$ to the loop and (C, h) is automatically d-well-spaced. A curve (C, h) is therefore well-spaced if and only if it is d-well-spaced for all $d > 0$ with $\mathbf{V}_d \neq \emptyset$.

Lemma 3.2.21
It holds

$$\dim V_d/V_{<d} \leq \#\mathbf{FS}_d - \#\mathbf{P}_d - 1$$

for all curves $(C, h) \in \mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ and for all $d > 0$ with $\mathbf{V}_d \neq \emptyset$. It is moreover true that

$$\dim V_0/V_{<0} \leq \#\mathbf{FS}_0 - \#\mathbf{P}_0.$$

Proof. Let $(C, h) \in \mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$, $d \geq 0$ and assume $\mathbf{V}_d \neq \emptyset$, i.e. there exists a vertex with distance d to the loop, which is always true for $d = 0$. Due to the balancing condition at all points $p \in \mathbf{P}_d(C)$ with distance d to the loop, it holds $\dim V_d/V_{<d} \leq \#\mathbf{FS}_d - \#\mathbf{P}_d$.
Assume that $d > 0$ and that $\dim V_d/V_{<d} = \#\mathbf{FS}_d - \#\mathbf{P}_d$, i.e. the direction vectors of the flags with distance d to the loop fulfill no relation modulo $V_{<d}$ in addition to the balancing condition at

the vertices $v \in \mathbf{V}_d$. Let $v \in \mathbf{V}_d$ be a vertex with distance d to the loop and let $F_1, F_2 \in \mathbf{F}_v$ with $F_1 \neq F_2$ be flags that lie directly behind v seen from the loop. Then it holds

$$\dim V(\mathbf{FS}_d \setminus \{F_1, F_2\})_{<d} < \dim V_d,$$

and there exists a hyperplane H that contains $V_{<d}$, all direction vectors $\mathrm{v}(F)$ of flag segments $F \in \mathbf{FS}_d \setminus \{F_1, F_2\}$ with distance d to the loop but not $\mathrm{v}(F_1)$ and $\mathrm{v}(F_2)$. This implies that $\#\mathbf{F}_H = 2$ and $(C, h) \notin \mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$. □

Lemma 3.2.22
Let $(C, h) \in \mathrm{P}_I(\Delta, \mathbb{R}^m)$, $d > 0$ such that $\mathbf{V}_d \neq \emptyset$. Then (C, h) is d-well-spaced if and only if there exist pairwise different $\lambda_F \in \mathbb{R}$, $F \in \mathbf{FS}_d$, that fulfill

$$\sum_{F \in \mathbf{FS}_d} \lambda_F \cdot \mathrm{v}^\omega(F) \in \mathbf{V}_{<d}.$$

Proof. Assume that there exist pairwise different $\lambda_F \in \mathbb{R}$, $F \in \mathbf{FS}_d$, such that

$$\sum_{F \in \mathbf{FS}_d} \lambda_F \cdot \mathrm{v}^\omega(F) \in \mathbf{V}_{<d}.$$

Assume moreover that (C, h) is not d-well-spaced. Then there exists a hyperplane $H \subset \mathbb{R}^m$, a vertex $v \in \mathbf{FS}_d$ with distance d to the loop and different flags $F_1, F_2 \in \mathbf{FS}_v$ such that

$$V(\mathbf{FS}_d \setminus \{F_1, F_2\})_{<d} \subset H$$

and $\mathrm{v}^\omega(F_1), \mathrm{v}^\omega(F_2) \notin H$, i.e. the weighted direction vectors of all flag segments whose distance to the loop is at most d are contained in H except the weighted direction vectors of F_1 and F_2. In particular, it holds $V_{<d} \subset H$. Due to the balancing condition at $v \in \mathbf{P}_d$ we can assume that $\lambda_{F_1} \neq \lambda_{F_2} = 0$. It follows

$$\mathrm{v}^\omega(F_1) - \sum_{F \in \mathbf{FS}_d \setminus \{F_1, F_2\}} \frac{\lambda_F}{\lambda_{F_1}} \mathrm{v}^\omega(F) \in V_{<d} \subset H$$

and hence $\mathrm{v}^\omega(F_1) \in H$. This is a contradiction.
Assume now that (C, h) is d-well-spaced. Assume moreover that there exist no pairwise different $\lambda_F \in \mathbb{R}$, $F \in \mathbf{FS}_d$, such that

$$\sum_{F \in \mathbf{FS}_d} \lambda_F \cdot \mathrm{v}^\omega(F) \in \mathbf{V}_{<d}.$$

Then there exists a vertex $v \in V_d$ with distance d to the loop and flags $F_1, \ldots, F_r \in \mathbf{FS}_v$ directly behind v, with $r \geq 2$, such that $\lambda_{F_i} = \lambda_{F_j}$ for all such equations and all $i, j \in [r]$. (If there exists an equation of this type with $\lambda^1_{G_1} \neq \lambda^1_{G_2}$ and an equation with $\lambda^2_{G_2} \neq \lambda^3_{G_3}$, by scaling and adding these two equations we see that there also exists an equation of this type where $\lambda_{G_1}, \lambda_{G_2}$ and λ_{G_3} are pairwise different.) Let $\{F_1, \ldots, F_r\} \subset \mathbf{FS}_d$ be the set containing all flags $F \in \mathbf{F}_v$ that lie directly behind the vertex $v \in \mathbf{V}_d$ and that fulfill $\lambda_F = \lambda_{F_1}$ for all equations of the type from above. Then it holds that the vectors $\mathrm{v}^\omega([F_1]), \ldots, \mathrm{v}^\omega([F_r])$ fulfill only one relation modulo $V(\mathbf{FS}_d \setminus \{F_1, \ldots F_r\})_{<d}$, namely the one given by the balancing condition

$$\sum_{i \in [r]} \mathrm{v}^\omega(F_i) \in V(\mathbf{FS}_d \setminus \{F_1, \ldots F_r\})_{<d},$$

in particular $[\mathrm{v}^\omega(F_1)], \ldots, [\mathrm{v}^\omega(F_{r-1})]$ are linearly independent in $V(\mathbf{FS}_d \setminus \{F_1, \ldots, F_r\})_{<d}$. Hence, there exists a hyperplane $H \subset \mathbb{R}^m$ that contains the linear space $V(\mathbf{FS}_d \setminus \{F_1, \ldots F_r\})_{<d}$ and the vectors $\mathrm{v}^\omega(F_1), \ldots, \mathrm{v}^\omega(F_{r-2})$ but not $\mathrm{v}^\omega(F_{r-1})$ and $\mathrm{v}^\omega(F_r)$. (Remember that $r \geq 2$.) This is a contradiction to the d-well-spacedness of (C, h).

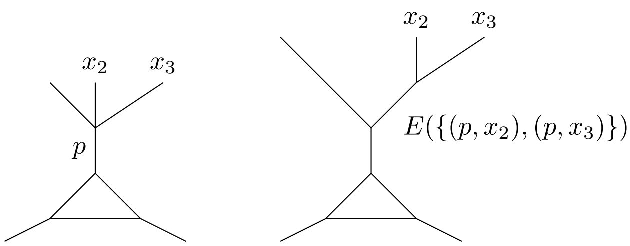
□

The following constructions allow to find a well-spaced resolution of a well-spaced curve if condition b) or c) is not fulfilled in the proposition above.

Construction 3.2.23 ($\Delta(F)$, $(\alpha_{\mathcal{P}}, \leq_{\mathcal{P}})$)
Let $(C,h) \in \mathrm{P}_I(\Delta, \mathbb{R}^m)$. Let (p,E) be a flag segment outside the loop that points away from the loop, i.e. $\mathbf{d}(q) \geq \mathbf{d}(p)$ for all $q \in E$. Denote by $\Delta(F) \subset \Delta \cup I$ the set of labels $i \in \Delta \cup I$ such that the leaf x_i lies behind F seen from the loop, i.e. the path from p to the leaf x_i labeled by $i \in \Delta(F)$ contains F as first flag segment. If the flag segments $F_1, \dots, F_r \in \mathbf{F}_p$ all lie behind the same point $p \in \mathbf{P}(C)$, we define

$$\Delta(F_1, \dots, F_r) = \bigcup_{i \in [r]} \Delta(F_i).$$

Given such a set $\Delta(F_1, \dots, F_r)$, we can add an edge $E(\{F_1, \dots, F_r\})$ of length one to C such that removing $E(\{F_1, \dots, F_r\})$ from C induces the partition $\{\Delta(F_1, \dots, F_r), \Delta \setminus \Delta(F_1, \dots, F_r)\}$ of Δ. We get a curve $C(\{F_1, \dots, F_r\})$. See the figure below for an example. (Remember that the length of the edge $E(\{(p, E_2), (p, E_3)\})$ was chosen to be one.)



Let now $\mathcal{P} = \{P_1, \dots, P_r\} \in \mathbf{P}(\{\mathbf{F}_v\}_{v \in \mathbf{P}_d})$, i.e. $\mathcal{P}$ is a partition of the set of flag segments of C that have distance d to the loop and $\mathcal{P}$ is finer than the partition given by the vertices with distance d to the loop. Let $(C_{\mathcal{P}}, h_{\mathcal{P}})$ be a curve that arises from (C,h) by adding the edges $E(P_i)$ of length one to C for all $i \in [r]$ and by choosing the position of the root vertex arbitrarily. See the figure below for an example.

If the fine combinatorial type of (C,h) is $(\alpha, \leq)$, we denote by $(\alpha_{\mathcal{P}}, \leq_{\mathcal{P}})$ the fine combinatorial type of $(C_{\mathcal{P}}, h_{\mathcal{P}})$ (which does not depend on the choice of the position of the root vertex of $(C_{\mathcal{P}}, h_{\mathcal{P}})$).

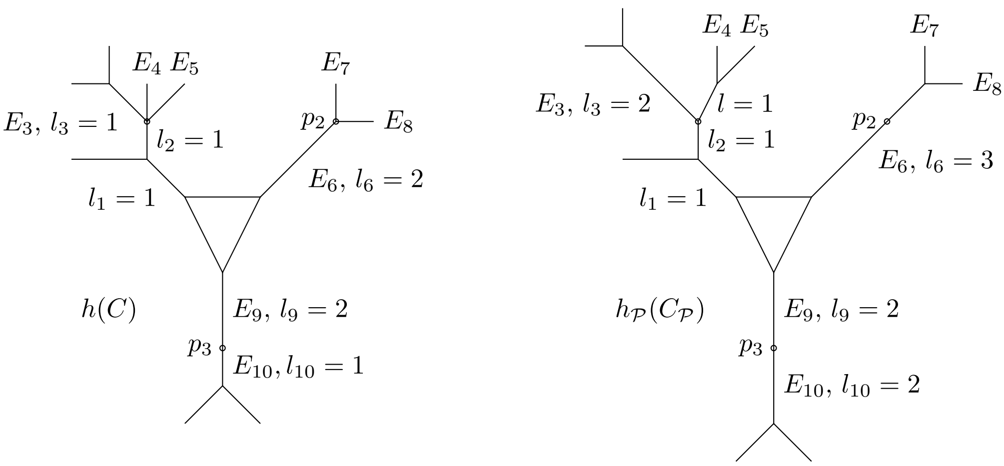


FIGURE 7. The fine combinatorial type of $(C_{\mathcal{P}}, h_{\mathcal{P}})$ on the right is constructed from the fine combinatorial type of (C,h) on the left using the partition $\mathcal{P} = \{\{(p_1, E_3)\}, \{(p_1, E_4), (p_1, E_5)\}, \{(p_2, E_7), (p_2, E_8)\}, \{(p_3, E_{10})\}\}$ of $\mathbf{FS}_2(C)$, the flags with distance two to the loop, where $p_1 = E_4 \cap E_5$.

Remark 3.2.24
Let $(C,h) \in \mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ be of fine combinatorial type $(\alpha, \leq)$ and let $\mathcal{P} = \{P_i\}_{i \in [r]}$ be a partition of $\mathbf{FS}_d$ that refines the partition $\{\mathbf{F}_p | p \in \mathbf{P}_d\}$ given by the points with distance d to the loop. In

the case $d > 0$, $(\alpha, \leq) = (\alpha_{\mathcal{P}}, \leq_{\mathcal{P}})$ if and only if $\#P_i = 1$ for all $i \in [r]$ (i.e. we enlarge the length of all flag segments in $\mathbf{FS}_d$ with distance d to the loop by one) or $\mathcal{P} = \{\mathbf{F}_p | p \in \mathbf{P}_d\}$ (i.e. we enlarge the length of all edges which lie in front of $p \in \mathbf{P}_d$ by one). In the case $d = 0$, $(\alpha, \leq) = (\alpha_{\mathcal{P}}, \leq_{\mathcal{P}})$ if and only if $\#P_i = 1$ for all $i \in [r]$.

Remark 3.2.25
Assume that $(C, h) \in \mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ and $\mathcal{P} \in \mathbf{P}(\{\mathbf{F}_p\}_{p \in \mathbf{P}_d})$. Let $d > 0$. Since (C, h) is well-spaced, a curve $(C_{\mathcal{P}}, h_{\mathcal{P}})$ (see previous construction) is well-spaced if and only if $(C_{\mathcal{P}}, h_{\mathcal{P}})$ is d-well-spaced. If $d = 0$, a curve $(C_{\mathcal{P}}, h_{\mathcal{P}})$ is well-spaced if and only if it is 1-well-spaced.

Let (C, h) be of fine combinatorial type α. We will define a set $\mathrm{Par}_d(C, h) \subset \mathbf{P}(\{\mathbf{F}_p(C, h)\}_{p \in \mathbf{P}_d(C,h)})$ of partitions of $\mathbf{FS}_d$ which are finer than the partition given by the points with distance d to the loop. $\mathrm{Par}_d(C, h)$ is of interest because, in the case that $d > 0$, a resolution $(\alpha_{\mathcal{P}}, \leq_{\mathcal{P}})$ of $(\alpha, \leq)$, where $\mathcal{P} \in \mathbf{P}(\{\mathbf{F}_p\}_{p \in \mathbf{P}_d})$, is well-spaced if and only if $\mathcal{P} \in \mathrm{Par}_d(C, h)$ (see the next corollary).

Construction 3.2.26 (Par_d)
Let $(C, h) \in \mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ and $d > 0$ such that there exists a vertex with distance d to the loop, i.e. $\#\mathbf{V}_d > 0$. An equation

$$\sum_{F \in \mathbf{FS}_d} \lambda_F \cdot \mathrm{v}^{\omega}(F) \in V_{<d}$$

defines a partition $\{\mathbf{F}_1, \ldots, \mathbf{F}_r\}$ of the flag segments with distance d to the loop that refines the partition $\{\mathbf{F}_p\}_{p \in \mathbf{P}_d}$, which is given by the vertices of distance d to the loop, in the following way:
Two flag segments F_1, F_2 which lie behind a point $p \in \mathbf{P}_d$ with distance d to the loop lie in the same set $\mathbf{F}_i$ if and only if $\lambda_{F_1} = \lambda_{F_2}$, i.e. for $p \in \mathbf{P}_d$ and $F_1, F_2 \in \mathbf{FS}_p$, there exists $i \in [r]$ with $F_1, F_2 \in \mathbf{F}_i$ if and only if $\lambda_{F_1} = \lambda_{F_2}$.
Define $\mathrm{Par}_d(C, h) \subset \mathbf{P}(\{\mathbf{F}_p(C, h)\}_{p \in \mathbf{P}_d(C,h)})$ (or just Par_d) as the set of partitions of $\mathbf{FS}_d$ that are defined in this way by an equation as above and that are finer than the partition of $\mathbf{FS}_d$ that is given by the points with distance d to the loop. If $d = 0$, denote the combinatorial type of (C, h) by α and define $\mathrm{Par}_0(\alpha)$ analogously.

Corollary 3.2.27
Let $(C, h) \in \mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ be of fine combinatorial type $(\alpha, \leq)$ and let $d > 0$ with $\#\mathbf{V}_d > 0$. Assume that $\mathcal{P} \in \mathbf{P}(\{\mathbf{F}_p(C)\}_{p \in \mathbf{P}_d(C)})$ is a partition of the flags with distance d to the loop of C that is finer than the one given by the vertices with the same distance to the loop.
Then the fine combinatorial type $(\alpha_{\mathcal{P}}, \leq_{\mathcal{P}})$ is a fine combinatorial type of well-spaced curves if and only if $\mathcal{P} \in \mathrm{Par}_d(C, h)$.

PROOF. Let $\mathcal{P} \in \mathrm{Par}_d(C, h)$ and let $(C_{\mathcal{P}}, h_{\mathcal{P}})$ be a curve of combinatorial type $(\alpha_{\mathcal{P}}, \leq_{\mathcal{P}})$ as in construction 3.2.23. It holds $\dot{\bigcup}_{P \in \mathcal{P}} P = \mathbf{FS}_d(C)$, i.e. the union of the elements of $\mathcal{P}$ is the set of flag segments with distance d to the loop of C. Since $\mathcal{P} \in \mathrm{Par}_d$, there exist pairwise different λ_P, $P \in \mathcal{P}$, such that

$$\sum_{P \in \mathrm{Par}_d} \lambda_P \cdot \left(\sum_{F \in P} \mathrm{v}^{\omega}(F) \right) \in V_{<d}.$$

With lemma 3.2.22 and remark 3.2.25, it follows that $(C_{\mathcal{P}}, h_{\mathcal{P}})$, and hence $(\alpha_{\mathcal{P}}, \leq_{\mathcal{P}})$, is well-spaced. □

Lemma 3.2.28
Let $(C, h) \in \mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$. If there exists $d > 0$ with $\mathbf{V}_d \neq \emptyset$ such that

$$\dim V_d / V_{<d} < \#\mathbf{FS}_d - \#\mathbf{P}_d - 1$$

or if

$$\dim V_0 / V_{<0} < \#\mathbf{FS}_0 - \#\mathbf{P}_0,$$

then the fine combinatorial type of (C, h) is not maximal.

Proof. Denote the fine combinatorial type of (C,h) by $(\alpha,\leq)$. Assume that there exists $d>0$ with $\mathbf{V}_d\neq\emptyset$ and

$$\dim(V_d/V_{>d}) < \#\mathbf{FS}_d - \#\mathbf{P}_d - 1.$$

Then the weighted direction vectors of the flag segments with distance d to the loop fulfill at least two independent relations in $V_{<d}$ of the type

$$\sum_{F\in\mathbf{FS}_d} \lambda_F\,\mathrm{v}^\omega(F)\in V_{<d}$$

in addition to balancing at the vertices with distance d to the loop. Hence, the set of partitions Par_d of $\mathbf{FS}_d$ from the construction above contains at least one element $\mathcal{P}\in\mathrm{Par}_d$ different from $\{\{F\}|F\in\mathbf{FS}_d\}$ and different from $\{\mathbf{FS}_p|p\in\mathbf{P}_d\}$ (the latter partition is induced by the equations given by the balancing condition at the points $p\in\mathbf{P}_d$ with distance d to the loop). According to the previous corollary and remark 3.2.24, it follows that $(\alpha_\mathcal{P},\leq_\mathcal{P})$ is a well-spaced resolution of $(\alpha,\leq)$ that is different from $(\alpha,\leq)$.

Let us turn to the case where $\dim V_0/V_{<0} < \#\mathbf{FS}_0-\#\mathbf{P}_0$: The weighted direction vectors $\mathrm{v}^\omega(F)$ of flag segments $F\in\mathbf{FS}_0$ at the loop fulfill at least one relation in addition to the balancing condition at the vertices $p\in\mathbf{FS}_0$, i.e. we find coefficients $0\neq\lambda_F\in\mathbb{R}$, $F\in\mathbf{FS}_0$, such that there exists $v\in\mathbf{P}_0$ and $F_1,F_2\in\mathbf{F}_v$ with $\lambda_{F_1}\neq\lambda_{F_2}=0$ and such that

$$\sum_{F\in\mathbf{FS}_0}\lambda_F\cdot\mathrm{v}^\omega(F)\in V_{<0}.$$

Define a partition of the set of flag segments $\mathbf{FS}_0$ at the loop via

$$\mathcal{P}=\{\{F\}|F\neq F_1,F_2\}\cup\{\{F_1,F_2\}\},$$

and let $(C_\mathcal{P},h_\mathcal{P})\in\mathrm{P}_I(\Delta,\mathbb{R}^m)$ be a curve of fine combinatorial type $(\alpha_\mathcal{P},\leq_\mathcal{P})\neq(\alpha,\leq)$ as in construction 3.2.23, see also remark 3.2.24.

$(C_\mathcal{P},h_\mathcal{P})$ is d-well-spaced for all $1\neq d>0$ because (C,h) is well-spaced. It is also 1-well-spaced: It holds $\lambda_1\neq0$ and

$$\mathrm{v}^\omega(F_1)=\sum_{F\in\mathbf{FS}_0\setminus\{F_1,F_2\}}\frac{\lambda_F}{\lambda_{F_1}}\cdot\mathrm{v}^\omega(F)\in V(C_\mathcal{P},h_\mathcal{P})_0.$$

Due to the balancing condition, it follows $\mathrm{v}^\omega(F_2)\in V(C_\mathcal{P},h_\mathcal{P})_0$ and hence

$$V(C_\mathcal{P},h_\mathcal{P})_1/V(C_\mathcal{P},h_\mathcal{P})_{<1}=V(C_\mathcal{P},h_\mathcal{P})_0/V(C_\mathcal{P},h_\mathcal{P})_0=\{0\}.$$

It follows that there do not exist hyperplanes $H\subset\mathbb{R}^m$ that fulfill $\mathbf{d}_H=1$. Hence, $(C_\mathcal{P},h_\mathcal{P})$ is well-spaced and the combinatorial type of (C,h) is not maximal in $\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)$. □

Proof of proposition 3.2.18. Let $(C,h)\in\mathcal{M}_{I,1}(\Delta,\mathbb{R}^m)$ be regular,

$$(\forall\, d>0:\mathbf{V}_d\neq\emptyset\Rightarrow\dim V_d/V_{<d}=\#\mathbf{FS}_d-\#\mathbf{P}_d-1)\ \text{ and }\ (\dim V_0/V_{<0}=\#\mathbf{FS}_0-\#\mathbf{P}_0)\,.$$

Assume that the fine combinatorial type of (C,h) is not maximal. Then there exists $(C,h)\leq(D,g)\in\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)$ such that the fine combinatorial types of the two curves are different. Denote by $\pi:\mathrm{supp}(D)\to\mathrm{supp}(C)$ the projection map. Due to the assumptions, there exists an edge $E\in\mathbf{E}(D)$ and a vertex $p\in\mathbf{V}(C)$ such that $\pi(E)=p$, i.e. $E\subset\mathrm{supp}(D)$ is contracted to the point $v\in\mathrm{supp}(C)$. E is bounded and we denote the two vertices contained in E by $p_1,p_2\in\mathbf{V}(D)$.

If $E\subset D_L$ is contained in the loop of D, denote the vertices in the loop of D by $p_1,\dots,p_r\in D_L$ with $p_1,p_2\in E$ in a way that $((p_1,E),(p_2,E_2),\dots,(p_r,E_r))$ is a path around the loop of D. Set $E_1\coloneqq E$. By lemma 3.2.15, there exist $\lambda_i\in\mathbb{R}_{>0}$ - with $\lambda_i<\lambda_j$ if $i>j$ - such that

$$\sum_{i\in[r-1]}\lambda_i\cdot\mathrm{v}^\omega_{(D,g)}(p_i)=\mathrm{v}^\omega_{(D,g)}(p_r,E_r)\in V_{<0}(D,g)\subset V_{<0}(C,h)$$

Hence, the weighted direction vectors $\mathrm{v}^\omega(F)$ of the flags $F \in \mathbf{FS}_0(C)$ at the loop of C fulfill a relation modulo $V(C,h)_{<0}$ in addition to the balancing condition at the vertices $v \in \mathbf{V}_0(C)$. This is a contradiction to

$$\dim V(C,h)_0/V_{<0}(C,h) = \#\mathbf{FS}_0(C,h) - \#\mathbf{P}_0(C,h).$$

Assume now that $E \not\subset D_L$, denote the two boundary points by $p_1, p_2 \in E$ and assume without loss of generality that $\mathbf{d}(p_1) < \mathbf{d}(p_2)$. It follows from lemma 3.2.21 that there exist $a_1, a_2 \in \mathbb{N}$ with $a_2 \geq 1$ and $a_1 \geq 0$ (a_1 can be zero if p_1 is contained in the loop) such that

$$\dim V(D,g)_{\mathbf{d}(p_1)}/V(D,g)_{<\mathbf{d}(p_1)} = \#\mathbf{FS}_{\mathbf{d}(p_1)}(D,g) - \#\mathbf{P}_{\mathbf{d}(p_1)}(D,g) - a_1$$

and

$$\dim V(D,g)_{\mathbf{d}(p_2)}/V(D,g)_{<\mathbf{d}(p_2)} = \#\mathbf{FS}_{\mathbf{d}(p_2)}(D,g) - \#\mathbf{P}_{\mathbf{d}(p_2)}(D,g) - a_2.$$

Since $\pi : \mathrm{supp}(D) \to \mathrm{supp}(C)$ fulfills $\pi(E) = \pi\{p_1\} = \pi\{p_2\} = p$, it follows that

$$\dim V(C,h)_{\mathbf{d}(p)}/V(C,h)_{<\mathbf{d}(p)} \leq \#\mathbf{FS}_{\mathbf{d}(p)}(C,h) - \#\mathbf{P}_{\mathbf{d}(p)}(C,h) - (a_1 + a_2).$$

If $\mathbf{d}(p_1) > 0$, it holds $a_1 \geq 1$ and $a_1 + a_2 \geq 2$. If $\mathbf{d}(p_1) = 0$, it holds $a_1 \geq 0$ and $a_1 + a_2 \geq 1$. Both statements contradict the prerequisites stated at the beginning of the proof. Thus, the fine combinatorial type of (C,h) is maximal in $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$.

It follows from lemma 3.2.13 and lemma 3.2.28 that curves in $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ of maximal fine combinatorial type fulfill the properties stated in the proposition. □

PROOF OF THEOREM 3.2.10. Using proposition 3.2.11, it remains to show that $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ is pure-dimensional of dimension $\Delta + \#I - \dim\langle j(\Delta)\rangle$. Let $(C,h) \in \mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ be of maximal fine combinatorial type $(\Gamma, \mathrm{v}, \leq)$. Denote the set of bounded edges of C by $\mathbf{E}^b$. From construction 3.1.9 and proposition 3.2.18 it follows that

$$\dim \mathrm{M}_1(\Gamma, \mathrm{v}, \leq) = \#\mathbf{E}^b + m - \dim V_{<0} - \sum_{d>0,\, \#\mathbf{V}_d>0} (\#\mathbf{V}_d - 1).$$

The terms $(\#\mathbf{V}_d - 1)$ stand for the condition that all $v \in \mathbf{V}_d$ have the same distance to the loop, m stands for the position of the root vertex and $\dim V_{<0}$ for the condition that the loop is closed. Since (C,h) is regular (i.e. C contains only vertices of genus zero and an "honest" loop), the number of bounded edges is given by

$$\mathbf{E}^b = \#\Delta + \#I - \sum_{v \in \mathbf{V}} (\mathrm{val}(v) - 3).$$

From lemma 3.2.18, it follows $\sum_{v \in \mathbf{P}_0} (\mathrm{val}(v) - 3) = \dim V_0/V_{<0}$ (the direction vectors of flags at the loop fulfill no relation modulo $V_{<0}$ in addition to balancing and there are two edges in the loop adjacent to each vertex $v \in \mathbf{V}_0$ in the loop) and for all $d > 0$ with $\#\mathbf{V}_d > 0$

$$\sum_{v \in \mathbf{P}_d} (\mathrm{val}(v) - 3) + \#\mathbf{P}_d - 1 = \sum_{v \in \mathbf{P}_d} (\mathrm{val}(v) - 2) - 1 = \dim V_d/V_{<d},$$

i.e. the direction vectors of flags with distance d to the loop fulfill precisely one relation modulo $V_{<d}$ in addition to balancing at each point $p \in \mathbf{P}_d$ and, for each $p \in \mathbf{P}_d$, there exists one flag segment $(p, E) \in \mathbf{FS}$ that points towards the loop.

We conclude

$$\begin{aligned} & \dim \mathrm{M}_1(\Gamma, \mathrm{v}, \leq) \\ = \quad & \#\Delta + \#I - \sum_{v \in \mathbf{V}} (\mathrm{val}(v) - 3) + m - \dim V_{<0} - \sum_{d>0, \mathbf{V}_d \neq \emptyset} (\#\mathbf{V}_d - 1) \\ = \quad & \#\Delta + \#I - \left(\dim V_0 / V_{<0} + \sum_{d>0, \mathbf{V}_d \neq \emptyset} (\dim V_d / V_{<d} - (\#\mathbf{V}_d - 1)) \right) \\ & + m - \dim V_{<0} - \sum_{d>0, \mathbf{V}_d \neq \emptyset} (\#\mathbf{V}_d - 1) \\ = \quad & \#\Delta + \#I + m - \dim \langle j(\Delta) \rangle. \end{aligned}$$

□

Definition 3.2.29
Let $(C, h) \in \mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ be of fine combinatorial type $(\Gamma, \mathrm{v}, \leq)$. We define the (co-)dimension of $(\Gamma, \mathrm{v}, \leq)$ as the (co-)dimension of the polyhedron $\mathrm{M}_1(\Delta, \mathrm{v}, \leq)$ in $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$. Its dimension is equal to

$$\#\Delta + \#I - \sum_{v \in \mathbf{V}} (\mathrm{val}(v) - 3) + m - \dim V_{<0} - \sum_{d>0, \mathbf{V}_d \neq \emptyset} (\#\mathbf{V}_d - 1),$$

where $\#\Delta + \#I - \sum_{v \in \mathbf{V}} (\mathrm{val}(v) - 3)$ is the number of bounded edges of Γ, m stands for the position of a root vertex, $(-\dim V_{<0})$ appears because the loop is closed and $(-\sum_{d>0, \mathbf{V}_d \neq \emptyset} (\#\mathbf{V}_d - 1))$ is the number of conditions on the edge lengths outside the loop imposed by the total preorder $\leq$ on the set of vertices outside the loop of Γ.

The (co-)dimension of (C, h) is defined as the (co-)dimension of $(\Gamma, \mathrm{v}, \leq)$

3.3. A tropical structure on $\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)^{\text{reg}}$

We equip the facets of $\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)$ with integer weights and construct a tropical atlas on a dense open subset $\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)^{\text{reg}}$ of $\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)$ containing only regular elliptic curves, which have an "honest" loop.

Weights on $\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)$.

Definition 3.3.1 (ind(α), $n(P)$, $\#\operatorname{Aut}(\alpha)$)
Let $(C,h)\in\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)$ be of combinatorial type α. For shortening notation, we define

$$\operatorname{ind}(\alpha) := \operatorname{ind}(L(\alpha)_0)) = [\operatorname{sat}(L(\alpha)_0) : L(\alpha)_0],$$

i.e. ind(α) is the index of the lattice spanned by the weighted direction vectors in and at the loop of α. For a finite set $P\neq\emptyset$, we define

$$n(P) := (-1)^{\#P-1}(\#P-1)!.$$

Moreover, define $\#\operatorname{Aut}(\alpha)$ as the order of the automorphism group of a curve (C,h) of combinatorial type α. $\#\operatorname{Aut}(\alpha)$ is one, except if α has only one edge that is contained in the loop - in this case all curves of combinatorial type α have two automorphisms and $\#\operatorname{Aut}(\alpha)$ is two, see corollary 1.3.13.

Definition 3.3.2 (The weighted abstract polyhedral complex $\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)$, $\mathbf{P}^0(\alpha)$)
If $(\alpha,\leq)$ is a maximal combinatorial type in $\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)$, we equip the facet $\overline{\mathrm{M}_1(\alpha,\leq)}$ containing all curves whose fine combinatorial type is a specialization of $(\alpha,\leq)$ with weight

$$\omega(\overline{\mathrm{M}_1(\alpha,\leq)}) := \omega(\alpha) := \frac{1}{\#\operatorname{Aut}(\alpha)}\cdot\sum_{\mathcal{P}\in\mathbf{P}(\{\mathbf{F}_v(\alpha)\}_{v\in\mathbf{V}_0(\alpha)})}\left[\prod_{P\in\mathcal{P}} n(P)\right]\cdot\operatorname{ind}(\alpha_{\mathcal{P}}),$$

i.e. the sum runs over all partitions $\mathcal{P}\in\mathbf{P}(\mathbf{FS}_0(\alpha))$ of the flags at the loop of α that are finer than the partition $\{\mathbf{F}_v(\alpha)\}_{v\in\mathbf{V}_0(\alpha)}$ given by the vertices in the loop of α. The combinatorial types $\alpha_{\mathcal{P}}$, $\mathcal{P}\in\{\mathbf{F}_v\}_{v\in\mathbf{V}_0}$, are constructed from α in 3.2.23.

For shortening notation, we set $\mathbf{P}^0(\alpha) := \mathbf{P}(\{\mathbf{F}_v(\alpha)\}_{v\in\mathbf{V}_0(\alpha)})$.

Example 3.3.3
Figure 8 illustrates a combinatorial type α and also the combinatorial types which are of the form $\alpha_{\mathcal{P}}$, $\mathcal{P}\in\mathbf{P}^0(\alpha)$. Assume that the fine combinatorial type $(\alpha,\leq)$ is maximal and let us calculate the weight $\omega(\alpha)$ of the facet $\overline{\mathrm{M}_1(\alpha,\leq)}$:

$$\begin{aligned}
\omega(\alpha) &= \sum_{\mathcal{P}\in\mathbf{P}^0(\alpha)}\left[\prod_{P\in\mathcal{P}}(-1)^{\#P-1}(\#P-1)!\right]\cdot\operatorname{ind}(\alpha_{\mathcal{P}})\\
&= \operatorname{ind}(\alpha)\\
&+ (-1)\cdot\sum_{i=1}^{4}\operatorname{ind}(\beta_i)\\
&+ (-1)^2\cdot\left(2!\cdot\operatorname{ind}(\gamma_1)+\sum_{i=2}^{4}\operatorname{ind}(\gamma_i)\right)\\
&+ (-1)^3\cdot 2!\cdot\operatorname{ind}(\delta)\\
&= \operatorname{ind}(\alpha)-\sum_{i=1}^{4}\operatorname{ind}(\beta_i)+2\cdot\operatorname{ind}(\gamma_1)+\sum_{i=2}^{4}\operatorname{ind}(\gamma_i)-2\cdot\operatorname{ind}(\delta)
\end{aligned}$$

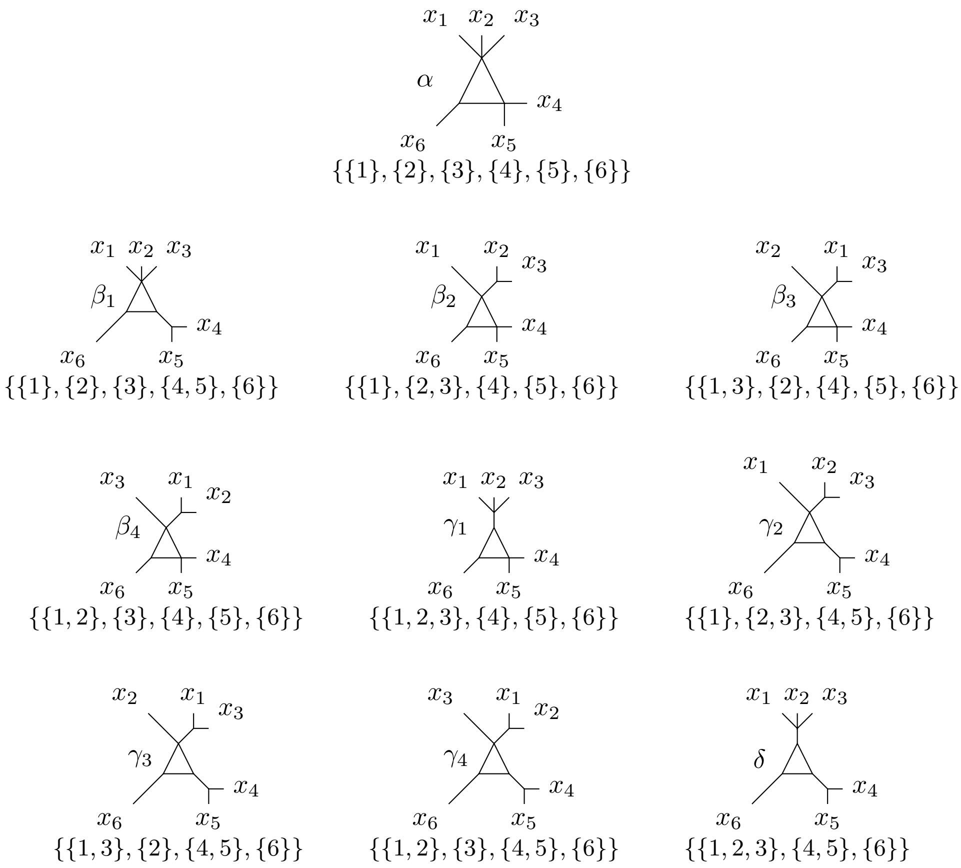


FIGURE 8. A list of the combinatorial types which are of the form $\alpha_{\mathcal{P}}$, $\mathcal{P} \in \mathbf{P}^0(\alpha)$. The corresponding partition of the set of flags at the loop of α is given below each curve, where we refer to a flag by $i \in [6]$ if it contains the leaf x_i.

Tropical atlas on $\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)^{\text{reg}}$. We will construct a tropical atlas on the open and dense subvariety $\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)^{\text{reg}}$ of $\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)$, which turns $\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)^{\text{reg}}$ into an abstract tropical variety. The target of the fan charts will be suitable moduli spaces of rational curves. In order to define the charts, we use the following construction that a assigns a rational curve (C_F, h_F) to a regular elliptic curve $(C,h) \in \mathrm{P}_I(\Delta,\mathbb{R}^m)$ and to a flag $F \in \mathbf{F}(C)$ in the loop of C that fulfills $\mathrm{v}^{\omega}(F) \neq 0$.

The idea is the same as for example in [**KM09**] and [**Her09**].

Construction 3.3.4 (Rational curve $(C_F, h_F) \in \mathcal{M}_{0,I'}(\Delta_{[F]},\mathbb{R}^m)$, $(\alpha_{[F]}, \leq_{[F]})$)
Let $(C,h) \in \mathrm{P}_I(\Delta,\mathbb{R}^m)$ be regular. Hence, there exists a flag $F = (p,E) \in \mathbf{F}(C)$ that is part of the loop and that fulfills $\mathrm{v}^{\omega}(F) \neq 0$. Denote the vertex contained in E which is not p by $q \in E$. Set $z := \omega(E)\,\mathrm{v}(p,E)$ to be the weighted direction vector of F. Define

$$\Delta_{[F]} := \Delta \cup \{z, -z\} \text{ and } I' := I \dot{\cup} \{A, B\}.$$

We

- cut the loop of C in the middle of the edge E,
- add a vertex q' at the new end that is adjacent to the vertex q and add two leaves at q', which we label by $-z$ and A,
- add a vertex p' at the new end that is adjacent to the vertex p and add two leaves at p', which we label by z and B.

See figure 9 for an example of the described construction. We denote the arising rational curve by C_F and define $h_F : C_F \to \mathbb{R}^m$ as the map which is induced by $h : C \to \mathbb{R}^m$, fulfills that the weighted direction vectors of the new unbounded flags (p', x_z) and (q', x_{-z}) are z and $-z$, respectively, and fulfills that the leaves x_A and x_B are contracted to a point. The curve (C_F, h_F) is an element of $\mathcal{M}_{0,I'}(\Delta_{[F]}, \mathbb{R}^m)$.

Let P be the unique path from p' to q' in C_F. Then we call the union of the edges that appear in this path together with the leaves x_A, x_B, x_z and x_{-z} the loop $(C_F)_L$ of C_F, i.e.

$$(C_F)_L := x_A \cup x_B \cup x_z \cup x_{-z} \cup \left(\bigcup E\right)$$

where the union runs over all flags (p, E) that appear in the path P from p' to q'. With this notation, all definitions and notations around the loop of (C, h) can be used analogously for (C_F, h_F), e.g. $V(C_F, h_F)_0$, $L(\alpha_{[F]})_0$ and $\mathbf{F}_H(C_F, h_F)$ for a hyperplane $H \subset \mathbb{R}^m$, and also the question if (C_F, h_F) is well-spaced makes sense.

In particular, (C_F, h_F) has a fine combinatorial type, and (C, h) is well-spaced if and only if (C_F, h_F) is well-spaced. If the fine combinatorial type of (C, h) is $(\alpha, \leq)$, we denote the fine combinatorial type of (C_F, h_F) by $(\alpha_{[F]}, \leq_{[F]})$.

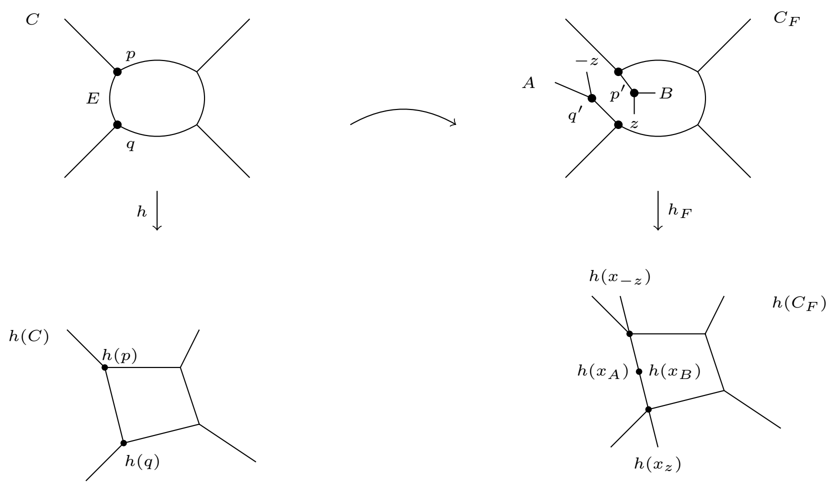


FIGURE 9. Example of construction 3.3.4. The first row shows the abstract curves C and C_F, the second row shows their images under h and h_F.

Definition 3.3.5 ($U(\alpha_{[F]}, \leq_{[F]}) \subset \mathcal{M}_{0,I'}(\Delta_{[F]}, \mathbb{R}^m)$)
Let $(\alpha, \leq)$ be a regular combinatorial type in $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ and let $[F]$ be a flag in the loop of α that fulfills $\mathrm{v}^\omega([F]) \neq 0$. Then we denote by $U(\alpha_{[F]}, \leq_{[F]}) \subset \mathcal{M}_{0,I'}(\Delta_{[F]}, \mathbb{R}^m)$ the subset of curves whose fine combinatorial type specializes to $(\alpha_{[F]}, \leq_{[F]})$ and which correspond to an elliptic curve in $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$, i.e. a curve $(C, h) \in U(\alpha_{[F]}, \leq_{[F]})$ that fulfills $h(x_A) = h(x_B)$, that the edges between the vertices p and p' and between the vertices q and q' have the same length (where we use the notation from the construction above) and that it is well-spaced.

Since the coordinates both on $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ and $\mathcal{M}_{0,I'}(\Delta_{[F]}, \mathbb{R}^m)$ are given by the edge lengths and the position of a root vertex, it follows from theorem 3.2.10 that $U(\alpha_{[F]}, \leq_{[F]})$ is a pure-dimensional open subcomplex of $\mathcal{M}_{0,I'}(\Delta, \mathbb{R}^m)$. We equip a facet of $U(\alpha_{[F]}, \leq_{[F]})$ containing curves of combinatorial type $\beta_{[F]}$ with the weight

$$\# \operatorname{Aut}(\beta) \cdot \omega(\beta),$$

where β is the fine combinatorial type of curves corresponding to $\beta_{[F]}$, i.e. the facet in $U(\alpha_{[F]}, \leq_{[F]})$ has $\# \operatorname{Aut}(\beta)$-times the weight of the corresponding facet in $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ and the factor $\frac{1}{\# \operatorname{Aut}(\beta)}$ appearing in $\omega(\beta)$ cancels out. Remember that $\operatorname{Aut}(\beta)$ is one if β has at least two edges in the loop.

By abuse of notation, we denote the arising weighted abstract polyhedral complex by $U(\alpha_{[F]}, \leq_{[F]})$, too.

Construction 3.3.6 ($\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)^{\text{reg}}$, fan charts $\psi^{[F]}_{\alpha,\leq}$)
Let $(\beta, \leq_\beta)$ be fine combinatorial types of regular elliptic curves that specializes to $(\alpha, \leq)$. Let $U(\alpha, \leq)$ be the polyhedral neighborhood of $M_1(\alpha, \leq)$ which contains all curves that have a fine combinatorial type that specializes to $(\alpha, \leq)$, where $M_1(\alpha, \leq)$ is the set of curves that have fine combinatorial type $(\alpha, \leq)$. In particular, it holds $M_1(\beta, \leq_\beta) \subset U(\alpha, \leq)$.

Define
$$\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)^{\text{reg}} = \bigcup U(\alpha, \leq),$$
where the sum runs over all fine combinatorial types $(\alpha, \leq)$ which have at least codimension one in $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ and at least two edges in the loop and over all maximal fine combinatorial types in $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ which have at least one edge in the loop. Note that $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)^{\text{reg}}$ contains only regular curves and note that a fine combinatorial type $(\alpha, \leq)$ in $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)^{\text{reg}}$ has at least two edges in the loop, if the codimension of $(\alpha, \leq)$ is at least one. We equip the facets of $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)^{\text{reg}}$ with the weight of the corresponding facet in $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$, and denote the arising weighted abstract polyhedral complex by $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)^{\text{reg}}$, too.

Choose a flag $[F] \in \mathbf{F}(\alpha)$ in the loop of α that fulfills $\mathrm{v}^\omega([F]) \neq 0$. If a curve (C, h) of combinatorial type α has no non-trivial automorphisms, there exists precisely one flag in the loop of β which is mapped to $[F]$ in the specialization process, see lemma 3.1.7. In particular, it has the same weighted direction vector as $[F]$. We denote this flag of β by $[F]$, too. If α is maximal and has only one edge in the loop, let (C, h) be of combinatorial type α. There exist two different flags $F_1 = (p, E_2), F_2 = (p, E_2)$ in the loop of C that fulfill $[F] = [F_1] = [F_2]$. Since $\mathrm{v}(F_1) = \mathrm{v}(F_2)$ and $\omega(E_1) = \omega(E_2)$ and since there are only two edges in the loop of C, it holds $(C_{F_1}, h_{F_1}) = (C_{F_2}, h_{F_2})$. Thus, the following map is well-defined if $(\alpha, \leq)$ is a fine combinatorial type in $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$:
$$\begin{array}{rcl} \psi^{[F]}_{\alpha,\leq} : U(\alpha, \leq) & \rightarrow & U(\alpha_{[F]}) \subset \mathcal{M}_{0,I'}(\Delta_{[F]}, \mathbb{R}^m) \\ (C, h) & \mapsto & (C_F, h_F), \end{array}$$
where $U(\alpha_{[F]})$ is the polyhedral neighborhood of $M_0(\alpha_{[F]})$ that contains all curves whose combinatorial type specializes to $\alpha_{[F]}$. The image consists precisely of the curves $(D, g) \in U(\alpha_{[F]})$ that satisfy

- $g(A) = g(B)$,
- that the lengths of the edges between the vertices p and p' and between the vertices q and q' coincide (in the notation of construction 3.3.4) and
- that are well-spaced.

Hence, it holds $\mathrm{im}\left(\psi^{[F]}_{\alpha,\leq}\right) = U(\alpha_{[F]}, \leq_{[F]})$.

Since the coordinates in the domain and the target space are the edge lengths and the position of a root vertex, $\psi^{[F]}_{\alpha,\leq}$ is injective and integer affine linear invertible on each polyhedron $M_1(\beta, \leq_\beta)$ in $U(\alpha, \leq)$. (There is only one coordinate in the target space for the lengths of the two cut edges because their length is equal. It is given by the sum of edge lengths of the cut edges.)

Let $U(\alpha, \leq) \cap U(\beta, \leq_\beta) \neq \emptyset$ and let $[G]$ be a flag in the loop of β. Then the concatenation $\psi^{[F]}_{\alpha,\leq} \circ (\psi^{[G]}_{\beta,\leq'})^{-1}$ is, where defined, the restriction of an integer affine linear invertible map because both on $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$, $\mathcal{M}_{0,I'}(\Delta_{[F]}, \mathbb{R}^m)$ and $\mathcal{M}_{0,I'}(\Delta_{[G]}, \mathbb{R}^m)$ the coordinates are given by the edge lengths and the position of a root vertex.

Remark 3.3.7
With the described construction, we can define fan charts $\psi^{[F]}_{\alpha,\leq}$ only on subvarieties of $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ which contain only regular curves. Non-regular fine combinatorial types $(\alpha, \leq)$ do not have any flags in the loop and the direction vectors of flags in the loop of their regular resolutions differ in general. Hence, in the case that α is non-regular it is not possible to identify a polyhedral neighborhood of a polyhedron $M_1(\alpha, \leq)$ with a open subvariety of a unique moduli space of rational curves.

Moreover, if there is only one edge in the loop of α and if $\alpha \leq \beta$ has more than one edge in the loop, there exist two edges in the loop of β that correspond to the unique edge in the loop of α, see the figure below. Hence, the fan chart $\psi^{[F]}_{\alpha,\leq}$ in the previous construction would not be well-defined. It follows that $\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)^{\text{reg}}$ is the maximal subset of $\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)$ on which we can define fan charts with the previous construction.

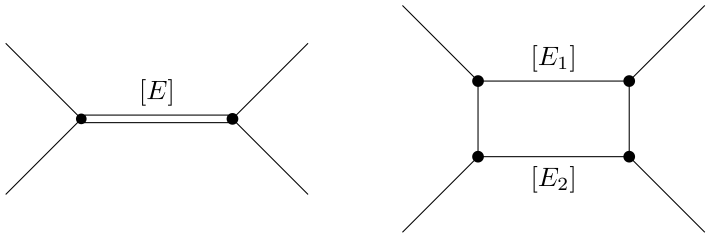


FIGURE 10. Assume $\omega([E_1]) = \omega([E_2])$. The combinatorial type β in $\mathbb{R}^2$ on the right specializes to the combinatorial type α on the left, whose only edge in the loop is $[E]$. Both edges $[E_1]$ and $[E_2]$ go to $[E]$ in the specialization process. If we cut the edges $[E_1]$ and $[E_2]$ in the middle, two different combinatorial types arise.

The aim of this section is to prove the following theorem.

Theorem 3.3.8
$\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)^{\text{reg}}$ together with the weights defined in 3.3.2 and together with the atlas $\left\{\psi^{[F]}_{\alpha,\leq}\right\}_{(\alpha,\leq)}$, where the fine combinatorial type $(\alpha,\leq)$ has at least codimension one, is an abstract tropical variety.

Remark 3.3.9
Note that the weights on $U(\alpha_{[F]},\leq_{[F]})$, the image of $\psi^{[F]}_{\alpha,\leq}$ coincide with the corresponding weights in $\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)^{\text{reg}}$ (although their definitions differs by a factor $\#\operatorname{Aut}(\alpha)$) because $\#\operatorname{Aut}(\alpha) = 1$ for all fine combinatorial types of codimension at least one in $M_{1,I}(\Delta,\mathbb{R}^m)^{\text{reg}}$.

In order to prove this theorem, we will show that $U(\alpha_{[F]},\leq_{[F]})$ is a tropical fan if $(\alpha,\leq)$ is a regular fine combinatorial type of codimension one in $\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)$, where "regular" means that α has only vertices of genus zero and an "honest" loop. This is sufficient to prove the theorem above because it holds

$$\operatorname{im}\left(\psi^{[F]}_{\alpha,\leq}\right) = U(\alpha_{[F]},\leq_{[F]})$$

if $(\alpha,\leq)$ is a fine combinatorial type in $\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)^{\text{reg}}$, i.e. if α has at least two edges in the loop. (In this case $\#\operatorname{Aut}(\alpha)$ is one.)

The following lemma describes the regular fine combinatorial types in $\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)$ which have codimension one.

Lemma 3.3.10
Let $(\alpha,\leq)$ be a regular fine combinatorial type in $\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)$ and let (C,h) be of fine combinatorial type $(\alpha,\leq)$. Then (C,h) has codimension one if and only if it fulfills one of the following properties:

a) (C,h) is regular and there exists $d > 0$ with $\#\mathbf{V}_d > 0$ such that
   (a) $\dim V_d/V_{<d} = \#\mathbf{FS}_d - \#\mathbf{P}_d - 2$,
   (b) $\dim V_0/V_{<0} = \#\mathbf{FS}_0 - \#\mathbf{P}_0$,
   (c) $\dim V_{d'}/V_{<d'} = \#\mathbf{FS}_{d'} - \#\mathbf{P}_{d'} - 1$ for all $d \neq d' > 0$ with $\#\mathbf{V}_{d'} \neq \emptyset$.
   In this case, in the specialization process from a curve of maximal combinatorial type to (C,h) an edge which does not intersect the loop is contracted.
b) (C,h) is regular and fulfills
   (a) $\dim V_0/V_{<0} = \#\mathbf{FS}_0 - \#\mathbf{P}_0 - 1$ and

(b) $\dim V_d/V_{<d} = \#\mathbf{FS}_d - \#\mathbf{P}_d - 1$ for all $d > 0$ with $\#\mathbf{V}_d \neq \emptyset$.

In this case, in the specialization process from a curve of maximal combinatorial type to (C, h) an edge in or at the loop is contracted.

Proof. The local coordinates on $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ are given by the edge lengths and the coordinates for the position of a root vertex in $\mathbb{R}^m$. A curve $(C, h) \in \mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ has codimension one if and only if it is a specialization of a curve $(D, g) \in \mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ of maximal combinatorial type such that in the specialization only one coordinate, an edge length, has become zero. It follows from proposition 3.2.18 that, if the edge whose edge length is set to zero does not intersect the loop of D a curve with the properties from a) arises. Using the same proposition we see that, if the contracted edge intersects the loop, (C, h) is of type b). □

We first deal with case a) of the lemma above. First, we show balancing of $U(\alpha_{[F]}, \leq_{[F]})$ in example cases in $\mathbb{R}^3$.

Example 3.3.11

A curve (C_F, h_F) of combinatorial type $(\alpha_{[F]}, \leq_{[F]})$ in $\mathbb{R}^3$ is shown in figure 11. (In order to have clearer pictures, we do not illustrate the cut flag $[F]$.) We assume $\dim V(C_F, h_F)_0 = 2$, that the distance of the unique vertex outside the loop to the loop is one and that $V(C_F, h_F)_1 = \mathbb{R}^3$. We denote the flag segments in $\mathbf{FS}_1(C, h)$ by F_i, $i \in [6]$.

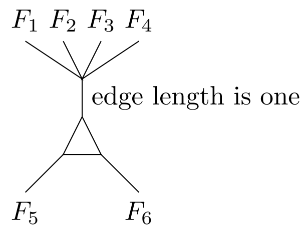


Figure 11.

- Assume that none of the direction vectors of the flags F_i, $i \in [4]$, lie in $V(\alpha_{[F]})_0$, i.e. $\langle V(\alpha_{[F]})_0 \cup \{\mathrm{v}(F_i)\}\rangle = \mathbb{R}^3$ for all $i \in [4]$. Then the well-spaced resolutions are given, in figure 12, by the combinatorial types on the left for any $i_1, i_2 \in [4]$ with $i_1 \neq i_2$. The sum over the representatives of respective normal vectors is given by the curve on the right (see lemma 1.3.16), which is an element of the vector space spanned by curves of fine combinatorial type $(\alpha, \leq)$ in which the loop is closed. We do not have to consider the weights of the facets in this calculation because they are all equal (since the weight of a facet only depends on the combinatorics in and at the loop of the corresponding curves).

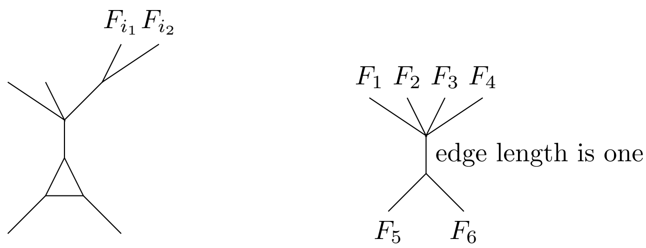
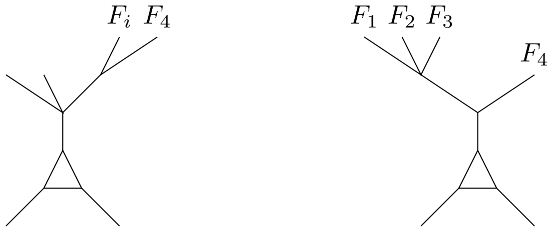


Figure 12.

- Assume that $\mathrm{v}(F_4) \in V(\alpha_{[F]})_0$, $\mathrm{v}(F_i) \notin V(\alpha_{[F]})_0$ for $i \in [3]$ and $\mathrm{v}(F_4) \in V(\alpha_{[F]})_0$. The well-spaced resolutions are shown below, where $i \in [3]$. The sum over the representatives of the normal vectors is the same as in the previous case, see again lemma 1.3.16. Again, the weights of the facets corresponding to the resolutions are all equal.



- Finally assume that $\mathrm{v}(F_3), \mathrm{v}(F_4) \in V(\alpha_{[F]})_0$ and $\mathrm{v}(F_1), \mathrm{v}(F_2) \notin V(\alpha_{[F]})_0$. In this case, $(\alpha_{[F]}, \leq_{[F]})$ is not well-spaced: Let the hyperplane $H \subset \mathbb{R}^m$ be given by $H = V(\alpha_{[F]})_{<0}$. Then it holds $\mathbf{F}_H = \{F_1, F_2\}$, $\#\mathbf{F}_H = 2$ and $(\alpha, \leq)$ is not well-spaced. Hence, we do not deal with this case.

Proposition 3.3.12
Let $(\alpha, \leq)$ be a regular fine combinatorial type of codimension one in $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ (i.e. α has only vertices of genus zero) and let (C, h) be of fine combinatorial type $(\alpha, \leq)$. Assume that there exists $d > 0$ with $\#\mathbf{V}_d > 0$ and $\dim V_d/V_{<d} = \#\mathbf{FS}_d - \#\mathbf{P}_d - 2$. Then, for all flags $[F]$ in the loop of α that fulfill $\mathrm{v}([F]) \neq 0$, the weighted subcomplex $U(\alpha_{[F]}, \leq_{[F]})$ of $\mathcal{M}_{0,I'}(\Delta_{[F]}, \mathbb{R}^m)$, whose support contains all curves whose fine combinatorial type specializes to $(\alpha_{[F]}, \leq_{[F]})$, is a tropical fan.

Proof. Since (C, h) is regular, there exists a flag F in the loop of (C, h) whose direction vector $\mathrm{v}(F)$ is non-zero. It follows from proposition 3.2.18 and lemma 3.2.27 that for all fine combinatorial types $(\alpha_{[F]}, \leq_{[F]}) \leq (\beta, \leq_\beta)$ in $\mathcal{M}_{0,I'}(\Delta_{[F]}, \mathbb{R}^m)$ there exists $\mathcal{P} \in \mathrm{Par}_d(C_F, h_F)$ such that

$$(\beta, \leq_\beta) = ((\alpha_{[F]})_\mathcal{P}, (\leq_{[F]})_\mathcal{P}).$$

Since all $((\alpha_{[F]})_\mathcal{P}, (\leq_{[F]})_\mathcal{P})$, $\mathcal{P} \in \mathrm{Par}_d$, have the same combinatorics in and at the loop as the fine combinatorial type $(\alpha_{[F]}, \leq_{[F]})$, all facets of $U(\alpha_{[F]}, \leq_{[F]})$ have the same weight

$$\mathrm{Aut}(\alpha) \cdot \omega(\alpha).$$

We claim that, for all pairs of flag segments $F_1, F_2 \in \mathbf{FS}_d(C, h)$ of distance d to the loop which both lie behind the same vertex $v \in \mathbf{P}_d$, there exists precisely one partition $\mathcal{P} \in \mathrm{Par}_d(C, h)$ that fulfills that F_1 and F_2 belong to the same set, i.e. that fulfills that there exists $P \in \mathcal{P}$ with $F_1, F_2 \in P$:
Remember that all partitions $\mathcal{P} \in \mathrm{Par}_d$ are strictly finer than the partition $\{\mathbf{F}_v\}_{v \in \mathbf{P}_d}$ given by the points with distance d to the loop. Assume that there are two different partitions $\mathcal{P}_1, \mathcal{P}_2 \in \mathrm{Par}_d$ that are strictly finer than the partition $\{\mathbf{F}_v\}_{v \in \mathbf{P}_d}$ given by the points with distance d to the loop such that F_1 and F_2 lie in the same set in both partitions, i.e. there exist sets $P_1 \in \mathcal{P}_1$ and $P_2 \in \mathcal{P}_2$ such that $F_1, F_2 \in P_1$ and $F_1, F_2 \in P_2$. Then, in addition to the balancing condition at points $p \in \mathbf{P}_d$ there exist two equations

$$\sum_{F \in \mathbf{FS}_d} \lambda^1_F \cdot \mathrm{v}^\omega(F) \in V_{<d} \text{ and } \sum_{F \in \mathbf{FS}_d} \lambda^2_F \cdot \mathrm{v}^\omega(F) \in V_{<d}$$

that induce different partitions $\mathcal{P}_1, \mathcal{P}_2$ such that $\mathrm{v}^\omega(F_1)$ and $\mathrm{v}^\omega(F_2)$ have the same coefficients in both equations, i.e. $\lambda^i_{F_1} = \lambda^i_{F_2}$ for $i \in \{1, 2\}$ (see 3.2.26 for the construction of the partitions $\mathcal{P}_1, \mathcal{P}_2$). In particular, the two equations are independent. Due to $\dim V_d/V_{<d} = \#\mathbf{FS}_d - \#\mathbf{P}_d - 2$ the weighted direction vectors $\mathrm{v}^\omega(F)$ of flags $F \in \mathbf{FS}_d$ with distance d to the loop fulfill altogether only two independent relations modulo $V_{<d}$ in addition to balancing. It follows that all such equations $\sum_{F \in \mathbf{FS}_d} \lambda_F \cdot \mathrm{v}^\omega(F) \in V_{<d}$ fulfill $\lambda_{F_1} = \lambda_{F_2}$. Using lemma 3.2.22, this is a contradiction to the d-well-spacedness of (C_F, h_F).
Next, we show that for two flag segments $F_1, F_2 \in \mathbf{FS}_d$ that lie behind one point $p \in \mathbf{P}_d$ there exists a partition $\mathcal{P} \in \mathrm{Par}_d$ which is strictly finer than the partition $\{\mathbf{F}_v\}_{v \in \mathbf{P}_d}$ given by the points with distance d to the loop and which fulfills that there exists $P \in \mathcal{P}$ with $F_1, F_2 \in P$. Since

$$\dim V_d/V_{<d} = \#\mathbf{FS}_d - \#\mathbf{P}_d - 2$$

there exist two independent relations

$$\sum_{F\in\mathbf{FS}_d}\lambda_F^1\cdot \mathrm{v}^\omega(F)\in V_{<d} \text{ and } \sum_{F\in\mathbf{FS}_d}\lambda_F^2\cdot \mathrm{v}^\omega(F)\in V_{<d}$$

in addition to balancing at points with distance d to the loop. (Balancing implies that, for all points $p\in\mathbf{P}_d$ with distance d to the loop, the sum over the weighted direction vectors of flags that lie directly behind p seen from the loop lies in $V_{<d}$, i.e. $\sum_{F\in\mathbf{F}_p}\mathrm{v}^\omega(F)\in V_{<d}$.) If $\lambda_{F_1}^1=\lambda_{F_2}^1$ or $\lambda_{F_1}^2=\lambda_{F_2}^2$, we define $\mathcal{P}$ as the partition corresponding to this equation, see construction 3.2.26. Otherwise, it holds $\lambda_{F_1}^1\neq\lambda_{F_2}^1$ and $\lambda_{F_1}^2\neq\lambda_{F_2}^2$. Due to balancing at the points $p\in\mathbf{P}_d$, we can assume without loss of generality that

$$\lambda_{F_1}^1=\lambda_{F_2}^2=1 \text{ and } \lambda_{F_2}^1=\lambda_{F_1}^2=0.$$

Hence, the coefficient both of $\mathrm{v}^\omega(F_1)$ and $\mathrm{v}^\omega(F_2)$ is one in

$$\sum_{F\in\mathbf{FS}_d}(\lambda_F^1+\lambda_F^2)\cdot \mathrm{v}^\omega(F)\in V_{<d}.$$

Assume that the partition defined by this equation is $\{\mathbf{F}_v\}_{v\in\mathbf{P}_d}$, see again construction 3.2.26. Then it holds $\lambda_{G_1}^1+\lambda_{G_1}^2=\lambda_{G_2}^1+\lambda_{G_2}^2$ for all pairs of flag segments $G_1,G_2\in\mathbf{FS}_d$ with distance d to the loop that lie behind one vertex $v\in\mathbf{P}_d$, i.e. there exists $v\in\mathbf{P}_d$ such that $G_1,G_2\in\mathbf{F}_v$. It follows that for all $p\in\mathbf{P}_d$ there exist $r_p\in\mathbb{R}$ such that $\lambda_F^2=r_p-\lambda_F^1$ for all $F\in\mathbf{F}_p$. This is a contradiction to the prerequisite that

$$\sum_{F\in\mathbf{FS}_d}\lambda_F^1\cdot \mathrm{v}^\omega(F)\in V_{<d} \text{ and } \sum_{F\in\mathbf{FS}_d}\lambda_F^2\cdot \mathrm{v}^\omega(F)\in V_{<d}$$

are two independent relations in addition to the balancing condition ($\sum_{F\in\mathbf{F}_p}\mathrm{v}^\omega(F)\in V_{<d}$) at the points with distance d to the loop (which is $\sum_{F\in\mathbf{F}_p}\mathrm{v}^\omega(F)\in V_{<d}$).

Let $\mathcal{P}\in\mathrm{Par}_d$. A representative $v_{\mathcal{P}}$ of the normal vector $u_{((\alpha_{[F]})_{\mathcal{P}},(\leq_{[F]})_{\mathcal{P}})/(\alpha_{[F]},\leq_{[F]})}$ is given by

$$v_{\mathcal{P}}:=\sum_{P\in\mathcal{P}}v_{\Delta(P)}$$

where $\Delta(P)\subset\Delta_{[F]}\cup(I\cup\{A,B\})$ is defined in construction 3.2.23 and where v_J is defined in 1.3.15 and 1.3.25 for a subset $J\subset\Delta_{[F]}\cup(I\cup\{A,B\})$. It follows from lemma 1.3.16 that

$$\begin{aligned}
v_{\Delta(P)} &= \sum_{j_1,j_2\in\Delta(P)}v_{\{j_1,j_2\}}\\
&= \sum_{\substack{F_1,F_2\in P,\\ F_1\neq F_2}}\ \sum_{\substack{j_1\in\Delta(F_1),\\ j_2\in\Delta(F_2)}}v_{\{j_1,j_2\}}+\sum_{F\in P}\ \sum_{j_1,j_2\in\Delta(F)}v_{\{j_1,j_2\}}\\
&= \sum_{F_1,F_2\in P}\big(v_{\Delta(F_1)\cup\Delta(F_2)}-v_{\Delta(F_1)}-v_{\Delta(F_2)}\big)+\sum_{F\in P}v_{\Delta(F)}
\end{aligned}$$

and hence

$$v_{\mathcal{P}}=\left(\sum_{P\in\mathcal{P}}\ \sum_{\substack{F_1,F_i\in P,\\ F_1\neq F_2}}(v_{\Delta(F_1)\cup\Delta(F_2)}-v_{\Delta(F_1)}-v_{\Delta(F_2)})\right)+\sum_{F\in\mathbf{FS}_d}v_{\Delta(F)}.$$

All $\mathcal{P}\in\mathrm{Par}_d$ are refinements of the partition $\{\mathbf{F}_v\}_{v\in\mathbf{P}_d}$ given by the vertices in the loop and for all pairs $F_1,F_2\in\mathbf{FS}_d$ of flag segments that lie behind one point $p\in\mathbf{P}_d$ there exists precisely one partition $\mathcal{P}\in\mathrm{Par}_d$ and one element $P\in\mathcal{P}$ with $F_1,F_2\in P$, see above. We conclude (again with lemma 1.3.16)

$$\begin{aligned}\sum_{\mathcal{P}\in \mathrm{Par}_d} v_{\mathcal{P}} &= \sum_{\mathcal{P}\in \mathrm{Par}_d}\left(\sum_{P\in\mathcal{P}}\sum_{\substack{F_1,F_2\in P,\\ F_1\neq F_2}}(v_{\Delta(F_1)\cup\Delta(F_2)}-v_{\Delta(F_1)}-v_{\Delta(F_2)})+\sum_{F\in \mathbf{FS}_d} v_{\Delta(F)}\right)\\ &= \sum_{p\in \mathbf{P}_d}\left(\sum_{\substack{F_1,F_2\in \mathbf{F}_p,\\ F_1\neq F_2}}(v_{\Delta(F_1)\cup\Delta(F_2)}-v_{\Delta(F_1)}-v_{\Delta(F_2)})\right)+\#\,\mathrm{Par}_d\cdot\sum_{F\in \mathbf{FS}_d} v_{\Delta(F)}\\ &= \sum_{p\in \mathbf{P}_d} v_{\Delta(\mathbf{F}_p)}+(\#\,\mathrm{Par}_d-1)\sum_{F\in FS_d} v_{\Delta(F)}\in \mathrm{W}(\alpha_{[F]},\leq_{[F]}),\end{aligned}$$

where $\mathrm{W}(\alpha_{[F]},\leq_{[F]})$ is the smallest linear space containing all curves of fine combinatorial type $(\alpha_{[F]},\leq_{[F]})$ which fulfill that the loop is closed, i.e. $\mathrm{ev}_A=\mathrm{ev}_B$ and $l_p=l_q$ in the notation of construction 3.3.4. □

In the rest of the section, we will deal with case b) of the lemma above and prove the following proposition.

Proposition 3.3.13
Let $(\alpha,\leq)$ be a regular fine combinatorial type of codimension one in $\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)$ that fulfills

$$\dim V_0/V_{<0}=\#\mathbf{FS}_0-\#\mathbf{P}_0-1.$$

Then, for all flags $[F]$ in the loop of α, the weighted subcomplex $U(\alpha_{[F]},\leq_{[F]})$ of $\mathcal{M}_{0,I'}(\Delta_{[F]}),\mathbb{R}^m)$, whose support contains all curves whose fine combinatorial type is a resolution of $(\alpha,\leq)$, is a tropical fan.

Assume in the rest of this section that α is a regular combinatorial type in $\mathrm{P}_I(\Delta,\mathbb{R}^m)$ that fulfills

$$\dim V_0/V_{<0}=\#\mathbf{FS}_0-\#\mathbf{P}_0-1$$

and that $F\in\mathbf{F}(\alpha)$ is a flag in the loop of α that fulfills $\mathrm{v}^\omega(F)\neq 0$. If we talk about $(\alpha,\leq)$, we mean a well-spaced combinatorial type in $\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)$ of codimension one that fulfills the same condition.

A maximal fine combinatorial type $(\beta,\leq_\beta)$ in $U(\alpha_{[F]},\leq_{[F]})$ that specializes to $(\alpha_{[F]},\leq_{[F]})$ can have either additional edges in the loop or additional edges outside the loop. Additional edges both outside and inside the loop are not possible because the codimension of $(\alpha,\leq)$ is one and because well-spacedness is a condition on the lengths of edges outside the loop only. We are going to study these kinds of resolutions separately, starting with those that have only additional edges inside the loop.

We will describe the intersection product

$$X:=(l_p-l_q)^*\{0\}\cdot(\mathrm{ev}_A-\mathrm{ev}_B)^*(V(\alpha_{[F]})_0^c)\cdot \mathrm{X}(\alpha_{[F]})$$

combinatorially, where the notation is the following.

Notation 3.3.14 ($V(\alpha_{[F]})_0^c$, l_p, l_q, ev_A, ev_B, $\mathrm{X}(\alpha_{[F]})$)
Denote by $V(\alpha_{[F]})_0^c$ a complement of the linear space $V(\alpha_{[F]})_0$ spanned by the direction vectors of the flags in and at the loop of $\alpha_{[F]}$. Remember that $U(\alpha_{[F]})\subset\mathcal{M}_{0,I'}(\Delta_{[F]},\mathbb{R}^m)$ is the open subvariety whose support consists of the curves whose combinatorial type specializes to $\alpha_{[F]}$, and all facets are equipped with weight one. Denote by $l_p,l_q:U(\alpha_{[F]})\to\mathbb{R}$ the maps that map a curve $(C,h)\in U(\alpha_{[F]})$ onto the lengths of the edges E (between p and p') and E' (between q and q') which we glue together to close the loop of $\alpha_{[F]}$, where we use the notation of construction 3.3.4. Moreover, define $\mathrm{ev}_A,\mathrm{ev}_B:U(\alpha_{[F]})\to\mathbb{R}^m$ as the evaluation maps of $A,B\in I'=I\cup\{A,B\}$, the marked points we glue together to close the loop of $\alpha_{[F]}$.

Denote by $\mathrm{X}(\alpha_{[F]})$ the open subvariety of $U(\alpha_{[F]})$ that consists precisely of the curves whose combinatorial type specializes to $\alpha_{[F]}$ and in the specialization process only edges in and at the loop are contracted, i.e. the combinatorics away from the loop are the same for all curves in $\mathrm{X}(\alpha_{[F]})$ and we resolve only vertices in the loop of $\alpha_{[F]}$.

We will describe the facets of

$$X := (l_p - l_q)^*\{0\} \cdot (\mathrm{ev}_A - \mathrm{ev}_B)^*(V(\alpha_{[F]})_0^c) \cdot \mathrm{X}(\alpha_{[F]})$$

combinatorially and will determine their weights. Our aim is proposition 3.3.25 which describes the balancing condition at the polyhedron of X containing curves of combinatorial type $\alpha_{[F]}$ combinatorially. (We will show that this polyhedron has codimension one in X.) The resulting equation will be used to prove balancing of $U(\alpha_{[F]}, \leq_{[F]})$ and thus to prove proposition 3.3.13.

Lemma 3.3.15
Denote the number of bounded edges of $\alpha_{[F]}$ that are outside the loop by $b_{\alpha_{[F]}}$. Then it holds

$$\dim \mathrm{X}(\alpha_{[F]}) = b_{\alpha_{[F]}} + \dim V_0 + m + \#\mathbf{P}_0 - \dim V_{<0} + 2$$

and

$$\dim X = b_{\alpha_{[F]}} + m + \#\mathbf{P}_0 - \dim V_{<0} + 1.$$

Proof. Since in $\mathrm{X}(\alpha_{[F]})$ we allow to resolve only the vertices in the loop of $\alpha_{[F]}$ and since combinatorial types in $\mathrm{X}(\alpha_{[F]})$ have at most $\#\mathbf{FS}_0 + 1$ edges in the loop (where one edge length in the loop is determined by $l_p = l_q$), it holds

$$\begin{aligned}
\dim \mathrm{X}(\alpha_{[F]}) &= b_{\alpha_{[F]}} + \#\mathbf{FS}_0 + 1 + m \\
&= b_{\alpha_{[F]}} + \#\mathbf{FS}_0 + 1 + m + \#\mathbf{P}_0 - \#\mathbf{P}_0 + \dim V_{<0} - \dim V_{<0} \\
&= b_{\alpha_{[F]}} + \dim V_0 + m + \#\mathbf{P}_0 - \dim V_{<0} + 2
\end{aligned}$$

because $\dim V_0/V_{<0} = \#\mathbf{FS}_0 - \#\mathbf{P}_0 - 1$. (The term $+1$, instead of -3 appears in the first line of the equation because of the four marked points x_A, x_B, x_z, x_{-z} in the loop of $\alpha_{[F]}$ that arise when cutting the flag $[F]$ of α in the construction of $\alpha_{[F]}$.) It follows

$$\dim X = b_{\alpha_{[F]}} + m + \#\mathbf{P}_0 - \dim V_{<0} + 1.$$

□

Lemma 3.3.16
It holds

$$\operatorname{supp} X \subset (\mathrm{ev}_A - \mathrm{ev}_B)^{-1}\{0\} \cap (l_p - l_q)^{-1}\{0\} \cap \{(D, g) \in \mathrm{X}(\alpha_{[F]}) | V(D, g)_0 = V(\alpha_{[F]})_0\}.$$

Proof. Using corollary 1.2.29 (which states that the pull-back is contained in the preimage), for $(D, g) \in \operatorname{supp}(X)$ it holds $(\mathrm{ev}_A - \mathrm{ev}_B)(D, g) \in V(\alpha_{[F]})_0^c$ and $l_p(D, g) = l_q(D, g)$. It also holds $(\mathrm{ev}_A - \mathrm{ev}_B)(D, g) \in V(\alpha_{[F]})_0$ due to the combinatorics of (D, g). (The direction vectors of all flags in a path around the loop from A to B are contained in $V(\alpha_{[F]})_0$.) We conclude

$$\mathrm{ev}_A(D, g) - \mathrm{ev}_B(D, g) \in V(\alpha_{[F]})_0 \cap V(\alpha_{[F]})_0^c = \{0\}.$$

Let us check that $V(D, g)_0 = V(\alpha_{[F]})_0$ for all $(D, g) \in \operatorname{supp}(X)$: It holds $V(D, g)_0 \subset V(\alpha_{[F]})$ for all $(D, g) \in \operatorname{supp}(X)$ since the combinatorial type of (D, g) is a resolution of $\alpha_{[F]}$. So let β be a combinatorial type such that $\operatorname{supp} X$ contains curves of combinatorial type β and assume that $V(\beta)_0 \subsetneq V(\alpha_{[F]})_0$. Then we choose a lattice basis $w_1, \ldots, w_s$ of $V(\alpha_{[F]})_0^c$, extend it to a lattice basis $w_1, \ldots, w_r$ of a complement $V(\beta)_0^c$ of $V(\beta)_0$, where s is strictly smaller than r, and to a lattice basis $w_1, \ldots, w_m$ of $\mathbb{R}^m$. For $r < i \leq m$, define the functions $\phi_i : \mathbb{R}^m \to \mathbb{R}$ by $\sum_{j \in [m]} \lambda_j \cdot w_j \mapsto \max\{\lambda_i, 0\}$, which fulfill

$$V(\alpha_{[F]})_0^c = \operatorname{supp}\left(\prod_{s < i \leq m} \phi_i \cdot \mathbb{R}^m\right).$$

It therefore holds

$$(\mathrm{ev}_A - \mathrm{ev}_B)^*(V(\alpha_{[F]})_0^c) \cdot \mathrm{X}(\alpha_{[F]}) = \prod_{s<i\leq m} (\mathrm{ev}_A - \mathrm{ev}_B)^*\phi_i \cdot \mathrm{X}(\alpha_{[F]}).$$

All curves (D, g) whose combinatorial type specializes to β fulfill $V(D, g)_0^c \subset V(\beta)_0^c$. With the same argument as above, wee see that $(\mathrm{ev}_A - \mathrm{ev}_B)(D, g) = 0$ if the fine combinatorial type of (D, g) specializes to β and if (D, g) is contained in the support of

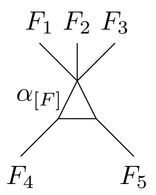
$$X' = (\mathrm{ev}_A - \mathrm{ev}_B)^*(V(\beta)_0^c) \cdot \mathrm{X}(\alpha_{[F]}) = \prod_{r<i\leq m} (\mathrm{ev}_A - \mathrm{ev}_B)^*\phi_i \cdot \mathrm{X}(\alpha_{[F]}).$$

Hence, for $s < i \leq r$, the restriction of $\phi_i \circ (\mathrm{ev}_A - \mathrm{ev}_B)$ to $U(\beta) \cap \mathrm{supp}(X')$ is the zero function. ($U(\beta)$ is the fan containing all curves whose combinatorial type specializes to β.) Since

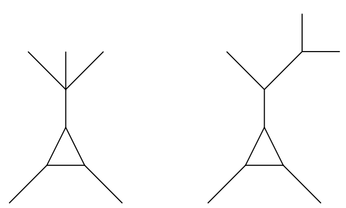
$$X = \prod_{s<i\leq r} (\mathrm{ev}_A - \mathrm{ev}_B)^*\phi_i \cdot X',$$

this implies that $U(\beta) \cap \mathrm{supp}(X)$ is empty, which is a contradiction. □

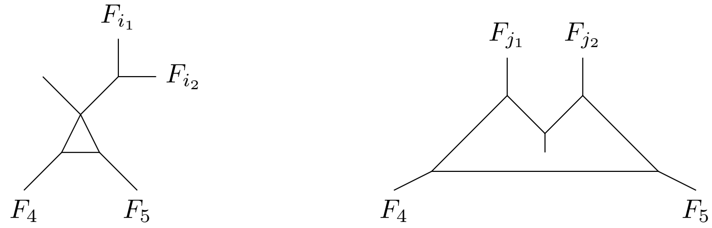
Example 3.3.17
We illustrate the result of the previous lemma. Consider the well-spaced combinatorial type $\alpha_{[F]}$ in $\mathbb{R}^3$. (Again we do not illustrate the cut flag $[F]$.) Assume that $V(\alpha_{[F]})_{<0} = \langle e_1, e_2 \rangle$, that $\mathrm{v}^\omega(F_i) \notin V(\alpha_{[F]})_{<0}$ for $i \in [3]$ (i.e. $V(\alpha_{[F]})_0 = \mathbb{R}^3$) and that the third coordinate of $\mathrm{v}^\omega(F_1)$ and $\mathrm{v}^\omega(F_2)$ is greater than zero. Hence, due to the balancing condition the third coordinate of $\mathrm{v}^\omega(F_3)$ is smaller than zero.



Since it holds, according to the last lemma, that $V(D, g)_0 = V(\alpha_{[F]})_0$ for all curves $(D, g) \in \mathrm{supp}(X)$, curves of the combinatorial type shown below do not appear in the support of X.



Since the loop is closed for all curves in $\mathrm{supp}(X)$ and since $\mathrm{v}(F_i) \notin V(\alpha_{[F]})_{<0}$ for $i \in [3]$, the curves in $\mathrm{supp}(X)$ have one of the following combinatorial types where $i_1, i_2 \in [3]$ and $i_1 \neq i_2$ and where $j_1, j_2 \in \{1, 2\}$ with $j_1 \neq j_2$. (Since the third coordinate of both $\mathrm{v}^\omega(F_1)$ and $\mathrm{v}^\omega(F_2)$ is greater than zero and since the loop is closed, it holds $j_1, j_2 \neq 3$ in the figure on the right.)



Notation 3.3.18 ($\sigma(\beta)$)
Let $\alpha_{[F]} \leq \beta$ be a combinatorial type of curves in $\mathrm{X}(\alpha_{[F]})$. Then we denote by $\sigma(\beta)$ the polyhedron which contains all curves (D,g) of combinatorial type β such that the loop is closed, i.e. $(\mathrm{ev}_A - \mathrm{ev}_B)(D,g) = (l_p - l_q)(D,g) = 0$. $\sigma(\beta)$ may be empty.

Corollary 3.3.19
$\sigma(\alpha_{[F]})$ has codimension one in X.

Proof. Due to lemma 3.3.16, $\sigma(\alpha_{[F]})$ is contained in $\mathrm{supp}(X)$. Due to lemma 3.3.15, it has codimension one. □

If $\alpha_{[F]} \leq \beta$ is a combinatorial type and if there exist curves of combinatorial type β in X, β has - compared to $\alpha_{[F]}$ - either additional edges inside or outside the loop. This is true because the intersection product X does not impose conditions on the edges outside the loop and because $\alpha_{[F]}$ has codimension one.

We study first the facets $\sigma(\beta)$ of X that fulfill that β has, compared to $\alpha_{[F]}$, only additional edges inside the loop.

Notation 3.3.20 ($\mathcal{A} \in \mathrm{Par}_0(\alpha_{[F]})$)
Due to

$$\dim V(\alpha_{[F]})_0 / V(\alpha_{[F]})_{<0} = \#\mathbf{FS}_0(\alpha_{[F]}) - \#\mathbf{P}_0(\alpha_{[F]}) - 1,$$

the weighted direction vectors of the flags $[F] \in \mathbf{FS}_0$ at the loop of $\alpha_{[F]}$ fulfill only one relation modulo $V_{<0}$ of the type

$$\sum_{[F]\in\mathbf{FS}_0} \lambda_{[F]} \cdot \mathrm{v}^{\omega}([F]) \in V_{<0}(\alpha_{[F]})$$

in addition to the balancing condition at the vertices $v \in \mathbf{V}_0$ in the loop of $\alpha_{[F]}$. It follows that $\mathrm{Par}_0(\alpha_{[F]})$ (see construction 3.2.26) contains precisely one partition that strictly refines the partition $\{\mathbf{F}_v\}_{v\in\mathbf{V}_0}$ given by the vertices in the loop of $\alpha_{[F]}$. Denote this unique element by

$$\mathcal{A} \in \mathrm{Par}_0(\alpha_{[F]}).$$

Lemma 3.3.21
Let $\alpha_{[F]} \neq \beta$ be a combinatorial type in the intersection product X that has, compared to $\alpha_{[F]}$, only additional edges inside the loop. Assume $\sigma(\beta) \neq \emptyset$. Then, the partition of the flags at the loop of β given by the vertices in the loop of β is $\mathcal{A}$, i.e.

$$\mathcal{A} = \{\mathbf{F}_v\}_{v\in\mathbf{V}_0(\beta)},$$

where we identify a flag at the loop of β with the corresponding flag at the loop of $\alpha_{[F]}$. Moreover, the weight of $\sigma(\beta)$ in X is

$$2 \cdot \mathrm{ind}(\alpha_{[F]}).$$

Proof. Due to $\dim V(\alpha_{[F]})_0 / V(\alpha_{[F]})_{<0} = \#\mathbf{FS}_0(\alpha_{[F]}) - \#\mathbf{P}_0(\alpha_{[F]}) - 1$, the weighted direction vectors of the flags at the loop of α fulfill only one relation modulo $V(\alpha_{[F]})_{<0}$ in a addition to balancing at the vertices in the loop, namely the relation that defines $\mathcal{A}$.

Since β has, compared to $\alpha_{[F]}$, additional edges only in the loop of $\alpha_{[F]}$, it holds $V(\beta)_0 = V(\alpha_{[F]})_0$. Moreover, each flag $[G_\beta]$ at the loop of β corresponds to a unique flag $[G]$ at the loop of $\alpha_{[F]}$. The two flags have the same weighted direction vector and we denote both of them by $[G]$.

There exist pairwise different $\lambda_{[v]} \in \mathbb{R}$ such that

$$\mathrm{v}^{\omega}([F]) = \sum_{[v]\in\mathbf{V}_0(\beta)} \left(\lambda_{[v]} \cdot \sum_{[(p,E)]\in\mathbf{F}_v(\beta)} \omega(E) \cdot \mathrm{v}([(p,E)]) \right) \in V(\alpha_{[F]})_{<0},$$

see lemma 3.2.15. Since $\beta \neq \alpha_{[F]}$, it follows

$$\mathcal{A} = \{\mathbf{F}_v(\beta)\}_{v\in\mathbf{V}_0(\beta)}$$

and the weighted direction vectors of the flags at the loop of β fulfill no relation modulo $V(\beta)_{<0}$ in addition to balancing at the vertices in the loop.

Let us calculate the weight of a facet $\sigma(\beta)$ of X. Since the weighted direction vectors of the flags at the loop of β fulfill no relation modulo $V(\beta)_{<0}$ in addition to balancing at the vertices in the loop, it holds

$$\dim V(\beta)_0/V(\beta)_{<0} = \#\mathbf{FS}_0(\beta) - \#\mathbf{P}_0(\beta).$$

Since the pull-back is contained in the preimage and since $V(\alpha_{[F]})_0 = V(\beta)_0$, for small generic $\epsilon > 0$, the intersection product

$$X_\epsilon = (l_p - l_q)^*\{0\} \cdot (\mathrm{ev}_A - \mathrm{ev}_B)^*(V(\alpha_{[F]})_0^c + \epsilon) \cdot \mathrm{X}(\alpha_{[F]})$$

is an intersection of hyperplanes in a unique facet of $\mathrm{X}(\alpha_{[F]})$, i.e. all vertices in the loop of β are 3-valent. Hence, the weight of the unique facet σ_ϵ of X_ϵ is given by the index of the restriction of the map $(l_p - l_q) \times (\mathrm{ev}_A - \mathrm{ev}_B)$ to σ, see lemma 1.2.9. The index of this map is

$$2 \cdot \mathrm{ind}(L(\beta)_{<0}),$$

where $L(\beta)_{<0}$ is the lattice spanned by the weighted direction vectors of the flags in the loop of β. This is true because we run around the loop of β in order to get from the marked point x_A to the marked point x_B and because the factor 2 comes up due to the condition $l_p = l_q$. It holds $L(\beta)_0 = L(\beta)_{<0}$ because the vertices in the loop of β are 3-valent. Since the weighted direction vectors of flags at the loop of β are the same as those of flags at the loop of $\alpha_{[F]}$, it holds moreover that

$$\mathrm{ind}(L(\beta)_0) = \mathrm{ind}(L(\alpha_{[F]})_0) = \mathrm{ind}(\alpha).$$

Since $V(\alpha_{[F]})_0^c$ and $\epsilon + V(\alpha_{[F]})_0^c$ are rationally equivalent, for small ϵ also X and X_ϵ are rationally equivalent, see 1.2.37. It follows that the weight of $\sigma(\beta)$ in X coincides with the weight of σ_ϵ in X_ϵ, which is $2 \cdot \mathrm{ind}(\alpha_{[F]})$. □

Next, we study the facets $\sigma(\beta)$ of X that fulfill that β has, compared to $\alpha_{[F]}$, only additional edges outside the loop. They will be given by

$$(\alpha_{[F]})_{\mathcal{P}'}$$

for $\mathcal{P}' \in \mathbf{P}_{\mathcal{P}_0(\alpha_{[F]})} \subset \mathbf{P}^0(\alpha_{[F]})$, where $\mathbf{P}_{\mathcal{P}_0(\alpha_{[F]})}$ is defined below. See also example 3.3.17.

Notation 3.3.22 ($I(\mathcal{P},\mathcal{P}')$, $\mathcal{P}_0(\alpha_{[F]})$, $\mathbf{P}_{\mathcal{P}_0(\alpha_{[F]})}$, $F(\mathcal{P}')_1$, $F(\mathcal{P}')_2$)
For partitions $\mathcal{P},\mathcal{P}' \in \mathbf{P}^0(\alpha_{[F]}) = \mathbf{P}(\{F_{[v]}\}_{[v]\in\mathbf{P}_0(\alpha_{[F]})})$, we denote by $I(\mathcal{P},\mathcal{P}')$ the coarsest common refinement of $\mathcal{P}$ and $\mathcal{P}'$, i.e.

$$I(\mathcal{P},\mathcal{P}') = \{P_1 \cap P_2 \neq \emptyset | P_1 \in \mathcal{P}, P_2 \in \mathcal{P}'\}.$$

Denote by $\mathcal{P}_0(\alpha_{[F]}) \in \mathbf{P}^0(\alpha_{[F]})$ the finest partition $\mathcal{P}_0(\alpha_{[F]}) = \{\{[F]\}|[F] \in \mathbf{FS}_0(\alpha_{[F]})\}$ of the set of flags at the loop of $\alpha_{[F]}$.

We define $\mathbf{P}_{\mathcal{P}_0(\alpha_{[F]})}$ as the set of partitions in $\mathbf{P}^0(\alpha_{[F]})$ that arise from the finest partition $\mathcal{P}_0(\alpha_{[F]})$ by uniting two flags which lie behind the same vertex in the loop of $\alpha_{[F]}$ and which are contained in different elements of $\mathcal{A}$, i.e.

$$\mathbf{P}_{\mathcal{P}_0(\alpha_{[F]})} = \{\mathcal{P} \in \mathbf{P}^0(\alpha_{[F]}) | \#I(\mathcal{P},\mathcal{P}_0(\alpha_{[F]})) = \#\mathcal{P}+1, \#I(\mathcal{P},\mathcal{A}) = \#\mathcal{P}+1\}.$$

For $\mathcal{P}' \in \mathbf{P}_{\mathcal{P}_0(\alpha_{[F]})}$, we denote by $F(\mathcal{P}')_1, F(\mathcal{P}')_2 \in \mathbf{FS}_0(\alpha_{[F]})$ the two different flags at the loop of $\alpha_{[F]}$ that fulfill $\{F(\mathcal{P}')_1, F(\mathcal{P}')_2\} \in \mathcal{P}'$. All other flags $F \in \mathbf{FS}_0(\alpha_{[F]})$ fulfill $\{F\} \in \mathcal{P}'$.

Lemma 3.3.23
Let $\alpha_{[F]} \neq \beta$ be a combinatorial type in $\mathrm{X}(\alpha_{[F]})$ that has, compared to $\alpha_{[F]}$, only additional edges outside the loop. Assume moreover $V(\beta)_0 = V(\alpha_{[F]})_0$. Then, there exists $\mathcal{P}' \in \mathbf{P}_{\mathcal{P}_0(\alpha_{[F]})}$ such that $\beta = (\alpha_{[F]})_{\mathcal{P}'}$, and it holds $V(\alpha_{\mathcal{P}})_0 = V(\alpha_{[F]})_0$ for all $\mathcal{P} \in \mathbf{P}_{\mathcal{P}_0(\alpha_{[F]})}$. Moreover, the weight of $\sigma(\beta)$ in X is

$$2 \cdot \mathrm{ind}(\beta) = 2 \cdot \mathrm{ind}((\alpha_{[F]})_{\mathcal{P}'}).$$

PROOF. First, we show that $V(\alpha_{\mathcal{P}})_0 = V(\alpha_{[F]})_0$ for all $\mathcal{P}' \in \mathbf{P}_{\mathcal{P}_0(\alpha_{[F]})}$:
Let $\mathcal{P}' \in \mathbf{P}_{\mathcal{P}_0(\alpha_{[F]})}$. It holds

$$\dim V(\alpha_{[F]})_0/V(\alpha_{[F]})_{<0} = \#\mathbf{FS}_0(\alpha_{[F]}) - \#\mathbf{P}_0(\alpha_{[F]}) - 1,$$

and the weighted direction vectors of flags at the loop of $\alpha_{[F]}$ fulfill only one relation modulo $V(\alpha_{[F]})_0$ in addition to balancing at the vertices in the loop of $\alpha_{[F]}$, namely the one that defines $\mathcal{A}$. There exist two different $A_1, A_2 \in \mathcal{A}$ such that $F(\mathcal{P}')_1 \in A_1$ and $F(\mathcal{P}')_2 \in A_2$. It follows that the weighted direction vectors

$$\mathrm{v}^\omega([F]) \text{ with } [F] \in \mathbf{FS}_0(\alpha_{[F]}) \setminus \{F(\mathcal{P}')_1, F(\mathcal{P}')_2\} \text{ and } (\mathrm{v}^\omega(F(\mathcal{P}')_1) + \mathrm{v}^\omega(F(\mathcal{P}')_2))$$

of flags at the loop of $\alpha_{\mathcal{P}'}$ fulfill no relation modulo $V((\alpha_{[F]})_{\mathcal{P}'})_{<0} \subset V(\alpha_{[F]})_{<0}$ in addition to balancing at the vertices in the loop of $(\alpha_{[F]})_{\mathcal{P}'}$, i.e.

$$\dim V((\alpha_{[F]})_{\mathcal{P}'})_0/V((\alpha_{[F]})_{\mathcal{P}'})_{<0} = \#\mathbf{FS}_0((\alpha_{[F]})_{\mathcal{P}'}) - \#\mathbf{P}_0((\alpha_{[F]})_{\mathcal{P}'}).$$

Due to $\#\mathbf{FS}_0((\alpha_{[F]})_{\mathcal{P}'}) = \#\mathbf{FS}_0(\alpha_{[F]}) - 1$, $\#\mathbf{P}_0((\alpha_{[F]})_{\mathcal{P}'}) = \#\mathbf{P}_0(\alpha_{[F]})$ and $V((\alpha_{[F]})_{\mathcal{P}'})_{<0} = V(\alpha_{[F]})_{<0}$, it follows

$$\dim V((\alpha_{[F]})_{\mathcal{P}'})_0 = \dim V(\alpha_{[F]})_{<0} + \#\mathbf{FS}_0(\alpha_{[F]}) - \#\mathbf{P}_0(\alpha_{[F]}) - 1 = \dim V(\alpha_{[F]})_0.$$

Since $V((\alpha_{[F]})_{\mathcal{P}'})_0 \subset V(\alpha_{[F]})_0$, we conclude $V((\alpha_{[F]})_{\mathcal{P}'})_0 = V(\alpha_{[F]})_0$.
Let now β be as in the statement of the lemma. Since β has, compared to $\alpha_{[F]}$, only additional edges outside the loop, there exists $\mathcal{P}' \in \mathbf{P}^0(\alpha_{[F]})$ such that $\beta = (\alpha_{[F]})_{\mathcal{P}'}$. Since $V(\beta)_0 = V(\alpha_{[F]})_0$ and

$$\dim V(\alpha_{[F]})_0/V(\alpha_{[F]})_{<0} = \#\mathbf{FS}_0(\alpha_{[F]}) - \#\mathbf{P}_0(\alpha_{[F]}) - 1,$$

it holds that $\#\mathcal{P}' = \#\mathbf{FS}_0(\alpha_{[F]}) - 1$ and that $\mathcal{P}'$ is not finer than $\mathcal{A} \in \mathrm{Par}_0(\alpha_{[F]})$. We conclude $\mathcal{P}' \in \mathbf{P}_{\mathcal{P}_0(\alpha_{[F]})}$.
With the argument with which we have already calculated the weight of a facet of X in the last lemma, it follows that the weight of $\sigma(\beta) = \sigma((\alpha_{[F]})_{\mathcal{P}'})$ in X is

$$2 \cdot \mathrm{ind}(\beta) = 2 \cdot \mathrm{ind}((\alpha_{[F]})_{\mathcal{P}'}).$$

□

Corollary 3.3.24
It holds

$$\operatorname{supp} X = (\mathrm{ev}_A - \mathrm{ev}_B)^{-1}\{0\} \cap (l_p - l_q)^{-1}\{0\} \cap \{(D,g) \in \mathrm{X}(\alpha_{[F]}) | V(D,g)_0 = V(\alpha_{[F]})_0\}.$$

PROOF. Since $\sigma(\alpha_{[F]})$ has codimension one in X (see corollary 3.3.19) and since the intersection product imposes only condition on the edges inside the loop, a maximal combinatorial type β in X has, compared to $\alpha_{[F]}$, either additional edges inside or outside the loop, but not both. Hence, the claim follows from lemmata 3.3.16, 3.3.21 and 3.3.23. □

The following proposition is a combinatorial description of the balancing condition at the polyhedron $\sigma(\alpha_{[F]})$ of X containing curves of combinatorial type $\alpha_{[F]}$ which fulfill that the loop is closed. Remember that $\mathrm{X}(\alpha_{[F]})$ contains all curves whose combinatorial type specializes to $\alpha_{[F]}$ and in the specialization no edges outside the loop are contracted. Remember moreover that it holds $\{F(\mathcal{P}')_1, F(\mathcal{P}')_2\} \in \mathcal{P}'$ (see notation 3.3.22) for all partitions $\mathcal{P}' \in \mathbf{P}_{\mathcal{P}_0(\alpha_{[F]})}$ of the flags at the loop of $\alpha_{[F]}$.

Proposition 3.3.25
Let $\alpha_{[F]} \leq \alpha_1, \ldots, \alpha_i$ be the combinatorial types in $\mathrm{X}(\alpha_{[F]})$

- that have, compared to $\alpha_{[F]}$, only additional edges inside the loop,
- that fulfill $\mathcal{A} = \{\mathbf{F}_v\}_{v \in \mathbf{V}_0(\alpha_j)}$ and
- that fulfill that the loop of α_j can be closed, i.e. there exists a curve (D,g) of combinatorial type α_j that fulfills $\mathrm{ev}_A(D,g) = \mathrm{ev}_B(D,g)$ and $l_p(D,g) = l_q(D,g)$.

For $j \in [i]$, let v^{α_j} be a representative of the primitive normal vector $u_{\sigma(\alpha_j)/\sigma(\alpha_{[F]})}$ that has no components outside the loop, i.e. only edges inside the loop. Then it holds

$$\operatorname{ind}(\alpha_{[F]}) \cdot \sum_{j=1}^{i} v^{\alpha_j} + \sum_{\mathcal{P}' \in \mathbf{P}_{\mathcal{P}_0(\alpha_{[F]})}} \operatorname{ind}((\alpha_{[F]})_{\mathcal{P}'}) \cdot \left(\sum_{\substack{j_1 \in \Delta_{[F]}(F(\mathcal{P}')_1), \\ j_2 \in \Delta_{[F]}(F(\mathcal{P}')_2)}} v_{\{j_1, j_2\}} \right) \in \mathrm{W}(\sigma(\overline{\alpha_{[F]}})),$$

where $\overline{\alpha_{[F]}}$ is the specialization of $\alpha_{[F]}$ in which precisely the edges outside the loop are contracted and where $\mathrm{W}(\sigma(\overline{\alpha_{[F]}}))$ is the smallest linear space containing all curves of combinatorial type $\overline{\alpha_{[F]}}$ such that the loop is closed.

Proof. The polyhedron $\sigma(\alpha_{[F]})$ has codimension one in X, see corollary 3.3.19. Hence, X is balanced at $\sigma(\alpha_{[F]})$. It follows from the two previous lemmata that the facets of X are given by $\sigma(\alpha_1), \ldots, \sigma(\alpha_i)$ and by $\sigma((\alpha_{[F]})_{\mathcal{P}'})$, $\mathcal{P}' \in \mathbf{P}_{\mathcal{P}_0(\alpha_{[F]})}$. Their weight is $2 \cdot \operatorname{ind}(\alpha_j) = 2 \cdot \operatorname{ind}(\alpha_{[F]})$ and $2 \cdot \operatorname{ind}((\alpha_{[F]})_{\mathcal{P}'})$, respectively. A representative of the normal vector $u_{\sigma((\alpha_{[F]})_{\mathcal{P}'})/\sigma(\alpha_{[F]})}$ is given by

$$v_{\Delta_{[F]}(F(\mathcal{P}')_1) \cup \Delta_{[F]}(F(\mathcal{P}')_2)},$$

where $F(\mathcal{P}')_1$ and $F(\mathcal{P}')_2$ are the two distinguished flags at the loop of $\alpha_{[F]}$ that fulfill

$$\{F(\mathcal{P}')_1, F(\mathcal{P}')_2\} \in \mathcal{P}',$$

see construction 3.2.23 and notation 3.3.22. Another representative is given by

$$\sum_{\substack{j_1 \in \Delta_{[F]}(F(\mathcal{P}')_1), \\ j_2 \in \Delta_{[F]}(F(\mathcal{P}')_2)}} v_{\{j_1, j_2\}},$$

since

$$v_{\Delta_{[F]}(F(\mathcal{P}')_1) \cup \Delta_{[F]}(F(\mathcal{P}')_2)} = \left(\sum_{\substack{j_1 \in \Delta_{[F]}(F(\mathcal{P}')_1), \\ j_2 \in \Delta_{[F]}(F(\mathcal{P}')_2)}} v_{\{j_1, j_2\}} \right) + v_{\Delta_{[F]}(F(\mathcal{P}')_1)} + v_{\Delta_{[F]}(F(\mathcal{P}')_2)}.$$

Since X is balanced at $\sigma(\alpha_{[F]})$, it follows that

$$s = \sum_{j \in [i]} \operatorname{ind}(\alpha_{[F]}) \cdot v_{\sigma(\alpha_i)/\sigma(\alpha_{[F]})} + \sum_{\mathcal{P}' \in \mathbf{P}_{\mathcal{P}_0(\alpha_{[F]})}} \operatorname{ind}((\alpha_{[F]})_{\mathcal{P}'}) \cdot \left(\sum_{\substack{j_1 \in \Delta_{[F]}(F(\mathcal{P}')_1), \\ j_2 \in \Delta_{[F]}(F(\mathcal{P}')_2)}} v_{\{j_1, j_2\}} \right) \in \mathrm{W}(\sigma(\alpha_{[F]})).$$

Since the curves v^{α_j}, which are representatives of the normal vectors $u_{\sigma(\alpha_j)/\sigma(\alpha_{[F]})}$, were chosen in a way that they contain no components outside the loop and since

$$\sum_{\substack{j_1 \in \Delta_{[F]}(F(\mathcal{P}')_1), \\ j_2 \in \Delta_{[F]}(F(\mathcal{P}')_2)}} v_{\{j_1, j_2\}}$$

contains no edges that appear outside the loop of $\alpha_{[F]}$, also s contains no edges outside the loop, i.e. $s \in \mathrm{W}(\sigma(\overline{\alpha_{[F]}}))$. □

As a next step, we study the set of maximal fine combinatorial types in $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)^{\mathrm{reg}}$ that have - compared to $(\alpha, \leq)$ - only additional edges outside the loop. Remember that

$$\dim V_0(\alpha)/V_{<0}(\alpha) = \#\mathbf{FS}_0(\alpha) - \#\mathbf{P}_0(\alpha) - 1,$$

that $(\alpha, \leq)$ is a regular fine combinatorial type of codimension one in $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ and that $[F]$ is a flag in the loop of α that fulfills $\mathrm{v}([F]) \neq 0$.

The following notation is used to describe the well-spaced resolutions of $(\alpha_{[F]}, \leq_{[F]})$ that have only additional edges outside the loop. For simplifying notation, we set $(\gamma, \leq) := (\alpha_{[F]}, \leq_{[F]})$.

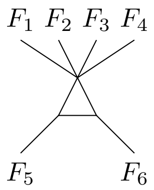
We are interested in the set $\mathbf{W}(\gamma) \subset \mathbf{P}^0(\gamma)$ defined below because a fine combinatorial type $(\gamma_{\mathcal{W}}, \leq_{\mathcal{W}})$ is well-spaced if and only if $\mathcal{W} \in \mathbf{W}(\gamma)$, see lemma 3.3.28.

Notation 3.3.26 ($\mathbf{W}(\gamma)$)
Remember that $\mathcal{A}$ is the unique element of $\text{Par}_0(\gamma)$ that is strictly finer than $\{\mathbf{F}_v\}_{v\in\mathbf{V}_0(\gamma)}$. Define

$$\mathbf{W}(\gamma) = \{\mathcal{W} \in \mathbf{P}^0(\gamma) | W \neq \mathcal{P}_0(\gamma), \#(A \cap W) \leq 1 \,\forall\, W \in \mathcal{W} \,\forall A \in \mathcal{A}\},$$

where $\mathcal{P}_0(\gamma) = \{\{[F]\}\}_{[F]\in\mathbf{FS}_0(\gamma)}$. We just write $\mathbf{W}$ if no confusion can occur.

Example 3.3.27
Here is an example of $\mathbf{W}(\gamma)$ for the following combinatorial type γ in $\mathbb{R}^4$ for which we assume $\dim V(\gamma)_{<0} = 2$ and $\dim V(\gamma)_0 = 4$.



- Assume that $\mathcal{A} = \{\{F_i\}|i \in [6]\}$ and that $\text{v}(F_i) \notin V(\gamma)_{<0})$ for $i \in [4]$. Since there do not exist vertices outside the loop, γ is a well-spaced combinatorial type and it has codimension one in the corresponding moduli space. It holds
$$\mathbf{W}(\gamma) = \mathbf{P}^0(\gamma) \setminus \{\mathcal{P}_0(\gamma)\}.$$
The fine combinatorial types $(\gamma_{\mathcal{W}}, \leq_{\mathcal{W}})$ for $\mathcal{W} \in \mathbf{W}(\gamma)$ are shown in figure 13, where $i_1, i_2 \in [4]$ with $i_1 \neq i_2$ and $i \in [4]$. The bounded edges outside the loop in the second curve from the left both have the same length.

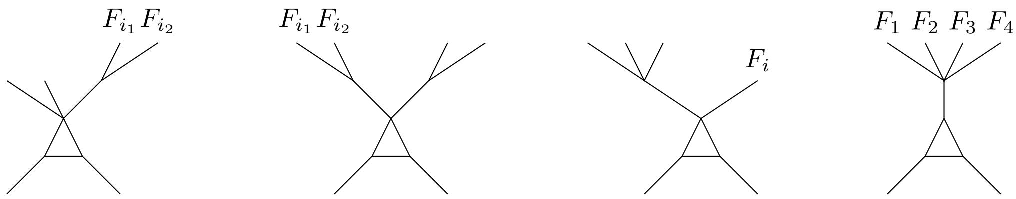


FIGURE 13.

- Assume that $\mathcal{A} = \{\{F_1, F_3\}, \{F_2, F_4\}, \{F_5\}, \{F_6\}\}$ and $\text{v}(F_i) \notin V(\alpha_{[F]})$ for $i \in [4]$. Again γ is well-spaced and has codimension one in the corresponding moduli space. The combinatorial types $(\gamma_{\mathcal{W}}, \leq_{\mathcal{W}})$ for $\mathcal{W} \in \mathbf{W}(\gamma)$ are shown in figure 14, where $i_1 \in \{1, 3\}$ and $i_2 \in \{2, 4\}$. The bounded edges outside the loop in the curve on the left both have the same length.

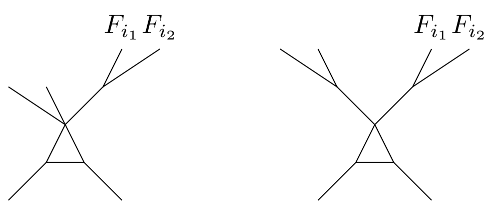


FIGURE 14.

Lemma 3.3.28
Let $\mathcal{W} \in \mathbf{P}^0(\gamma)$ be a partition of the flags at the loop of γ that is finer than the partition given by the vertices in the loop of γ. Then $(\gamma_{\mathcal{W}}, \leq_{\mathcal{W}})$ is well-spaced if and only if $\mathcal{W} \in \mathbf{W}(\gamma)$.

Proof. Let $(\beta, \leq_\beta)$ be a well-spaced fine combinatorial type of curves that specializes to $(\gamma, \leq)$ and that has only additional edges outside the loop compared to $(\gamma, \leq)$. Then there exists a unique $\mathcal{P} \in \mathbf{P}^0(\gamma)$ such that $(\beta, \leq_\beta) = (\gamma_{\mathcal{P}}, \leq_{\mathcal{P}})$. Assume that there exists $P \in \mathcal{P}$ and $A \in \mathcal{A}$ with $\#(A \cap P) \geq 2$. Denote by $[F_1], [F_2]$ two different elements of $A \cap P$. Since $\dim V(\gamma)_0/V(\gamma)_{<0} = \#\mathbf{FS}_0(\gamma) - \#\mathbf{P}_0(\gamma) - 1$ and since there exists a vertex $[v] \in \mathbf{V}_0$ such that $[F_1], [F_2] \in \mathbf{F}_{[v]}$, it holds

$$\left\langle V(\gamma)_{<0} \cup \{\mathrm{v}([F]) | [F] \in \mathbf{FS}_0(\gamma) \setminus \{[F_1], [F_2]\}\}\right\rangle \subsetneq V(\gamma)_0.$$

Hence, there exists a hyperplane $H \subset \mathbb{R}^m$ that contains all $\mathrm{v}([F])$, $[F] \in \mathbf{FS}_0(\gamma)$, except $\mathrm{v}([F_1])$ and $\mathrm{v}([F_2])$. This is a contradiction to the well-spacedness of $(\gamma_{\mathcal{P}}, \leq_{\mathcal{P}})$.
Let now $\mathcal{W} \in \mathbf{W}(\gamma)$ and let $(C_{\mathcal{W}}, h_{\mathcal{W}})$ be of fine combinatorial type $(\gamma_{\mathcal{W}}, \leq_{\mathcal{W}})$. If there exists no vertex outside the loop of $\gamma_{\mathcal{W}}$, $(\gamma_{\mathcal{W}}, \leq_{\mathcal{W}})$ is well-spaced. So assume that there exists a vertex outside the loop of $\gamma_{\mathcal{W}}$ and let the minimal distance of a vertex in $C_{\mathcal{W}}$ to the loop be d. The combinatorics of $C_{\mathcal{W}}$ at distance d to the loop is given by the partition $\mathcal{W}$. A point $P \in \mathrm{supp}(C_{\mathcal{W}})$ with distance d to the loop corresponds to an element $W \in \mathcal{W}$ and the flags behind the point p correspond to the flags $[F] \in W$, in particular these corresponding flags have the same weighted direction vector.
There exist pairwise different λ_A, $A \in \mathcal{A}$, such that

$$\sum_{A \in \mathcal{A}} \lambda_A \cdot \sum_{[F] \in A} \mathrm{v}^\omega([F]) \in V(\gamma)_{<0}.$$

It holds $\#A \cap W \leq 1$ for all $A \in \mathcal{A}$ and all $W \in \mathcal{W}$. Due to the balancing condition at the points $p \in \mathbf{V}_d(C_{\mathcal{W}})$ $(\sum_{F \in \mathbf{F}_p} \mathrm{v}^\omega(F) \in V_{<d}(C_{\mathcal{W}}))$, there exist hence pairwise different $\lambda_F \in \mathbb{R}$ such that

$$\sum_{[F] \in \mathbf{FS}_0(\gamma)} \lambda_W \cdot \sum_{[F] \in W} \mathrm{v}^\omega([F]) \in V(C_{\mathcal{W}}, h_{\mathcal{W}})_{<d} \supset V(\gamma)_{<0}.$$

It follows that $(C_{\mathcal{W}}, h_{\mathcal{W}})$ and $(\alpha_{\mathcal{W}}, \leq_{\mathcal{W}})$ are well-spaced. ☐

Lemma 3.3.29
Let $l, k, a \in \mathbb{N}$ with $1, a \leq l \leq k$. Then it holds

$$\sum_{i=0}^{l} (-1)^i \binom{l}{i} \frac{k!}{(k-i)!} (k+l-i-a)! = \begin{cases} 0, & \text{if } a > 0 \\ k! \cdot l!, & \text{if } a = 0. \end{cases}$$

Proof. We denote the constant term of a Laurent polynomial by $\mathrm{CT}(\cdot)$. It holds

$$\begin{aligned}
&\sum_{i=0}^{l} (-1)^i \binom{l}{i} \frac{k!}{(k-i)!} (k+l-i-a)! = \sum_{i=0}^{l} (-1)^i \binom{l}{i} \binom{k+l-i-a}{k-i} \cdot k! \cdot (l-a)! \\
= \;& \mathrm{CT}\left(\sum_{i=0}^{l} (-1)^i \binom{l}{i} \frac{(1+y)^{k+l-i-a}}{y^{k-i}} \right) \cdot k! \cdot (l-a)! \\
= \;& \mathrm{CT}\left(\sum_{i=0}^{l} (-1)^i \binom{l}{i} \frac{y^i}{(1+y)^i} \frac{(1+y)^{k+l-a}}{y^k} \right) \cdot k! \cdot (l-a)! \\
= \;& \mathrm{CT}\left(\left(1 - \frac{y}{(1+y)}\right)^l \frac{(1+y)^{k+l-a}}{y^k} \right) \cdot k! \cdot (l-a)! \\
= \;& \mathrm{CT}\left(\left(\frac{1}{1+y}\right)^l \frac{(1+y)^{k+l-a}}{y^k} \right) \cdot k! \cdot (l-a)! = \mathrm{CT}\left(\left(\frac{(1+y)^{k-a}}{y^k} \right) \right) \cdot k! \cdot (l-a)! \\
= \;& \begin{cases} 0, & \text{if } a > 0 \\ k! \cdot l!, & \text{if } a = 0. \end{cases}
\end{aligned}$$

□

The following notation appears in the proof of proposition 3.3.13 and in the next two lemmata needed for this proof.

Notation 3.3.30 ($n(\mathcal{W},P)$, $p(\mathcal{W},P)$, $\omega(\mathcal{W},\mathcal{P})$)
Let $\mathcal{W} \leq \mathcal{P} \in \mathbf{P}^0(\gamma)$. For $P \in \mathcal{P}$ we define

a) $n(\mathcal{W},P) = \#\{W \in \mathcal{W} | W \subset P\}$ as the number of elements of $\mathcal{W}$ contained in P,
b) $p(\mathcal{W},P) = (-1)^{n(\mathcal{W},P)-1}(n(\mathcal{W},P)-1)!$.
c) $\omega(\mathcal{W},\mathcal{P}) = \prod_{P\in\mathcal{P}} p(\mathcal{W},P)$.

Lemma 3.3.31
Let $\mathcal{P} \in \mathbf{P}^0(\gamma)$ such that $\#I(\mathcal{A},\mathcal{P}) > \#\mathcal{P}+1$, i.e. the coarsest common refinement of $\mathcal{P}$ and of the distinguished partition $\mathcal{A} \in \text{Par}_0$ has at least two elements more than $\mathcal{P}$. Choose two flags $F_1, F_2 \in \mathbf{FS}_0$ at the loop of γ. Then it holds

$$\sum_{\substack{\mathcal{W}\in\mathbf{W}(\gamma):\mathcal{W}\leq\mathcal{P},\\ \exists W\in\mathcal{W}:F_1,F_2\in W}} \omega(\mathcal{W},\mathcal{P}) = 0,$$

i.e. the sum runs over all partitions $\mathcal{W} \in \mathbf{W}$ which are finer than $\mathcal{P}$ and which fulfill that there exists an element $W \in \mathcal{W}$ that contains F_1 and F_2.

Example 3.3.32
Let us illustrate the previous lemma with the example of the combinatorial type γ in $\mathbb{R}^4$ shown in figure 15. Assume $\dim V(\gamma)_{<0} = 2$ and $V(\gamma)_0 = \mathbb{R}^4$. Assume moreover

$$\mathcal{A} = \{\{F_1,F_3\},\{F_i\}|i \in \{2,4,5,6\}\} \text{ and } \mathcal{P} = \{\{F_1,F_2,F_3,F_4\},\{F_5\},\{F_6\}\}.$$

It holds $I(\mathcal{A},\mathcal{P}) = \mathcal{A}$ and hence $\#I(\mathcal{A},\mathcal{P}) = 5 = \#\mathcal{P}+2$. Let $\mathbf{W}_\mathcal{P} \subset \mathbf{W}(\gamma)$ be the set of partitions

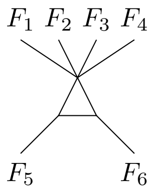
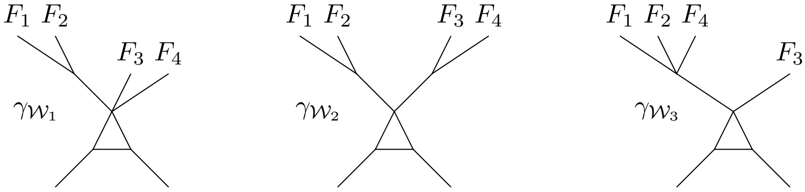


FIGURE 15.

$\mathcal{W} \in \mathbf{W}(\gamma)$ that are finer than $\mathcal{P}$ (i.e. $\mathcal{W} \leq \mathcal{P}$) and that fulfill that there exists $W \in \mathcal{W}$ with $F_1, F_2 \in W$. It holds $\mathbf{W}_\mathcal{P} = \{\mathcal{W}_i\}_{i\in[3]}$ with

- $\mathcal{W}_1 = \{\{F_1,F_2\}\} \cup \{\{F_i\}|i \in \{3,4,5,6\}\}$,
- $\mathcal{W}_2 = \{\{F_1,F_2\},\{F_3,F_4\},\{F_5\},\{F_6\}\}$ and
- $\mathcal{W}_3 = \{\{F_1,F_2,F_4\}\} \cup \{\{F_i\}|i \in \{3,5,6\}\}$.

The combinatorial types $\gamma_{\mathcal{W}_i}$ are illustrated below.



The only element $P \in \mathcal{P}$ with $\#P > 1$ is $P = \{1, 2, 3, 4\}$. Thus, it holds

$$\begin{aligned}\sum_{\mathcal{W} \in \mathbf{W}_{\mathcal{P}}} \omega(\mathcal{W}, \mathcal{P}) &= \sum_{i \in [3]} \prod_{P \in \mathcal{P}} \Big((-1)^{n(\mathcal{W}_i, P)-1} (n(\mathcal{W}_i, P) - 1)! \Big) \\ &= (-1)^{3-1}(3-1)! + 2 \cdot (-1)^{2-1}(2-1)! \\ &= 0\end{aligned}$$

In the notation of the proof below, it holds $Q = \{F_1, F_2, F_3, F_4\}$, $A_1 = \{F_1, F_3\}$, $A_2 = \{F_2\}$ and $A = \{F_4\}$.

Proof of 3.3.31. Denote by $\mathbf{W}_{\mathcal{P}}$ the subset of $\mathbf{W}(\gamma)$ that consists of all $\mathcal{W} \in \mathbf{W}(\gamma)$ which refine $\mathcal{P}$ and for which there exists $W \in \mathcal{W}$ with $F_1, F_2 \in W$, i.e. $\mathbf{W}_{\mathcal{P}}$ is the set over which the sum in the statement of the lemma runs. Let $\mathcal{W} \in \mathbf{W}_{\mathcal{P}} \subset \mathbf{W}(\gamma)$. Then it holds $\#(A \cap W) \leq 1$ for all $A \in \mathcal{A}$ and all $W \in \mathcal{W}$. Since there exists $W \in \mathcal{W}$ with $F_1, F_2 \in W$, there exist different $A_1, A_2 \in \mathcal{A}$ such that $F_1 \in A_1$ and $F_2 \in A_2$. Due to $\#I(\mathcal{A}, \mathcal{P}) > \#\mathcal{P} + 1$, there exist moreover $A_1, A_2 \neq A \in \mathcal{A}$ and $Q \in \mathcal{P}$ such that $\emptyset \neq A \cap Q \subsetneq Q$. Define

$$\mathbf{W}_{A \cap Q} = \Big\{ \mathcal{W} \in \mathbf{W}_{\mathcal{P}} | \forall F \in A \cap Q : \{F\} \in \mathcal{W} \Big\},$$

i.e. all elements of $A \cap Q$ are contained in separate sets of order one. For $\mathcal{W} \in \mathbf{W}_{A \cap P}$ define $\mathbf{W}^{\mathcal{W}}$ as the set

$$\Big\{ \mathcal{W} \leq \mathcal{Y} \in \mathbf{W}_{\mathcal{P}} | \forall Y \in \mathcal{Y} : (\exists W \in \mathcal{W} \text{ and } F \in A \cap Q : Y = W \cup \{F\} \text{ or } Y = W \text{ or } Y = \{F\}) \Big\},$$

i.e. $\mathcal{W}^{\mathcal{W}}$ contains the partitions $\mathcal{W} \leq \mathcal{Y} \leq \mathcal{P}$ contained in $\mathbf{W}_{\mathcal{P}}$ which are constructed from $\mathcal{W} \in \mathbf{W}_{A \cap Q}$ by uniting elements of $\mathcal{W}$ contained in $Q \setminus A$ with elements of $\mathcal{W}$ contained in $A \cap Q$. Remember that for $\mathcal{W}' \in \mathbf{W}_{\mathcal{P}} \subset \mathbf{W}(\gamma)$ it holds $\#(A' \cap W) \leq 1$ for all $A' \in \mathcal{A}$ and all $W \in \mathcal{W}'$. Since all elements of $\mathbf{W}_{\mathcal{P}}$ are finer than $\mathcal{P}$ and since $F_1, F_2 \notin A$, it holds

$$\mathbf{W}_{\mathcal{P}} = \dot{\bigcup}_{\mathcal{W} \in \mathbf{W}_{A \cap Q}} \mathbf{W}^{\mathcal{W}}.$$

Fix an arbitrary $\mathcal{W} \in \mathbf{W}_{A \cap Q}$. It holds $l := \#(A \cap Q) > 0$ and $k := \{W \in \mathcal{W} | W \subset Q\} - l \geq 1$ because $\emptyset \subsetneq A \cap Q \subsetneq Q$. k is the number of elements of $\mathcal{W}$ that are contained in Q and whose intersection with A is empty and $k + l$ elements of $\mathcal{W}$ are contained in Q. Assume $l \leq k$. All elements of $\mathbf{W}^{\mathcal{W}}$ are of the following form:

Choose $0 \leq i \leq l$ elements $\{F_1\}, \ldots, \{F_i\} \in \mathcal{W}$ with $\{F_1, \ldots, F_i\} \subset A \cap Q$ and take the union $\{F_j\} \cup W_j$ with pairwise different $W_1, \ldots W_i \in \mathcal{W}$ that fulfill $W_j \subset Q$ and $W_j \cap A = \emptyset$ $(j \in [i])$.

There are $\binom{l}{i} \cdot \frac{k!}{(k-i)!}$ possibilities to do this. The resulting partition $\mathcal{Y} \in \mathbf{W}^{\mathcal{W}}$ has precisely $k + l - i$ elements which are subsets of Q. We get

$$\begin{aligned}&\sum_{\mathcal{Y} \in \mathbf{W}^{\mathcal{W}}} \omega(\mathcal{Y}, \mathcal{P}) \\ = \;& \sum_{\mathcal{Y} \in \mathbf{W}^{\mathcal{W}}} \prod_{P \in \mathcal{P}} (-1)^{n(\mathcal{Y}, P)-1} (n(\mathcal{Y}, P) - 1)! \\ = \;& \sum_{i=0}^{l} \binom{l}{i} \frac{k!}{(k-i)!} \cdot \left[(-1)^{(k+l-i)-1} \cdot ((k+l-i)-1)! \left(\prod_{\substack{P \in \mathcal{P}, \\ P \neq Q}} p(\mathcal{W}, P) \right) \right] \\ = \;& (-1)^{k+l-1} \cdot \left(\prod_{\substack{P \in \mathcal{P}, \\ P \neq Q}} p(\mathcal{W}, P) \right) \cdot \sum_{i=0}^{l} \left((-1)^i \cdot \binom{l}{i} \frac{k!}{(k-i)!} \cdot (k - l - i - 1)! \right) \\ \overset{3.3.29}{=} \;& 0.\end{aligned}$$

If $k \leq l$, we reverse the roles: We choose $0 \leq i \leq k$ elements $W_1, \ldots, W_i \in \mathcal{W}$ that fulfill $W_1, \ldots, W_i \subset Q$ and $W_1, \ldots, W_i \cap (A \cap Q) = \emptyset$ and take the union $W_j \cup \{F_j\}$ with i elements $\{F_1\}, \ldots, \{F_i\} \in W$ that fulfill $F_j \in A \cap Q$, $j \in [i]$. Then we get the same result.
The claim follows because

$$\mathbf{W}_{\mathcal{P}} = \dot{\bigcup}_{\mathcal{W} \in \mathbf{W}_{A \cap Q}} \mathbf{W}^{\mathcal{W}}.$$

□

Remark 3.3.33
Assume that $\mathcal{P}' \in \mathbf{P}^0(\gamma)$ and $\#I(\mathcal{P}', \mathcal{A}) = \mathcal{P}' + 1$. Then there exist two distinguished elements $P_1, P_2 \in I(\mathcal{P}', \mathcal{A})$ and different $A_1, A_2 \in \mathcal{A}$ which fulfill $(P_1 \cup P_2) \in \mathcal{P}'$, $P_1 \subset A_1$ and $P_2 \subset A_2$. For all elements $P_1, P_2 \neq P \in I(\mathcal{A}, \mathcal{P}')$, there exist $P' \in \mathcal{P}'$ and $A \in \mathcal{A}$ with $P = P' \subset A$.

Lemma 3.3.34
Let $\mathcal{P}' \in \mathbf{P}^0(\gamma)$ such that $\#I(\mathcal{A}, \mathcal{P}') = \#\mathcal{P}' + 1$. Denote the two distinguished elements of $I(\mathcal{A}, \mathcal{P}')$ by P_1 and P_2, see the remark above. Let $F_1 \in P_1$ and $F_2 \in P_2$. Then it holds

$$\sum_{\substack{\mathcal{W} \in \mathbf{W}, \mathcal{W} \leq \mathcal{P}', \\ \exists W \in \mathcal{W}: F_1, F_2 \in W}} \omega(\mathcal{W}, \mathcal{P}') = \prod_{P \in I(\mathcal{A}, \mathcal{P}')} (-1)^{\#P-1}(\#P-1)!.$$

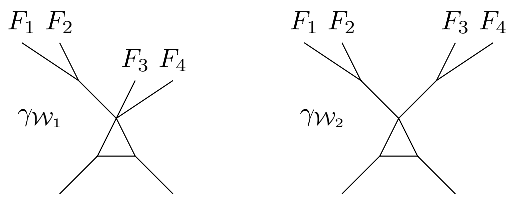
Example 3.3.35
Let us illustrate the previous lemma with the example of the combinatorial type γ in $\mathbb{R}^4$ from example 3.3.32. Assume again $\dim V(\gamma)_{<0} = 2$ and $V(\gamma)_0 = \mathbb{R}^4$. Assume this time that

$$\mathcal{A} = \{\{F_1, F_3\}, \{F_2, F_4\}, \{F_5\}, \{F_6\}\} \text{ and } \mathcal{P}' = \{\{F_1, F_2, F_3, F_4\}, \{F_5\}, \{F_6\}\}.$$

It holds $I(\mathcal{A}, \mathcal{P}') = \mathcal{A}$ and hence $\#I(\mathcal{A}, \mathcal{P}') = 4 = \#\mathcal{P}' + 1$.
Let $\mathbf{W}_{\mathcal{P}'} \subset \mathbf{W}(\gamma)$ be the set of partitions in $\mathcal{W} \in \mathbf{W}(\gamma)$ that are finer than $\mathcal{P}'$ (i.e. $\mathcal{W} \leq \mathcal{P}'$) and that fulfill that there exists $W \in \mathcal{W}$ with $F_1, F_2 \in W$. It holds $\mathcal{W}_{\mathcal{P}'} = \{\mathcal{W}_i\}_{i \in [2]}$ with

$$\mathcal{W}_1 = \{\{1,2\}\} \cup \{\{F_i\} | i \in \{3,4,5,6\}\} \text{ and } \mathcal{W}_2 = \{\{F_1, F_2\}, \{F_3, F_4\}, \{F_5\}, \{F_6\}\}.$$

The combinatorial types $\gamma_{\mathcal{W}_1}$, $\gamma_{\mathcal{W}_2}$ are illustrated below.



The only element $P \in \mathcal{P}'$ with $\#P > 1$ is $P = \{1,2,3,4\}$. It holds

$$\begin{aligned}
\sum_{\mathcal{W} \in \mathbf{W}_{\mathcal{P}'}} \omega(\mathcal{W}, \mathcal{P}') &= \sum_{i \in [2]} \prod_{P \in \mathcal{P}'} \Big((-1)^{n(\mathcal{W}_i, P)-1}(n(\mathcal{W}_i, P) - 1)!\Big) \\
&= (-1)^{3-1}(3-1)! + (-1)^{2-1}(2-1)! \\
&= 1 \\
&= \prod_{P \in I(\mathcal{A}, \mathcal{P}')} (-1)^{\#P-1}(\#P-1)!
\end{aligned}$$

In the notation of the proof below, it holds $Q = \{F_1, F_2, F_3, F_4\}$, $P_1 \subset A_1 = \{F_1, F_3\}$ and $P_2 \subset A_2 = \{F_2, F_4\}$.

Proof of lemma 3.3.34. Define

$$\mathbf{W}_{\mathcal{P}'} := \Big\{\mathcal{W} \in \mathbf{W} | \mathcal{W} \leq \mathcal{P}', \ \exists W \in \mathcal{W} : F_1, F_2 \in W\Big\}$$

as the set over which the sum in the lemma runs. Define $Q = P_1 \cup P_2 \in \mathcal{P}'$ and let $A_1, A_2 \in \mathcal{A}$ be the elements that fulfill $P_1 \subset A_1$ and $P_2 \subset A_2$.

According to lemma 3.3.28, it holds that $\#A \cap W \leq 1$ for all $W \in \mathcal{W} \in \mathbf{W}_{\mathcal{P}'}$ and $A \in \mathcal{A}$. Let $\mathcal{W} \in \mathbf{W}_{\mathcal{P}'}$. Since $\mathcal{W} \leq \mathcal{P}'$ and since for all $Q \neq P \in \mathcal{P}'$ there exists $A \in \mathcal{A}$ with $P \subset A$, it holds for all flags $F \in \mathbf{FS}_0(\gamma) \setminus Q$ at the loop of γ that are not contained in Q that $\{F\} \in \mathcal{W}$. Set $l_1 := \#P_1 - 1$, $l_2 := \#P_2 - 1$, i.e. $\#Q = l_1 + l_2 + 2$. Assume without loss of generality $l_1 \leq l_2$. All elements $\mathcal{W} \in \mathbf{W}_{\mathcal{P}'}$ are of the following form:

We start with the partition

$$\{\{F\} | F \in \mathbf{FS}_0 \setminus \{F_1, F_2\}\} \cup \{\{F_1, F_2\}\} \in \mathcal{W}_{\mathcal{P}'}.$$

Then we choose $0 \leq i \leq l_1$ elements

$$F_1^1, \ldots, F_i^1 \in P_1 \setminus \{F_1\} = (A_1 \cap Q) \setminus \{F_1\}$$

and unite them with pairwise different elements

$$F_1^2, \ldots, F_i^2 \in P_2 \setminus \{F_2\} = (A_2 \cap Q) \setminus \{F_2\}.$$

There are $\binom{l_1}{i} \cdot \frac{l_2!}{(l_2-i)!}$ possibilities to do this. The resulting partition has $l_1 + l_2 + 1 - i$ elements that are contained in Q. It follows

$$\begin{aligned}
&= \sum_{\mathcal{W} \in \mathbf{W}_{\mathcal{P}'}} \omega(\mathcal{W}, \mathcal{P}') \\
&= \sum_{i=0}^{l_1} \binom{l_1}{i} \frac{l_2!}{(l_2 - i)!} \left((-1)^{l_1 + l_2 - i} \cdot (l_1 + l_2 - i)! \prod_{P \in \mathcal{P}', Q \neq P} p(\mathcal{W}, P) \right) \\
&\overset{3.3.29}{=} (-1)^{l_1 + l_2} \cdot l_1! \cdot l_2! \cdot \prod_{P \in \mathcal{P}', Q \neq P} (-1)^{\#P - 1} (\#P - 1)! \\
&= \prod_{P \in I(\mathcal{A}, \mathcal{P}')} (-1)^{\#P - 1} (\#P - 1)!
\end{aligned}$$

□

PROOF OF PROPOSITION 3.3.13. Remember that $(\alpha, \leq)$ is a regular fine combinatorial type in $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ and that $[F] \in \mathbf{F}(\alpha)$ is a flag in the loop of α which fulfills $\mathrm{v}([F]) \neq 0$. $(\alpha, \leq)$ has codimension one and it holds

$$\dim V(\alpha)_0 / V(\alpha)_{<0} = \#\mathbf{FS}_0(\alpha) - \#\mathbf{P}_0(\alpha) - 1.$$

Remember that we denote $(\alpha_{[F]}, \leq_{[F]})$ by $(\gamma, \leq)$, for simplifying notation.

The set of maximal fine combinatorial types in $U(\gamma, \leq)$ that specialize to $(\gamma, \leq)$ and that, compared to $(\gamma, \leq)$, have only additional edges outside the loop, is given by

$$\{(\gamma_{\mathcal{W}}, \leq_{\mathcal{W}}) | \mathcal{W} \in \mathbf{W}(\gamma)\},$$

see lemma 3.3.28. The weight of a facet of $U(\gamma, \leq)$ containing curves of fine combinatorial type $(\gamma_{\mathcal{W}}, \leq_{\mathcal{W}})$ is $\omega(\gamma_{\mathcal{W}})$, see definition 3.3.2. For $\mathcal{W} \in \mathbf{W}(\gamma)$, a representative of the normal vector $u_{(\gamma_{\mathcal{W}}, \leq_{\mathcal{W}})/(\gamma, \leq)}$ is given by

$$\sum_{W \in \mathcal{W}} v_{\Delta_{[F]}(W)},$$

where $\Delta_{[F]}(W)$ is the set of labels in $\Delta_{[F]}$ that lie behind the flags $G \in W$ seen from the loop and where $v_{\Delta_{[F]}(W)}$ is a curve that has only one edge of length one and such that all leaves labeled by $i \in \Delta_{[F]}(W)$ sit at one vertex of the edge and all leaves labeled by $\Delta_{[F]} \setminus \Delta_{[F]}(W)$ sit at the other vertex, see definition 1.3.15 and construction 3.2.23.

For a partition $\mathcal{P}' \in \mathbf{P}^0(\gamma)$ of the flags at the loop of γ, $\#I(\mathcal{A}, \mathcal{P}') = \#\mathcal{P}'$ implies that $\mathcal{P}'$ is finer than $\mathcal{A}$. Moreover, $(\gamma_{\mathcal{P}'}, \leq_{\mathcal{P}'})$ is not well-spaced if $\mathcal{P}'$ is finer than $\mathcal{A}$ (except if $\mathcal{P}' = \{\{F\} | F \in \mathbf{FS}_0(\gamma)\} \notin \mathbf{W}(\gamma)$, but then $(\gamma_{\mathcal{P}'}, \leq_{\mathcal{P}'}) = (\gamma, \leq)$). We conclude that $\#I(\mathcal{A}, \mathcal{P}') \leq \#\mathcal{P}' + 1$ implies $\#I(\mathcal{A}, \mathcal{P}') = \#\mathcal{P}' + 1$ if there exists a "well-spaced" partition $\mathcal{W} \in \mathbf{W}(\gamma)$ that is finer than $\mathcal{P}'$, see lemma 3.3.28.

For $\mathcal{P} \in \mathbf{P}^0(\gamma)$ with $\mathcal{P} \leq \mathcal{A}$ define

$$\mathbf{P}_{\mathcal{P}}(\gamma) := \{\mathcal{P}' \in \mathbf{P}^0(\gamma) | \#I(\mathcal{A},\mathcal{P}') = \#\mathcal{P}'+1, \#I(\mathcal{P}',\mathcal{P}) = \#\mathcal{P}+1\},$$

i.e. $\mathbf{P}_{\mathcal{P}}(\gamma)$ consists of the partitions $\mathcal{P}' \in \mathbf{P}^0(\gamma)$ that arise from $\mathcal{P}$ by uniting two elements $P_1, P_2 \in \mathcal{P}$ whose flags all lie behind one vertex and for which exist two different $A_1, A_2 \in \mathcal{A}$ with $P_1 \subset A_1$ and $P_2 \subset A_2$. In particular the partitions in $\mathbf{P}_{\mathcal{P}}(\gamma)$ are finer than the partition $\{\mathbf{F}_{[v]}\}_{[v]\in\mathbf{V}_0(\gamma)}$ given by the vertices in the loop of γ.

For $\mathcal{P}' \in \mathbf{P}_{\mathcal{P}}(\gamma)$ we denote the two distinguished elements of $\mathcal{P}$ by $P(\mathcal{P}')_1$ and $P(\mathcal{P}')_2$ - they fulfill that there exist different $A_1, A_2 \in \mathcal{A}$ with $P(\mathcal{P}')_1 \subset A_1$ and $P(\mathcal{P}')_2 \subset A_2$ and that there exists a vertex $[v] \in \mathbf{V}_0(\gamma)$ such that $A_1, A_2 \subset \mathbf{F}_{[v]}(\gamma)$. Moreover, it holds $I(\mathcal{A},\mathcal{P}') = \mathcal{P}$ for all $\mathcal{P}' \in \mathbf{P}_{\mathcal{P}}(\gamma)$ and

$$\dot{\bigcup}_{\mathcal{P}\in\mathbf{P}^0(\gamma),\mathcal{P}\leq\mathcal{A}} \mathbf{P}_{\mathcal{P}}(\gamma) = \{\mathcal{P}' \in \mathbf{P}^0(\gamma) | \#I(\mathcal{A},\mathcal{P}') \leq \#\mathcal{P}'+1,\ \exists \mathcal{W} \in \mathbf{W} : \mathcal{W} \leq \mathcal{P}\}.$$

With $v_{\Delta(F_1,F_2)} := v_{\Delta_{[F]}(F_1)\cup\Delta_{[F]}(F_2)} - v_{\Delta_{[F]}(F_1)} - v_{\Delta_{[F]}(F_2)}$ for $F_1, F_2 \in \mathbf{FS}_0$ we get:

$$\begin{aligned}
&\sum_{\mathcal{W}\in\mathbf{W}} \#\operatorname{Aut}(\gamma)\cdot\omega(\gamma_{\mathcal{W}})\cdot\left(\sum_{W\in\mathcal{W}} v_{\Delta_{[F]}(W)}\right)\\
=\ &\sum_{\mathcal{W}\in\mathbf{W}} \sum_{\substack{\mathcal{P}\in\mathbf{P}^0(\gamma),\\ \mathcal{W}\leq\mathcal{P}}} \omega(\mathcal{W},\mathcal{P})\cdot\operatorname{ind}(\gamma_{\mathcal{P}})\cdot\left(\sum_{W\in\mathcal{W}} v_{\Delta_{[F]}(W)}\right)\\
\overset{1.3.16}{=}\ &\sum_{\substack{F_1,F_2\in\mathbf{FS}_0\\ F_1\neq F_2}} \sum_{\substack{\mathcal{P}\in\mathbf{P}^0(\gamma),\\ \#I(\mathcal{A},\mathcal{P})>\#\mathcal{P}+1}} \operatorname{ind}(\gamma_{\mathcal{P}})\left(\sum_{\substack{\mathcal{W}\in\mathbf{W},\mathcal{W}\leq\mathcal{P},\\ \exists W\in\mathcal{W}:F_1,F_2\in W}} \omega(\mathcal{W},\mathcal{P})\right)\cdot v_{\Delta(F_1,F_2)}\\
&+\sum_{\substack{F_1,F_2\in\mathbf{FS}_0\\ F_1\neq F_2}} \sum_{\substack{\mathcal{P}\in\mathbf{P}^0(\gamma),\\ P\leq\mathcal{A}}} \sum_{\mathcal{P}'\in\mathbf{P}_{\mathcal{P}}(\gamma)} \operatorname{ind}(\gamma_{\mathcal{P}'})\left(\sum_{\substack{\mathcal{W}\in\mathbf{W},\mathcal{W}\leq\mathcal{P}',\\ \exists W\in\mathcal{W}:F_1,F_2\in W}} \omega(\mathcal{W},\mathcal{P}')\right)\cdot v_{\Delta(F_1,F_2)}\\
&+\sum_{[G]\in\mathbf{FS}_0}\left(\sum_{\mathcal{P}\in\mathbf{P}^0(\gamma)} \sum_{\substack{\mathcal{W}\in\mathbf{W},\\ \mathcal{W}\leq\mathcal{P}}} \omega(\mathcal{W},\mathcal{P})\right)\cdot v_{\Delta_{[F]}([G])}
\end{aligned}$$

The term

$$\sum_{[G]\in\mathbf{FS}_0}\left(\sum_{\mathcal{P}\in\mathbf{P}^0(\gamma)} \sum_{\substack{\mathcal{W}\in\mathbf{W},\\ \mathcal{W}\leq\mathcal{P}}} \omega(\mathcal{W},\mathcal{P})\right)\cdot v_{\Delta_{[F]}([G])}$$

lies in the vector space $\mathrm{W}(\gamma,\leq)$ spanned by the curves of combinatorial type $(\gamma,\leq)$ for which the loop is closed. Moreover, it follows from lemma 3.3.31 that

$$\sum_{\substack{F_1,F_2\in\mathbf{FS}_0\\ F_1\neq F_2}} \sum_{\substack{\mathcal{P}\in\mathbf{P}^0(\gamma),\\ \#I(\mathcal{A},\mathcal{P})>\#\mathcal{P}+1}} \operatorname{ind}(\gamma_{\mathcal{P}})\left(\sum_{\substack{\mathcal{W}\in\mathbf{W},\mathcal{W}\leq\mathcal{P},\\ \exists W\in\mathcal{W}:F_1,F_2\in W}} \omega(\mathcal{W},\mathcal{P})\right)\cdot v_{\Delta_{[F]}(F_1,F_2)} = 0.$$

For a partition $\mathcal{P} \in \mathbf{P}^0(\gamma)$ that is finer than $\mathcal{A}$, i.e. $\mathcal{P} \leq \mathcal{A}$, let $\mathcal{P}' \in \mathbf{P}_{\mathcal{P}}(\gamma)$. Hence, it holds $\#I(\mathcal{A},\mathcal{P}') = \#\mathcal{P}'+1$ and $I(\mathcal{A},\mathcal{P}') = \mathcal{P}$. Remember that we denote the two distinguished elements of $\mathcal{P}$ whose union is an element of $\mathcal{P}'$ by $P(\mathcal{P}')_1, P(\mathcal{P}')_2 \in \mathcal{P}$, i.e. $\{P(\mathcal{P}')_1, P(\mathcal{P}')_2\} \in \mathcal{P}'$. For $F_1 \in P(\mathcal{P}')_1$ and $F_2 \in P(\mathcal{P}')_2$ we get with lemma 3.3.34

$$\left(\sum_{\substack{\mathcal{W}\in\mathbf{W},\mathcal{W}\leq\mathcal{P}',\\ \exists W\in\mathcal{W}:F_1,F_2\in W}} \omega(\mathcal{W},\mathcal{P})\right)\cdot v_{\Delta(F_1,F_2)} = \left(\prod_{P\in\mathcal{P}} n(P)\right)\cdot v_{\Delta(F_1,F_2)}.$$

Otherwise, if two flags $F_1, F_2 \in \mathbf{FS}_0(\gamma)$ do not fulfill that they are contained in different elements of $\{P(\mathcal{P}')_1, P(\mathcal{P}')_2\} \subset \mathcal{P}$, there does not exist $\mathcal{W} \in \mathbf{W}(\gamma)$ and $W \in \mathcal{W}$ such that $F_1, F_2 \in W$. Therefore, it holds in this case

$$\left(\sum_{\substack{\mathcal{W}\in\mathbf{W}, \mathcal{W}\leq\mathcal{P}',\\ \exists W\in\mathcal{W}: F_1,F_2\in W}} \omega(\mathcal{W},\mathcal{P}) \right) \cdot v_{\Delta(F_1,F_2)} = 0.$$

It follows

$$\sum_{\substack{F_1,F_2\in\mathbf{FS}_0\\ F_1\neq F_2}} \sum_{\substack{\mathcal{P}\in\mathbf{P}^0(\gamma),\\ P\leq\mathcal{A}}} \sum_{\mathcal{P}'\in\mathbf{P}_{\mathcal{P}}(\gamma)} \operatorname{ind}(\gamma_{\mathcal{P}'}) \left(\sum_{\substack{\mathcal{W}\in\mathbf{W}, \mathcal{W}\leq\mathcal{P}',\\ \exists W\in\mathcal{W}: F_1,F_2\in W}} \omega(\mathcal{W},\mathcal{P}') \right) \cdot v_{\Delta(F_1,F_2)}$$

$$= \sum_{\substack{\mathcal{P}\in\mathbf{P}^0(\gamma),\\ \mathcal{P}\leq\mathcal{A}}} \sum_{\mathcal{P}'\in\mathbf{P}_{\mathcal{P}}(\gamma)} \left(\prod_{P\in\mathcal{P}} n(P) \right) \cdot \operatorname{ind}(\gamma_{\mathcal{P}'}) \cdot \left(\sum_{\substack{F_1\in P(\mathcal{P}')_1,\\ F_2\in P(\mathcal{P}'_2)}} v_{\Delta(F_1,F_2)} \right).$$

Let $(\gamma^1, \leq^1), \ldots, (\gamma^i, \leq^i)$ be the maximal combinatorial types that are coarser than $(\gamma, \leq)$ and that have only additional edges in the loop, $i \in \mathbb{N}$. For $j \in [i]$, let v_{γ^j} be representatives of the normal vectors $u_{(\gamma^j,\leq^j)/(\gamma,\leq)}$ that have no components outside the loop.
We conclude that

$$\sum_{j\in[i]} \#\operatorname{Aut}(\gamma^j) \cdot \omega(\gamma^j) \cdot v_{\gamma^j} + \sum_{\mathcal{W}\in\mathbf{W}(\gamma)} \#\operatorname{Aut}(\gamma_{\mathcal{W}}) \cdot \omega(\gamma_{\mathcal{W}}) \cdot \left(\sum_{W\in\mathcal{W}} v_{\Delta_{[F]}(W)} \right)$$

$$= \sum_{\substack{\mathcal{P}\in\mathbf{P}^0(\gamma),\\ \mathcal{P}\leq\mathcal{A}}} \left(\prod_{P\in\mathcal{P}} n(P) \right) \cdot \left[\operatorname{ind}(\gamma_{\mathcal{P}}) \cdot \left(\sum_{j=1}^{i} v_{\gamma^j} \right) + \sum_{\mathcal{P}'\in\mathbf{P}_{\mathcal{P}}(\gamma)} \operatorname{ind}(\gamma_{\mathcal{P}'}) \cdot \left(\sum_{\substack{F_1\in P(\mathcal{P}')_1,\\ F_2\in P(\mathcal{P}')_2}} v_{\Delta(F_1,F_2)} \right) \right]$$

$$+ \sum_{[G]\in\mathbf{FS}_0(\gamma)} \left(\sum_{\mathcal{P}\in\mathbf{P}^0(\gamma)} \sum_{\substack{\mathcal{W}\in\mathbf{W},\\ \mathcal{W}\leq\mathcal{P}}} \omega(\mathcal{W},\mathcal{P}) \right) \cdot v_{\Delta_{[F]}([G])}$$

Note that for all $\mathcal{P} \leq \mathcal{A}$ and $\mathcal{P}' \in \mathbf{P}_{\mathcal{P}'}$, it holds in the last term above

$$\sum_{\substack{F_1\in P(\mathcal{P}')_1,\\ F_2\in P(\mathcal{P}')_2}} v_{\Delta(F_1,F_2)} = \sum_{\substack{j_1\in\Delta_{[F]}(P(\mathcal{P}')_1),\\ j_2\in\Delta_{[F]}(P(\mathcal{P}')_2)}} v_{\{j_1,j_2\}}.$$

We will apply proposition 3.3.25 to $\gamma_{\mathcal{P}}$ for $\mathcal{P} \leq \mathcal{A}$ in order to show that

$$\operatorname{ind}(\gamma_{\mathcal{P}}) \cdot \left(\sum_{j=1}^{i} v_{\gamma^j} \right) + \sum_{\mathcal{P}'\in\mathbf{P}_{\mathcal{P}}(\gamma)} \operatorname{ind}(\gamma_{\mathcal{P}'}) \cdot \left(\sum_{\substack{F_1\in P(\mathcal{P}')_1,\\ F_2\in P(\mathcal{P}')_2}} v_{\Delta(F_1,F_2)} \right)$$

is an element of $\mathrm{W}(\gamma, \leq)$, which is the linear space spanned by the curves of fine combinatorial type $(\gamma, \leq)$ in which the loop is closed. The prerequisites of proposition 3.3.25 are fulfilled because the flags at the loop of γ fulfill only one relation in addition to the balancing condition at vertices in the loop of γ and because, due to $\mathcal{P} \leq \mathcal{A}$, the flags at the loop of $\gamma_{\mathcal{P}}$ fulfill only one relation modulo $V(\gamma_{\mathcal{P}})_{<0}$ in addition to balancing at the vertices in the loop of $\gamma_{\mathcal{P}}$. So let $\mathcal{P} \leq A$.

v_{γ^j} is not only a representative of the normal vector $u_{(\gamma^j,\leq^j)/(\gamma,\leq)}$ but also of the normal vector $u_{\sigma(\gamma^j_{\mathcal{P}})/\sigma(\gamma_{\mathcal{P}})}$, where we use the notation of 3.3.18.

For an example of the following constructions, see figure 16. For each $P \in \mathcal{P}$, $\gamma_{\mathcal{P}}$ has a corresponding flag at the loop, which we denote by $F_P \in \mathbf{FS}_0(\gamma_P)$. It holds $\Delta_{[F]}(P) = \Delta_{[F]}(F_P)$ for all

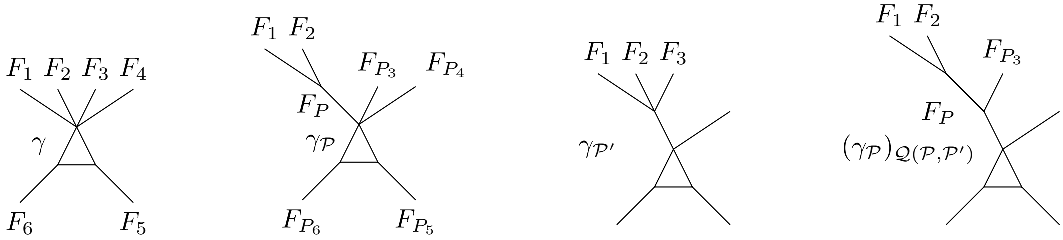


FIGURE 16. We denote the combinatorial type of the curve on the left by γ. F_i, $i \in [6]$ are the flags at the loop of γ. We set

$$\mathcal{P} = \{\{F_1, F_2\}, \{F_i\} | i = 3, \ldots, 6\},$$

$P = \{F_1, F_2\}$ and $P_i = \{F_i\}$, $i \in [6]$. The partition

$$\mathcal{P}' = \{\{F_1, F_2, F_3\}, \{F_i\} | i = 4, 5, 6\}$$

is coarser than $\mathcal{P}$. It holds $\mathcal{Q}(\mathcal{P}, \mathcal{P}') = \{\{F_P, F_{P_3}\}, \{F_{P_i}\} | i = 4, 5, 6\}$.

$P \in \mathcal{P}$. A partition $\mathcal{P}' \in \mathbf{P}_{\mathcal{P}}(\gamma)$ that is coarser than $\mathcal{P}$ induces a partition $\mathcal{Q}(\mathcal{P}, \mathcal{P}')$ of the set of flags $\{F_P | P \in \mathcal{P}\}$ at the loop of $\gamma_{\mathcal{P}}$, via

$$\mathcal{Q}(\mathcal{P}, \mathcal{P}') = \{\{F_P | P \subset P'\}\}_{P' \in \mathcal{P}'}.$$

It holds $\mathcal{Q}(\mathcal{P}, \mathcal{P}'_1) \neq \mathcal{Q}(\mathcal{P}, \mathcal{P}'_2)$ for different partitions $P'_1, \mathcal{P}'_2 \in \mathbf{P}_{\mathcal{P}}(\gamma)$ and

$$\mathbf{P}_{\mathcal{P}_0(\gamma_{\mathcal{P}})} = \{\mathcal{Q}(\mathcal{P}, \mathcal{P}') | \mathcal{P}' \in \mathbf{P}_{\mathcal{P}}(\gamma)\},$$

where $\mathbf{P}_{\mathcal{P}_0(\gamma_{\mathcal{P}})}$ is the notation from proposition 3.3.22, which we recall here:

$\mathcal{P}_0(\gamma_{\mathcal{P}})$ is the finest partition of the flags at the loop of $\gamma_{\mathcal{P}}$ and it holds $\mathcal{P}_0(\gamma_{\mathcal{P}}) = \mathcal{Q}(\mathcal{P}, \mathcal{P})$. Then $\mathbf{P}_{\mathcal{P}_0(\gamma_{\mathcal{P}})}$ is the set of partitions $\mathcal{Q} \in \mathbf{P}^0(\gamma_{\mathcal{P}})$ of the flags of the loop of $\gamma_{\mathcal{P}}$ which are finer than the one given by the vertices in the loop of $\gamma_{\mathcal{P}}$ and which fulfill $\#I(\mathcal{Q}, \mathcal{Q}(\mathcal{P}, \mathcal{P})) = \#\mathcal{Q} + 1$ and $\#I(\mathcal{Q}, \mathcal{Q}(\mathcal{P}, \mathcal{A})) = \#\mathcal{Q} + 1$.

Moreover, for $\mathcal{P}' \in \mathbf{P}_{\mathcal{P}}$, it is true that

$$\mathrm{ind}(\gamma_{\mathcal{P}'}) = \mathrm{ind}((\gamma_{\mathcal{P}})_{\mathcal{Q}(\mathcal{P},\mathcal{P}')}).$$

It hence follows with proposition 3.3.25 that

$$\begin{aligned}
& \mathrm{ind}(\gamma_{\mathcal{P}}) \cdot \left(\sum_{j=1}^{i} v_{\gamma^j} \right) + \sum_{\mathcal{P}' \in \mathbf{P}_{\mathcal{P}}(\gamma)} \mathrm{ind}(\gamma_{\mathcal{P}'}) \cdot \left(\sum_{\substack{j_1 \in \Delta_{[F]}(P(\mathcal{P}')_1), \\ j_2 \in \Delta_{[F]}(P(\mathcal{P}')_2)}} v_{\{j_1, j_2\}} \right) \\
= \; & \mathrm{ind}(\gamma_{\mathcal{P}}) \cdot \left(\sum_{j=1}^{i} v_{\gamma^j} \right) + \sum_{\mathcal{P}' \in \mathbf{P}_{\mathcal{P}}(\gamma)} \mathrm{ind}((\gamma_{\mathcal{P}})_{\mathcal{Q}(\mathcal{P},\mathcal{P}')}) \cdot \left(\sum_{\substack{j_1 \in \Delta_{[F]}(F_{P(\mathcal{P}')_1}), \\ j_2 \in \Delta_{[F]}(F_{P(\mathcal{P}')_2})}} v_{\{j_1, j_2\}} \right) \\
= \; & \mathrm{ind}(\gamma_{\mathcal{P}}) \cdot \left(\sum_{j=1}^{i} v_{\gamma^j} \right) + \sum_{\mathcal{P}' \in \mathbf{P}_{\mathcal{P}_0(\gamma_{\mathcal{P}})}} \mathrm{ind}((\gamma_{\mathcal{P}})_{\mathcal{P}'}) \cdot \left(\sum_{\substack{j_1 \in \Delta_{[F]}(F(\mathcal{P}')_1), \\ j_2 \in \Delta_{[F]}(F(\mathcal{P}')_2)}} v_{\{j_1, j_2\}} \right)
\end{aligned}$$

is an element of $\mathrm{W}(\sigma(\overline{\gamma_{\mathcal{P}}}))$, where $\overline{\gamma_{\mathcal{P}}}$ is the specialization of $\gamma_{\mathcal{P}}$ in which precisely the edges outside the loop are contracted, where $\sigma(\overline{\gamma_{\mathcal{P}}})$ is the set of curves of combinatorial type $\gamma_{\mathcal{P}}$ for which the loop is closed and where $\mathrm{W}(\sigma(\overline{\gamma_{\mathcal{P}}}))$ is the smallest linear space containing $\sigma(\overline{\gamma})$.

Since it holds $\sigma(\overline{\gamma}) = \sigma(\overline{\gamma_{\mathcal{P}}})$ for all $\mathcal{P} \in \mathbf{P}^0(\gamma)$ (where $\overline{\gamma}$ is the specialization of γ in which precisely the edges outside the loop are contracted), we conclude

$$\begin{aligned} & \sum_{j\in[i]} \#\operatorname{Aut}(\gamma^j)\cdot\omega(\gamma^j)\cdot v_{\gamma^j} + \sum_{\mathcal{W}\in\mathbf{W}(\gamma)} \#\operatorname{Aut}(\gamma_{\mathcal{W}})\cdot\omega(\gamma_{\mathcal{W}})\cdot\left(\sum_{W\in\mathcal{W}} v_{\Delta_{[F]}(W)}\right) \\ \in \quad & \mathrm{W}(\sigma(\overline{\gamma})) + \left\langle \sum_{[G]\in\mathbf{FS}_0(\gamma)} v_{\Delta_{[F]}([G])}\right\rangle \subset \mathrm{W}(\gamma,\leq), \end{aligned}$$

where $\mathrm{W}(\gamma,\leq)$ is the smallest linear space that contains all curves of fine combinatorial type $(\gamma,\leq)$ which fulfill that the loop is closed. □

Proof of theorem 3.3.8. Combine theorem 3.2.10 and propositions 3.3.13 and 3.3.12. □

Definition 3.3.36 (Evaluation maps)
If x_i is a leaf of $(C,h)\in\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)$ labeled by $i\in I$, it is contracted by h and mapped to a point. Hence, the map

$$\begin{aligned} \mathrm{ev}_i : \mathcal{M}_{1,I}(\Delta,\mathbb{R}^m) &\to \mathbb{R}^m \\ (C,h) &\mapsto h(x_i) \end{aligned}$$

is well-defined. It is moreover affine linear on every polyhedron in $\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)$ and it follows from proposition 4.8 in [**GKM09**] and the previous theorem that the restriction of ev_i to $\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)^{\text{reg}}$ is an open tropical morphism.

3.4. Counting well-spaced elliptic curves in $\mathbb{R}^m$

Throughout this section we assume that $m \in \mathbb{N}$ is a natural number with $m \geq 2$. We fix a degree $j : \Delta \to \mathbb{R}^m$ of elliptic curves in $\mathbb{R}^m$ and fan varieties L_i in $\mathbb{R}^m$ that can be cut out by rational functions and that fulfill

$$\sum_{i=1}^{n} \operatorname{codim} L_i = \#\Delta + n + m - \dim\langle\Delta\rangle = \dim \mathcal{M}_{1,n}(\Delta, \mathbb{R}^m).$$

Set $\mathcal{L} := \{L_1, \ldots, L_n\}$. Then the intersection product

$$\prod_{i\in[n]} \operatorname{ev}_i^*(a_i + L_i) \cdot \mathcal{M}_{1,n}(\Delta, \mathbb{R}^m)^{\text{reg}}$$

is well-defined and zero-dimensional for all $(a_1, \ldots, a_n) \in (\mathbb{R}^m)^n$, where $a_i + L_i$ denotes the translation of L_i by a_i, see definition 1.2.30 and theorem 3.3.8.

The aim of this section is to prove that the degree of

$$\prod_{i\in[n]} \operatorname{ev}_i^*(a_i + L_i) \cdot \mathcal{M}_{1,n}(\Delta, \mathbb{R}^m)^{\text{reg}}$$

does not depend on $(a_1, \ldots, a_m) \in \mathbb{R}^m$ as long as $a_1 + L_1, \ldots, a_n + L_n$ is in general position, which is defined below. The approach is the similar to e.g. [**GM07a**] and [**KM09**].

We set

$$\operatorname{ev} := \prod_{i\in[n]} \operatorname{ev}_i : \mathcal{M}_{1,n}(\Delta, \mathbb{R}^m) \to (\mathbb{R}^m)^n$$

to be the total evaluation map. Denote by $\operatorname{Lin}(L_i)$ the lineality space of L_i and by $L_i/\operatorname{Lin}(L_i)$ the fan variety in $\mathbb{R}^m/\operatorname{Lin}(L_i)$ in which we mod out $\operatorname{Lin}(L_i)$ from the support of L_i. $\prod_{i\in[n]} \mathbb{R}^m/\operatorname{Lin}(L_i)$ parametrizes the set of incidence conditions $(a_i + L_i)_{i\in[n]}$ via

$$([a_1], \ldots, [a_n]) \mapsto (a_i + L_i)_{i\in[n]},$$

where $(a_i + L_i)$ does not depend on the choice of $a_i \in [a_i]$. Let $q : (\mathbb{R}^m)^n \to \prod_{i\in[n]} \mathbb{R}^m/\operatorname{Lin}(L_i)$ be the quotient map.

We will study a parameter space of I-marked well-spaced elliptic curves of degree Δ in $\mathbb{R}^m$ together with incidence conditions $a_i + L_i$, $i \in [n]$ which they fulfill. Consider $\mathcal{M}_{1,n}(\Delta, \mathbb{R}^m) \times \prod_{i\in[n]} \mathbb{R}^m/\operatorname{Lin}(L_i) \times \prod_{i\in[n]} L_i/\operatorname{Lin}(L_i)$. The subcomplex with support

$$\{((C,h),[a],[b]) | q \circ \operatorname{ev}(C,h) = [a] + [b]\} = \{((C,h), q \circ \operatorname{ev}(C,h) - [b], [b])\}$$

contains the set of triples $((C,h),[a],[b])$ such that $\operatorname{ev}_i(C,h) \in \operatorname{supp}(a_i + L_i)$ and such that there exist $c \in \prod_{i\in[n]} \operatorname{Lin}(L_i)$ with $\operatorname{ev}(C,h) = a + b + c$. It is a pure-dimensional polyhedral complex that can be identified with

$$X := \mathcal{M}_{1,n}(\Delta, \mathbb{R}^m) \times \prod_{i\in[n]} L_i/\operatorname{Lin}(L_i)$$

and that has dimension $m \cdot n - \sum_{i\in[n]} \dim(\operatorname{Lin}(L_i))$ (due to $\sum_{i\in[n]} \operatorname{codim}(L_i) = \dim \mathcal{M}_{1,n}(\Delta, \mathbb{R}^m)$), which is also the dimension of

$$Y := \prod_{i\in[n]} \mathbb{R}^m/\operatorname{Lin}(L_i).$$

Define

$$\begin{array}{rcl} \operatorname{ev}' : X & \to & Y \\ ((C,h),[b]) & \mapsto & q(\operatorname{ev}(C,h)) - [b], \end{array}$$

which is continuous and affine linear on each polyhedron.

We will define dense open subsets $\mathcal{G}, \mathcal{G}_1 \subset Y$, where $\mathcal{G}_1$ is also connected. An element $[a] \in Y$ which is contained in $\mathcal{G}$ will be called in general position, and we will show, as mentioned above, that the degree of

$$\prod_{i\in[n]} \operatorname{ev}_i^*(a_i + L_i) \cdot \mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)^{\text{reg}}$$

does not depend on $(a_1, \dots, a_n) \in (\mathbb{R}^m)^n$ as long as $([a_1], \dots, [a_n]) \in \mathcal{G} \subset Y$.

Definition 3.4.1 (General position, $\mathcal{G}$, $\mathcal{G}_1$)
We say that $(a_i + L_i)_{i\in[n]}$ is in general position if $([a_1], \dots, [a_n]) \in Y$ is an element of

$$\mathcal{G} := Y \setminus \bigcup \operatorname{ev}' \left(\overline{\mathrm{M}_1(\alpha, \leq)} \times \prod_{i\in[n]} L_i / \operatorname{Lin}(L_i) \right)$$

where the union is taken over all fine combinatorial types $(\alpha, \leq)$ of positive codimension. Remember that $\overline{M_1(\alpha, \leq)}$ is the set of curves whose fine combinatorial type is a specialization of $(\alpha, \leq)$. Moreover, set

$$\mathcal{G}_1 := Y \setminus \bigcup \operatorname{ev}' \left(\overline{\mathrm{M}_1(\alpha, \leq)} \times \prod_{i\in[n]} \sigma_i \right)$$

where the union is taken over all fine combinatorial types $(\alpha, \leq)$ and polyhedra $\sigma_i \in \operatorname{pol}(L_i / \operatorname{Lin}(L_i))$ such that

- $(\alpha, \leq)$ has codimension at least two and σ is a facet,
- $(\alpha, \leq)$ has codimension one, $\operatorname{ev}|_{\mathrm{M}_1(\alpha,\leq)}$ is not injective and σ is a facet or
- $(\alpha, \leq)$ has codimension one and σ has codimension one.

Since $\dim X = \dim Y$, it holds that $\mathcal{G}$ is dense in Y and that $\mathcal{G}_1$ is dense and connected in Y.

Definition 3.4.2 ($N_{\Delta,\mathcal{L}}$)
Define the map

$$\begin{array}{rccc} N_{\Delta,\mathcal{L}}: & \mathcal{G} & \to & \mathbb{N} \\ & ([a_i])_{i\in[n]} & \mapsto & \deg\left(\prod_{i\in[n]} \operatorname{ev}_i^*(a_i + L_i) \cdot \mathcal{M}_{1,n}(\Delta, \mathbb{R}^m)^{\text{reg}}\right). \end{array}$$

We state the theorem which we are going to prove in this section.

Theorem 3.4.3
The map $N_{\Delta,\mathcal{L}}$ is constant, i.e. the degree of the intersection product

$$\deg\left(\prod_{i\in[n]} \operatorname{ev}_i^*(a_i + L_i) \cdot \mathcal{M}_{1,n}(\Delta, \mathbb{R}^m)^{\text{reg}}\right)$$

does not depend on $(a_1, \dots, a_n) \in (\mathbb{R}^m)^n$ as long as $(a_1, \dots, a_n) \in q^{-1}(\mathcal{G})$, which is dense in $(\mathbb{R}^m)^n$.

Remark 3.4.4 (Enumerative relevance of the map $N_{\Delta,\mathcal{L}}$)
As in the case of rational curves, it follows from lemma 2.1.5, that there exists a dense open subset $D \subset (\mathbb{R}^m)^n$ such that for all $(a_1, \dots, a_n) \in D$ the degree of the intersection product

$$\prod_{i\in[n]} \operatorname{ev}_i^*(a_i + L_i) \cdot \mathcal{M}_{1,n}(\Delta, \mathbb{R}^m)^{\text{reg}}$$

is equal to the number of well-spaced elliptic n-marked curves of degree Δ, counted with the intersection-theoretic multiplicity, that fulfill $\operatorname{ev}_i(C, h) \in (a_i + L_i)$ for all $i \in [n]$.

Since $\mathcal{G}$ is dense in Y and since $\mathcal{G}_1$ is dense and connected in Y, $N_{\Delta,\mathcal{L}}$ is constant if and only if all $a \in \mathcal{G}_1$ have a neighborhood $\mathcal{U}(a) \subset Y$ such that $N_{\Delta,\mathcal{L}}|_{\mathcal{U}(a)\cap\mathcal{G}}$ is constant. ev' is continuous and the sets $\mathcal{U}(a)$ can be chosen such that

a) each connected component $W_1, \dots, W_l$ of $(\operatorname{ev}')^{-1}(\mathcal{U}(a))$ contains exactly one connected component of $(\operatorname{ev}')^{-1}\{a\}$

b) $W_i \cap W_j \neq \emptyset$ for $i, j \in [l]$ with $i \neq j$ and
c) $W_j \subset \bigcup_{p \in (\mathrm{ev}')^{-1}\{a\}} U(p)$,

where $U(p)$ are polyhedral neighborhoods of Y at p. Denote the tropical subvariety of $\mathcal{M}_{1,n}(\Delta, \mathbb{R}^m)$ corresponding to $W_j \subset \mathcal{M}_{1,n}(\Delta, \mathbb{R}^m) \times \prod_{i\in[n]} L_i/\operatorname{Lin}(L_i)$ by M_j. Then $N_{\Delta,\mathcal{L}}$ is constant if and only if

$$\begin{aligned} N_a^{W_j} : \mathcal{U}(a) \cap \mathcal{G} &\to \mathbb{N} \\ b &\mapsto \deg\left(\prod_{i\in[n]} \mathrm{ev}_i^*(b_i + L_i) \cdot M_j\right) \end{aligned}$$

is constant for all $a \in \mathcal{G}_1$ and for all $j \in [l]$.

Fix $a = ([a_1], \ldots, [a_n]) \in \mathcal{G}_1$. According to the definition of $\mathcal{G}_1$, W_j fulfills one of the following properties, where

$$\pi_1 : X \to \mathcal{M}_{1,n}(\Delta, \mathbb{R}^m) \text{ and } \pi_2 : X \to \prod_{i\in[n]} L_i/\operatorname{Lin}(L_i)$$

are the projections on the first and second factor, respectively:

(A) $(\mathrm{ev}')^{-1}\{a\} \cap W_j$ is at least one-dimensional.
(B) $\#\left((\mathrm{ev}')^{-1}\{a\} \cap W_j\right) = 1$, we denote the unique element by $((C,h),x) \in X$ and it holds
  (i) (C,h) is contained in $\mathcal{M}_{1,n}(\Delta, \mathbb{R}^m)^{\mathrm{reg}}$.
  (ii) (C,h) has codimension one and precisely two edges in the loop that have the same weight, and x is contained in a facet of $\prod_{i\in[n]} L_i/\operatorname{Lin}(L_i)$.
  (iii) (C,h) has codimension one and is non-regular, and x is contained in a facet of $\prod_{i\in[n]} L_i/\operatorname{Lin}(L_i)$.

Proposition 3.4.5
For all $a \in \mathcal{G}_1$, the map $N_a^{W_j}$ is constant if we are in case (A), in the first case of (B) or in the second case of (B).

Proof. In case (A), property b) of W_j means that W_j is contained in a polyhedral neighborhood of $(\mathrm{ev}')^{-1}\{a\} \cap W_j$, which implies that $(\mathrm{ev}')^{-1}\{b\} \cap W_j$ is at least one-dimensional for all $b \in \mathcal{U}(a) \cap \mathcal{G}$. Hence, ev' is injective nowhere on W_j and $N_a^{W_j}$ is the constant zero function.

In the first case of (B), it holds $\operatorname{supp}(M_j) \subset \mathcal{M}_{1,n}(\Delta, \mathbb{R}^m)^{\mathrm{reg}}$, which is an abstract tropical variety. $(b_i + L_i)_{i\in[n]} \in \mathcal{U}(a)$ is rationally equivalent to $(a_i + L_i)_{i\in[n]}$, and if $\mathcal{U}(a)$ is small enough,

$$\prod_{i\in[n]} \mathrm{ev}_i^*(b_i + L_i) \cdot M_j \text{ and } \prod_{i\in[n]} \mathrm{ev}_i^*(a_i + L_i) \cdot M_j$$

are hence also rationally equivalent, see lemma 1.2.37. Their degree is therefore equal and $N_a^{W_j}$ is constant.

Let us deal with the second case of (B) and let F be a flag in the loop of C. Denote the combinatorial type of (C,h) by $(\alpha, \leq)$ and consider the abstract tropical variety

$$Z_j := U(\alpha_{[F]}, \leq_{[F]}) \cap j_{\alpha,\leq}^{[F]}(M_j),$$

equipped with the weights of $U(\alpha_{[F]}, \leq_{[F]})$, whose support contains all curves whose combinatorial type specializes to $(\alpha_{[F]}, \leq_{[F]})$, which fulfill that the loop is closed and which correspond to curves in M_j, see also definition 3.3.5. Remember that a facet of $U(\alpha_{[F]}, \leq_{[F]})$ containing curves of combinatorial type $\beta_{[F]}$ is equipped with weight

$$\#\operatorname{Aut}(\beta) \cdot \omega(\beta).$$

$\#\operatorname{Aut}(\beta)$ is two if there is only one edge in the loop of β, and $\#\operatorname{Aut}(\beta)$ is one if there is more than one edge in the loop of β. Proposition 3.3.13 states that $U(\alpha_{[F]}, \leq_{[F]})$ is balanced at the polyhedron containing curves of fine combinatorial type $(\alpha_{[F]}, \leq_{[F]})$. Hence, also Z_j is balanced.

If $\mathcal{U}(a)$ is small enough,

$$\prod_{i\in[n]} \operatorname{ev}_i^*(b_i + L_i) \cdot Z_j \text{ and } \prod_{i\in[n]} \operatorname{ev}_i^*(a_i + L_i) \cdot Z_j$$

are hence rationally equivalent for all $([b_1], \ldots, [b_n]) \in \mathcal{U}(a)$, and their degrees coincide.

If $(\alpha, \leq) \leq (\beta, \leq_\beta)$ is a maximal fine combinatorial type in $\mathcal{M}_{1,n}(\Delta, \mathbb{R}^m)$, there exist precisely

$$\frac{2}{\# \operatorname{Aut}(\beta)}$$

fine combinatorial types in Z_j that correspond to $(\beta, \leq_\beta)$, see the figure above for an example.

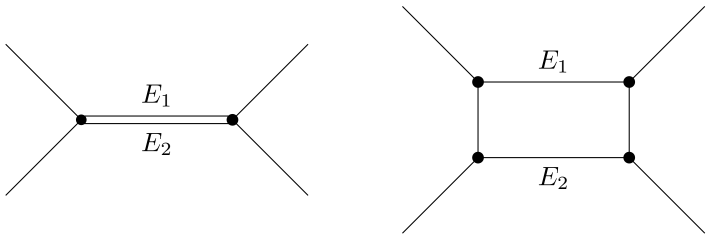


FIGURE 17. Assume $\omega(E_1) = \omega(E_2)$. If we cut one of the edges E_1 and E_2 in the curve on the left for constructing a rational curve, the resulting combinatorial types are the same. However, if we cut one of the edges E_1 and E_2 in the curve on the right, the resulting combinatorial types differ.

Since the weights on a facet of Z_j containing curves of fine combinatorial type $(\beta_{[F]}, \leq_{[F]})$ are chosen to be $\# \operatorname{Aut}(\beta)$ times the corresponding weight in $\mathcal{M}_{1,n}(\Delta, \mathbb{R}^m)$, it follows for all $([b_1], \ldots, [b_n]) \in \mathcal{U}(a)$ that

$$N_a^{W_j}([b_1], \ldots, [b_n]) = \frac{1}{2} \deg \left(\prod_{i\in[n]} \operatorname{ev}_i^*(a_i + L_i) \cdot Z_j \right)$$

and the map $N_a^{W_j}$ is constant. □

The only remaining case is the third case of (B). We will deal with this case in the rest of this section:

So let $a \in \mathcal{G}_1$, let $\mathcal{U}(a)$ be a neighborhood and let W be a connected component of $(\operatorname{ev}')^{-1}\{\mathcal{U}(a)\}$ such that $\# \left[(\operatorname{ev}')^{-1}\{a\} \cap W\right] = 1$, say $((C, h), x) \in (\operatorname{ev}')^{-1}\{a\} \cap W$, and

a) (C, h) has codimension one and is non-regular and
b) x is contained in a facet of $\prod_{i\in[n]} L_i / \operatorname{Lin}(L_i)$.

We assume moreover that for all $i \in [n]$ the support of L_i is an affine linear subspace of $\mathbb{R}^m$. (This is no restriction because the element x of the support of $\prod_{i\in[n]} L_i / \operatorname{Lin}(L_i)$ is contained in a facet and because intersection products are calculated locally.) Furthermore, we assume without loss of generality $m \geq 3$ because non-regular curves in $\prod_{i\in[n]} \operatorname{ev}_i^* L_i \cdot \mathcal{M}_{1,n}(\Delta, \mathbb{R}^2)$ have codimension greater than one.

We denote the fine combinatorial type of (C, h) by $(\alpha, \leq)$.

3.4.1. Invariance in a special case. In this subsection, we prove in a special case that the map N_a^W is constant under the assumptions of the third case of (B). The subsequent subsection deals with the general case, and the proof that N_a^W is constant in the general case is a corollary of the result 3.4.7 of this section.

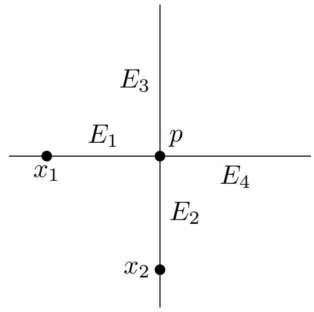


FIGURE 18. The flags F_i are given by (p, E_i), where p is the unique vertex of genus one. The marked points x_1 and x_2 are adjacent to the leaves that have direction $\mathrm{v}(F_1)$ and $\mathrm{v}(F_2)$, respectively.

Definition 3.4.6
Let $(D, g) \in \mathcal{M}_{1,n}(\Delta, \mathbb{R}^m)$. We call (D, g) a star curve if there exist affine linear maps $f_i : [0, \infty) \to \mathbb{R}^m$, $i \in [\#\Delta]$ such that $f_i(0) = f_j(0)$ for all $i, j \in [\#\Delta]$ and

$$g(D) = \bigcup_{i \in [\#\Delta]} f_i[0, \infty).$$

Moreover, we demand of star curves that for all $i \in [\#\Delta]$ there exists at most one marked point x_j with $j \in [n]$ such that $g(x_j) \in f_j[0, \infty)$. This means that there is at most one marked point on each of the rays of $g(D)$.

Proposition 3.4.7
Assume that we are in the third case of (B) and that

a) the number of marked points is $n = 2$,
b) the unique element (C, h) of $\pi_1((\mathrm{ev}')^{-1}\{a\} \cap W)$ is a star curve,
c) $\#\Delta = m + 1$ and $\langle j(\Delta)\rangle = \mathbb{R}^m$,
d) both L_1 and L_2 are spanned by elements of $j(\Delta)$.

Then the map N_a^W is constant.

We assume in this subsection that the assumptions of the proposition above are valid. Let $p \in \mathrm{supp}(C)$ be the unique vertex of genus one and let $\{F_1, \dots, F_{m+1}\} \in \mathbf{FS}_0(\alpha)$ be the set of flags at the loop of α. For $i \in [m + 1]$ let

$$y_i = \omega(E_i) \cdot \mathrm{v}(F_i)$$

be the weighted direction vector of the flag $F_i = (p, E_i)$ at the loop of α. Assume that the marked point x_i is adjacent to the leaf of (C, h) that has direction y_i, $i \in \{1, 2\}$, see the figure below for an example of the notation.

Since the dimension of $\mathcal{M}_{1,2}(\Delta, \mathbb{R}^m)$ is $m + 3$, it holds moreover $\dim(L_1) + \dim(L_2) = m - 3$. If $\mathrm{Lin}(L_1) \cap \mathrm{Lin}(L_2) \neq \{0\}$ or if one of the weighted direction vectors y_1 and y_2 is an element of L_1 or L_2, the map $q \circ \mathrm{ev} : \mathcal{M}_{1,2}(\Delta, \mathbb{R}^m) \to \mathbb{R}^m/\mathrm{Lin}(L_1) \times \mathbb{R}^m/\mathrm{Lin}(L_2)$ is not injective and N_a^W is the constant zero function. So we may assume (by reordering) that

$$L_1 \oplus L_2 = \langle y_5, \dots, y_{m+1}\rangle.$$

If $m = 3$, this condition means that both L_1 and L_2 are points in $\mathbb{R}^3$.

Notation 3.4.8 (FD, $\mathcal{P}(a_1)$, $\beta(a_1)$, $\mathrm{ind}(\mathcal{P})$, $z(i, j, a_1)$, $z(a_1)$)
Define

$$\mathrm{FD} := \mathbb{Z}^m \cap \{\sum_{i=2}^{m+1} \lambda_i \cdot y_i \mid 0 \leq \lambda_i < 1\},$$

which is a fundamental domain of the lattice spanned by $y_2, \ldots, y_{m+1}$. An element

$$a_1 = \sum_{i=1}^{m+1} \lambda_i \cdot y_i \in \mathrm{FD}$$

with $\lambda_1 = 0$ defines an ordered partition $\mathcal{O}(a_1) = (P_1, \ldots, P_r)$ of the set of flags $\mathbf{FS}_0(\alpha)$ at the loop of α and thus a resolution $\beta(a_1)$ of α, which has only additional edges in the loop, see also construction 3.2.17, lemma 3.2.15 and figure 19:

Set $P_1 := \{F_i | \lambda_i = 0, i \in [m+1]\} \subset \mathbf{FS}_0(\alpha)$ and let $r := \#\{\lambda_i | i \in [m]\}$ be the number of different coefficients that appear in the linear combination $a_1 = \sum_{i=1}^{m} \lambda_i \cdot y_i$. Note that $F_1 \in P_1$. For all $i_1, i_2 \in \{2, \ldots, m+1\}$ it holds that there exists $j_1, j_2 \in \{2, \ldots, r\}$ with $j_1 < j_2$ and $F_{i_1} \in P_{j_1}$, $F_{i_2} \in P_{j_2}$ if and only if $\lambda_{i_1} > \lambda_{i_2} > 0$. This ordered partition defines a combinatorial type $\beta(a_1)$ which has only additional edges in the loop compared to α, and the order of the flags $F_1, \ldots, F_{m+1}$ in the loop of $\beta(a_1)$ is reflected by $\mathcal{O}(a_1)$. Moreover, a_1 is the weighted direction vector of a flag in the loop of $\beta(a_1)$ adjacent to the flag F_1 if we run around the loop of $\beta(a_1)$ as specified by $\mathcal{O}(a_1)$.

For all resolutions β of α, there exists at least one $a_1 \in \mathrm{FD}$ such that $\beta = \beta(a_1)$. Since α is the combinatorial type of a star curve, all curves of combinatorial type α and $\beta(a_1)$ are well-spaced. We denote the fine combinatorial type corresponding to $a_1 \in \mathrm{FD}$ which is a resolution of $(\alpha, \leq)$ by $(\beta(a_1), \leq_{a_1})$.

For $a_1 \in \mathrm{FD}$, denote the unordered partition of the set of flags $\mathbf{FS}_0(\alpha)$ at the loop of α corresponding to $\mathcal{O}(a_1)$ by

$$\mathcal{P}(a_1) \in \mathbf{P}^0(\alpha),$$

where $\mathbf{P}^0(\alpha)$ is the set of partitions of the flags at the loop of α. If $\mathcal{P} \in \mathbf{P}^0(\alpha)$ and $P \in \mathcal{P}$, we put

$$y_P = \sum_{i \in P} y_i \text{ and } \operatorname{ind}(\mathcal{P}) = \operatorname{ind}(y_P | P \in \mathcal{P}).$$

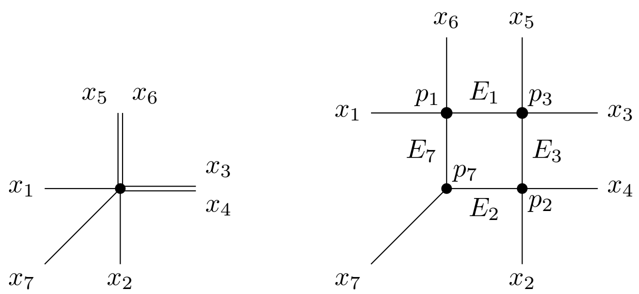


FIGURE 19. The combinatorial type on the right is $\beta(a_1)$ where a_1 is the weighted direction vector of the flag (p_1, E_1). The ordered partition of the flags at the loop of the combinatorial type α on the left which induces $\beta(a_1)$ is $\mathcal{O}(a_1) = (\{F_1, F_6\}, \{F_3, F_5\}, \{F_2, F_4\}, \{F_7\})$ where $F_i = (p, x_i)$ for $i \in [7]$.

For $i_1, i_2 \in [m+1]$ with $i_1 < i_2$ and for $a_1 \in \mathrm{FD}$ we set

$$Z(i_1, i_2, a_1) := |\det(y_1, \ldots, y_{i_1-1}, y_{i_1+1}, \ldots, y_{i_2-1}, y_{i_2+1}, \ldots, y_{m+1}, a_1)|$$

and

$$Z := |\det(y_1, \ldots, y_m)|.$$

It holds $Z = |\det(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{m+1})|$ for all $k \in [m+1]$ because $\sum_{j=1}^{m+1} y_j = 0$.

Define for $i_1, i_2 \in [m+1]$ and $i_1 \neq i_2$ and for $a_1 \in \mathrm{FD}$

$$z(i_1, i_2, a_1) := Z(i_1, i_2, a_1) \cdot (Z - Z(i_1, i_2, a_1))$$

and

$$z(a_1) := z(1, 3, a_1) + z(2, 4, a_1) - z(1, 4, a_1) - z(2, 3, a_1).$$

$z(i_1, i_2, a_1)$ is (up to a constant multiple) a product of volumes in the parallelotope spanned by $y_2, \dots, y_{m+1}$ (in which the direction vector $y_1 = -\sum_{i=2}^{m+1} y_i$ of the flag F_1 is a diagonal), see the figure below.

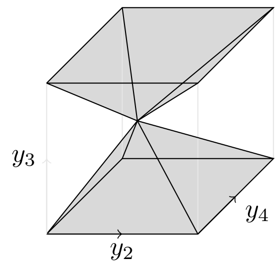


FIGURE 20. Example of $z(i_1, i_2, a_1)$. In this case in $\mathbb{R}^3$ it holds $i_1 = 1, i_2 = 3$ and a_1 is the point in the middle of the cube where the two pyramids meet. The geometric interpretation of $z(i_1, i_2, a_1)$ is four times the product of the volumes of the two gray pyramids.

We will show the following statements, which are sufficient to prove proposition 3.4.7, see the proof below:

a) For all $a_1 \in \mathrm{FD}$ there exists a matrix $M(a_1)$ and for all $b \in (\mathbb{R}^m)^2$ such that $q(b) \in \mathcal{G}$ is in general position, there exists $s(b) \in \{-1, 1\}$ such that the following holds: There exists a curve of fine combinatorial type $(\beta(a_1), \leq_{a_1})$ in the support of
$$K(b) = \prod_{i \in [2]} \mathrm{ev}^*(b_i + L_i) \cdot \mathcal{M}_{1,2}(\Delta, \mathbb{R}^m)$$
if and only if
$$s(b) \cdot \det(M(a_1)) > 0.$$
Therefore, the sign of $\det(M(a_1))$ determines whether a curve of combinatorial type $\beta(a_1)$ appears in $K(b)$.

b) The weight of a curve $(D, g) \in \mathrm{supp}(K(b))$ of fine combinatorial type $(\beta(a_1), \leq_{a_1})$ is given by
$$\frac{\omega(\beta(a_1)) \cdot \mathrm{ind}(\mathcal{P}(a_1))}{\mathrm{ind}(\beta(a_1)) \cdot \mathrm{ind}(\mathcal{P}_0) \cdot \mathrm{ind}(y_5, \dots, y_{m+1})} \cdot |\det(M(a_1))|,$$
where $\mathcal{P}_0 = \{\{F_i\} | i \in [m+1]\}$ is finest partition of the flags at the loop of α.

c) For all maximal fine combinatorial types $(\alpha, \leq) \leq (\beta, \leq_\beta)$, there exist precisely $\frac{2}{\#\,\mathrm{Aut}(\beta)}$ elements $a_1 \in \mathrm{FD}$ such that
$$(\beta, \leq_\beta) = (\beta(a_1), \leq_{a_1}).$$
In this case, set $\det(M(\beta)) := \det(M(a_1))$.

d) For all $a_1 \in \mathrm{FD}$, it is true that $\det(M(a_1)) = \frac{1}{2} z(a_1)$.

e) It holds
$$\sum_{a_1 \in \mathrm{FD}} \#\,\mathrm{Aut}(\beta(a_1)) \cdot \frac{\omega(\beta(a_1)) \cdot \mathrm{ind}(\mathcal{P}(a_1))}{\mathrm{ind}(\beta(a_1)) \cdot \mathrm{ind}(\mathcal{P}_0)} \cdot z(a_1) = 0.$$

PROOF OF PROPOSITION 3.4.7 USING THE STATEMENTS FROM ABOVE. Let $(\alpha, \leq) \leq (\beta, \leq_\beta)$ be a maximal fine combinatorial type. Then there exists at least one $a_1 \in \mathrm{FD}$ which fulfills
$$(\beta, \leq_\beta) = (\beta(a_1), \leq_{a_1}),$$
see lemma 3.2.15. For $b \in (\mathbb{R}^m)^2$, there exists a curve of fine combinatorial type $(\beta(a_1), \leq_{a_1})$ in the support of
$$K(b) = \prod_{i \in [2]} \mathrm{ev}^*(b_i + L_i) \cdot \mathcal{M}_{1,2}(\Delta, \mathbb{R}^m)$$

if and only if $s(b) \cdot \det(M(a_1)) > 0$ (see a) above). Hence, for $a_1 \in \mathrm{FD}$ the sign of $\det(M(a_1))$ determines whether a curve of fine combinatorial type $(\beta(a_1), \leq_{a_1})$ appears in the support of $K(b)$, and the restriction of N_a^W to

$$\mathcal{U}(a)^+ = \{q(b) \in \mathcal{U}(a) | s(b) > 0\} \text{ and } \mathcal{U}(a)^- = \{q(b) \in \mathcal{U}(a) | s(b) < 0\}$$

is constant, where $q : (\mathbb{R}^m)^2 \to \mathbb{R}^m / \operatorname{Lin}(L_1) \times \mathbb{R}^m / \operatorname{Lin}(L_2)$ is the projection.

For all $([b_1], [b_2]) \in \mathcal{U}(a)$, the weight of a curve of combinatorial type β in $K(b_1, b_2)$ is given by

$$\frac{\omega(\beta) \cdot \operatorname{ind}(\mathcal{P}_\beta)}{\operatorname{ind}(\beta) \cdot \operatorname{ind}(\mathcal{P}_0) \cdot \operatorname{ind}(y_5, \dots, y_{m+1})} \cdot |\det(M(\beta))|,$$

where $\mathcal{P}_\beta \in \mathbf{P}^0(\alpha)$ is the partition of the set of flags $\mathbf{FS}_0(\alpha)$ at the loop of α given by the vertices in the loop of β (see b) above).

Assume that $b_1, b_2 \in \mathcal{U}(a) \cap \mathcal{G}$ such that $s(b_1) > 0$ and $s(b_2) < 0$. Since for all maximal fine combinatorial types $(\alpha, \leq) \leq (\beta, \leq_\beta)$ there exist precisely $\frac{2}{\# \operatorname{Aut}(\beta)}$ elements $a_1 \in \mathrm{FD}$ such that $(\beta, \leq_\beta) = (\beta(a_1), \leq_{a_1})$ (see c) above), it follows from d) and e) that

$$\begin{aligned}
& N_a^W(b_1) \;-\; N_a^W(b_2) \\
= \;& \sum_{\substack{(\alpha,\leq)<(\beta,\leq_\beta),\\ \det(M(\beta))>0}} \frac{\omega(\beta) \cdot \operatorname{ind}(\mathcal{P}_\beta)}{\operatorname{ind}(\beta) \cdot \operatorname{ind}(\mathcal{P}_0) \cdot \operatorname{ind}(y_5, \dots, y_{m+1})} \cdot |\det(M(\beta))| \\
- \;& \sum_{\substack{(\alpha,\leq)<(\beta,\leq_\beta),\\ \det(M(\beta))<0)}} \frac{\omega(\beta) \cdot \operatorname{ind}(\mathcal{P}_\beta)}{\operatorname{ind}(\beta) \cdot \operatorname{ind}(\mathcal{P}_0) \cdot \operatorname{ind}(y_5, \dots, y_{m+1})} \cdot |\det(M(\beta))| \\
= \;& \sum_{(\alpha,\leq)<(\beta,\leq_\beta)} \frac{\omega(\beta) \cdot \operatorname{ind}(\mathcal{P}_\beta)}{\operatorname{ind}(\beta) \cdot \operatorname{ind}(\mathcal{P}_0) \cdot \operatorname{ind}(y_5, \dots, y_{m+1})} \cdot \det(M(\beta)) \\
= \;& \frac{1}{\operatorname{ind}(y_5, \dots, y_{m+1})} \cdot \sum_{a_1 \in \mathrm{FD}} \frac{\# \operatorname{Aut}(\beta(a_1))}{2} \cdot \frac{\omega(\beta(a_1)) \cdot \operatorname{ind}(\mathcal{P}(a_1))}{\operatorname{ind}(\beta(a_1)) \cdot \operatorname{ind}(\mathcal{P}_0)} \cdot \det(M(a_1)) \\
= \;& \frac{1}{4 \cdot \operatorname{ind}(y_5, \dots, y_{m+1})} \cdot \sum_{a_1 \in \mathrm{FD}} \# \operatorname{Aut}(\beta(a_1)) \cdot \frac{\omega(\beta(a_1)) \cdot \operatorname{ind}(\mathcal{P}(a_1))}{\operatorname{ind}(\beta(a_1)) \cdot \operatorname{ind}(\mathcal{P}_0)} \cdot z(a_1) \\
= \;& 0.
\end{aligned}$$

It follows that N_a^W is constant. □

In the rest of this section, we prove the statements in the list above and we start with part b).

Lemma 3.4.9
The weight of a curve of combinatorial type $\beta(a_1)$ in

$$K(b) = \prod_{i \in [2]} \operatorname{ev}^*(b_i + L_i) \cdot \mathcal{M}_{1,2}(\Delta, \mathbb{R}^m)$$

is given by

$$\frac{\omega(\beta(a_1)) \cdot \operatorname{ind}(\mathcal{P}(a_1))}{\operatorname{ind}(\beta(a_1)) \cdot \operatorname{ind}(\mathcal{P}_0) \cdot \operatorname{ind}(y_5, \dots, y_{m+1})} \cdot |\det(M(a_1))|,$$

where the matrix $M(a_1)$ is defined below.

Construction 3.4.10 $(M(a_1), a_k, c_k)$
For $a_1 \in \mathrm{FD}$ the matrix $M(a_1)$ is given as

$$M(a_1) = \begin{pmatrix} y_1 & y_2 & Y_5 & a_1 & & c_3 & c_4 & C_5 \\ & & & a_1 & a_2 & a_3 & a_4 & A_5 \end{pmatrix}$$

where $Y_5 = \begin{pmatrix} y_5 & \dots & y_{m+1} \end{pmatrix}$, $C_5 = \begin{pmatrix} c_5 & \dots & c_{m+1} \end{pmatrix}$, $A_5 = \begin{pmatrix} a_5 & \dots & a_{m+1} \end{pmatrix}$ and c_k, a_k are defined in the following:

Let $\mathcal{O}(a_1) = (P_1, \ldots, P_{l(a_1)})$ be the ordered partition of the flags at the loop of α that defines $(\beta(a_1), \leq_{a_1})$, see notation 3.4.8. For $i \in [m+1]$, we denote by $j_{a_1}(i)$ the index of the element

$$P_{j_{a_1}(i)} \in \mathcal{P}(a_1)$$

in the ordered partition $\mathcal{O}(a_1) = (P_1, \ldots, P_{l(a_1)})$ which contains the flag F_i, i.e. it holds $j_{a_1}(i) = k \in [l(a_1)]$ if and only if $F_i \in P_k$. Remember that it always holds $F_1 \in P_1$ and hence $j_{a_1}(1) = 1$. For $r \in [\#\mathcal{P}(a_1)]$, denote the vertex of $\beta(a_1)$ that is adjacent to the flags contained in $P_r \in \mathcal{P}(a_1)$ by $p_{m_{a_1}(r)}$ if $m_{a_1}(r) \in [m+1]$ is the lowest index of a flag contained in P_r, i.e.

$$m_{a_1}(r) = \min\{j | F_j \in P_r\}.$$

Denote the edge in the loop of $\beta(a_1)$ that lies behind the vertex $p_{m_{a_1}(r)}$ when we run around the loop according to the ordered partition $\mathcal{O}(a_1)$ by $E_{m_{a_1}(r)}$. For an example of the notation see figure 19 on the right.

For $k \in [m+1]$ define a_k as the weighted direction vector of the flag (p_k, E_k) in the loop of $\beta(a_1)$ if an edge with this index exists, otherwise set a_k to be minus the weighted direction vector $-y_k$ of the flag $F_k \in \mathbf{FS}_0(\alpha)$ at the loop of α, i.e.:

$$a_k := \begin{cases} \omega(E_k) \cdot \mathrm{v}(p_k, E_k), & \exists\, j \in [l(a_1)] : \ \min P_j = k \\ -y_k, & \text{else.} \end{cases}$$

For $k \in [m+1]$ define c_k as the weighted direction vector of the flag $([p_k], [E_k])$ in the loop of $\beta(a_1)$ if an edge with this index exists and if the flag F_k appears in front of F_2 when we run around the loop as specified by the ordered partition $\mathcal{O}(a_1)$. Otherwise, we set c_k to be zero, i.e.:

$$c_k := \begin{cases} a_k, & \exists\, j_{a_1}(2) > j_{a_1}(k) \in [l(a_1)] : \ \min P_j = k \\ 0, & \text{else} \end{cases}$$

If the weight of a curve of combinatorial type $\beta(a_1)$ in $K(b)$ is non-zero, it always holds $a_1 = c_1$, $c_2 = 0$ (because in this case the flags F_1 and F_2 are adjacent to different vertices p_1, p_2 in the loop of $\beta(a_1)$) and

$$a_k = \omega(E_k) \cdot \mathrm{v}(p_k, E_k) = a_1 - \sum_{\substack{i \in [l]: \\ 1 < j_{a_1}(i) \leq j_{a_i}(k)}} y_i$$

if E_k is an edge in the loop of $\beta(a_1)$, i.e. if there exists $l \in [l(a_1)]$ with $\min P_l = k$.

Example 3.4.11

For an example of the matrix $M(a_1)$ we consider the curve in figure 19. Let $v_k \in \mathbb{R}^m$ be the weighted direction vector $\omega(E_k) \cdot \mathrm{v}(p_k, E_k)$ if E_k is an edge in the loop of $\beta(a_1)$. Then it holds

$$\begin{aligned} M(a_1) &= \begin{pmatrix} y_1 & y_2 & y_5 & y_6 & y_7 & c_1 & 0 & c_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \end{pmatrix} \\ &= \begin{pmatrix} y_1 & y_2 & y_5 & y_6 & y_7 & v_1 & 0 & v_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & v_1 & v_2 & v_3 & -y_4 & -y_5 & -y_6 & v_7 \end{pmatrix} \end{aligned}$$

Proof of 3.4.9. Let $([b_1], [b_2]) \in \mathcal{U}(a)$ and

$$(D, g) \in \operatorname{supp}\left(\prod_{i \in [2]} \mathrm{ev}_i^*(b_i + L_i) \cdot \mathcal{M}_{1,2}(\Delta, \mathbb{R}^m)^{\mathrm{reg}} \right)$$

of fine combinatorial type $(\beta(a_1), \leq_{a_1})$. Choose a complement $V(\beta(a_1))_0^c$ of $V(\beta(a_1))_0$ in $\mathbb{R}^m$ (where $V(\beta(a_1))_0$ is the linear space spanned by the direction vectors of all flags in and at the loop of $\beta(a_1)$) and let $q' : \mathbb{R}^m \to \mathbb{R}^m / V(\beta(a_1))_0^c$ be the quotient map. Let $F = (p_2, E_2)$ be the flag in the loop of D such that p_2 is adjacent to the leaves in the element $P \in \mathcal{P}(a_1)$ which contains F_2. We assume that the direction vector $\mathrm{v}(F)$ of the flag F is non-zero. (Otherwise the weight of a curve of combinatorial type $\beta(a_1)$ in $K(b)$ would be zero.) Define $Y(\alpha_{[F]}, \leq_{[F]})$ as the subvariety of $\mathcal{M}_{0,[2]\cup\{A,B\}}(\Delta_{[F]}, \leq_{[F]})$, which contains all well-spaced curves whose combinatorial type specializes to $(\alpha_{[F]}, \leq_{[F]})$, and set

$$\mathrm{ev}_0 := (q\circ \mathrm{ev})\times(q'\circ(\mathrm{ev}_A-\mathrm{ev}_B))\times(l_p-l_q) : Y(\alpha_{[F]}, \leq_{[F]}) \to \prod_{i\in[2]} \mathbb{R}^m/\operatorname{Lin}(L_i)\times\mathbb{R}^m/V(\beta(a_1))_0^c\times\mathbb{R},$$

where we use the notation from 3.3.14.

It follows from lemmata 3.3.21, 3.3.23 and 1.2.9 and from theorem 3.3.8 that the weight of (D, g) in $\prod_{i\in[2]} \mathrm{ev}_i^*(b_i + L_i)\cdot\mathcal{M}_{1,2}(\Delta,\mathbb{R}^m)^{\mathrm{reg}}$ is given by $\frac{\omega(\beta(a_1))}{2\cdot\operatorname{ind}(\beta(a_1))}$ times the weight of $(D_{[F]}, g_{[F]})$ in

$$\prod_{i\in[2]} \mathrm{ev}_i^*(b_i + L_i)\cdot(\mathrm{ev}_A-\mathrm{ev}_B)^* V(\beta(a_1))_0^c\cdot(l_p-l_q)^*\{0\}\cdot Y(\alpha_{[F]}, \leq_{[F]}),$$

which is $\operatorname{ind}(\mathrm{ev}_0|_{M_0(\beta(a_1)_{[F]})})$, where $M_0(\beta(a_1)_{[F]})$ is the set of curves of combinatorial type $\beta(a_1)_{[F]}$. In order to determine this index as the determinant of a matrix, we choose the standard coordinates on $\mathbb{Z}^m$ and the edge lengths as lattice coordinates on $\mathcal{M}_{0,[2]\cup\{A,B\}}(\Delta_{[F]},\mathbb{R}^m)$. Remember that the marked points x_1 and x_2 are adjacent to the leaves of α whose weighted direction vector is y_1 and y_2, respectively, and that a_k is the weighted direction vector of the flag (p_k, E_k) in the loop of $\beta(a_1)$ if such a flag exists. Then it holds

$$\operatorname{ind}(\mathrm{ev}_0|_{M_0(\beta(a_1)_{[F]})}) = \frac{\operatorname{ind}(\mathcal{P}(a_1))}{\operatorname{ind}(\mathcal{P}_0)\cdot\operatorname{ind}(y_5,\dots,y_{m+1})}\cdot|\det(M'(a_1))|$$

and

$$M'(a_1) = \begin{pmatrix} y_1 & y_2 & Y_5 & a_1 & & & c_3 & c_4 & C_5\\ & & & a_1 & a_2 & a_2 & a_3 & a_4 & A_5\\ & & & & 1 & -1 & & & \end{pmatrix}.$$

The entries 1 and -1 stand for the condition that $l_p = l_q$. The factor $\frac{\operatorname{ind}(\mathcal{P}(a_1))}{\operatorname{ind}(\mathcal{P}_0)}$ appears because of the entries $a_k = -y_k \notin V(\beta(a_1))_{<0}$ in the case that there does not exist $P_i \in \mathcal{P}(a_1)$ with $\min\{j|F_j \in P_i\} = k$, i.e. in the case that there is no edge in the loop of $\beta(a_1)$ labeled by E_k. It follows that the weight of (D, g) in $K(b)$ is

$$\frac{\omega(\beta(a_1)\cdot\operatorname{ind}(\mathcal{P}(a_1))}{\operatorname{ind}(\beta(a_1))\cdot\operatorname{ind}(\mathcal{P}_0)\cdot\operatorname{ind}(y_5,\dots,y_{m+1})}\cdot|\det(M(a_1))|.$$

□

Remark 3.4.12
Note that $\det(M(a_1)) = 0$ if there exists $P \in \mathcal{P}(a_1)$ and pairwise different flags $F_{i_1}, F_{i_2}, F_{i_3} \in \{F_1, F_2, F_3, F_4\}$ that are adjacent to the same vertex in the loop of $\beta(a_1)$: This property either means that the flags F_1 and F_2 are adjacent to the same vertex p_1 or that the direction vectors of flags on the path from x_1 to x_2 together with the generators $y_5,\dots,y_{m+1}$ of $L_1 + L_2$ (which appear in the first row of the matrix $M(a_1)$) do not span $\mathbb{R}^m$.

We proceed with part a) from the list above.

Lemma 3.4.13
For $b = (b_1, b_2) \in (\mathbb{R}^m)^2$ and for $a_1 \in \mathrm{FD}$, we denote by $M(a_1, b)$ the matrix that arises from $M(a_1)$ by replacing the m-th column by $(b_2 - b_1, 0) \in (\mathbb{R}^m)^2$, i.e.

$$M(a_1, b) = \begin{pmatrix} y_1 & y_2 & Y_5 & b_2-b_1 & & c_3 & c_4 & C_5\\ & & & 0 & a_2 & a_3 & a_4 & A_5\end{pmatrix}.$$

If $([b_1], [b_2]) \in \mathcal{G}$, there exists $s(b) \in \{-1, 1\}$ such that the sign of $\det(M(a_1, b))$ is $s(b)$ for all $a_1 \in \mathrm{FD}$. Moreover, there exists a curve of combinatorial type $\beta(a_1)$ in the support of

$$K(b) = \prod_{i\in[2]} \mathrm{ev}^*(b_i + L_i)\cdot\mathcal{M}_{1,2}(\Delta,\mathbb{R}^m)$$

if and only if

$$s(b)\cdot\det(M(a_1)) > 0.$$

PROOF. Let $\mathcal{O}(a_1) = (P_1, \dots, P_{l(a_1)})$ be the ordered partition of the flags at the loop of α induced by $a_1 \in \text{FD}$. In order to determine whether there exists a curve of combinatorial type $\beta(a_1)$ in supp $\left(\prod_{i\in[2]} \text{ev}^*(b_i + L_i) \cdot \mathcal{M}_{1,n}(\Delta, \mathbb{R}^m)\right)$, we have to solve the system of linear equations

$$M(a_1) \cdot v(a_1, b) = \begin{pmatrix} b_2 - b_1 \\ 0 \end{pmatrix},$$

where the entry $v(a_1, b)_{m-1+k}$ stands for the length of the edge E_k with weighted direction vector a_k if such an edge exists, i.e. if there exists $j \in [l(a_1)]$ with $\min P_j = k$. Using Cramer's rule, $v(a_1, b)_m$, which is the length of the edge E_1, can be calculated as

$$v(a_1, b)_m = \frac{1}{\det(M(a_1))} \cdot \det(M(a_1, b)).$$

If and only if $v(a_1, b)_m > 0$, i.e. the edge length of E_1 is greater than zero, the lengths of all edges in the loop of $\beta(a_1)$ are greater than zero and there exists a curve of combinatorial type $\beta(a_1)$ in $K(b)$. (This is true because the direction vectors of flags at the loop of $\beta(a_1)$ fulfill no relation in $V(\beta(a_1))_0 / V(\beta(a_1))_{<0}$ in addition to balancing at the vertices in the loop of $\beta(a_1)$.)
For every edge E_k of $\beta(a_1)$ it holds

$$a_k = a_1 - \sum_{\substack{i\in[m+1]: \\ 1<j_{a_1}(i)\le j_{a_1}(k)}} y_i,$$

where $j_{a_1}(i) \in [l(a_1)]$ is defined such that $F_i \in P_{j_{a_1}(i)}$, see construction 3.4.10. (We subtract all weighted direction vectors y_i of flags F_i in the loop of $\beta(a_1)$ that appear after the flag F_1 and in front of the flag F_2 when we run around the loop of $\beta(a_1)$ according to the ordered partition $\mathcal{O}(a_1)$.
Assume that $F_2 \in P_{l(a_1)}$. Then we conclude

$$\begin{aligned} \det(M(a_1), b) &= \det \begin{pmatrix} y_1 & y_2 & Y_5 & b_2 - b_1 & & & \\ & & & & a_2 & \cdots & a_{m+1} \end{pmatrix} \\ &= \det \begin{pmatrix} y_1 & y_2 & Y_5 & b_2 - b_1 & & & \\ & & & & a_1 - y_2 & -y_3 & \cdots & -y_{m+1} \end{pmatrix}. \end{aligned}$$

Since there exists $\lambda_i \in [0, 1)$, $i \in \{2, \dots, m+2\}$, with $a_1 = \sum_{i=2}^{m+1} \lambda_i \cdot y_i$, it follows that the sign of $\det(M(a_1, b))$ is equal to the sign $s(b)$ of the determinant of

$$M(b) := \begin{pmatrix} y_1 & y_2 & Y_5 & b_2 - b_1 & & & \\ & & & & -y_2 & -y_3 & \cdots & -y_{m+1} \end{pmatrix}$$

and does not depend on $a_1 \in \text{FD}$. If $\det(M(b))$ is zero, $([b_1], [b_2]) \in \mathcal{U}(a)$ is not in general position.
If F_2 is an element of an arbitrary $P_r \in \mathcal{P}(a_1)$, $r \in [l(a_1)]$, and not of $P_{l(a_1)} \in \mathcal{P}(a_1)$, we get the same result with the same argument but more technical effort. □

Next, we deal with part c) of the list above.

Lemma 3.4.14
Let $(\alpha, \le) \le (\beta, \le_\beta)$ be a maximal fine combinatorial type. Then there exist precisely $\frac{2}{\#\,\text{Aut}(\beta)}$ elements $a_1 \in \text{FD}$ such that

$$(\beta, \le_\beta) = (\beta(a_1), \le_{a_1}).$$

PROOF. Let $p_1 \in V(\beta)$ be the vertex that is adjacent to the leaf with direction y_1 and let $\overline{E}_1, \overline{E}_2 \in \mathbf{E}(\beta)$ be the edges in the loop that are adjacent to p (where possibly $\overline{E}_1 = \overline{E}_2$). The vectors $a_1^1 := \omega(\overline{E}_1) \cdot \text{v}(p, \overline{E}_1)$, $a_1^2 := \omega(\overline{E}_2) \cdot \text{v}(p, \overline{E}_2)$ (see the figure below for an illustration) are the only elements of FD that fulfill

$$(\beta, \le_\beta) = (\beta(a_1^1), \le_{a_1^1}) = (\beta(a_1^2), \le_{a_1^2}),$$

where possibly $a_1^1 = a_1^2$, see lemma 3.2.15. Since $(\alpha, \le)$ has codimension one and is non-regular, $a_1^1 = a_1^2$ is true if and only if the loop of β contains only one edge, i.e. $\#\,\text{Aut}(\beta) = 2$. Otherwise, it holds $\#\,\text{Aut}(\beta) = 1$. □

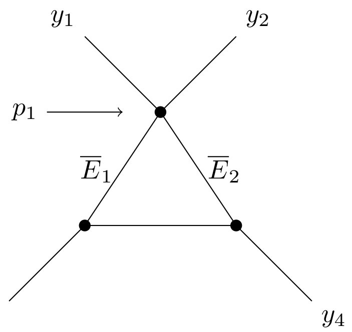


FIGURE 21. Illustration of the nomenclature used in the lemma above. It holds $\omega(\overline{E}_1) \cdot \mathrm{v}(p, \overline{E}_1) = a_1^1$, $\overline{E}_2 \cdot \mathrm{v}(p, \overline{E}_2) = a_1^2$ and $P_1 = \{1, 2\}$.

Here is the proof of part d) from the list above.

Proposition 3.4.15
Let $a_1 \in \mathrm{FD}$. Then it holds $\det(M(a_1)) = \frac{1}{2} z(a_1)$.

PROOF. Let $a_1 \in \mathrm{FD}$ and $\mathcal{O}(a_1) = (P_1, \ldots, P_{l(a_1)})$ be the ordered partition that determines the order of the leaves in the loop, in particular $F_1 \in P_1$. Remember that the direction vector of an edge E_k in the loop of $\beta(a_1)$ is given by

$$a_k = a_1 - \sum_{\substack{i \in [m]: \\ 1 < j_{a_1}(i) \leq j_{a_1}(k)}} y_i$$

and that $j_{a_1}(i) = k \in [l(a_1)]$ means $F_i \in P_k$.
Assume that $1 < j_{a_1}(3) < j_{a_1}(4) \leq j_{a_1}(2)$, i.e. in $\beta(a_1)$ the flags F_1, F_3 and F_4 are adjacent to different vertices and if we run around the loop of $\beta(a_1)$ according to the ordered partition $\mathcal{O}(a_1)$ starting at the flag F_1, we first pass the flag F_3, then F_4 and then F_2 (where F_2 and F_4 might be adjacent to the same vertex). We get (a_2 is replaced by a_1 in the last term because there exist $i_1, \ldots, i_k \in [m+1] \setminus \{3, 4\}$ such that $a_2 = a_1 + \sum_{j=1}^{k} y_{i_j}$):

$$\begin{aligned}
& \det(M(a_1)) \\
= \; & \det\begin{pmatrix} y_1 & y_2 & Y_5 & a_1 & & c_3 & c_4 & C_5 \\ & & & a_1 & a_2 & a_3 & a_4 & A_5 \end{pmatrix} \\
= \; & \det\begin{pmatrix} y_1 & y_2 & Y_5 & a_1 & & -y_3 & -y_4 & -Y_5 \\ & & & a_1 & a_2 & -y_3 & -y_4 & -Y_5 \end{pmatrix} \\
= \; & \det\begin{pmatrix} y_1 & y_2 & Y_5 & & -a_2 & & & \\ & & & a_1 & a_2 & -y_3 & -y_4 & -Y_5 \end{pmatrix} \\
= \; & \det\begin{pmatrix} y_1 & y_2 & Y_5 & & -a_2 & & & \\ & & & a_1 & & -y_3 & -y_4 & -Y_5 \end{pmatrix} \\
= \; & (-1)^{2m} \cdot \det\begin{pmatrix} y_1 & y_2 & Y_5 & a_2 & & & & \\ & & & & y_3 & y_4 & Y_5 & a_1 \end{pmatrix} \\
= \; & \det\begin{pmatrix} y_1 & y_2 & Y_5 & a_1 & & & & \\ & & & & y_3 & y_4 & Y_5 & a_1 \end{pmatrix}.
\end{aligned}$$

There exist $\lambda_i \in [0, 1)$ with $a_1 = \sum_{i=2}^{m+1} \lambda_i \cdot y_i$. Due to $1 = j_{a_1}(1) < j_{a_1}(3) < j_{a_1}(4) \leq j_{a_1}(2)$, it holds $\lambda_3 > \lambda_4 \geq \lambda_2$. It follows $Z(2, 4, a_1) = |\det(y_1, y_3, Y_5, a_1)|$ and

$$\det(y_1, y_3, Y_5, a_1) = \det(y_1, y_3, Y_5, \lambda_2 y_2 + \lambda_4 y_4) = \det(y_1, y_3, Y_5, (\lambda_2 - \lambda_4) \cdot y_2).$$

Moreover, it holds

$$\det(y_1, y_3, Y_5, a_1 + y_2) = \det(y_1, y_3, Y_5, y_2) - \det(y_1, y_3, Y_5, (\lambda_4 - \lambda_2) \cdot y_2)$$

(with $1 > \lambda_4 - \lambda_2 \geq 0$) and hence

$$(Z - Z(2, 4, a_1)) = |\det(y_1, y_3, Y_5, a_1 + y_2)|.$$

We conclude

$$z(2,4,a_1) = Z(2,4,a_1)\cdot(Z - Z(2,4,a_1)) = -\det\begin{pmatrix} y_1 & y_3 & Y_5 & a_1 & & & & \\ & & & & y_1 & y_3 & Y_5 & a_1+y_2 \end{pmatrix}.$$

With similar considerations for $z(1,3,a_1)$, $z(2,3,a_1)$ and $z(1,4,a_1)$ we get

$$\begin{aligned}
& z(a_1) \\
= \;& z(2,4,a_1)+z(1,3,a_1)-z(2,3,a_1)-z(1,4,a_1) \\
= \;& -\det\begin{pmatrix} y_1 & y_3 & Y_5 & a_1 & & & & \\ & & & & y_1 & y_3 & Y_5 & a_1+y_2 \end{pmatrix} - \det\begin{pmatrix} y_2 & y_4 & Y_5 & a_1 & & & & \\ & & & & y_2 & y_4 & Y_5 & a_1-y_3 \end{pmatrix} \\
& +\det\begin{pmatrix} y_1 & y_4 & Y_5 & a_1 & & & & \\ & & & & y_1 & y_4 & Y_5 & a_1-y_3 \end{pmatrix} + \det\begin{pmatrix} y_2 & y_3 & Y_5 & a_1 & & & & \\ & & & & y_2 & y_3 & Y_5 & a_1-y_4 \end{pmatrix}.
\end{aligned}$$

With $y_1+y_3+\sum_{i=5}^{m+1} y_i = -y_2-y_4$ and $y_1+y_4+\sum_{i=5}^{m+1} y_i = -y_2-y_3$ and by splitting up the first two determinants in the expression above into four summands each and the last two determinants into three summands each we get:

$$\begin{aligned}
& z(a_1) \\
= \;& z(2,4,a_1)+z(1,3,a_1)-z(2,3,a_1)-z(1,4,a_1) \\
= \;& -\det\begin{pmatrix} y_1 & y_3 & Y_5 & a_1 & & & & \\ & & & & y_1 & y_3 & Y_5 & a_1+y_2 \end{pmatrix} - \det\begin{pmatrix} y_2 & y_4 & Y_5 & a_1 & & & & \\ & & & & y_2 & y_4 & Y_5 & a_1-y_3 \end{pmatrix} \\
& +\det\begin{pmatrix} y_1 & y_4 & Y_5 & a_1 & & & & \\ & & & & y_1 & y_4 & Y_5 & a_1-y_3 \end{pmatrix} + \det\begin{pmatrix} y_2 & y_3 & Y_5 & a_1 & & & & \\ & & & & y_2 & y_3 & Y_5 & a_1-y_4 \end{pmatrix} \\
= \;& -\det\begin{pmatrix} -y_4 & y_3 & Y_5 & a_1 & & & & \\ & & & & y_1 & -y_2 & Y_5 & a_1+y_2 \end{pmatrix} - \det\begin{pmatrix} -y_4 & y_3 & Y_5 & a_1 & & & & \\ & & & & y_1 & -y_4 & Y_5 & a_1+y_2 \end{pmatrix} \\
& -\det\begin{pmatrix} -y_2 & y_3 & Y_5 & a_1 & & & & \\ & & & & y_1 & -y_2 & Y_5 & a_1+y_2 \end{pmatrix} - \det\begin{pmatrix} -y_2 & y_3 & Y_5 & a_1 & & & & \\ & & & & y_1 & -y_4 & Y_5 & a_1+y_2 \end{pmatrix} \\
& -\det\begin{pmatrix} y_2 & -y_1 & Y_5 & a_1 & & & & \\ & & & & -y_3 & y_4 & Y_5 & a_1-y_3 \end{pmatrix} - \det\begin{pmatrix} y_2 & -y_1 & Y_5 & a_1 & & & & \\ & & & & -y_1 & y_4 & Y_5 & a_1-y_3 \end{pmatrix} \\
& -\det\begin{pmatrix} y_2 & -y_3 & Y_5 & a_1 & & & & \\ & & & & -y_3 & y_4 & Y_5 & a_1-y_3 \end{pmatrix} - \det\begin{pmatrix} y_2 & -y_3 & Y_5 & a_1 & & & & \\ & & & & -y_1 & y_4 & Y_5 & a_1-y_3 \end{pmatrix} \\
& +\det\begin{pmatrix} y_1 & -y_2 & Y_5 & a_1 & & & & \\ & & & & y_1 & y_4 & Y_5 & a_1-y_3 \end{pmatrix} + \det\begin{pmatrix} -y_2 & -y_3 & Y_5 & a_1 & & & & \\ & & & & y_1 & y_4 & Y_5 & a_1-y_3 \end{pmatrix} \\
& +\det\begin{pmatrix} -y_4 & -y_3 & Y_5 & a_1 & & & & \\ & & & & y_1 & y_4 & Y_5 & a_1-y_3 \end{pmatrix} + \det\begin{pmatrix} y_2 & y_3 & Y_5 & a_1 & & & & \\ & & & & -y_4 & y_3 & Y_5 & a_1-y_4 \end{pmatrix} \\
& +\det\begin{pmatrix} y_2 & y_3 & Y_5 & a_1 & & & & \\ & & & & -y_1 & -y_4 & Y_5 & a_1+y_2 \end{pmatrix} + \det\begin{pmatrix} y_2 & y_3 & Y_5 & a_1 & & & & \\ & & & & -y_1 & -y_2 & Y_5 & a_1+y_2 \end{pmatrix}.
\end{aligned}$$

In the term above with fourteen determinants all summands - except those two determinants in which the pair y_1 and y_2 and the pair y_3 and y_4 appears - have a partner with the opposite sign and cancel out. It follows

$$z(a_1) = 2\det\begin{pmatrix} y_1 & y_2 & Y_5 & a_1 & & & & \\ & & & & y_3 & y_4 & Y_5 & a_1 \end{pmatrix} = 2\cdot\det(M(a_1)).$$

Assume now that $1 < j_{a_1}(3) \le j_{a_1}(2) \le j_{a_1}(4)$ and $j_{a_1}(3) < j_{a_1}(4)$. It follows (a_2 is replaced by $a_1 - y_3$ and a_1 in the last two determinants because there exist $i_1,\dots,i_k \in [m+1]\setminus\{1,2,3\}$ such that $a_2 = a_1 + y_1 + \sum_{j=1}^{k} y_{i_k}$)

$$\begin{aligned}
& \det(M(a_1)) \\
= \;& \det\begin{pmatrix} y_1 & y_2 & Y_5 & a_1 & & a_3 & & C_5 \\ & & & a_1 & a_2 & a_3 & a_4 & A_5 \end{pmatrix} \\
= \;& \det\begin{pmatrix} y_1 & -y_3 & Y_5 & a_1 & & -y_3 & & \\ & & & a_1 & a_2 & -y_3 & -y_4 & -Y_5 \end{pmatrix} + \det\begin{pmatrix} y_1 & -y_4 & Y_5 & a_1 & & -y_3 & -y_4 & \\ & & & a_1 & a_2 & -y_3 & -y_4 & -Y_5 \end{pmatrix} \\
= \;& \det\begin{pmatrix} y_1 & -y_3 & Y_5 & a_1 & & & & \\ & & & a_1 & a_2 & y_1 & -y_4 & -Y_5 \end{pmatrix} + \det\begin{pmatrix} y_1 & -y_3 & Y_5 & a_1 & & & & \\ & & & a_1 & a_2 & y_2 & -y_4 & -Y_5 \end{pmatrix} \\
& +\det\begin{pmatrix} y_1 & -y_4 & Y_5 & & -a_2 & & & Y_5 \\ & & & a_1 & a_2 & -y_3 & y_1 & -Y_5 \end{pmatrix} + \det\begin{pmatrix} y_1 & -y_4 & Y_5 & & -a_2 & & & Y_5 \\ & & & a_1 & a_2 & -y_3 & y_2 & -Y_5 \end{pmatrix} \\
= \;& (-1)^{2m}\det\begin{pmatrix} y_1 & y_3 & Y_5 & a_1 & & & & \\ & & & & y_2 & y_4 & Y_5 & a_1-y_3 \end{pmatrix} - (-1)^{2m}\det\begin{pmatrix} y_1 & y_4 & Y_5 & a_2 & & & & \\ & & & & y_2 & y_3 & Y_5 & a_1 \end{pmatrix} \\
= \;& \det\begin{pmatrix} y_1 & y_3 & Y_5 & a_1 & & & & \\ & & & & y_2 & y_4 & Y_5 & a_1-y_3 \end{pmatrix} - \det\begin{pmatrix} y_1 & y_4 & Y_5 & a_1 & & & & \\ & & & & y_2 & y_3 & Y_5 & a_1 \end{pmatrix}.
\end{aligned}$$

For $a_1 = \sum_{i=2}^{m+1} \lambda_i y_i$, it holds $\lambda_3 \geq \lambda_2 \geq \lambda_4$ and $\lambda_3 > \lambda_4$ because of $1 < j_{a_1}(3) \leq j_{a_1}(2) \leq j_{a_1}(4)$ and $j_{a_1}(3) < j_{a_1}(4)$. With the same considerations as in the first case, we conclude:

$$
\begin{aligned}
& z(a_1) \\
= \; & z(1,3,a_1) + z(2,4,a_1) - z(1,4,a_1) - z(2,4,a_1) \\
= \; & -\det\begin{pmatrix} y_1 & y_3 & Y_5 & a_1 \\ y_1 & y_3 & Y_5 & a_1 + y_4 \end{pmatrix} - \det\begin{pmatrix} y_2 & y_4 & Y_5 & a_1 \\ y_2 & y_4 & Y_5 & a_1 - y_3 \end{pmatrix} \\
& + \det\begin{pmatrix} y_1 & y_4 & Y_5 & a_1 \\ y_1 & y_4 & Y_5 & a_1 - y_3 \end{pmatrix} + \det\begin{pmatrix} y_2 & y_3 & Y_5 & a_1 \\ y_2 & y_3 & Y_5 & a_1 + y_1 \end{pmatrix}.
\end{aligned}
$$

By splitting up each of the four determinants above into three summands it follows:

$$
\begin{aligned}
& z(a_1) \\
= \; & -\det\begin{pmatrix} y_1 & y_3 & Y_5 & a_1 \\ -y_4 & -y_2 & Y_5 & a_1 - y_3 \end{pmatrix} - \det\begin{pmatrix} y_1 & y_3 & Y_5 & a_1 \\ -y_3 & -y_2 & Y_5 & a_1 - y_3 \end{pmatrix} \\
& - \det\begin{pmatrix} y_1 & y_3 & Y_5 & a_1 \\ y_1 & -y_4 & Y_5 & a_1 + y_4 \end{pmatrix} - \det\begin{pmatrix} -y_3 & -y_1 & Y_5 & a_1 \\ y_2 & y_4 & Y_5 & a_1 - y_3 \end{pmatrix} \\
& - \det\begin{pmatrix} -y_3 & -y_2 & Y_5 & a_1 \\ y_2 & y_4 & Y_5 & a_1 - y_3 \end{pmatrix} - \det\begin{pmatrix} -y_1 & y_4 & Y_5 & a_1 \\ y_2 & y_4 & Y_5 & a_1 - y_3 \end{pmatrix} \\
& + \det\begin{pmatrix} y_1 & y_4 & Y_5 & a_1 \\ -y_3 & -y_2 & Y_5 & a_1 - y_3 \end{pmatrix} + \det\begin{pmatrix} y_1 & y_4 & Y_5 & a_1 \\ -y_3 & -y_1 & Y_5 & a_1 - y_3 \end{pmatrix} \\
& + \det\begin{pmatrix} y_1 & y_4 & Y_5 & a_1 \\ -y_2 & y_4 & Y_5 & a_1 - y_3 \end{pmatrix} + \det\begin{pmatrix} y_2 & y_3 & Y_5 & a_1 \\ -y_4 & -y_1 & Y_5 & a_1 + y_1 \end{pmatrix} \\
& + \det\begin{pmatrix} y_2 & y_3 & Y_5 & a_1 \\ -y_4 & -y_2 & Y_5 & a_1 - y_3 \end{pmatrix} + \det\begin{pmatrix} y_2 & y_3 & Y_5 & a_1 \\ -y_1 & y_3 & Y_5 & a_1 + y_1 \end{pmatrix}.
\end{aligned}
$$

The determinants in which the pair y_1 and y_3 and the pair y_2 and y_3 appear in the same row and the determinants in which the pair y_1 and y_3 and the pair y_1 and y_4 appear in the same row have a partner with the opposite sign and they cancel out. It follows

$$
\begin{aligned}
& z(a_1) \\
= \; & 2 \det\begin{pmatrix} y_1 & x_3 & Y_5 & a_1 \\ y_2 & y_4 & Y_5 & a_1 - y_3 \end{pmatrix} - 2 \cdot \det\begin{pmatrix} y_1 & y_4 & Y_5 & a_1 \\ y_2 & y_3 & Y_5 & a_1 \end{pmatrix} \\
= \; & 2 \det(M(a_1)).
\end{aligned}
$$

The remaining cases can be proven accordingly. □

Finally, we deal with part e) from the list above.

Proposition 3.4.16
It holds

$$\sum_{a_1 \in \mathrm{FD}} \# \operatorname{Aut}(\beta(a_1)) \cdot \frac{\omega(\beta(a_1)) \cdot \operatorname{ind}(\mathcal{P}(a_1))}{\operatorname{ind}(\beta(a_1)) \cdot \operatorname{ind}(\mathcal{P}_0)} \cdot z(a_1) = 0,$$

where $\mathcal{P}_0 = \{\{F_i\} | i \in [m+1]\}$ is the finest partition of the flags at the loop of α.

For the proof, we need some lemmata:

Lemma 3.4.17
Let $m \in \mathbb{N}$ and $c_1, \ldots, c_m \in \mathbb{N}$ and $Q := \prod_{i \in [m]} c_i$. Then it holds:

$$\frac{1}{6}(Q^3 - Q) = \sum_{k=1}^{m} \left(\prod_{j=1}^{k-1} c_j \right) \left(\sum_{i=0}^{c_k} (c_k - i) \cdot i \right) \cdot \left(\prod_{j=k+1}^{m} c_j \right)^3$$

Proof. We proof the claim by induction on $m \in \mathbb{N}$. The case $m = 1$ can be seen using the formulae for the summation of the first c_1 integers and the first c_1 squares. Assume that the claim is true for some $m \in \mathbb{N}$. It holds

$$\begin{aligned}\frac{1}{6}(Q^3 - Q) &= \frac{1}{6}\left(\frac{Q}{c_1}\right)^3((c_1)^3 - c_1) + \frac{c_1}{6}\left(\left(\frac{Q}{c_1}\right)^3 - \frac{Q}{c_1}\right) \\ &= \left(\sum_{i=0}^{c_1}(c_1 - i)\cdot i\right)\left(\frac{Q}{c_1}\right)^3 + c_1 \cdot \sum_{k=2}^{m}\left(\prod_{j=2}^{k-1} c_j\right)\left(\sum_{i=0}^{c_k}(c_k - i)\cdot i\right)\left(\prod_{j=k+1}^{m} c_j\right)^3 \\ &= \sum_{k=1}^{m}\left(\prod_{j=1}^{k-1} c_j\right)\left(\sum_{i=0}^{c_k}(c_k - i)\cdot i\right)\cdot\left(\prod_{j=k+1}^{m} c_j\right)^3,\end{aligned}$$

where the second equation follows by the induction hypothesis. □

Notation 3.4.18 ($\mathcal{P}^{(i,j)}$, $\mathcal{P}(\mathcal{S})_k$, $\mathrm{FD}(\mathcal{P})$)
Let $\mathcal{P} \in \mathbf{P}^0(\alpha)$ and $i_1, i_2 \in [m+1]$. We set

$$\mathcal{P}^{(i_1,i_2)} := \left(\mathcal{P} \setminus \{P | F_{i_1} \in P \text{ or } F_{i_2} \in P\}\right) \cup \left\{ \bigcup_{\substack{P\in\mathcal{P}: \\ (F_{i_1}\in P \text{ or } F_{i_2}\in P)}} P \right\},$$

i.e. we unite the elements of $\mathcal{P}$ containing the flags F_{i_1} and F_{i_2}.
For an m-tuple $\mathcal{S} = ((i_1^1, i_2^1), \ldots, (i_1^m, i_2^m))$ with $i_k^l \in [m+1]$ ($k \in \{1,2\}$ and $l \in [m]$) and for $j \in [m]$, we define the partition $\mathcal{P}(\mathcal{S})_j \in \mathbf{P}^0(\alpha)$ recursively via $\mathcal{P}(\mathcal{S})_0 = \mathcal{P}_0$ and

$$\mathcal{P}(\mathcal{S})_j = (\mathcal{P}(\mathcal{S})_{j-1})^{\mathcal{S}_j},$$

i.e. we unite subsets of $\mathcal{P}_0$ as prescribed by the first j entries of $\mathcal{S}$. If $\mathcal{P} \in \mathbf{P}^0(\alpha)$, we define

$$\mathrm{FD}(\mathcal{P}) := \{a_1 \in \mathrm{FD}\,|\mathcal{P} \leq \mathcal{P}(a_1)\},$$

i.e. $a_1 \in \mathrm{FD}$ is an element of $\mathrm{FD}(\mathcal{P})$ if the partition $\mathcal{P}(a_1)$ of the flags at the loop of α given by the vertices in the loop of $\beta(a_1)$ is coarser than the partition $\mathcal{P}$. It holds $\mathcal{P}_1 \leq \mathcal{P}_2$ if and only if $\mathrm{FD}(\mathcal{P}_2) \subset \mathrm{FD}(\mathcal{P}_1)$. $\mathrm{FD}(\mathcal{P})$ is the intersection of

$$\mathrm{FD} = \{\sum_{i=2}^{m+1} \lambda_i y_i\} \cap \mathbb{Z}$$

with the vector space spanned by the vectors

$$y_P = \sum_{F_i \in P} y_i$$

for $P \in \mathcal{P}$. Note that

$$\langle y_P | P \in \mathcal{P}\rangle \subset \langle y_P | P \in \mathcal{P}'\rangle$$

for all $\mathcal{P}, \mathcal{P}' \in \mathbf{P}^0(\alpha)$ which fulfill $\mathcal{P}' \leq \mathcal{P}$.

Remark 3.4.19
Note that $(\mathcal{P}^{T_1})^{T_2} = (\mathcal{P}^{T_2})^{T_1}$ for all $\mathcal{P} \in \mathbf{P}^0(\alpha)$ and all pairs $T_1, T_2 \in [m+1]^2$.

Lemma 3.4.20
Let $\mathcal{S} = ((i_1^1, i_2^1), \ldots, (i_1^m, i_2^m))$ be an m-tuple as above that fulfills that $\mathcal{P}(\mathcal{S})_m = \{\mathbf{FS}_0(\alpha)\}$ is the coarsest partition of the flags at the loop of α and that $i_1^j < i_2^j$ for all $j = 1, \ldots, m$. Then it holds

$$\sum_{j=1}^{m} \sum_{a_1 \in \mathrm{FD}(\mathcal{P}(\mathcal{S})_j)} z(i_1^j, i_2^j, a_1) \cdot \frac{\mathrm{ind}(\mathcal{P}(\mathcal{S})_j)}{\mathrm{ind}(\mathcal{P}_0)} = \frac{1}{6}(\mathrm{ind}(\mathcal{P}_0)^3 - \mathrm{ind}(\mathcal{P}_0)).$$

Proof. The key to the proof is the previous lemma. For all $j \in [m]$ let $b_j \in \mathbb{R}^m$ be a vector that complements a lattice basis of

$$\langle y_i \mid i \in [m+1] \setminus \{(\mathcal{S}_j)_1, (\mathcal{S}_j)_2\}\rangle \cap \mathbb{Z}^m$$

to a lattice basis of $\mathbb{Z}^m$. Put

$$c_j := \frac{\operatorname{ind}(\mathcal{P}(\mathcal{S})_{j-1})}{\operatorname{ind}(\mathcal{P}(\mathcal{S})_j)},$$

then it holds

$$\prod_{j\in[m]} c_j = \operatorname{ind}(\mathcal{P}_0)$$

because of $\mathcal{P}(\mathcal{S})_m = \{\mathbf{FS}_0(\alpha)\}$ and $\operatorname{ind}(\mathbf{FS}_0(\alpha)) = 1$ (remember that $\mathbf{FS}_0(\alpha) = \{F_1, \dots, F_{m+1}\}$ and $\sum_{i\in[m+1]} y_i = 0$). With the previous lemma we conclude the following:

$$\begin{aligned} & \frac{1}{6}(\operatorname{ind}(\mathcal{P}_0)^3 - \operatorname{ind}(\mathcal{P})) \\ = \; & \sum_{j=1}^{m} \left(\prod_{k=1}^{j-1} c_k\right) \cdot \left(\sum_{i=1}^{c_j} (c_j - i)\cdot i\right) \cdot \left(\prod_{k=j+1}^{m} c_k\right)^3 \cdot \frac{\prod_{k=1}^{j-1} c_k}{\prod_{k=1}^{j-1} c_k} \end{aligned}$$

It holds for all $j \in [m]$ and $i \in \mathbb{N}$ that

$$\begin{aligned} & Z(i_1^j, i_2^j, i\cdot b_j) \\ = \; & |\det(y_1, \dots, y_{i_1^j-1}, y_{i_1^j+1}, \dots, y_{i_2^j-1}, y_{i_2^j+1}, \dots, y_{m+1}, i\cdot b_j)| \\ = \; & \left(\prod_{k=1}^{j-1} c_k\right) \cdot i \cdot \left(\prod_{k=j+1}^{m} c_k\right). \end{aligned}$$

and

$$Z = |\det(y_1, \dots, y_{i_1^j-1}, y_{i_1^j+1}, \dots, y_{i_2^j-1}, y_{i_2^j+1}, \dots, y_{m+1}, c_j\cdot b_j)|$$

We conclude:

$$\begin{aligned} & \frac{1}{6}(\operatorname{ind}(\mathcal{P}_0)^3 - \operatorname{ind}(\mathcal{P})) \\ = \; & \sum_{j=1}^{m}\sum_{i=1}^{c_j} Z(i_1^j, i_2^j, i\cdot b_j)\cdot (Z - Z(i_1^j, i_2^j, i\cdot b_j)) \cdot \frac{\prod_{i=j+1}^{m} c_i}{\prod_{i=1}^{j-1} c_i} \\ = \; & \sum_{j=1}^{m}\sum_{i=1}^{c_j} Z(i_1^j, i_2^j, i\cdot b_j)\cdot (Z - Z(i_1^j, i_2^j, i\cdot b_j)) \cdot \operatorname{ind}(\mathcal{P}(\mathcal{S})_{j+1}) \cdot \left(\frac{\operatorname{ind}(\mathcal{P}(\mathcal{S})_0)}{\operatorname{ind}(\mathcal{P}(\mathcal{S})_j)}\right)^{-1} \\ = \; & \sum_{j=1}^{m}\sum_{a_1\in \operatorname{FD}(\mathcal{P}_j)} Z(i_1^j, i_2^j, a_1)\cdot (Z - Z(i_1^j, i_2^j, a_1)) \cdot \frac{\operatorname{ind}(\mathcal{P}(\mathcal{S})_j)}{\operatorname{ind}(\mathcal{P}_0)} \end{aligned}$$

□

Construction 3.4.21 ($\mathbf{R}^0(\alpha)$, $\mathbf{T}$)
Let $\mathcal{P} \in \mathbf{P}^0(\alpha)$ be a partition of the set of flags at the loop of α such that F_1, F_2, F_3, F_4 are elements of pairwise different elements P_1, P_2, P_3, $P_4 \in \mathcal{P}$. Denote the set of such partitions $\mathcal{P}$ by $\mathbf{R}^0(\alpha)$ and for $\mathcal{P} \in \mathbf{R}^0(\alpha)$ set

$$\mathcal{P}_{[4]} = (\mathcal{P} \setminus \{P \mid \exists i \in [4] : i \in P\}) \cup \{\bigcup_{\substack{P\in\mathcal{P}:\\ (\exists i\in[4]: i\in P)}} P\},$$

i.e. we unite the sets containing $1, 2, 3$ and 4, respectively. Define

$$T_1 = (1,3), T_2 = (1,4), T_3 = (2,4), T_4 = (2,3) \text{ and } \mathbf{T} = \{T_1, T_2, T_3, T_4\}.$$

It holds $z(a_1) = \sum_{i=1}^{4}(-1)^{i+1} z(T_i, a_1)$.

Lemma 3.4.22
Let $\mathcal{P} \in \mathbf{R}^0(\alpha)$, i.e. the flags $F_1, \ldots, F_4$ are contained in four different elements of $\mathcal{P}$. Then:

$$\begin{aligned} 0 \quad = \quad & \sum_{a_1 \in \mathrm{FD}(\mathcal{P})} z(a_1) \cdot \frac{\mathrm{ind}(\mathcal{P})}{\mathrm{ind}(\mathcal{P}_0)} \\ & - \sum_{i \in [4]} \sum_{a_1 \in \mathrm{FD}(\mathcal{P}^{T_i})} z(a_1) \cdot \frac{\mathrm{ind}(\mathcal{P}^{T_i})}{\mathrm{ind}(\mathcal{P}_0)} \\ & + \sum_{i=1}^{2} \sum_{a_1 \in \mathrm{FD}((\mathcal{P}^{T_i})^{T_{i+2}})} z(a_1) \cdot \frac{\mathrm{ind}((\mathcal{P}^{T_i})^{T_{i+2}})}{\mathrm{ind}(\mathcal{P}_0)} \end{aligned}$$

PROOF. Fix $\mathcal{P} \in \mathbf{R}^0(\alpha)$ and an m-tuple

$$\mathcal{S}_{\mathcal{P}} = ((i_1^{\mathcal{P},1}, i_2^{\mathcal{P},1}), \ldots, (i_1^{\mathcal{P},m}, i_2^{\mathcal{P},m}))$$

that fulfills that there exists $d(\mathcal{P}) \in [m]$ with

a) $\mathcal{P}((\mathcal{S}_{\mathcal{P}})_{d(\mathcal{P})}) = \mathcal{P}$, i.e. after uniting $d(\mathcal{P})$ times elements of $\mathcal{P}_0$ as prescribed by $\mathcal{S}_{\mathcal{P}}$ we get $\mathcal{P}$, and
b) $\mathcal{P}((\mathcal{S}_{\mathcal{P}})_{d(\mathcal{P})+3}) = \mathcal{P}_{[4]}$, i.e. after uniting $(d(\mathcal{P})+3)$ times elements of $\mathcal{P}_0$ as prescribed by $\mathcal{S}_{\mathcal{P}}$ we get $\mathcal{P}_{[4]}$ and
c) $\mathcal{P}((\mathcal{S}_{\mathcal{P}})_m) = \{\mathbf{FS}_0(\alpha)\}$.

Moreover, for any different $i, j \in [4]$ fix $c(j,i) = c(i,j) \in [4] \setminus \{i, j\}$ and define the m-tuple of index pairs

$$S_{\mathcal{P}}^{i,j} = \Big((i_1^{\mathcal{P},1}, i_2^{\mathcal{P},1}), \ldots, (i_1^{\mathcal{P},d(\mathcal{P})}, i_2^{\mathcal{P},d(\mathcal{P})}), T_i, T_j, T_{c(i,j)}, (i_1^{\mathcal{P},d(\mathcal{P})+4}, i_2^{\mathcal{P},d(\mathcal{P})+4}), \ldots, (i_1^{\mathcal{P},m}, i_2^{\mathcal{P},m})\Big),$$

which fulfills $\mathcal{P}((\mathcal{S}_{\mathcal{P}}^{i,j})_m) = \{\mathbf{FS}_0(\alpha)\}$. With the last lemma, we conclude

$$\mathcal{G}_{\mathcal{P}}^{i,j} := \sum_{k=1}^{m} \sum_{a_1 \in \mathrm{FD}(P(S_{\mathcal{P}}^{i,j})_{k-1})} z((S_{\mathcal{P}}^{i,j})_k, a_1) \cdot \frac{\mathrm{ind}(\mathcal{P}(S_{\mathcal{P}}^{i,j})_k)}{\mathrm{ind}(\mathcal{P}_0)} = \frac{1}{6}(\mathrm{ind}(\mathcal{P}_0)^3 - \mathrm{ind}(\mathcal{P}_0)).$$

There are twelve of these expression $\mathcal{G}_{\mathcal{P}}^{i,j}$ in the following sum, six with a positive and six with a negative sign, hence

$$0 = \sum_{i \in [4]} (-1)^{i+1} \sum_{\substack{j \in [4],\\ j \neq i}} (-1)^{i+j} \cdot \mathcal{G}_{\mathcal{P}}^{i,j}.$$

The summands of $\mathcal{G}_{\mathcal{P}}^{i,j}$ differ for different $i, j \in [4]$ only for $k \in \{d(\mathcal{P})+1, d(\mathcal{P})+2, d(\mathcal{P})+3\}$ and the remaining summands cancel out in the sum above which defines $\mathcal{G}_{\mathcal{P}}^{i,j}$. Moreover, for all $i \in [4]$ and $a_1 \in \mathrm{FD}((\mathcal{P}_0)^{T_i})$ (i.e. a_1 is an element of the hyperplane spanned by $\{y_j | j \neq (T_i)_1, j \neq (T_i)_2\}$), it holds $z(T_i, a_1) = 0$. Therefore, it follows with remark 3.4.19 and due to

$z(a_1) = \sum_{i=1}^{4}(-1)^{i+1}z(T_i, a_1)$ that

$$\begin{aligned}
0 &= \sum_{i\in[4]}(-1)^{i+1}\sum_{\substack{j\in[4],\\ j\neq i}}(-1)^{i+j}\cdot\mathcal{G}_{\mathcal{P}}^{i,j}\\
&= \sum_{a_1\in\mathrm{FD}(\mathcal{P})}\sum_{i=1}^{4}(-1)^{i}z(T_i,a_1)\cdot\frac{\mathrm{ind}(\mathcal{P})}{\mathrm{ind}(\mathcal{P}_0)}\\
&+ \sum_{i=1}^{4}\sum_{a_1\in\mathrm{FD}(\mathcal{P}^{T_i})}\sum_{j\in[4]\setminus\{i\}}(-1)^{j+1}z(T_j,a_1)\cdot\frac{\mathrm{ind}(\mathcal{P}^{T_i})}{\mathrm{ind}(\mathcal{P}_0)}\\
&+ \sum_{i=1}^{4}\sum_{j\in[4]\setminus\{i\}}\sum_{a_1\in\mathrm{FD}((\mathcal{P}^{T_i})^{T_j})}(-1)^{j+1}z(T_{c(i,j)},a_1)\cdot\frac{\mathrm{ind}((\mathcal{P}^{T_i})^{T_j})}{\mathrm{ind}(\mathcal{P}_0)}\\
&= \sum_{a_1\in\mathrm{FD}(\mathcal{P})}(-z(a_1))\cdot\frac{\mathrm{ind}(\mathcal{P})}{\mathrm{ind}(\mathcal{P}_0)}\\
&+ \sum_{i=1}^{4}\sum_{a_1\in\mathrm{FD}(\mathcal{P}^{T_i})}\sum_{j=1}^{4}(-1)^{j+1}z(T_j,a_1)\cdot\frac{\mathrm{ind}(\mathcal{P}^{T_i})}{\mathrm{ind}(\mathcal{P}_0)}\\
&+ \sum_{i=1}^{2}\sum_{a_1\in\mathrm{FD}((\mathcal{P}^{T_i})^{T_{i+2}})}2\cdot(-1)^{i+1}z(T_{c(i,i+2)},a_1)\cdot\frac{\mathrm{ind}((\mathcal{P}^{T_i})^{T_{i+2}})}{\mathrm{ind}(\mathcal{P}_0)}
\end{aligned}$$

The last equation is true because $c(i,j) = c(j,i)$. Moreover, it holds $z(T_j, a_1) = z(T_{j+2}, a_1)$ for $a_1 \in \mathrm{FD}((\mathcal{P}^{T_i})^{T_{i+2}})$ if $i, j \in [2]$ with $i \neq j$ and therefore

$$2\cdot(-1)^{i+1}z(T_{c(i,i+2)},a_1) = \sum_{k=1}^{4}(-1)^{k}z(T_k,a_1) = -z(a_1).$$

□

Corollary 3.4.23
Assume that $m = 3$. Then the map N_a^W is constant under the assumptions of proposition 3.4.7.

PROOF. It remains to check that

$$\sum_{a_1\in\mathrm{FD}}\#\,\mathrm{Aut}(\beta(a_1))\cdot\frac{\omega(\beta(a_1))\cdot\mathrm{ind}(\mathcal{P}(a_1))}{\mathrm{ind}(\beta(a_1))\cdot\mathrm{ind}(\mathcal{P}_0)}\cdot z(a_1) = 0.$$

Let $a_1 \in \mathrm{FD}$ and note that it holds $\#\mathbf{FS}_0(\alpha) = m + 1 = 4$. With remark 3.4.12 it follows that $\det(M(a_1)) = \frac{1}{2}z(a_1) = 0$, if there exists $P \in \mathcal{P}(a_1)$ which contains more than three flags at the loop of α, i.e.

$$\#P = \#(P \cap \{F_1, F_2, F_3, F_4\}) \geq 3.$$

We conclude that it holds $\#P \leq 2$ if $P \in \mathcal{P}(a_1)$ and $z(a_1) \neq 0$. If $z(a_1) \neq 0$ it hence follows

$$n(\mathcal{P}(a_1)) := \left[\prod_{P\in\mathcal{P}(a_1)}(-1)^{\#P-1}(\#P-1)!\right] \in \{-1, +1\}.$$

More precisely, it holds

- $n(\mathcal{P}(a_1)) = 1$ if $\mathcal{P}(a_1) = \mathcal{P}_0$ is the finest partition of $\mathbf{FS}_0(\alpha)$,
- $n(\mathcal{P}(a_1)) = -1$ if there exists $i_1, i_2, i_3, i_4 \in [4]$ with $\mathcal{P}(a_1) = \{\{F_{i_1}, F_{i_2}\}, \{F_{i_3}\}, \{F_{i_4}\}\}$ and
- $n(\mathcal{P}(a_1)) = 1$ if there exist $i_1, i_2, i_3, i_4 \in [4]$ with $\mathcal{P}(a_1) = \{\{F_{i_1}, F_{i_2}\}, \{F_{i_3}, F_{i_4}\}\}$.

Moreover, it holds $z(a_1) = 0$ if there exists $P \in \mathcal{P}(a_1)$ with $\{F_1, F_2\} \subset P$ or $\{F_3, F_4\} \subset P$, i.e. if the flags F_1 and F_2 or the flags F_3 and F_4 sit at the same vertex in the loop of $\beta(a_1)$.

Due to

$$\omega(\beta(a_1)) = \frac{1}{\mathrm{Aut}(\beta(a_1))} \cdot \sum_{\mathcal{P} \le \mathcal{P}(a_1)} n(\mathcal{P}) \cdot \mathrm{ind}(\beta(a_1)_{\mathcal{P}})$$

we conclude with the previous lemma that

$$\begin{aligned}
& \sum_{a_1 \in \mathrm{FD}} \# \mathrm{Aut}(\beta(a_1)) \cdot \frac{\omega(\beta(a_1)) \cdot \mathrm{ind}(\mathcal{P}(a_1))}{\mathrm{ind}(\beta(a_1)) \cdot \mathrm{ind}(\mathcal{P}_0)} \cdot z(a_1) \\
= & \sum_{a_1 \in \mathrm{FD}} \sum_{\mathcal{P} \le \mathcal{P}(a_1)} n(\mathcal{P}) \frac{\mathrm{ind}(\beta(a_1)_{\mathcal{P}}) \cdot \mathrm{ind}(\mathcal{P}(a_1))}{\mathrm{ind}(\beta(a_1)) \cdot \mathrm{ind}(\mathcal{P}_0)} \cdot z(a_1) \\
= & \sum_{a_1 \in \mathrm{FD}} \sum_{\mathcal{P} \le \mathcal{P}(a_1)} n(\mathcal{P}) \cdot \frac{\mathrm{ind}(\mathcal{P})}{\mathrm{ind}(\mathcal{P}_0)} \cdot z(a_1) \\
= & \sum_{\mathcal{P} \le \mathcal{P}_0} \sum_{a_1 \in \mathrm{FD}(\mathcal{P})} n(\mathcal{P}) \cdot \frac{\mathrm{ind}(\mathcal{P})}{\mathrm{ind}(\mathcal{P}_0)} \cdot z(a_1) \\
= & \sum_{a_1 \in \mathrm{FD}(\mathcal{P}_0)} z(a_1) \cdot \frac{\mathrm{ind}(\mathcal{P})}{\mathrm{ind}(\mathcal{P}_0)} \\
& - \sum_{i \in [4]} \sum_{a_1 \in \mathrm{FD}(\mathcal{P}_0^{T_i})} z(a_1) \cdot \frac{\mathrm{ind}(\mathcal{P}^{T_i})}{\mathrm{ind}(\mathcal{P}_0)} \\
& + \sum_{i=1}^{2} \sum_{a_1 \in \mathrm{FD}((\mathcal{P}_0^{T_i})^{T_{i+2}})} z(a_1) \cdot \frac{\mathrm{ind}((\mathcal{P}^{T_i})^{T_{i+2}})}{\mathrm{ind}(\mathcal{P}_0)} \\
= & \ 0
\end{aligned}$$

□

Proof of proposition 3.4.16. For any $\mathcal{P} \in \mathbf{P}^0(\alpha)$, it holds $a_1 \in \mathrm{FD}(\mathcal{P})$ if and only if $\mathcal{P} \le \mathcal{P}(a_1)$. Therefore, it follows from the previous lemma that

$$\begin{aligned}
0 = & \sum_{\mathcal{P} \in \mathbf{R}^0(\alpha)} \prod_{P \in \mathcal{P}} n(P) \cdot \Bigg[\sum_{a_1 \in \mathrm{FD}(\mathcal{P})} \cdot z(a_1) \cdot \frac{\mathrm{ind}(\mathcal{P})}{\mathrm{ind}(\mathcal{P}_0)} \\
& - \sum_{i \in [4]} \sum_{a_1 \in \mathrm{FD}(\mathcal{P}^{T_i})} \cdot z(a_1) \cdot \frac{\mathrm{ind}(\mathcal{P}^{T_i})}{\mathrm{ind}(\mathcal{P}_0)} \\
& + \sum_{i=1}^{2} \sum_{a_1 \in \mathrm{FD}((P^{T_i})^{T_{i+2}})} \cdot z(a_1) \cdot \frac{\mathrm{ind}((\mathcal{P}^{T_i})^{T_{i+2}})}{\mathrm{ind}(\mathcal{P}_0)} \Bigg] \\
= & \sum_{a_1 \in \mathrm{FD}} \sum_{\substack{\mathcal{P} \in \mathbf{R}^0(\alpha): \\ \mathcal{P} \le \mathcal{P}(a_1)}} \left(\prod_{P \in \mathcal{P}} n(P) \right) \cdot z(a_1) \cdot \frac{\mathrm{ind}(\mathcal{P})}{\mathrm{ind}(\mathcal{P}_0)} \\
& - \sum_{i=1}^{4} \sum_{\substack{a_1 \in \mathrm{FD}: \\ (\exists \mathcal{R} \in \mathbf{R}^0(\alpha): a_1 \in \mathrm{FD}(\mathcal{R}^{T_i}))}} \sum_{\substack{\mathcal{P} \in \mathbf{R}^0(\alpha): \\ \mathcal{P}^{T_i} \le \mathcal{P}(a_1)}} \left(\prod_{P \in \mathcal{P}} n(P) \right) \cdot z(a_1) \cdot \frac{\mathrm{ind}(\mathcal{P}^{T_i})}{\mathrm{ind}(\mathcal{P}_0)} \\
& + \sum_{i=1}^{2} \sum_{\substack{a_1 \in \mathrm{FD}: \\ (\exists \mathcal{R} \in \mathbf{R}^0(\alpha): a_1 \in \mathrm{FD}((\mathcal{R}^{T_i})^{T_{i+2}}))}} \sum_{\substack{\mathcal{P} \in \mathbf{R}^0(\alpha): \\ (\mathcal{P}^{T_i})^{T_{i+2}} \le \mathcal{P}(a_1)}} \left(\prod_{P \in \mathcal{P}} n(P) \right) \cdot z(a_1) \cdot \frac{\mathrm{ind}((\mathcal{P}^{T_i})^{T_{i+2}})}{\mathrm{ind}(\mathcal{P}_0)}
\end{aligned}$$

Let $T = (i_1, i_2) \in \mathbf{T}$, $\mathcal{R} \in \mathbf{R}^0(\alpha)$ and set $\mathcal{A} := \mathcal{R}^T$. Let us calculate

$$\sum_{\substack{\mathcal{P}\in\mathbf{R}^0(\alpha),\\ \mathcal{P}^T=\mathcal{A}}} \prod_{P\in\mathcal{P}} n(P).$$

Assume that the element of $\mathcal{A}$ containing i_1 and i_2 has cardinality $k+2 \in \mathbb{N}$ and let $0 \le l \le k$. Then there are $\binom{k}{l}$ elements $\mathcal{P} \in \mathbf{R}^0(\alpha)$ with the following properties:

- The element of $\mathcal{P}$ containing i_1 has $l+1$ elements.
- The element of $\mathcal{P}$ containing i_2 has $k-l+1$ elements.
- $\mathcal{P}^T = \mathcal{A}$.

Therefore it holds:

$$\begin{aligned}
&\sum_{\substack{\mathcal{P}\in\mathbf{R}^0(\alpha)\\ \mathcal{P}^T=\mathcal{A}}} \prod_{P\in\mathcal{P}} (-1)^{\#P-1}(\#P-1)! \\
= \;& \sum_{k=0}^{l} \binom{k}{l} \left(\prod_{\substack{P\in\mathcal{A},\\ i_1,i_2\notin P}} (-1)^{\#P-1}(\#P-1)! \right) \cdot l! \cdot (k-l)! \cdot (-1)^{l+k-l} \\
= \;& (-1)^k \cdot (k+1)! \prod_{\substack{P\in\mathcal{A},\\ i_1,i_2\notin P}} (-1)^{\#P-1}(\#P-1)! \\
= \;& -\prod_{P\in\mathcal{A}} (-1)^{\#P-1}(\#P-1)!
\end{aligned}$$

Let now $i \in \{1,2\}$, $\mathcal{R} \in \mathbf{R}^0(\alpha)$ and set $\mathcal{A} := (\mathcal{R}^{T_i})^{T_{i+2}}$. We will calculate

$$\sum_{\substack{\mathcal{P}\in\mathbf{R}^0(\alpha)\\ (\mathcal{P}^{T_i})^{T_j}=\mathcal{A}}} \prod_{P\in\mathcal{P}} n(P).$$

Assume that the element of $\mathcal{A}$ containing $(T_i)_1$ and $(T_i)_2$ has cardinality k_1+2 and that the element of $\mathcal{A}$ containing $(T_{i+2})_1$ and $(T_{i+2})_2$ has cardinality k_2+2. Let $0 \le l_1 \le k_1$ and $0 \le l_2 \le k_2$. Then there exist $\binom{k_1}{l_1} \cdot \binom{k_2}{l_2}$ elements $\mathcal{P} \in \mathbf{R}^0(\alpha)$ with the following properties:

- The element of $\mathcal{P}$ containing $(T_i)_1$ has l_1+1 elements.
- The element of $\mathcal{P}$ containing $(T_i)_2$ has k_1-l_1+1 elements.
- The element of $\mathcal{P}$ containing $(T_{i+2})_1$ has l_2+1 elements.
- The element of $\mathcal{P}$ containing $(T_{i+2})_2$ has k_2-l_2+1 elements.
- $(\mathcal{P}^{T_i})^{T_{i+2}} = (\mathcal{P}^{T_{i+2}})^{T_i} = \mathcal{A}$.

It follows:

$$\begin{aligned}
&\sum_{\substack{\mathcal{P}\in\mathbf{R}^0(\alpha)\\ (\mathcal{P}^{T_i})^{T_j}=\mathcal{A}}} \prod_{P\in\mathcal{P}} (-1)^{\#P-1}(\#P-1)! \\
= \;& \sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2} \binom{k_1}{l_1} \cdot \binom{k_2}{l_2} \cdot \left(\prod_{\substack{P\in\mathcal{A},\\ [4]\cap P=\emptyset}} (-1)^{\#P-1}(\#P-1)! \right) \cdot (k_1-l_1)! \cdot (k_2-l_2)! \cdot (-1)^{k_1+k_2} \\
= \;& (k_1+1)! \cdot (k_2+1)! \cdot \prod_{\substack{P\in\mathcal{A},\\ [4]\cap P=\emptyset}} (-1)^{\#P-1}(\#P-1)! \\
= \;& \prod_{P\in\mathcal{A}} (-1)^{\#P-1}(\#P-1)!
\end{aligned}$$

Since it holds $\det(M(a_1)) = \frac{1}{2}z(a_1) = 0$ if there exists $P \in \mathcal{P}(a_1)$ containing three of the flags $F_1, F_2, F_3, F_4 \in \mathbf{FS}_0(\alpha)$ (see remark 3.4.12), we conclude:

$$\begin{aligned}
0 &= \sum_{a_1 \in \mathrm{FD}} \sum_{\mathcal{A} \le \mathcal{P}(a_1) : \mathcal{A} \in \mathbf{R}^0(\alpha)} \left(\prod_{A \in \mathcal{A}} n(A) \right) \cdot z(a_1) \cdot \frac{\mathrm{ind}(\mathcal{A})}{\mathrm{ind}(\mathcal{P}_0)} \\
&\quad + \sum_{i=1}^{4} \sum_{\substack{a_1 \in \mathrm{FD}: \\ (\exists \mathcal{R} \in \mathbf{R}^0(\alpha) : a_1 \in \mathrm{FD}(\mathcal{R}^{T_i}))}} \sum_{\substack{\mathcal{A} \le \mathcal{P}(a_1): \\ \left(\exists \mathcal{P} \in \mathbf{R}^0(\alpha) : \mathcal{A} = \mathcal{P}^{T_i}\right)}} \left(\prod_{A \in \mathcal{A}} n(A) \right) \cdot z(a_1) \cdot \frac{\mathrm{ind}(\mathcal{A})}{\mathrm{ind}(\mathcal{P}_0)} \\
&\quad + \sum_{i=1}^{2} \sum_{\substack{a_1 \in \mathrm{FD}: \\ (\exists \mathcal{R} \in \mathbf{R}^0(\alpha) : a_1 \in \mathrm{FD}((\mathcal{R}^{T_i})^{T_{i+2}}))}} \sum_{\substack{\mathcal{A} \le \mathcal{P}(a_1): \\ \left(\exists \mathcal{P} \in \mathbf{R}^0(\alpha) : \mathcal{A} = (\mathcal{P}^{T_i})^{T_{i+2}}\right)}} \left(\prod_{A \in \mathcal{A}} n(A) \right) \cdot z(a_1) \cdot \frac{\mathrm{ind}(\mathcal{A})}{\mathrm{ind}(\mathcal{P}_0)} \\
&= \sum_{a_1 \in \mathrm{FD}} \sum_{\mathcal{P} \le \mathcal{P}(a_1)} \left(\prod_{P \in \mathcal{P}} n(P) \right) \cdot z(a_1) \cdot \frac{\mathrm{ind}(\mathcal{P})}{\mathrm{ind}(\mathcal{P}_0)} \\
&= \sum_{a_1 \in \mathrm{FD}} \sum_{\mathcal{P} \le \mathcal{P}(a_1)} \left(\prod_{P \in \mathcal{P}} n(P) \right) \cdot \frac{\mathrm{ind}(\beta(a_1)_{\mathcal{P}}) \cdot \mathrm{ind}(\mathcal{P}(a_1))}{\mathrm{ind}(\beta(a_1)) \cdot \mathrm{ind}(\mathcal{P}_0)} \cdot z(a_1) \\
&= \sum_{a_1 \in \mathrm{FD}} \# \mathrm{Aut}(\beta(a_1)) \cdot \frac{\omega(\beta(a_1)) \cdot \mathrm{ind}(\mathcal{P}(a_1))}{\mathrm{ind}(\beta(a_1)) \cdot \mathrm{ind}(\mathcal{P}_0)} \cdot z(a_1)
\end{aligned}$$

□

3.4.2. Invariance in the general case. We generalize proposition 3.4.7 and prove the main theorem 3.4.3 which states that the degree of

$$\prod_{i \in [n]} \mathrm{ev}_i^*(a_i + L_i) \cdot \mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)^{\mathrm{reg}}$$

does not depend on $(a_1, \ldots, a_n) \in (\mathbb{R}^m)^n$ as long as $([a_1], \ldots, [a_n]) \in \mathcal{G}$ is in general position.

Proposition 3.4.24
Assume that we are in the third case of (B). Then the map N_a^W is constant.

This proposition is proven as a corollary of proposition 3.4.16. Again $(C, h) \in \mathcal{M}_{1,n}(\Delta, \mathbb{R}^m)$ is the unique element of $\pi_1((\mathrm{ev}')^{-1}\{a\} \cap W)$ and has fine combinatorial type $(\alpha, \le)$. (C, h) is non-regular and has a vertex of genus one.

As in the special case treated in the previous subsection, denote the weighted direction vectors of the flags $F_1, \ldots, F_s \in \mathbf{FS}_0(\alpha)$ at the loop of α by $y_1, \ldots, y_s$. Since $(\alpha, \le)$ has codimension one and is non-regular, it holds that the vectors $y_1, \ldots, y_{s-1} \in \mathbb{R}^m$ are linearly independent and $\sum_{i=1}^{s} y_i = 0$. It holds $\dim\langle y_1, \ldots, y_s \rangle \ge 3$ and hence $s \ge 4$ because otherwise a non-regular fine combinatorial type $(\alpha, \le)$ would have codimension at least two.

Define

$$\mathrm{FD}(y_1, \ldots, y_s) = \left\{ \sum_{i=2}^{s} \lambda_i \cdot y_i | 0 \le \lambda_i < 1 \; \forall \, i = 2, \ldots, s \right\} \cap \mathbb{Z}^m$$

as the intersection of $\mathbb{Z}^m$ with a fundamental domain of the lattice spanned by $y_2, \ldots, y_s$. As in the special case, for all maximal fine combinatorial types $(\alpha, \le) \le (\beta, \le_\beta)$ in $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ there exist $\frac{2}{\mathrm{Aut}(\beta(a_1))}$ elements $a_1 \in \mathrm{FD}(y_1, \ldots, y_s)$ such that

$$(\beta, \le_\beta) = (\beta(a_1), \le_{a_1}).$$

Similar to the special case, the weight of a curve of fine combinatorial type $(\beta(a_1), \le_{a_1})$ in

$$K(b) = \prod_{i \in [n]} \mathrm{ev}_i^*(b_i + L_i) \cdot \mathcal{M}_{1,n}(\Delta, \mathbb{R}^m)^{\mathrm{reg}}$$

is given by $\frac{\omega(\beta(a_1))}{2\cdot \mathrm{ind}(\beta(a_1))}$ times the weight of the corresponding curve in

$$\prod_{i\in[n]} \mathrm{ev}_i^*(b_i+L_i)\cdot(\mathrm{ev}_A-\mathrm{ev}_B)^*V(\beta(a_1))_0^c\cdot(l_p-l_q)^*\{0\}\cdot Y(\alpha_{[F]},\leq_{[F]}),$$

where $V(\beta(a_1))_0^c$ is a complement of $V(\beta(a_1))_0$, $Y(\alpha_{[F]},\leq_{[F]})$ is the variety containing all well-spaced curves in $\mathcal{M}_{0,I\cup\{A,B\}}(\Delta_{[F]},\mathbb{R}^m)$ whose fine combinatorial type specializes to $(\alpha_{[F]},\leq_{[F]})$ and all facets are equipped with weight one. See construction 3.3.4 for the remaining notation. This weight is given by the index of the map ev_0 given by

$$(q\circ \mathrm{ev})\times(q'\circ(\mathrm{ev}_A-\mathrm{ev}_B))\times(l_p-l_q): Y(\alpha_{[F]},\leq_{[F]})\to\prod_{i\in[n]}\mathbb{R}^m/\operatorname{Lin}(L_i)\times\mathbb{R}^m/V(\beta(a_1))_0^c\times\mathbb{R}$$

restricted to the polyhedron containing curves of combinatorial type $\beta(a_1)_{[F]}$, where

$$q:(\mathbb{R}^m)^n\to\prod_{i\in[n]}\mathbb{R}^m/\operatorname{Lin}(L_i) \text{ and } q':\mathbb{R}^m\to\mathbb{R}^m/V(\beta(a_1))_0^c$$

are the projections. Similarly to the special case, the weight of a curve of fine combinatorial type $(\beta(a_1),\leq_{a_1})$ in $K(b)$ is therefore given by

$$\frac{\omega(\beta(a_1))\cdot\mathrm{ind}(\mathcal{P}(a_1))}{\mathrm{ind}(\beta(a_1))\cdot\mathrm{ind}(\mathcal{P}_0)}\cdot|\det(M(a_1))|,$$

where $\mathcal{P}(a_1)$ is the partition of the set of flags $\mathbf{FS}_0(\alpha)$ at the loop of α given by $\beta(a_1)$, where $\mathcal{P}_0$ is the finest partition of $\mathbf{FS}_0(\alpha)$ and where the matrix $M(a_1)$ has the following form:

$$M(a_1)=\begin{pmatrix} R'_0 & R'_2 & \cdots & R'_q & R'_1 & & \\ & & & & R_{1,2} & & \\ & & & & \vdots & & \\ & & & & R_{1,r_1} & & \\ & R_{2,1} & & & R_1 & C_2(a_1) & \\ & \vdots & & & \vdots & \vdots & \\ & R_{2,r_2} & & & R_1 & C_2(a_1) & \\ & & \ddots & & \vdots & \vdots & \\ & & & R_{q,1} & R_1 & C_q(a_1) & \\ & & & \vdots & \vdots & \vdots & \\ & & & R_{q,r_q} & R_1 & C_q(a_1) & \\ & & & & & A(a_1) & B \end{pmatrix}.$$

The notation is explained below. The idea of this representation of the matrix $M(a_1)$ is that the submatrices with a capital R stand for the conditions $\beta(a_1)$ outside the loop (in the "rational" part of $\beta(a_1)$, which does not depend on a_1) and that the submatrices $C_j(a_1)$ and $A(a_1)$ encode the conditions in the loop of $\beta(a_1)$.

In order to calculate the index of the map ev_0 defined above, we choose edge lengths and a root vertex as coordinates on the polyhedron containing curves of combinatorial type $\beta(a_1)$. Without loss of generality we assume that all marked points of α lie behind the flags $F_1,\dots,F_q$ at the loop of α, seen from the loop, with $2\leq q\leq s$. Since $(\alpha,\leq)$ has codimension one in $\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)$ and since α is non-regular, it follows that there are no marked points in the loop of α (which is a single point). Otherwise α had codimension at least two. Hence, also the regular resolutions of α do not have marked points in the loop. If there existed only one flag F_1 at the loop behind which lie marked points, the map ev_0 would not be injective on the polyhedron containing curves of fine combinatorial type $(\alpha_{[F]},\leq_{[F]})$ and N_a^W would be the constant zero function. For $j\in[q]$, assume that there are r_q marked points that lie in α behind the flag $F_q\in\mathbf{FS}_0$. In particular, it holds $\sum_{j=1}^q r_q=n$ because we have altogether n marked points on α. Choose as root vertex a marked point, which we denote by x_1, that lies behind the flag F_1. We denote the linear part of the incidence condition that is fulfilled by the k-th marked point that lies behind F_j by $L_j^k \in \{L_1, \dots, L_n\}$, $j \in [q]$ and $k \in [r_j]$.

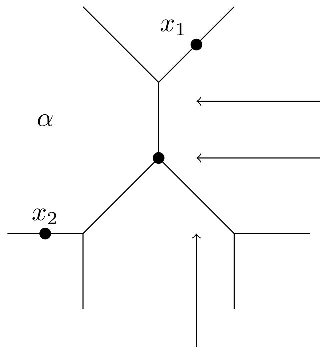


FIGURE 22. Example of a non-regular elliptic combinatorial type α. In a curve whose fine combinatorial type is a resolution of $(\alpha, \leq)$, the length of a flag at the loop behind which lies no marked point is determined by the partial order $\leq$ on the vertices of α given by well-spacedness (at least if the weight of the curve is non-zero).

By $R_{1,2}, \dots, R_{1,r_1}$ we denote the matrices given by a path from x_1 to one of the remaining $r_1 - 1$ marked points that lie behind F_1 together with columns that span a lattice basis of L_1^1 and of L_1^k, $k = 2, \dots, r_1$. The columns of these matrices lie in $\mathbb{R}^m$. A column is the zero vector if the corresponding edge that lies behind F_1 does not appear in the path from x_1 to the considered other marked point behind F_1, and the number of columns of each of these matrices equals the number of bounded edges that lie behind the flag F_1 plus the sum over the dimensions of the incidence conditions which lie behind the flag F_1.

Let the matrix R_1 (with columns in $\mathbb{R}^m$) encodes the path from x_1 to the loop of α: The columns of this matrix are either zero or one of the weighted direction vectors that appear in the path from x_1 to the loop or one of the vectors that span a lattice basis of L_1^1 (which is the incidence condition corresponding to the root vertex x_1). The number of columns of R_1 coincides with the number of columns of each of the matrices $R_{1,2}, \dots, R_{1,r_1}$.

For $j = 2, \dots, q$ and $i = 1, \dots, r_j$, we define $R_{j,i}$ as the matrices which encode the path from F_j to the r_j marked points of α that lie behind the flag F_j, i.e. the columns of these matrices are either zero, a weighted direction vector of a flag that lies in the part of α behind F_j or a vector that is part of a lattice basis of an incidence condition that lies behind F_j. The columns of $R_{j,i}$ lie in $\mathbb{R}^m$, and the number of columns of each $R_{j,i}$, $i \in [r_j]$, equals the number of bounded edges behind F_j plus the sum over the dimension of the incidence conditions behind F_j.

Up till now, the submatrices encode the incidence conditions. Now we come to the conditions given by well-spacedness and the total preorder $\leq$ on the vertices of α.

By $R'_0, \dots, R'_q$ we denote the matrices that stand altogether for the conditions on the edge lengths of bounded edges in α which are imposed by the total preorder $\leq$ on the vertices of α. A matrix R'_j has the same number of columns as each of the matrices $R_{j,i}$ (where $j \in [q]$ and $i \in [r_j]$). The columns of R'_0 stand for the bounded edges of α which lie behind one of the flags $F_{q+1}, \dots, F_s$ at the loop of α behind which lies no marked point.

Note that up till now the described submatrices of $M(a_1)$ only depend on α and not on the resolution $\beta(a_1)$.

We define the k-th column of the matrix $A(a_1) \in \mathbb{R}^{m \times s}$ as the weighted direction vector a_k of the edge E_k if an edge with this label exists in the loop of $\beta(a_1)$ and we define a_k otherwise to be zero. (Remember that we label a vertex p_k in the loop of $\beta(a_1)$ by the minimal index k of a flag at the loop in $\{F_1, \dots, F_s\} = \mathbf{FS}_0(\alpha)$ that is adjacent to p_k, and that we label the edge in the loop of $\beta(a_1)$ that lies behind the vertex p_k when we run around the loop according to the ordered partition $\mathcal{O}(a_1)$ by E_k.) This notation $a_1, \ldots, a_s$ is similar to the special case.

In the spirit of the definition of the submatrix $C \in \mathbb{R}^{m\times(m+1)}$ of $M(a_1)$ in the special case, for $j \in [q]$, we define $C_j(a_1) \in \mathbb{R}^{m\times s}$ as the matrix that encodes the path in the loop of $\beta(a_1)$ from the flag F_1 to the flag F_j, $j \in [q]$, i.e. $(C_j(a_1))_k = a_k$ if the path from F_1 to F_j passes the edge E_k (where the direction of this path in the loop of $\beta(a_1)$ is given by the ordered partition $\mathcal{O}(a_1)$ of $\mathbf{FS}_0(\alpha)$) and otherwise $(C_j(a_1))_k = 0$.)

The columns of the matrix $B \in \mathbb{R}^{m\times(m-s+1)}$ are a lattice basis of a complement of the vector space spanned by the weighted direction vectors $y_1, \ldots, y_s$ of the flags at the loop of α, i.e. B does not depend on $a_1 \in \mathrm{FD}(y_1, \ldots, y_s)$.

Remark 3.4.25
Let $a_1 \in \mathrm{FD}(y_1, \ldots, y_s)$. Similarly to the special case we see (using Cramer's rule in one of the columns of $M(a_1)$ with the matrices $C_j(a_1)$ and $A(a_1)$) that the sign of the determinant of $M(a_1)$ determines whether there appears a curve of fine combinatorial type $(\beta(a_1), \leq_{a_1})$ in the intersection product $K(b) = \prod_{i=1}^n \mathrm{ev}_i^*(b_i + L_i)\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$.

The key to the proof of proposition 3.4.24 is the following corollary of proposition 3.4.16.

Corollary 3.4.26
Let $z_1, \ldots, z_{m-1} \in \mathbb{R}^m$ and set $X = (z_1, \ldots, z_{m-1}) \in \mathbb{R}^{m\times(m-1)}$. Then it holds for all $j \in \{2, \ldots, q\}$ that

$$\sum_{a_1\in\mathrm{FD}(y_1,\ldots,y_s)} \#\,\mathrm{Aut}(\beta(a_1)) \cdot \frac{\omega(\beta(a_1)) \cdot \mathrm{ind}(\mathcal{P}(a_1))}{\mathrm{ind}(\beta(a_1)) \cdot \mathrm{ind}(\mathcal{P}_0)} \cdot \det\begin{pmatrix} X & C_j(a_1) & \\ & A(a_1) & B \end{pmatrix} = 0,$$

where $C_j(a_1)$, $A(a_1)$ and B are the submatrices of $M(a_1)$ defined above.

Proof. Let $a_1 \in \mathrm{FD}(y_1, \ldots, y_s)$, $j \in \{2, \ldots, q\}$ and consider the matrix

$$N(X, a_1) = \begin{pmatrix} X & C_j(a_1) & \\ & A(a_1) & B \end{pmatrix} \in \mathbb{R}^{2m\times 2m}.$$

We first consider a special type of matrix $X = (z_1, \ldots, z_{m-1}) \in \mathbb{R}^{m\times(m-1)}$. Assume that there exists $i_1, \ldots, i_{s-2} \in [s]$ such that $z_{i_1}, \ldots, z_{i_{s-2}} \in \{y_1, \ldots, y_s\}$. Assume moreover that $y_1, y_j \in \{z_{i_1}, \ldots, z_{i_{s-2}}\}$ where $j \in \{2, \ldots, q\}$ was fixed above. Let the remaining vectors $z_k \in \mathbb{R}^m$ with $k \in [m-1]$ and $k \notin \{i_1, \ldots, i_{s-2}\}$ be given by the columns of the matrix B.

Remember that it holds by the choice of B that $\mathrm{im}\, B \oplus \langle y_1, \ldots, y_s\rangle = \mathbb{R}^m$ and $\dim\langle y_1, \ldots, y_s\rangle = s-1$. Since the weighted direction vectors of all flags in the loop of $\beta(a_1)$ are contained in $\langle y_1, \ldots, y_s\rangle$, it follows that all columns of X, $A(a_1)$, $C_j(a_1)$ and B are contained either in $\langle y_1, \ldots, y_s\rangle$ or in $\mathrm{im}(B)$. We project all columns of X, $C_j(a_1)$ and $A(a_1)$ to $\langle y_1, \ldots, y_s\rangle$, choose lattice coordinates on $\langle y_1, \ldots, y_s\rangle$ and get a matrix

$$N'(a_1) = \begin{pmatrix} X' & C_j'(a_1) \\ & A'(a_1) \end{pmatrix} \in \mathbb{R}^{2(s-1)\times 2(s-1)},$$

where $\dim\langle y_1, \ldots, y_s\rangle = s-1$.

Since the columns of B are a lattice basis of a complement of $\langle y_1, \ldots, y_s\rangle$, there exists $e \in \{-1, 1\}$ such that $\det(N'(a_1)) = e \cdot \det(N(X, a_1))$ for all $a_1 \in \mathrm{FD}(y_1, \ldots, y_s)$. It holds moreover that the columns in the matrix X' stand for $s-2$ of the vectors $y_1, \ldots, y_s$ including y_1 and y_j. We conclude with propositions 3.4.15 and 3.4.16 that

$$\sum_{a_1\in\mathrm{FD}(y_1,\ldots,y_s)} \#\,\mathrm{Aut}(\beta(a_1)) \cdot \frac{\omega(\beta(a_1)) \cdot \mathrm{ind}(\mathcal{P}(a_1))}{\mathrm{ind}(\beta(a_1)) \cdot \mathrm{ind}(\mathcal{P}_0)} \cdot \det(N(X, a_1)) = 0.$$

Let now $z_1, \ldots, z_{m-1} \in \mathbb{R}^m$ be arbitrary. Using the multilinearity of the determinant in the first $m-1$ rows of the matrix $N(a_1)$, there exists $r \in \mathbb{N}$ and matrices $X_1, \ldots, X_r \in \mathbb{R}^{m\times(m-1)}$ of the special type studied above in this proof such that we can express $\det(N(X, a_1))$ as a linear combination of the determinants of $N(X_1, a_1), \ldots, N(X_r, a_1)$. This linear combination can be chosen independently of $a_1 \in \mathrm{FD}(y_1, \ldots, y_s)$, and the claim follows. □

Corollary 3.4.27
It holds

$$\sum_{a_1 \in \mathrm{FD}(y_1,\ldots,y_s)} \#\,\mathrm{Aut}(\beta(a_1)) \cdot \frac{\omega(\beta(a_1)) \cdot \mathrm{ind}(\mathcal{P}(a_1))}{\mathrm{ind}(\beta(a_1)) \cdot \mathrm{ind}(\mathcal{P}_0)} \cdot \det(M(a_1)) = 0,$$

Proof. Let $a_1 \in \mathrm{FD}(y_1,\ldots,y_s)$. We will show that $\det(M(a_1))$ is a linear combination of determinants of matrices of the form

$$N(X,a_1) = \begin{pmatrix} X & C_j(a_1) & \\ & A(a_1) & B \end{pmatrix} \in \mathbb{R}^{m\times m}$$

(with $X \in \mathbb{R}^{m\times(m-1)}$) that appear in the lemma above. It holds moreover that the coefficients of this linear combination do not depend on $a_1 \in \mathrm{FD}(y_1,\ldots,y_s)$.
The determinant of $M(a_1)$, defined at the beginning of this subsection, is equal to the determinant of (use row operations)

$$M'(a_1) = \begin{pmatrix}
R'_0 & R'_2 & \cdots & R'_q & R'_1 & & \\
 & & & & R_{1,2} & & \\
 & & & & \vdots & & \\
 & & & & R_{1,r_1} & & \\
 & R_{2,1} & & & R_1 & C_2(a_1) & \\
 & R_{2,2}-R_{2,1} & & & 0 & 0 & \\
 & \vdots & & & \vdots & \vdots & \\
 & R_{2,r_2}-R_{2,1} & & & 0 & 0 & \\
 & & \ddots & & \vdots & \vdots & \\
 & & & R_{q,1} & R_1 & C_q(a_1) & \\
 & & & R_{q,2}-R_{q,1} & 0 & 0 & \\
 & & & \vdots & \vdots & \vdots & \\
 & & & R_{q,r_q}-R_{q,1} & 0 & 0 & \\
 & & & & & A(a_1) & B(a_1)
\end{pmatrix}.$$

By using Laplace's formula in all rows of $M'(a_1)$ but those with the matrices $C_2(a_1),\ldots,C_q(a_1)$ and $A(a_1)$, we get that $\det(M'(a_1))$ (and hence $\det(M(a_1))$) is a linear combination of determinants of matrices of the form

$$N'(a_1) = \begin{pmatrix}
S_2 & & & S & C_2(a_1) & \\
 & \ddots & & \vdots & \vdots & \\
 & & S_q & S & C_q(a_1) & \\
 & & & & A(a_1) & B
\end{pmatrix} \in \mathbb{R}^{mq\times mq}.$$

The matrix S comes from the submatrix R_1 of $M'(a_1)$ by deleting columns and the matrices S_j come from $R_{j,1}$ for $j = 2,\ldots,q$ by deleting columns. In particular, this linear combination is independent of $a_1 \in \mathrm{FD}(y_1,\ldots,y_s)$.
Assume that $q > 2$ and assume that there exists $j \in \{2,\ldots,q\}$ such that the matrix S_j has m columns, say $j = 2$, then the determinant of $N'(a_1)$ is equal to a constant multiple (which does not depend on a_1) of the determinant of

$$N''(a_1) = \begin{pmatrix}
S_3 & & & S & C_3(a_1) & \\
 & \ddots & & S & \vdots & \\
 & & S_q & S & C_q(a_1) & \\
 & & & & A(a_1) & B
\end{pmatrix} \in \mathbb{R}^{(m-1)q\times(m-1)q}.$$

Hence, we have reduced the number of blocks in the matrix.
If $q > 2$ and if all submatrices $S_2,\ldots,S_q$ of $N'(a_1)$ have less than m columns, it follows that the submatrix S of $N'(a_1)$ has at least one column:

The matrix $(A(a_1)|B)$ has $s + (m - s + 1) = m + 1$ columns and the matrices $S_2, \ldots, S_q$ at most $(q-1)(m-1)$ columns altogether (because all matrices $S_2, \ldots, S_q$ have strictly less than m columns). Hence, S has at least

$$qm - (m+1) - (q-1)(m-1) = q - 2 > 0$$

columns.

Because S has at least one column, we may use the the multilinearity of the determinant in the columns of $N''(a_1)$ with the matrix S and get (by permuting the columns with the submatrix S) that the determinant of $N''(a_1)$ is a linear combination of determinants of matrices of the form

$$N'''(a_1) = \begin{pmatrix} S_3' & & & C_3(a_1) & \\ & \ddots & & \vdots & \\ & & S_q' & C_q(a_1) & \\ & & & A(a_1) & B \end{pmatrix} \in \mathbb{R}^{(m-1)q \times (m-1)q}.$$

Again this linear combination does not depend on $a_1 \in \mathrm{FD}$. Now, there exists at least one matrix S_j' which has m columns, and we can reduce the number of blocks as in the case above.

It follows that we can express $N'(a_1)$ (and hence $M(a_1)$) as a linear combination - that can be chosen independently of $a_1 \in \mathrm{FD}(y_1, \ldots, y_s)$ - of matrices of the form

$$N(X, a_1) = \begin{pmatrix} Y_2 & Y & C_j(a_1) & \\ & & A(a_1) & B \end{pmatrix} = \begin{pmatrix} X & C_j(a_1) & \\ & A(a_1) & B \end{pmatrix} \in \mathbb{R}^{2m \times 2m},$$

where $j \in \{2, \ldots, q\}$.

This means that there exists $n_j \in \mathbb{N}$ and $\lambda_i^j \in \mathbb{R}$, $X_i^j \in \mathbb{R}^{m \times m-1}$ for $j \in \{2, \ldots, q\}$ and $i \in [n_j]$ such that it holds for all $a_1 \in \mathrm{FD}(y_1, \ldots, y_s)$

$$\det(M(a_1)) = \sum_{j=2}^{q} \sum_{i=1}^{n_q} \lambda_i^j \cdot \det \begin{pmatrix} X_i^j & C_j(a_1) & \\ & A(a_1) & B \end{pmatrix}.$$

Now the claim follows with the previous corollary. □

PROOF OF 3.4.24. As in the special case with two marked points treated in the previous subsection, for all maximal $(\alpha, \leq) \leq (\beta, \leq_\beta)$, there exist $\frac{2}{\mathrm{Aut}(\beta)}$ elements $a_1 \in \mathrm{FD}(y_1, \ldots, y_s)$ that fulfill

$$(\beta, \leq_\beta) = (\beta(a_1), \leq_{a_1}).$$

The weight of a curve of fine combinatorial type $(\beta(a_1), \leq_{a_1})$ is given by

$$\frac{\omega(\beta(a_1)) \cdot \mathrm{ind}(\mathcal{P}(a_1))}{\mathrm{ind}(\beta(a_1)) \cdot \mathrm{ind}(\mathcal{P}_0)} \cdot |\det(M(a_1))|,$$

and the sign of $\det(M(a_1))$ determines whether a curve of fine combinatorial type $(\beta(a_1), \leq_{a_1})$ appears in the intersection product

$$K(b) = \prod_{i=1}^{n} \mathrm{ev}_i^*(b_i + L_i) \mathcal{M}_{1,I}(\Delta, \mathbb{R}^m).$$

If $(\beta, \leq_\beta) = (\beta(a_1), \leq_{a_1})$, we set $\det(M(\beta)) = \det(M(a_1))$ and $\mathcal{P}(\beta) = \mathcal{P}(a_1)$.

With the previous corollary it follows that

$$\begin{aligned} & \sum_{(\alpha, \leq) < (\beta, \leq_\beta)} \frac{\omega(\beta) \cdot \mathrm{ind}(\mathcal{P}(\beta))}{\mathrm{ind}(\beta) \cdot \mathrm{ind}(\mathcal{P}_0)} \cdot \det(M(\beta)) \\ = & \sum_{a_1 \in \mathrm{FD}(y_1, \ldots, y_2)} \frac{\mathrm{Aut}(\beta(a_1))}{2} \cdot \frac{\omega(\beta(a_1)) \cdot \mathrm{ind}(\mathcal{P}(a_1))}{\mathrm{ind}(\beta(a_1)) \cdot \mathrm{ind}(\mathcal{P}_0)} \cdot \det(M(a_1)) \\ = & \ 0, \end{aligned}$$

and similarly to the proof of proposition 3.4.7 we see the map N_a^W is constant. □

Proof of theorem 3.4.3. In proposition 3.4.5 we have shown that the map N_a^W is constant in case (A) and in the first two cases of (B). Proposition 3.4.24 states that N_a^W is constant also in the third case of (B). Using the argument at the beginning of this section, we conclude that the map

$$\begin{array}{rccc} N_{\Delta,\mathcal{L}}: & \mathcal{G} & \to & \mathbb{N} \\ & ([a_i])_{i\in[n]} & \mapsto & \deg\left(\prod_{i\in[n]} \mathrm{ev}_i^*(a_i+L_i)\cdot \mathcal{M}_{1,n}(\Delta,\mathbb{R}^m)^{\mathrm{reg}}\right) \end{array}$$

is constant, i.e. the degree of the intersection product does not depend on the position of the translated fans $a_1+L_1,\ldots,a_n+L_n$ as long as $q(a_1,\ldots,a_n)\in\mathcal{G}$ is in general position. □

Index of notations

Polyhedral complexes and tropical varieties.

- Let τ be a general polyhedron in a vector space $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ with lattice Λ. Then we denote by
$$\mathrm{W}(\tau)$$
the smallest linear space in $\Lambda_{\mathbb{R}}$ which contains $x - y$ for all $x, y \in \tau$. By τ° we denote the relative interior of τ.
- If $\mathcal{X} = (X, Y, \{\phi_\sigma\})$ is a polyhedral complex, we denote its polyhedral structure X by $\mathrm{pol}(\mathcal{X})$, its support Y by $\mathrm{supp}(\mathcal{X})$ and call the maps ϕ_σ polyhedral charts. For $\sigma \in \mathrm{pol}(\mathcal{X})$ we set
$$\mathrm{W}(\sigma) = \mathrm{W}(\phi_\sigma(\sigma)) \text{ and } U_{\mathcal{X}}(\sigma) = \bigcup_{\sigma \subset \tau} \tau^\circ.$$

Tropical curves.

- Let C be an I-marked curve where I is an index set labeling the leaves of C. By
$$\mathbf{V}(C), \mathbf{E}(C), \mathbf{F}(C) \text{ and } \mathbf{FS}(C)$$
we denote the set of vertices, edges, flags and flag segments of C. Vertices, edges, flags an flag segments are defined in 1.3.1. For an abstract combinatorial type Γ and a combinatorial type α, we denote the vertices, edges, flags and flag segments in the same way.
- We define
$$\omega(E)$$
as the index of the map $h \circ \phi_E^{-1}|_{\phi_E(E)}$, where (C, h) is a parametrized curve in $\mathbb{R}^m$ and $E \in \mathbf{E}(C)$ an edge of C with polyhedral chart $\phi_E : E \to \mathbb{R}$.
- By
$$\mathrm{v}_{(C,h)}(p, E) \in \mathbb{R}^m \text{ and } \mathrm{v}^\omega_{(C,h)}(p, E) = \omega(E) \cdot \mathrm{v}_{(C,h)}(p, E)$$
we denote the direction vector and the weighted direction vector of the flag $(p, E) \in \mathbf{F}(C)$ of a parametrized curve (C, h).

The moduli space of rational tropical curves.

- The general polyhedron
$$M_0(\alpha)$$
is the set of curves in $\mathcal{M}_{0,I}(\Delta, \mathbb{R}^m)$ which have combinatorial type α. The set $\overline{M_0(\alpha)}$ contains all curves whose combinatorial type is equal to or finer than α.
- The linear space
$$\mathrm{W}(\alpha) := \mathrm{W}(\overline{M_0(\alpha)})$$
is the smallest linear space containing $\overline{M_0(\alpha)}$.
- For $J \subset \Delta \cup I$ with $\#J \geq 2$ and $\#(\Delta \cup I) \setminus J \geq 2$, we denote by
$$v_J \in \mathcal{M}_{0,I}(\Delta, \mathbb{R}^m)$$
a curve which has only one bounded edge and the leaves with label $i \in J$ sit at one vertex and the leaves with label $i \in (\Delta \cup I) \setminus J$ sit at the other vertex. The position of the root vertex is chosen arbitrarily.

Elliptic Curves. Let $(C, h) \in \mathrm{P}_I(\Delta, \mathbb{R}^m)$ be an elliptic curve in $\mathbb{R}^m$ of combinatorial type α.

- We denote the set of curves in $\mathrm{P}_I(\Delta, \mathbb{R}^m)$ which have combinatorial type α by
$$\mathrm{P}(\alpha).$$
Its closure $\overline{\mathrm{P}(\alpha)}$ contains all curves whose combinatorial type is α or a specialization of α.
- The number
$$\#\,\mathrm{Aut}(\alpha)$$
is defined as the number of automorphisms of a curve (C, h) of combinatorial type α.
- Let $p \in \mathrm{supp}(C)$ be a point outside the loop of C. Then we define
$$\mathbf{d}_C(p)$$
as the distance of p to the loop of C, i.e. as the length of the path (which contains no flag in the loop of C) from p to a vertex $v_p \in C_L$ in the loop of C. If $p \in \mathrm{supp}(C)$ is a vertex in the loop of C, we set $\mathbf{d}_C(p) = 0$, and if $p \in \mathrm{supp}(C)$ is a point in the loop which is not a vertex, we set $\mathbf{d}_C(p) = -1$.
- Let $d \in \mathbb{R}$. By
$$\mathbf{V}_d(C) = \{v \in \mathbf{V}(C) |\, \mathbf{d}(v) = d\}$$
we denote the set of vertices of C which have distance d to the loop. The set
$$\mathbf{P}_d = \{p \in \mathrm{supp}(C) |\, \mathbf{d}(p) = d\}$$
contains the points in $\mathrm{supp}(C)$ which have distance d to the loop. By
$$\mathbf{FS}_d(C) = \{(p, E) \in \mathbf{FS}(C) |\, \mathbf{d}(p) = d,\ \mathbf{d}(q) \geq d\ \forall\ q \in E\}$$
we denote the set of flag segments which have distance d to the loop. If $d = 0$, we define
$$\mathbf{V}_0(\alpha), \mathbf{P}_0(\alpha) \text{ and } \mathbf{FS}_0(\alpha)$$
analogously.
- Let $p \in \mathrm{supp}(C)$ be a point with $\mathbf{d}(p) \geq 0$. Then we define
$$\mathbf{F}_p = \{(v, E) \in \mathbf{FS}_{\mathbf{d}(p)} | p = v\}$$
as the set of flag segments that lie directly behind p seen from the loop. We define
$$\mathrm{v}(p) = \sum_{F \in \mathbf{FS}_p(C)} \mathrm{v}^\omega(F)$$
as the sum of the weighted direction vectors of flag segments that lie directly behind p seen from the loop. We define $\mathbf{F}_{[p]}$ and $\mathrm{v}([p])$ analogously for $[p] \in \mathbf{V}(\alpha)$.
- Let $d \geq 0$. We define
$$V(C, h)_d \text{ and } V(C, h)_{<d}$$
as the vector spaces spanned by the direction vectors of flag segments $F \in \mathbf{FS}(C)$ whose distance to the loop is at most d and less than d, respectively. We define $V(\alpha)_0$ and $V(\alpha)_{<0}$ analogously.
- We define
$$L(C, h)_0 \text{ and } L(C, h)_{<0}$$
as the lattices spanned by the weighted direction vectors of flag segments $F \in \mathbf{FS}(C)$ in and at the loop of C and in the loop of C, respectively. We define $L(\alpha)_0$ and $L(\alpha)_{<0}$ analogously.
- The index
$$\mathrm{ind}(\alpha)$$
is defined as the index $\mathrm{ind}(L(\alpha)_0)$ of the lattice $L(\alpha)_0$.

- Let $F = (p, E) \in \mathbf{F}_p(C)$ be a flag segment of C outside the loop that points away from the loop. Then we define
$$\Delta(F) \subset \Delta \cup I$$
as the set of labels of leaves that lie behind F seen from the loop, i.e. the path from p to the marked point x_i with $i \in \Delta(F)$ has F as first flag and does not pass the loop. If $W \subset \mathbf{FS}(C)$ is a set of flag segments outside the loop of C that point away from the loop, we define
$$\Delta(W) = \bigcup_{F \in W} \Delta(F).$$
If $p \in \operatorname{supp}(C)$ is a point with $\mathbf{d}(p) \geq 0$ we define
$$\Delta(p) = \bigcup_{F \in \mathbf{FS}_p} \Delta(F)$$
as set of labels of leaves the lie behind p seen from the loop.
- Let $(C, h) \in \mathrm{P}_I(\Delta, \mathbb{R}^m)$ and let $H \subset \mathbb{R}^m$ be a hyperplane. We define
  - the vertices of (C, h) closest to the loop at which a flag runs out of H as
$$\mathbf{V}_H(C, h) = \{v \in \operatorname{supp}(C) | V(C, h)_{<\mathbf{d}(v)} \subset H,\ \exists (v, E) \in \mathbf{F}_v : \mathrm{v}(v, E) \notin H\},$$
  - the distance
$$\mathbf{d}_H(C, h)$$
of H to the loop of C as the distance of a vertex closest to the loop at which a flag runs out of H, i.e. if $\mathbf{V}_H(C, h) \neq \emptyset$, we define $\mathbf{d}_H(C, h) := \mathbf{d}(v)$ for an arbitrary vertex $v \in \mathbf{V}_H(C, h)$, otherwise we set $\mathbf{d}_H(C, h) = 0$,
  - the flags closest to the loop which run out of H as
$$\mathbf{F}_H(C, h) := \{(p, E) \in \mathbf{F}(C) | \, \mathbf{d}(p) = \mathbf{d}_H(C, h), \mathrm{v}(p, E) \notin H\}.$$

Rational curves corresponding to elliptic curves. To a regular curve $(C, h) \in \mathrm{P}_I(\Delta, \mathbb{R}^m)$ of fine combinatorial type $(\alpha, \leq)$ and a flag $F \in \mathbf{F}(C)$ in the loop of C, we associate a rational curve
$$(C_F, h_F) \in \mathcal{M}_{0, I \cup \{A, B\}}(\Delta_{[F]}, \mathbb{R}^m)$$
of fine combinatorial type
$$(\alpha_{[F]}, \leq_{[F]}),$$
see 3.3.4 for the constructions.
  - We denote by
$$U(\alpha_{[F]}) \subset \mathcal{M}_{0, I \cup \{A, B\}}(\Delta_{[F]}, \mathbb{R}^m)$$
the set of curves whose combinatorial type specializes to $\alpha_{[F]}$ (or is equal to $\alpha_{[F]}$).
  - The support of the weighted polyhedral complex
$$U(\alpha_{[F]}, \leq_{[F]})$$
contains all curves which correspond to elliptic curves whose fine combinatorial type specializes to $(\alpha, \leq)$. The weight on a facet of $U(\alpha_{[F]}, \leq_{[F]})$ containing curves of combinatorial type $\beta_{[F]}$ is given by $\mathrm{Aut}(\beta) \cdot \omega(\beta)$.
  - The open subvariety
$$X(\alpha_{[F]})$$
of $\mathcal{M}_{0, I \cup \{A, B\}}(\Delta_{[F]}, \mathbb{R}^m)$ contains all curves whose combinatorial type $\beta_{[F]}$ specializes to $\alpha_{[F]}$ and in the specialization process only edges in and at the loop of $\beta_{[F]}$ are contracted. The weight on each facet is one.
  - The specialization of $\alpha_{[F]}$ in which precisely the bounded edges outside the loop are contracted is denoted by
$$\overline{\alpha_{[F]}}.$$
  - The maps
$$\mathrm{ev}_A, \mathrm{ev}_B, l_p \text{ and } l_q$$
are defined in 3.3.14.

Partitions and elliptic curves. Assume that $(O_1,\dots,O_s)$ is an s-tuple of pairwise disjoint sets. Then we define

$$\mathbf{O}(O_1,\dots,O_s)$$

as the set of ordered partitions $(P_1,\dots,P_r)$ of $\dot{\bigcup}_{i=1}^s O_i$ that are finer than $(O_1,\dots,O_s)$, i.e. there exist $i_1,\dots,i_s=r\in[r]$ such that $O_j=\dot{\bigcup}_{i=i_{j-1}+1}^{i_j} P_i$ for all $j\in[r]$ (where $i_0=0$). By

$$\mathbf{P}(\{O_1,\dots,O_s\})$$

we denote the set of partitions $\{P_1,\dots,P_r\}$ of $\dot{\bigcup}_{i=1}^s O_i$ that are finer than $\{O_1,\dots,O_s\}$.

- Let Γ be an abstract combinatorial type of elliptic curves and let $\mathcal{O}\in\mathbf{O}(\mathbf{F}_{[v_1]},\dots,\mathbf{F}_{[v_s]})$ be an ordered partition of the flags at the loop of Γ, where $(([v_1],[E_1]),\dots,([v_s],[E_s])$ is a path around the loop of Γ if Γ is regular and where $s=1$ and $[v_1]$ is the unique vertex of genus one if Γ is non-regular. Then
$$\Gamma_{\mathcal{O}}$$
is a resolution of Γ which is constructed in 3.2.17.
- Let $(C,h)\in \mathrm{P}_I(\Delta,\mathbb{R}^m)$ a curve of fine combinatorial type $(\alpha,\leq)$ and let $\mathcal{P}\in\mathbf{P}(\{\mathbf{F}_v\}_{v\in\mathbf{P}_d})$ be a partition of the set of flag segments with distance d to the loop of C that is finer than the one given by the points with distance d to the loop of C. Then $\mathcal{P}$ defines resolutions
$$(C_{\mathcal{P}},h_{\mathcal{P}}) \text{ and } (\alpha_{\mathcal{P}},\leq_{\mathcal{P}})$$
of (C,h) and $(\alpha,\leq)$, see 3.2.23. If (C,h) is regular and $F\in\mathbf{F}(C)$ a flag in the loop of C with $\mathrm{v}(F)\neq 0$, we define $((C_F)_{\mathcal{P}},(h_F)_{\mathcal{P}})$ and $((\alpha_{[F]})_{\mathcal{P}},(\leq_{[F]})_{\mathcal{P}})$ analogously.
- For shortening notation define
$$\mathbf{P}^0(\alpha)=\mathbf{P}(\{\mathbf{F}_{[v]}\}_{[v]\in\mathbf{P}_0(\alpha)}).$$
- The subsets
$$\mathrm{Par}_d(C,h)\subset\mathbf{P}(\{\mathbf{F}_p\}_{p\in\mathbf{P}_d(C)})$$
and
$$\mathrm{Par}_0(\alpha)\subset\mathbf{P}(\{\mathbf{F}_{[v]}\}_{[v]\in\mathbf{P}_0(\alpha)})$$
are defined in 3.2.26.
- Let $(\Gamma,\mathrm{v},\leq)$ be a non-regular fine combinatorial type in $\mathcal{M}_{1,I}(\Delta,\mathbb{R}^m)$, let $F_1,\dots,F_s$ be the flags at the loop of $\alpha=(\Gamma,\mathrm{v})$ with weighted direction vectors $y_1,\dots,y_s$. Then an element $a_1\in\mathbb{Z}^m$ of
$$\mathrm{FD}(y_1,\dots,y_s)=\{\sum_{i=1}^s\lambda_i y_i|0\leq\lambda_i<1,\lambda_1=0\}\cap\mathbb{Z}^m$$
defines an ordered partition
$$\mathcal{O}(a_1)=(P_1,\dots,P_r)\in\mathbf{O}(\{\mathbf{FS}_0(\alpha)\})$$
of the flags at the loop of α as described in 3.4.10. The corresponding unordered partition is denoted by
$$\mathcal{P}(a_1)=\{P_1,\dots,P_r\}.$$
We denote by
$$(\beta(a_1),\leq_{a_1})$$
with $\beta(a_1)=(\Gamma_{\mathcal{O}(a_1)},\mathrm{v}(a_1))$ the well-spaced resolution of $(\alpha,\leq)$ where the abstract combinatorial type is given by $\Gamma_{\mathcal{O}(a_1)}$, $\mathrm{v}(a_1)$ is specified by the weighted direction vector $a_1\in\mathbb{Z}^m$ of the flag in the loop of $\Gamma_{\mathcal{O}(a_1)}$ that runs from the vertex adjacent to the flags in $P_1\in\mathcal{P}(a_1)$ to the vertex adjacent to the flags in $P_2\in\mathcal{P}(a_1)$ and the total preorder $\leq_{a_1}$ on the set of vertices of $\Gamma_{\mathcal{O}(a_1)}$ is induced by the total preorder $\leq$ on the set of vertices of α. For a partition $\mathcal{P}\in\mathbf{P}^0(\alpha)$ and $P\in\mathcal{P}$, we set
$$y_P=\sum_{i\in[s]:F_i\in P} y_i \text{ and } \mathrm{ind}(\mathcal{P})=\mathrm{ind}(y_P|P\in\mathcal{P}).$$

Moreover, we set

$$\mathrm{FD}(\mathcal{P}) = \{a_1 \in \mathrm{FD}(y_1, \dots, y_s) | \mathcal{P} \leq \mathcal{P}(a_1)\}$$

and we define

$$\mathcal{P}_0(\alpha) = \{\{F_1\}, \dots, \{F_s\}\}$$

as the finest partition of the flags at the loop of α. Assume that $s = m + 1$ and $\langle y_1, \dots, y_{m+1}\rangle = \mathbb{R}^m$.

- We set

  $$\mathrm{FD} = \mathrm{FD}(y_1, \dots, y_{m+1}),$$

  and

  $$z(i_1, i_2, a_1), z(a_1) \text{ and } M(a_1)$$

  are defined in 3.4.8 and 3.4.10.
- Let $i_1, i_2 \in [s] = [m + 1]$ and $\mathcal{P} \in \mathbf{P}^0(\alpha)$. We define

  $$\mathcal{P}^{(i_1,i_2)} = \left(\mathcal{P} \setminus \{P \in \mathcal{P} : F_{i_1} \in P \text{ or } F_{i_2} \in P\}\right) \cup \left\{ \bigcup_{\substack{P\in\mathcal{P}:\\ F_{i_1}\in P \text{ or } F_{i_2}\in P}} P \right\},$$

  i.e. we take the union of elements of $\mathcal{P}$ containing the flags F_{i_1} and F_{i_2}.
- By

  $$\mathbf{R}^0(\alpha) \subset \mathbf{P}^0(\alpha)$$

  we denote the set of partitions $\mathcal{P} \in \mathbf{P}^0(\alpha)$ such that the flags F_1, F_2, F_3 and F_4 are contained in pairwise different elements of $\mathcal{P}$.
- We set $T_1 = (1, 3)$, $T_2 = (1, 4)$, $T_3 = (2, 4)$, $T_4 = (2, 3)$ and

  $$\mathbf{T} = \{T_1, T_2, T_3, T_4\}.$$

Well-spaced elliptic curves of codimension one in $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$. Let $(\alpha, \leq)$ be a fine combinatorial type in $\mathcal{M}_{1,I}(\Delta, \mathbb{R}^m)$ of codimension one that fulfills

$$\dim V(\alpha)_0/V(\alpha)_{<0} = \#\mathbf{FS}_0(\alpha) - \#\mathbf{P}_0(\alpha) - 1.$$

Let $[F] \in \mathbf{F}(\alpha)$ be a flag in the loop that fulfill $\mathrm{v}([F]) \neq 0$.

- For shortening notation, we denote the fine combinatorial type $(\alpha_{[F]}, \leq_{[F]})$ of curves in $\mathcal{M}_{0,I\cup\{A,B\}}(\Delta_{[F]}, \mathbb{R}^m)$ by

  $$(\gamma, \leq).$$

- For $\mathcal{P}, \mathcal{P}' \in \mathbf{P}^0(\gamma)$ we define

  $$I(\mathcal{P}, \mathcal{P}') = \{P \cap P' \neq \emptyset | P \in \mathcal{P}, P' \in \mathcal{P}'\}.$$

- Due to the assumptions, $\mathrm{Par}_0(\gamma)$ contains exactly one partition of the set of flag segments $\mathbf{FS}_0(\gamma)$ that strictly refines the partition $\{F_{[v]}\}_{[v]\in\mathbf{P}_0(\gamma)}$ of $\mathbf{FS}_0(\gamma)$ given by the vertices in the loop of γ. We denote this distinguished partition by

  $$\mathcal{A} \in \mathrm{Par}_0(\gamma).$$

- We set

  $$\mathbf{W}(\gamma) = \{\mathcal{W} \in \mathbf{P}^0(\gamma) | \mathcal{W} \neq \mathcal{P}_0(\gamma), \#(A \cap W) \leq 1 \;\forall\, A \in \mathcal{A} \text{ and } \forall\, W \in \mathcal{W}\},$$

  where

  $$\mathcal{P}_0(\gamma) = \{\{[F]\}\}_{[F]\in\mathbf{FS}_0(\gamma)}$$

  is the finest partition of the flags at the loop of γ.
- We define

  $$\mathbf{P}_{\mathcal{P}_0(\gamma)} = \{\mathcal{P} \in \mathbf{P}^0(\gamma) | \#I(\mathcal{A}, \mathcal{P}) = \#\mathcal{P} + 1, \#I(\mathcal{P}_0(\gamma), \mathcal{P}) = \#\mathcal{P} + 1\}.$$

  The set $\mathbf{P}_{\mathcal{P}_0(\gamma)}$ contains all partitions that arise from $\mathcal{P}_0(\gamma)$ by uniting two flags $F_1, F_2 \in \mathbf{F}_{[v]}$ that lie directly behind one vertex $[v] \in \mathbf{V}_0(\gamma)$ in the loop of γ and that are contained in different elements $A_1, A_2 \in \mathcal{A}$.

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