COCYCLES ON TROPICAL VARIETIES VIA PIECEWISE POLYNOMIALS

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ABSTRACT. We use piecewise polynomials to define tropical cocycles generalising the well-known notion of Cartier divisors to higher codimensions. We also introduce an intersection product of cocycles with tropical cycles and prove that this gives rise to a Poincaré duality in some cases.

1. INTRODUCTION

Piecewise polynomials have been studied for their close relation to equivariant Chow cohomology theory on toric varieties ([B, B2, P, KP]). In [KP] the authors describe a method to assign a Minkowski weight in a complete fan Δ to a piecewise polynomial on Δ and therefore suggest to use piecewise polynomials in tropical geometry. If Δ is unimodular (i.e. corresponds to a smooth toric variety), their assignment is even an isomorphism.

We show in the second section that the assignment of [KP] agrees with the known (inductive) intersection product of rational functions. This motivates us to use piecewise polynomials as local ingredients for tropical cocycles. It turns out that each piecewise polynomial on an arbitrary tropical fan is a sum of products of rational functions; this can be used to intersect cocycles with tropical cycles. Finally, in theorem 2.25 we deduce a Poincaré duality on the cycle $\mathbb{R}^n$ from the isomorphism between the groups of piecewise polynomials and Minkowski weights on complete unimodular fans.

In the third section we focus on matroid varieties and smooth tropical varieties (that means cycles which locally look like matroid varieties). Thereby we prove that each subcycle of a matroid variety (modulo lineality space) can be cut out by a cocycle (theorem 3.1) and show a Poincaré duality in codimension 1 and dimension 0 for smooth varieties (corollary 3.8).

A similar construction to piecewise polynomials on fans has recently been introduced independently in [E]: Esterov defines tropical varieties with (degree k) polynomial weights and their (codimension 1) corner loci which are tropical varieties with (degree $k-1$) polynomial weights.

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2. PIECEWISE POLYNOMIALS AND TROPICAL COCYCLES

To set notations we start this section by recalling the definition of a piecewise polynomial on a (not necessarily tropical) fan F.

Definition 2.1. Let $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ be the real vector space corresponding to a lattice Λ. Let S be a union of cones in V. We define $\mathrm{P}^k(S)$ to be the set of functions $g : S \to \mathbb{R}$ that extend to a homogeneous polynomial of degree k on V_S having integer coefficients. Here V_S denotes the smallest linear space containing S. A piecewise polynomial of degree k on a fan F in V is a continuous function $f : |F| \to \mathbb{R}$ on the support of F such that the restriction $f_{|\sigma} \in \mathrm{P}^k(\sigma)$ for each cone $\sigma \in F$. The group of piecewise polynomials of degree k on the fan F is denoted by $\mathrm{PP}^k(F)$. We call $\mathrm{PP}^*(F) := \oplus_{k\in\mathbb{N}} \mathrm{PP}^k(F)$ the graded ring of piecewise polynomials on F. Finally, we define $\mathrm{L\,PP}^{k-1}(F) := \langle\{l \cdot f : l \text{ linear}, f \in \mathrm{PP}^{k-1}(F)\}\rangle$ to be the subgroup of $\mathrm{PP}^k(F)$ generated by linear functions.

Notation 2.2. Let $\mathcal{X}$ be a tropical fan (that means a weighted fan in a real vector space (containing a lattice) which satisfies the balancing condition (see for example [AR, definition 2.6])). We denote by $Z_k(\mathcal{X})$ the group of k-dimensional Minkowski weights in $\mathcal{X}$ (that means its elements are k-dimensional tropical subfans of $\mathcal{X}$).
A (tropical) fan cycle is the cycle associated to some tropical fan $\mathcal{X}$. We denote the group of dimension d fan subcycles of fan cycle X by $Z_d^{\text{fan}}(X)$.

We are ready to state the result of [KP] mentioned in the introduction.

Definition and Theorem 2.3. *Let Δ be a complete unimodular fan in $\mathbb{R}^n$. For two cones $\tau < \sigma \in \Delta^{(n)}$, with σ generated by $v_1, \ldots, v_n$, let $e_{\sigma,\tau} := \prod_{i:v_i \notin \tau} \frac{1}{v_i^*}$, where $v_1^*, \ldots, v_n^* \in \mathrm{P}^1(\mathbb{R}^n)$ form the dual basis of $v_1, \ldots, v_n$. Let $f \in \mathrm{PP}^k(\Delta)$. For a maximal cone $\sigma \in \Delta$, f_σ denotes the polynomial on $\mathbb{R}^n$ which agrees with f on σ. Then for any $\tau \in \Delta^{(n-p)}$*

$$c_{f\cdot\Delta}(\tau) := \sum_{\sigma>\tau,\ \sigma\in\Delta^{(n)}} e_{\sigma,\tau} f_\sigma$$

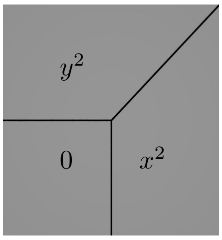
is a homogeneous integer polynomial of degree $k-p$. In particular, $c_{f\cdot\mathbb{R}^n}(\tau)$ is 0 if the codimension of τ is greater than k, and an integer if $\tau \in \Delta^{(n-k)}$. Furthermore,

$$f \cdot \Delta := \left(\bigcup_{i \le n-k} \Delta^{(i)}, c_{f\cdot\Delta} \right)$$

is tropical fan. Finally, $\mathrm{PP}^k(\Delta)/\mathrm{L\,PP}^{k-1}(\Delta) \to Z_{n-k}(\Delta)$, given by $f \mapsto f \cdot \Delta$ is an isomorphism. The fan cycle associated to $f \cdot \Delta$ is independent under refinement of Δ and is denoted by $f \cdot \mathbb{R}^n$.

Proof. We refer to chapter 1, proposition 1.2, theorem 1.4 of [KP] for a proof. □

Example 2.4. Let $f \in \mathrm{PP}^2(\Delta)$ the piecewise polynomial shown in the picture.



Then $f \cdot \Delta$ is the origin with weight

$$c_{f\cdot\Delta}(\{0\}) = \frac{x^2}{x(x-y)} + \frac{y^2}{y(y-x)} + \frac{0}{(-x)(-y)} = \frac{yx^2 - xy^2}{xy(x-y)} = 1.$$

Note that f is the square of the rational function $\max\{x, y, 0\}$ and that $\max\{x, y, 0\} \cdot \max\{x, y, 0\} \cdot \Delta$ (as product of rational functions with cycles defined in [AR, definition 3.4]) gives the origin with weight 1 too.

If a piecewise polynomial on a complete fan Δ is a product of rational (i.e. piecewise linear) functions φ_i, then there are two ways of defining its intersection product with Δ: We can either intersect inductively with the rational functions φ_i (cf. [AR, definition 3.4]) or use the formula of theorem 2.3. In the previous example both ways led to the same result. We show in the following proposition that this is true in general:

Proposition 2.5. *Let Δ be a complete unimodular fan in $\mathbb{R}^n$, and let $\varphi_1, \dots, \varphi_k$ be rational functions on $\mathbb{R}^n$ which are linear on every cone of Δ. Let $f = \varphi_1 \cdots \varphi_k \in \mathrm{PP}^k(\Delta)$. Then $f \cdot \mathbb{R}^n = \varphi_1 \cdots \varphi_k \cdot \mathbb{R}^n$, where the products on the right hand side are products of rational functions with cycles.*

Proof. To ease the notations we assume that each cone in Δ is generated by its rays. Let $\tau \in \Delta^{(n-k)}$ be an arbitrary codimension k cone in Δ. By adding an appropriate linear function l, we can assume that the restriction $\varphi_{1|\tau}$ is identically zero. This does not change $f \cdot \mathbb{R}^n$ since $l \cdot \varphi_2 \cdots \varphi_k$ is in $\mathrm{L\,PP}^{k-1}(\Delta)$. Set $g := \varphi_2 \cdots \varphi_k$. In the following r, r_i denote rays of Δ with respective primitive integral vector v, v_i. As $\varphi_{1|\tau} = 0$ the definition of intersecting with a rational function (cf. [AR, definition 3.4]) implies that τ has weight

$$\omega_{\varphi_1 \cdots \varphi_k \cdot \Delta}(\tau) = \sum_{r: \tau + r \in \Delta^{(n-k+1)}} \omega_{\varphi_2 \cdots \varphi_k \cdot \Delta}(\tau + r) \cdot \varphi_1(v)$$

in $\varphi_1 \cdots \varphi_k \cdot \Delta$. By induction on the degree of f this is equal to

$$\begin{aligned}
& \sum_{r: \tau + r \in \Delta^{(n-k+1)}} c_{(\varphi_2 \cdots \varphi_k) \cdot \Delta}(\tau + r) \cdot \varphi_1(v) \\
= & \sum_{r: \tau + r \in \Delta^{(n-k+1)}} \sum_{\sigma > \tau + r} e_{\sigma, \tau + r} \cdot g_\sigma \cdot \varphi_1(v) \\
= & \sum_{\substack{\sigma > \tau \text{ in } \Delta^{(n)} \\ \sigma = \tau + r_1 + \ldots + r_k}} \sum_{i=1}^{k} \varphi_1(v_i) \cdot e_{\sigma, \tau + r_i} \cdot g_\sigma \\
= & \sum_{\substack{\sigma > \tau \text{ in } \Delta^{(n)} \\ \sigma = \tau + r_1 + \ldots + r_k}} \sum_{i=1}^{k} \varphi_1(v_i) \cdot v_i^* \cdot e_{\sigma, \tau} \cdot g_\sigma \\
= & \sum_{\substack{\sigma > \tau \text{ in } \Delta^{(n)} \\ \sigma = \tau + r_1 + \ldots + r_k}} e_{\sigma, \tau} \cdot \left(g \cdot \sum_{i=1}^{k} \varphi_1(v_i) v_i^* \right)_\sigma .
\end{aligned}$$

Since $\varphi_{1|\tau} = 0$ the above agrees with

$$\sum_{\substack{\sigma > \tau \text{ in } \Delta^{(n)} \\ \sigma = \tau + r_1 + \ldots + r_k}} e_{\sigma, \tau} \cdot (g \cdot \varphi_1)_\sigma = c_{f \cdot \Delta}(\tau).$$

□

So far $\mathbb{R}^n$ is the only fan cycle admitting an intersection product with piecewise polynomials (cf. theorem 2.3). Therefore, our next aim is to define an intersection product for arbitrary fan cycles. The idea is to write piecewise polynomials as sums of products of rational fan functions and use these representations to define an intersection product. We introduce some more notation:

Notation 2.6. The group of piecewise polynomials of degree k on a fan cycle X is defined to be $\mathrm{PP}^k(X) := \{f : f \in \mathrm{PP}^k(\mathcal{X}) \text{ for some fan structure } \mathcal{X} \text{ of } X\}$. We set $\mathrm{L\,PP}^{k-1}(X) := \langle \{l \cdot f : l \text{ linear}, f \in \mathrm{PP}^{k-1}(X)\} \rangle$.

Notation 2.7. Let F be a unimodular fan such that each cone is generated by its rays. Let $v_1, \dots, v_m$ be primitive integral vectors of the rays $r_1, \dots, r_m$ of F. Then $\Psi_{r_i} := \Psi_i \in \mathrm{PP}^1(F)$ is the unique function which is linear on the cones of F and satisfies $\Psi_i(v_j) = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta function. For a cone $\tau \in F$ we have a piecewise polynomial $\Psi_\tau := \prod_{i:v_i\in\tau}\Psi_i \in \mathrm{PP}^{\dim\tau}(F)$. Note that Ψ_τ vanishes away from $\bigcup_{\sigma>\tau}\sigma$.

As mentioned in [B] we can show that the functions Ψ_τ generate the ring of piecewise polynomials.

Proposition 2.8. *Let $f \in \mathrm{PP}^k(F)$ be a piecewise polynomial of degree k on a unimodular fan F whose cones are generated by their rays. Then there exists a representation $f = \sum_{\sigma\in F^{(\leq k)}} a_\sigma \Psi_\sigma$, where the a_σ are homogeneous integer polynomials of degree $k-\dim(\sigma)$. In particular, piecewise polynomials on tropical fan cycles are sums of products of rational functions.*

Proof. We use induction on the dimension of F, the case $\dim F = 0$ being obvious. We know by induction hypothesis that there are (homogeneous) polynomials a_σ such that $f_{||F_1|} = \sum_{\sigma\in F_1^{(\leq k)}} a_\sigma\Psi_\sigma$, where $F_1 := \{\sigma : \sigma\in F^{(p)} \text{ with } p<\dim F\}$. Thus, it suffices to show the claim for $g := f - \sum_{\sigma\in F_1^{(\leq k)}} a_\sigma\Psi_\sigma \in \mathrm{PP}^k(F)$. Now we use induction on the number r of maximal cones in F. Let $r=1$ and σ be the unique maximal cone in F. By [B, section 1.2], we know that the following sequence is exact:

$$0 \to \Psi_\sigma\,\mathrm{P}^{k-\dim F}(F) \hookrightarrow \mathrm{PP}^k(F) \overset{\text{rest.}}{\to} \mathrm{PP}^k(F\setminus\{\sigma\}) \to 0.$$

Since $g_{||F\setminus\{\sigma\}|} = g_{||F_1|} = 0$, it follows that there is a polynomial a_σ such that $g = a_\sigma\Psi_\sigma$. Now let $r>1$ and $\sigma\in F$ a maximal cone. By the induction hypothesis, there are polynomials b_τ such that $g_{||F\setminus\{\sigma\}|} = \sum_{\tau\in F\setminus\{\sigma\}^{(\leq k)}} b_\tau\Psi_\tau$. Since the restriction of $g-\sum_{\tau\in F\setminus\{\sigma\}^{(\leq k)}} b_\tau\Psi_\tau$ to $F\setminus\{\sigma\}$ is 0, the claim follows from the exactness of the above sequence. As every fan can be refined to a unimodular fan whose cones are generated by their rays (cf. [R, proposition 1.1.2]) this also implies the "in particular" statement. □

It is clear that the representation of a piecewise polynomial as a sum of products of rational functions is not unique. Therefore, we need to ensure that the intersection product will not depend on the chosen representation.

Proposition 2.9. *Let $\varphi_1^i,\dots,\varphi_k^i$, $\gamma_1^j,\dots,\gamma_k^j$ with $k\leq d$ be rational fan functions on a fan cycle $X\in Z_d^{\text{fan}}(V)$ such that $f := \sum_{i\in I}\varphi_1^i\cdots\varphi_k^i = \sum_{j\in J}\gamma_1^j\cdots\gamma_k^j \in \mathrm{PP}^k(X)$. Then we have the following equation of intersection products of rational functions with cycles (cf.* [AR, definition 3.4]*):*

$$\sum_{i\in I}\varphi_1^i\cdots\varphi_k^i\cdot X = \sum_{j\in J}\gamma_1^j\cdots\gamma_k^j\cdot X.$$

The proof of the proposition makes use of the following technical lemma:

Lemma 2.10. *Let $c_{b_1\dots b_s}$ be real numbers such that $\sum_{b_1+\ldots+b_s=k} a_1^{b_1}\cdots a_s^{b_s}\cdot c_{b_1\dots b_s} = 0$ for all $a_i\geq 0$. Then all $c_{b_1\dots b_s}$ are 0.*

Proof. For $a_1\in\{1,\dots,k+1\}$ and any $a_2,\dots,a_s$ we have

$$0 = \sum_{b_1+\ldots+b_s=k} a_1^{b_1}\cdots a_s^{b_s}\cdot c_{b_1\dots b_s} = \sum_{b_1=0}^{k} a_1^{b_1}\sum_{b_2+\ldots+b_s=k-b_1} a_2^{b_2}\cdots a_s^{b_s}c_{b_1\dots b_s}.$$

Since the Vandermonde matrix $(i^j)_{i=1,\dots,k+1,j=0,\dots,k}$ is regular, it follows that

$$\sum_{b_2+\ldots+b_s=k-b_1} a_2^{b_2}\cdots a_s^{b_s}c_{b_1\dots b_s} = 0$$

for all $a_2,\dots,a_s\geq 0$ and all $b_1\in\{0,\dots,k\}$. Hence the claim follows by induction. □

Proof of proposition 2.9. We choose a unimodular fan structure $\mathcal{X}$ of X such that every cone is generated by its rays and all φ_p^i, ψ_p^j are linear on every cone of $\mathcal{X}$. Let $v_1, \ldots, v_m$ be the primitive integral vectors of the rays $r_1, \ldots, r_m$ of $\mathcal{X}$. Since $\varphi_p^i = \sum_{s=1}^m \varphi_p^i(v_s) \cdot \Psi_s$, we have

$$\begin{aligned} f = \sum_{i \in I} \varphi_1^i \cdots \varphi_k^i &= \sum_{i \in I} \left(\sum_{s=1}^m \varphi_1^i(v_s) \cdot \Psi_s \right) \cdots \left(\sum_{s=1}^m \varphi_k^i(v_s) \cdot \Psi_s \right) \\ &= \sum_{i \in I} \sum_{1 \le s_1 \le \ldots \le s_k \le m} \sum_{\sigma \in S_k} \varphi_1^i(v_{s_{\sigma(1)}}) \cdots \varphi_k^i(v_{s_{\sigma(k)}}) \cdot \Psi_{s_1} \cdots \Psi_{s_k} \\ &= \sum_{1 \le s_1 \le \ldots \le s_k \le m} \underbrace{\sum_{\sigma \in S_k} \sum_{i \in I} \varphi_1^i(v_{s_{\sigma(1)}}) \cdots \varphi_k^i(v_{s_{\sigma(k)}})}_{=: \lambda_{s_1 \ldots s_k} \in \mathbb{Z}} \cdot \Psi_{s_1} \cdots \Psi_{s_k}. \end{aligned}$$

The commutativity of intersecting with rational functions ([AR, proposition 3.7]) implies

$$\sum_{i \in I} \varphi_1^i \cdots \varphi_k^i \cdot X = \sum_{1 \le s_1 \le \ldots \le s_k \le m} \lambda_{s_1 \ldots s_k} \cdot \Psi_{s_1} \cdots \Psi_{s_k} \cdot X.$$

Analogously we find $\mu_{s_1 \ldots s_k} \in \mathbb{Z}$ such that

$$\sum_{j \in J} \gamma_1^j \cdots \gamma_k^j \cdot X = \sum_{1 \le s_1 \le \ldots \le s_k \le m} \mu_{s_1 \ldots s_k} \cdot \Psi_{s_1} \cdots \Psi_{s_k} \cdot X.$$

It follows that

$$\sum_{i \in I} \varphi_1^i \cdots \varphi_k^i \cdot X - \sum_{j \in J} \gamma_1^j \cdots \gamma_k^j \cdot X = \sum_{1 \le s_1 \le \ldots \le s_k \le m} \underbrace{(\lambda_{s_1 \ldots s_k} - \mu_{s_1 \ldots s_k})}_{=: c_{s_1 \ldots s_k}} \cdot \Psi_{s_1} \cdots \Psi_{s_k} \cdot X.$$

As $\Psi_{w_1} \cdots \Psi_{w_k} \cdot X = 0$ if the cone $\langle w_1, \ldots, w_k \rangle \notin \mathcal{X}$ (this can be showed in the same way as [A, lemma 1.4]) the above is equal to

$$\sum_{\sigma = \langle v_{s_1}, \ldots, v_{s_k} \rangle \in \mathcal{X}} c_{s_1 \ldots s_k} \cdot \Psi_{s_1} \cdots \Psi_{s_k} \cdot X.$$

It suffices thus to prove that each $c_{s_1 \ldots s_k}$ occurring in the above sum is equal to 0: Let $p \le k$ and $\langle v_{t_1}, \ldots, v_{t_p} \rangle \in \mathcal{X}^{(p)}$. For all $a_1, \ldots, a_p \ge 0$ we have

$$\begin{aligned} 0 &= (f - f)(a_1 v_{t_1} + \ldots + a_p v_{t_p}) \\ &= \left(\sum_{1 \le s_1 \le \ldots \le s_k \le m} c_{s_1 \ldots s_k} \cdot \Psi_{s_1} \cdots \Psi_{s_k} \right) (a_1 v_{t_1} + \ldots + a_p v_{t_p}) \\ &= \sum_{\substack{1 \le s_1 \le \ldots \le s_k \le m \\ \{s_1, \ldots, s_k\} \subset \{t_1, \ldots, t_p\}}} c_{s_1 \ldots s_k} \prod_{i=1}^p a_i^{|\{j : s_j = t_i\}|} \\ &= \sum_{b_1 + \ldots + b_p = k} c_{\underbrace{t_1 \ldots t_1}_{b_1 \text{ times}} \ldots \underbrace{t_p \ldots t_p}_{b_p \text{ times}}} a_1^{b_1} \cdots a_p^{b_p}. \end{aligned}$$

It follows from lemma 2.10 that all $c_{t_1 \ldots t_1 \ldots t_p \ldots t_p}$ are 0. □

The previous propositions together with the well-known intersections with rational functions enable us to define an intersection product of piecewise polynomials with tropical fan cycles. Later we will use this to construct an intersection product of cocycles with arbitrary cycles.

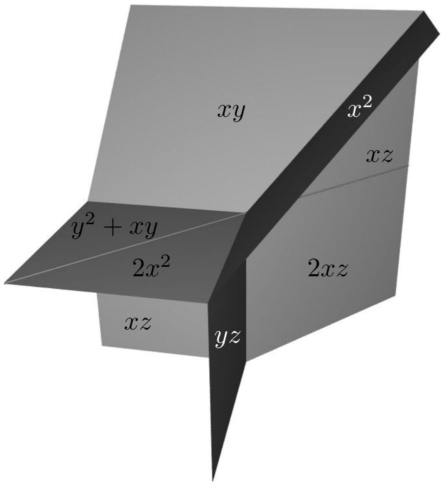
Definition 2.11. Let $X \in Z_d^{\text{fan}}(V)$ be a tropical fan cycle and let $f \in \text{PP}^k(X)$ be a piecewise polynomial on X. By proposition 2.8 we can choose rational functions φ_j^i such that $f = \sum_{i=1}^s \varphi_1^i \cdots \varphi_k^i \in \text{PP}^k(X)$. This allows us to define the intersection of f with the cycle X to be

$$f \cdot X := \sum_{i=1}^{s} \varphi_1^i \cdots \varphi_k^i \cdot X \in Z_{d-k}^{\text{fan}}(X).$$

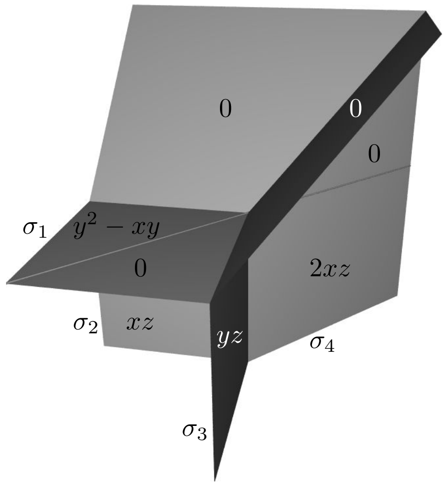
Note that this does not depend on the choice of rational functions by proposition 2.9.

Remark 2.12. It is clear that the intersection product is linear and that $f \cdot (g \cdot X) = (f \cdot g) \cdot X$ for two piecewise polynomials f, g on a fan cycle X. Furthermore, it follows straight from definition that $f \cdot X = 0$ if $f \in \text{L}\,\text{PP}^{k-1}(X)$.

Example 2.13. Let $f \in \text{PP}^2(L_2^3)$ be the piecewise polynomial on the tropical fan cycle $L_2^3 := \max\{x, y, z, 0\} \cdot \mathbb{R}^3$ shown in the following picture. Let $\mathcal{X}$ be the corresponding fan structure of L_2^3.



$f \in \text{PP}^2(\mathcal{X}) \subsetneq \text{PP}^2(L_2^3)$



$f - 2x \cdot \Psi_a - x \cdot \Psi_b$

We want to compute $f \cdot L_2^3$. Therefore, we use the idea of the proof of proposition 2.8 to obtain a representation of f as a sum of products of rational functions: We first make f vanish on the rays of $\mathcal{X}$ by adding appropriate (linear) multiples of the rational functions Ψ_r (with r ray of $\mathcal{X}$). Doing this we obtain $f - 2x \cdot \Psi_a - x \cdot \Psi_b$, where $a = (-1, -1, 0)$ and $b = (1, 1, 1)$. Now it is easy to see that

$$f - 2x \cdot \Psi_a - x \cdot \Psi_b = -\Psi_{\sigma_1} + \Psi_{\sigma_2} + \Psi_{\sigma_3} - 2 \cdot \Psi_{\sigma_4}.$$

As $\Psi_{\sigma_i} \cdot L_2^3 = 1 \cdot \{0\}$ for all i (cf. lemma 3.6) we obtain by definition 2.11 and remark 2.12 that $f \cdot L_2^3 = (-1 + 1 + 1 - 2) \cdot \{0\} = -1 \cdot \{0\}$.

Remark 2.14. Let X be a tropical cycle in a vector space V (that means the cycle associated to some balanced weighted polyhedral complex $\mathcal{X}$ in V (cf. [R, definition 1.1.8])). Let p be a point in X. Recall that in [R, section 1.2.3] the star $\text{Star}_X(p)$ is defined to be the tropical fan cycle in V associated to $\text{Star}_{\mathcal{X}}(p)$, where $\mathcal{X}$ is a polyhedral structure of X containing the cell $\{p\}$. That means $\text{Star}_X(p)$ is the fan cycle whose support consists of vectors v such that $p + \epsilon v \in |X|$ for small (positive) ϵ and whose weights are inherited from X.
A piecewise polynomial $f \in \text{PP}^k(X)$ on a fan cycle X induces a piecewise polynomial $f^p \in \text{PP}^k(\text{Star}_X(p))$ obtained by restricting f to a small neighbourhood of p and then extending it in the obvious way to $\text{Star}_X(p)$. As $f = \sum_{i=1}^s \varphi_1^i \cdots \varphi_k^i$ implies that $f^p = \sum_{i=1}^s (\varphi_1^i)^p \cdots (\varphi_k^i)^p$, it follows from [R, proposition 1.2.12] that

$$f^p \cdot \text{Star}_X(p) = \text{Star}_{f \cdot X}(p).$$

Our next aim is to use piecewise polynomials to define higher codimension cocycles on tropical cycles X. Prior to that we give a definition of (abstract) tropical cycles consistent with the definition of smooth tropical varieties in chapter 6 of [FR] (to which we refer for further details). Recall that a topological space is called weighted if each point from a dense open subset is equipped with a non-zero integer weight which is locally constant (in the dense open subset). A cycle X in a vector space can be made weighted by assigning to each interior point of a maximal cell $\sigma \in \mathcal{X}$ the weight of σ, where $\mathcal{X}$ is a polyhedral structure of X.

Definition 2.15. An (abstract) tropical cycle is a weighted topological space X together with an open cover $\{U_i\}$ and homeomorphisms

$$\phi_i : U_i \to W_i \subseteq |X_i|$$

such that

- each W_i is an (euclidean) open subset of $|X_i|$ for some tropical fan cycle X_i (in some vector space)
- for each pair i, k, the transition map
$$\phi_k \circ \phi_i^{-1} : \phi_i(U_i \cap U_k) \to \phi_k(U_i \cap U_k)$$
is the restriction of an affine $\mathbb{Z}$-linear map, i.e. the composition of a translation by a real vector and a $\mathbb{Z}$-linear map
- the weight of a point $p \in U_i$ is equal to the weight of $\phi_i(p)$ in X_i (if both are defined).

If all X_i can be chosen to be matroid varieties [FR, section 2] modulo lineality spaces, then we call X a smooth tropical variety. Recall that in [FR, definition 6.2] a subcycle C of X is defined as a weighted subset of $|X|$ such that for all i the induced weighted set $\phi_i(C \cap U_i)$ agrees with the intersection of W_i and a tropical cycle in X_i.

Definition 2.16. Let X be a fan cycle (in a vector space V) and U an open subset in $|X|$. A continuous function $f : U \to \mathbb{R}$ is called piecewise polynomial of degree k on U if it is locally around each point $p \in U$ a finite sum $\sum_j (f_p^j \circ T_p^j)$ of compositions of (restrictions of) piecewise polynomials $f_p^j \in \mathrm{PP}^k(\mathrm{Star}_X(p))$ and translations T_p^j. We define $f_p \in \mathrm{PP}^k(\mathrm{Star}_X(p))$ to be the (uniquely defined) sum of the f_p^j. The group of piecewise polynomials of degree k on U is denoted $\mathrm{PP}^k(U)$. Furthermore, $\mathrm{L}\,\mathrm{PP}^{k-1}(U)$ is the group of piecewise polynomials f (of degree k) on U such that $f_p \in \mathrm{L}\,\mathrm{PP}^{k-1}(\mathrm{Star}_X(p))$ for all p.

We now generalise the notion of Cartier divisors (i.e. codimension 1 cocycles) introduced in [AR, definition 6.1] by using piecewise polynomials (instead of piecewise linear functions) as local descriptions:

Definition 2.17. A representative of a codimension k cocycle on the cycle X is defined as a set $\{(V_1, f_1), \ldots, (V_p, f_p)\}$ satisfying

- $\{V_i\}$ is an open cover of $|X|$
- $(f_j \circ \phi_i^{-1})_{|\phi_i(U_i \cap V_j)} \in \mathrm{PP}^k(\phi_i(U_i \cap V_j))$ for all i, j
- $((f_j - f_k) \circ \phi_i^{-1})_{|\phi_i(U_i \cap V_j \cap V_k)} \in \mathrm{L}\,\mathrm{PP}^{k-1}(\phi_i(U_i \cap V_j \cap V_k))$ for all i, j, k.

The sum of two (representatives of) codimension k cocycles $\{(V_j, f_j)\}$ and $\{(V'_k, f'_k)\}$ is defined to be $\{(V_j \cap V'_k), f_j + f'_k)\}$. We call two representatives of codimension k cocycles $\{(V_j, f_j)\}$ and $\{(V'_k, f'_k)\}$ equivalent (and identify them) if we have for all i, s that

$$(g_s \circ \phi_i^{-1})_{|\phi_i(U_i \cap K_s)} \in \mathrm{L}\,\mathrm{PP}^{k-1}(\phi_i(U_i \cap K_s)),$$

where $\{(K_s, g_s)\} := \{(V_j, f_j)\} - \{(V'_k, f'_k)\}$.
The group of codimension k cocycles on X is denoted $C^k(X)$. The multiplication of two cocycles can be defined in the same way as the addition; therefore, there is a graded ring $C^*(X) := \oplus_{k\in\mathbb{N}} C^k(X)$ called ring of piecewise polynomials.

Example 2.18. For any cycle X, $C^1(X)$ is the group of Cartier divisors $\operatorname{Div}(X)$ introduced in [AR, definition 6.1].

Example 2.19. Vector bundles $\pi : F \to X$ of degree r on tropical cycles X have been introduced in [A2, definition 5.1.5]. A rational section $s : X \to F$ with open cover $U_1, \ldots, U_s$ induces rational functions $s_{ij} := p_j^{(i)} \circ \Phi_i \circ s : U_i \to \mathbb{R}$ (cf. [A2, definition 5.1.18]). Here the Φ_i are homeomorphisms identifying $\pi^{-1}(U_i)$ with $U_i \times \mathbb{R}^r$ and the $p_j^{(i)} : U_i \times \mathbb{R}^r \to \mathbb{R}$ are projections to the j-th component of $\mathbb{R}^r$. For any $k \le r$ one obtains the cocycle $s^{(k)} := \{(U_i, \sum_{1\le j_1 \le \ldots \le j_k \le r} s_{ij_1} \cdots s_{ij_k})\} \in C^k(X)$ (see [A2, definition 5.2.1]).

We are now ready to construct an intersection product of cocycles with tropical cycles.

Definition and Construction 2.20. Let $f = \{(V_j, f_j)\} \in C^k(X)$ be a codimension k cocycle on a tropical cycle X. For a point p in X we choose i, j such that $p \in U_i \cap V_j$. By definition $(f_j \circ \phi_i^{-1})_p \in \operatorname{PP}^k(\operatorname{Star}_{X_i}(\phi_i(p))$ is a piecewise polynomial on the star around $\phi_i(p)$. Thus we can define the local intersection $(f_j \circ \phi_i^{-1}) \cdot X_i$ by

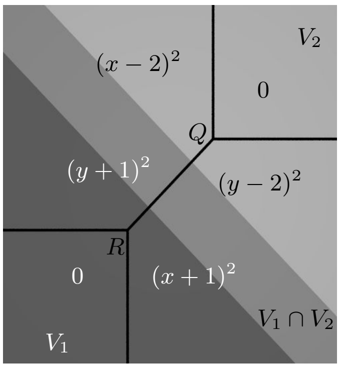
$$\operatorname{Star}_{(f_j\circ\phi_i^{-1})\cdot X_i}(\phi_i(p)) := (f_j \circ \phi_i^{-1})_p \cdot \operatorname{Star}_{X_i}(\phi_i(p)).$$

As $\phi_k \circ \phi_i^{-1}$ induces an isomorphism between the stars $\operatorname{Star}_{X_i}(\phi_i(p))$ and $\operatorname{Star}_{X_k}(\phi_k(p))$, the definition does not depend on the choice of open set U_i.
We can glue together the local intersections to a subcycle $f \cdot X \in Z_{\dim X - k}(X)$ of X: If $p \in U_i \cap V_j \cap V_s$, then $((f_j - f_s) \circ \phi_i^{-1})_p \in \operatorname{L}\operatorname{PP}^{k-1}(\operatorname{Star}_{X_i}(\phi_i(p))$. Therefore, it follows by remark 2.12 that the local intersections agree on the overlaps.

Remark 2.21. In the same way we can also intersect cocycles on X with any subcycle of X. Hence, definition 2.20 gives rise to an intersection product

$$C^k(X) \times Z_l(X) \to Z_{l-k}(X), \ \ (f, C) \mapsto f \cdot C.$$

Example 2.22. The following picture shows a cocycle $f = \{(V_1, f_1), (V_2, f_2)\} \in C^2(\mathbb{R}^2)$



with $R = (-1, -1)$, $Q = (2, 2)$. Note that for $p = (t, t)$ with $-1 < t < 2$ we have

$$(f_1 - f_2)_p = (y + x) \cdot \max\{x - y, y - x\} \in \operatorname{L}\operatorname{PP}^1(\operatorname{Star}_p(\mathbb{R}^2));$$

hence f is indeed a cocycle. As $(f_1)_R$ is the piecewise polynomial of example 2.4 we conclude that the multiplicity of R in $f \cdot \mathbb{R}^2$ is 1. We can deduce from an analogous argument for the point Q that $f \cdot \mathbb{R}^2 = R + Q$.

As in the case of rational functions and Cartier divisors ([AR, proposition 4.7 and 7.6]), we can define pull-backs of piecewise polynomials and cocycles along morphisms.

Definition 2.23. If $h : Y \to X$ is a morphism of fan cycles and $f \in \mathrm{PP}^k(X)$ is a piecewise polynomial on X, then we define the pull-back $h^* f \in \mathrm{PP}^k(Y)$ of f along the morphism h as $h^* f := f \circ h$.
Analogously the pull-back $h^* f \in C^k(Y)$ of a codimension k cocycle $f = \{(V_j, f_j)\} \in C^k(X)$ along a morphism $h : Y \to X$ of any cycles is defined to be $\{(h^{-1}(V_j), f_j \circ h)\}$.

Proposition 2.24. *The following properties hold for cocycles $f \in C^k(X)$ and $g \in C^l(X)$ on a cycle X.*

1. $C^k(X) \times Z_l(X) \to Z_{l-k}(X),\ \ (b, C) \mapsto b \cdot C$ *is bilinear.*
2. $f \cdot (g \cdot X) = (f \cdot g) \cdot X = g \cdot (f \cdot X)$.
3. $f \cdot (h_* E) = h_*(h^* f \cdot E)$ *for a morphism $h : Y \to X$ and a subcycle E of Y.*
4. *If $X \in Z_d(V_X), Y \in Z_e(V_Y)$ are contained in vector spaces (in the sense of* [R, definition 1.1.8]*), then $(f \cdot X) \times Y = \pi^* f \cdot (X \times Y)$, where $\pi : V_X \times V_Y \to V_X$ maps (x, y) to x.*
5. *If D is rationally equivalent to 0 on X (cf.* [AR2, definition 1]*), then so is $f \cdot D$.*

If X and Y are smooth and C, D are subcycles of X, then intersection products and pull-backs ([FR, definition 6.4 and 8.1]*) have the following additional properties:*

6. *If $D = f \cdot X$, then $D \cdot_X C = f \cdot C$.*
7. *If b is a cocycle on C, then $(b \cdot C) \cdot D = b \cdot (C \cdot D)$.*
8. *If $D = f \cdot X$ and $h : Y \to X$ is a morphism, then $h^* D = h^* f \cdot Y$.*

Proof. We first notice that all statements except (5) can be verified locally (that means for piecewise polynomials on fan cycles). But the local statements are either trivial or follow directly from the respective property of rational functions ([AR, 4.8, 9.6, 9.7, 9.8], [FR, 4.5, 8.2]). Using (3) the proof of (5) is the same as the proof of [AR2, lemma 2(b)]. □

After having listed the main properties of intersections with cocycles we now focus on cocycles on the cycle $\mathbb{R}^n$. We use theorem 2.3 to establish a Poincaré duality for this case:

Theorem 2.25. *For any $n \geq k$, the following is a group isomorphism:*

$$C^k(\mathbb{R}^n) \to Z_{n-k}(\mathbb{R}^n),\ \ f \mapsto f \cdot \mathbb{R}^n.$$

Proof. We first consider the corresponding local statement: Since every fan cycle in $\mathbb{R}^n$ has a fan structure lying in a complete unimodular fan ([AR2, lemma 5] and [R, proposition 1.1.2]), we can use theorem 2.3 to conclude that

$$\mathrm{PP}^k(\mathbb{R}^n)/\,\mathrm{L}\,\mathrm{PP}^{k-1}(\mathbb{R}^n) \to Z^{\mathrm{fan}}_{n-k}(\mathbb{R}^n), \quad g \mapsto g \cdot \mathbb{R}^n$$

is an isomorphism.
For the global case we start by proving the surjectivity. So let $C \in Z_{n-k}(\mathbb{R}^n)$ be an arbitrary subcycle of $\mathbb{R}^n$ and let $\mathcal{C}$ be a polyhedral structure of C. We choose an open cover $\{V_j\}$ of $\mathbb{R}^n$ and translation functions T_j such that $T_j(\mathcal{C} \cap V_j)$ is an open tropical fan (cf. [AR, definition 5.3]) for all j. By the local statement we can choose for each j a piecewise polynomial f_j whose intersection with $\mathbb{R}^n$ is the tropical fan associated to $T_j(\mathcal{C} \cap V_j)$. Then $f = \{(V_j, f_j \circ T_j)\} \in C^k(\mathbb{R}^n)$ is a cocycle satisfying $f \cdot \mathbb{R}^n = C$. Note that by construction the difference of two of these local functions gives a zero intersection on the overlaps of two open sets; therefore, the local statement implies that the third condition of definition 2.17 is fulfilled and f is indeed a cocycle on $\mathbb{R}^n$.
The injectivity follows immediately from the local statement. □

3. Cocycles on matroid varieties

In this section we analyse cocycles on smooth varieties. As mentioned in definition 2.15 a tropical cycle is smooth if its local building blocks are matroid varieties modulo lineality spaces (denoted by $\mathrm{B}(M)/L$). If M is a (loopfree) matroid whose ground set $E(M)$ has n elements, then the support of the corresponding matroid variety $\mathrm{B}(M)$ is the set $\{p \in \mathbb{R}^n : M_p \text{ is still loopfree}\}$. Here the matroid M_p is given by its set of bases

$$\{B : B \text{ basis of } M \text{ with } \sum_{i \in B} p_i \text{ minimal}\}.$$

Alternatively one can express a canonical fan structure of $\mathrm{B}(M)$ in terms of the flats (i.e. closed sets) of M (cf. for example [FR, section 2]).
The following theorem states that every subcycle of a matroid variety can be cut out by a cocycle. The idea of the proof is to delete elements of the matroid in order to make use of the $\mathbb{R}^n$-case. If an element i of $E(M)$ is not a coloop, then the deletion of i (see for example [FR, section 3]) corresponds to a projection. This means that the push-forward of $\mathrm{B}(M)$ along the projection $\pi_i : \mathbb{R}^n \to \mathbb{R}^{n-1}$ forgetting the i-th coordinate is equal to the matroid variety $\mathrm{B}(M \setminus \{i\})$ corresponding to the deletion matroid ([FR, lemma 3.8]).

Theorem 3.1. *For any $k \leq d := \dim(\mathrm{B}(M)/L)$, the following morphism is surjective:*

$$C^k(\mathrm{B}(M)/L) \to Z_{d-k}(\mathrm{B}(M)/L), \quad f \mapsto f \cdot \mathrm{B}(M)/L.$$

Proof. We first consider the case where $L = \{0\}$ and $\{a\}$ is a flat for every $a \in E(M)$. We use induction on the codimension of $\mathrm{B}(M)$: The induction start ($\mathrm{B}(M) = \mathbb{R}^n$) was proved in theorem 2.25. Let C be an arbitrary subcycle of $\mathrm{B}(M)$ of codimension k. After renaming the elements, we can assume that $\{1, \ldots, p\}$ is the set of elements of $E(M)$ which are not coloops. For $i \in \{1, \ldots, p\}$ we set

$$C_0 := C, \quad C_i := C_{i-1} - \pi_i^* \pi_{i*} C_{i-1},$$

where the $\pi_i : \mathrm{B}(M) \to \mathrm{B}(M \setminus \{i\})$ denote the projections forgetting the i-th coordinate. The induction hypothesis allows us to choose cocycles $f_i \in C^k(\mathrm{B}(M \setminus \{i\}))$ such that $f_i \cdot \mathrm{B}(M \setminus \{i\}) = \pi_{i*} C_{i-1}$ for $i \in \{1, \ldots, p\}$. [FR, Lemma 9.3] implies that $\pi_{i*} C_p = 0$ for all i; thus $C_p = 0$ by [FR, lemma 9.4]. It follows that

$$C = \sum_{i=1}^{p} \pi_i^* \pi_{i*} C_{i-1} = \sum_{i=1}^{p} \pi_i^* (f_i \cdot \mathrm{B}(M \setminus \{i\})) = \sum_{i=1}^{p} (\pi_i^* f_i) \cdot \mathrm{B}(M).$$

As $\pi_R : \mathrm{B}(M) \to \mathrm{B}(M \setminus R)$ is an isomorphism for $R = \mathrm{cl}_M(\{a\}) \setminus \{a\}$ this also implies the claim for arbitrary matroid varieties $\mathrm{B}(M)$.
Now let C be a subcycle of $\mathrm{B}(M)/L$. Since $\mathrm{B}(M) \cong \mathrm{B}(M)/L \times L$ we can choose a cocycle f with $f \cdot (\mathrm{B}(M)/L \times L) = C \times L$. It follows that $f \cdot (\mathrm{B}(M)/L \times \{0\}) = C \times \{0\}$. Therefore, we can conclude that $s^* f \cdot \mathrm{B}(M)/L = C$, where $s : \mathrm{B}(M)/L \to \mathrm{B}(M)/L \times L$ maps x to $(x, 0)$. □

Remark 3.2. It follows in the same way that each fan cycle $D \in Z^{\text{fan}}_{d-k}(\mathrm{B}(M)/L)$ is cut out by a piecewise polynomial $f \in \mathrm{PP}^k(\mathrm{B}(M)/L)$.

Remark 3.3. An alternative proof (in the case of a trivial lineality space $L = \{0\}$) has recently been found by Esterov in [E, corollary 4.2].

The rest of the section is devoted to show that the (surjective) morphism of theorem 3.1 is an isomorphism in some cases. Unfortunately, so far we have not been able prove this in general.

Proposition 3.4. *Let $d := \dim(\mathrm{B}(M)/L)$. Then the following is an isomorphism:*

$$\mathrm{PP}^1(\mathrm{B}(M)/L)/\,\mathrm{L}\,\mathrm{PP}^0(\mathrm{B}(M)/L) \to Z^{\text{fan}}_{d-1}(\mathrm{B}(M)/L),\ \ f \mapsto f \cdot \mathrm{B}(M)/L.$$

Proof. It remains to prove injectivity. We can assume without loss of generality that $\{a\}$ is a flat in M for every $a \in E(M)$. By successively deleting elements which are not coloops, we see that $\mathrm{B}(M)$ is obtained from $\mathbb{R}^{|\dim \mathrm{B}(M)|}$ by a series of modifications (cf. [FR, proposition 3.10]). Thus it follows from induction and [A2, theorem 4.2.6] that the above morphism is injective if the lineality space is trivial. The $\mathrm{B}(M)/L$ case follows immediately from the $\mathrm{B}(M)$ case. □

Proposition 3.5. *Let X be a locally irreducible fan cycle of dimension d which is connected in codimension 1 (cf.* [R, definition 1.2.27, lemma 1.2.29]*). Then the morphism of groups*

$$\mathrm{PP}^d(X)/\,\mathrm{L}\,\mathrm{PP}^{d-1}(X) \to Z^{\text{fan}}_0(X) = \mathbb{Z},\ f \mapsto f \cdot X$$

is injective. As matroid varieties modulo lineality spaces are locally irreducible and connected in codimension 1 (this follows from [FR, lemma 2.4]*), the above is an isomorphism of groups if $X = \mathrm{B}(M)/L$.*

For a proof we need the following two lemmas:

Lemma 3.6. *Let $\mathcal{X}$ be a unimodular fan structure of a fan cycle X of dimension d (such that every cone in $\mathcal{X}$ is generated by its rays). Let $\sigma \in \mathcal{X}$ be a maximal cone. Then $\Psi_\sigma \cdot X = \omega_{\mathcal{X}}(\sigma) \cdot \{0\}$. Here $\omega_{\mathcal{X}}$ denotes the weight function of $\mathcal{X}$.*

Proof. Let $v_1, \dots, v_d$ be the primitive integral vectors generating the rays of σ. It follows from the definition of Ψ_{v_i} and the intersection product with a rational function that the weight of the cone $\langle v_1, \dots v_{i-1}\rangle$ in $\Psi_{v_i} \cdots \Psi_{v_d} \cdot \mathcal{X}$ is equal to the weight of $\langle v_1, \dots v_i\rangle$ in $\Psi_{v_{i+1}} \cdots \Psi_{v_d} \cdot \mathcal{X}$. This implies the claim. □

Lemma 3.7. *Let $\mathcal{X}$ be a unimodular fan structure of a fan cycle X of dimension d (such that every cone is generated by its rays). Let $\sigma_1, \sigma_2 \in \mathcal{X}^{(d)}$ having a common face $\tau \in \mathcal{X}^{(d-1)}$. If X is locally irreducible then*

$$\omega_{\mathcal{X}}(\sigma_2) \cdot \Psi_{\sigma_1} - \omega_{\mathcal{X}}(\sigma_1) \cdot \Psi_{\sigma_2} = l \cdot \Psi_\tau,$$

for some linear function l on X.

Proof. Let $\sigma_3, \dots, \sigma_k$ be the remaining facets adjacent to τ. Let $v_1, \dots, v_{d-1}, w_1, \dots, w_k$ be the primitive integral vectors such that $\tau = \langle v_1, \dots, v_{d-1}\rangle$ and $\sigma_i = \langle v_1, \dots, v_{d-1}, w_i\rangle$. As

$$\omega_{\mathcal{X}}(\sigma_2) \cdot \Psi_{\sigma_1} - \omega_{\mathcal{X}}(\sigma_1) \cdot \Psi_{\sigma_2} = \Psi_\tau \cdot (\omega_{\mathcal{X}}(\sigma_2) \cdot \Psi_{w_1} - \omega_{\mathcal{X}}(\sigma_1) \cdot \Psi_{w_2}),$$

we need a linear function l satisfying

$$l_{|\sigma_1} = \omega_{\mathcal{X}}(\sigma_2) \cdot (\Psi_{w_1})_{|\sigma_1},\ l_{|\sigma_2} = -\omega_{\mathcal{X}}(\sigma_1) \cdot (\Psi_{w_2})_{|\sigma_2} \text{ and } l_{|\sigma_i} = 0 \text{ for } i \geq 3.$$

The local irreducibility of X implies that $v_1, \dots, v_d, w_3, \dots, w_k, w_1$ are linearly independent. Thus there exists a linear function l such that $l(w_1) = \omega_{\mathcal{X}}(\sigma_2)$ and $l(v) = 0$ for $v \in \{v_1, \dots, v_{d-1}, w_3, \dots, w_k\}$. By the balancing condition $l(w_2) = -\omega_{\mathcal{X}}(\sigma_1)$; hence l satisfies the above conditions. □

Proof of proposition 3.5. Let $f \in \mathrm{PP}^d(X)$ with $f \cdot X = 0$. We choose a unimodular fan structure $\mathcal{X}$ of X such that every cone in $\mathcal{X}$ is generated by its rays and $f \in \mathrm{PP}^d(\mathcal{X})$. Then there exist $a_\sigma \in \mathbb{Z}$ such that $\overline{f} = \sum_{\sigma \in \mathcal{X}^{(d)}} a_\sigma \cdot \overline{\Psi_\sigma}$ in $\mathrm{PP}^d(X)/\,\mathrm{L}\,\mathrm{PP}^{d-1}(X)$. Fix a maximal cone $\alpha \in \mathcal{X}$. Since $\mathcal{X}$ is connected in codimension 1 it follows by lemma 3.7 that $\overline{\Psi_\sigma} = \frac{\omega_{\mathcal{X}}(\sigma)}{\omega_{\mathcal{X}}(\alpha)} \cdot \overline{\Psi_\alpha}$ for all maximal cells σ. Hence we see that $\overline{f} = \left(\sum_{\sigma \in \mathcal{X}^{(d)}} a_\sigma \cdot \frac{\omega_{\mathcal{X}}(\sigma)}{\omega_{\mathcal{X}}(\alpha)}\right) \overline{\Psi_\alpha}$. Therefore, lemma 3.6 implies that $\overline{f} = 0$. □

We can prove the following corollary in a similar way as theorem 2.25.

Corollary 3.8. *Let X be a smooth tropical cycle and $k \in \{1, \dim X\}$. Then the following is an isomorphism of groups:*

$$C^k(X) \to Z_{\dim X - k}(X), \;\; f \mapsto f \cdot X.$$

Proof. The injectivity follows directly from the local statement (proposition 3.4 resp.3.5). Let $C \in Z_{\dim X-k}(X)$. We choose an open cover $\{V_i^j\}$ of X such that for all i, j we have $V_i^j \subseteq U_i$ and the weighted set $\phi_i(C \cap V_i^j)$ corresponds to (the translation of) an open tropical fan in $\phi_i(V_i^j)$. As the tropical fan associated to $\phi_i(V_i^j)$ is a matroid variety modulo lineality space, the local statement ensures that we can find piecewise polynomials $f_i^j \in \mathrm{PP}^k(\phi_i(V_i^j))$ cutting out $\phi_i(C \cap V_i^j)$. Then $f = \{(V_i^j, f_i^j \circ \phi_i)\} \in C^k(X)$ is a cocycle with $f \cdot X = C$. Note that the difference of two of these local functions gives a zero intersection on the overlaps of the open sets, so the local statement implies that f is indeed a cocycle. □

Remark 3.9. Proving the injectivity of

$$\mathrm{PP}^k(\mathrm{B}(M)/L)/\,\mathrm{L}\,\mathrm{PP}^{k-1}(\mathrm{B}(M)/L) \to Z^{\text{fan}}_{\dim \mathrm{B}(M)/L-k}(\mathrm{B}(M)/L)$$

is all that remains to be done in order to generalise corollary 3.8 to arbitrary codimensions k. Note that we also needed the injectivity of intersecting with piecewise polynomials to prove the surjectivity in the preceding proof.

Remark 3.10. Let C be a codimension k subcycle of a dimension d cycle Y satisfying $C^k(Y) \cong Z_{d-k}(Y)$. Let $h : X \to Y$ be a morphism. We can define the pull-back of C along h to be $h^*C := h^*f \cdot X$, where f is the (unique) cocycle satisfying $f \cdot Y = C$. If X and Y are smooth, this coincides with the pull-back of cycles defined in [FR, definition 8.1]. Furthermore, pull-backs defined in this way clearly have the properties listed in [FR, example 8.2, theorem 8.3]. In particular, we can define pull-backs of points and codimension 1 cycles if Y is smooth, as well as pull-backs of arbitrary cycles if $Y = \mathbb{R}^n$.

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