INTERSECTIONS ON TROPICAL MODULI SPACES

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ABSTRACT. This article tries to answer the question: How far can the algebro-geometric theory of rational descendant Gromov-Witten invariants be carried over to the tropical world? Given the fact that our moduli spaces are non-compact, the answer is surprisingly positive: We discuss universal families and the string, divisor and dilaton equations, we prove a splitting lemma describing the intersection with a "boundary" divisor and we give two criteria that suffice to prove the tropical version of a particular WDVV or topological recursion equation. Discussing these criteria in the case of curves in $\mathbb{R}^1$ or $\mathbb{R}^2$, we prove, for example, that for the toric varieties $\mathbb{P}^1$, $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{F}_1$, $\mathcal{B}l_2(\mathbb{P}^2)$, $\mathcal{B}l_3(\mathbb{P}^2)$ and with Psi-conditions only in combination with point conditions, the tropical and conventional descendant Gromov-Witten invariants coincide. In particular, we can unify and simplify the proofs of the previous tropical enumerative results.

INTRODUCTION

Over the last few years, the list of results in tropical enumerative geometry became quite long. However, lacking an appropriate tropical intersection theory, most existing results are obtained by

- relating the tropical numbers directly to the conventional ones (cf. [Mi03]) and then using the algebro-geometric theory,
- or by involved ad hoc computations (eg. [GM05], [KM06], [FM], [MR08], [CJM08]), which moreover have to be repeated for each new class of enumerative problem.

On the other hand, based on [Mi06], the basic constructions of tropical intersection theory are now developed in [AR07]. Furthermore, in [GKM07] the authors show that the moduli spaces of rational tropical curves are tropical cycles in the sense of [AR07], hence we can apply intersection theory to them. Moreover, in [Mi07] G. Mikhalkin proposes the definition of tropical Psi-divisors, which, as it is shown in [KM07], can also be integrated into the approach of [AR07]. Under these circumstances the obvious program is: Along the lines of the algebro-geometric theory of descendant Gromov-Witten invariants, construct a tropical copy of this theory — as far as possible. First attempts in this spirit are contained in [GKM07], [KM06] and [MR08]. This article tries to carry out this program consequently and in detail.

The "ready for use" main theorems 5.18 and 5.20 state that for $\mathbb{P}^1$ as well as for any complete smooth toric surface and certain degrees, and with Psi-conditions only in combination with point conditions, the tropical and conventional descendant Gromov-Witten invariants coincide. In particular, this unifies and simplifies the proofs of the previous tropical enumerative results for rational curves.

Tropical geometry, as far as it is explored now, can be regarded as an image of the geometry inside the big open torus of a toric variety. This has the advantage that, in a sense, tropical geometry merges the geometry of all toric varieties simultaneously. The other side of the coin is that there is no tropical counterpart of curves with components in the boundary of the toric variety in question, and therefore the moduli spaces of tropical curves are non-compact. Fortunately, tropical intersection theory works on such spaces, but still, we will see that this fact causes the main differences to the conventional theory, which become manifest in 3.13 and in the two criteria that we will need to prove the WDVV equations and the topological recursion (cf. 5.4 and 5.7).

The article contains the following parts: In section 1 we provide the necessary intersection-theoretic tools. In particular, in subsection 1.8 we show that the fan displacement rule for Minkowski weights

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defined in [FS94] coincides with the intersection product of tropical cycles introduced in [AR07, section 9]. In section 2 we cover the case of abstract curves. We prove the universal family property of the forgetful map ft_0, the dilaton equation and reprove the main theorem of [KM07]: The degrees of top-dimensional intersection products of Psi-classes are equal to their conventional counterparts. For this purpose, we introduce tropical analogues of boundary divisors and analyze the intersections, push-forwards and pull-backs of boundary and Psi-divisors. Section 3 extends this to parametrized curves. Moreover, we define tropical descendant Gromov-Witten invariants and verify general string and divisor equations. Section 4 deals with the intersection of a one-dimensional family of curves with a boundary divisor. Using the concept of general position, we derive a splitting lemma which states that this intersection can be decomposed into products of intersections on smaller moduli spaces — if a certain condition on the directions appearing in some fans is fulfilled. Therefore, a discussion of this condition follows. Finally, in section 5 we put things together. We first state the WDVV equations and the topological recursion, requiring two conditions: The just mentioned condition concerning certain directions, and the other one previously known as "the existence of a contracted bounded edge". After an analysis of this contracted edge condition, the final subsection 5.4 proves the equality of the tropical and conventional descendant Gromov-Witten invariants as mentioned before.

1. Intersection theory

This section is devoted to providing us with the intersection-theoretic tools we will need to attack the problems of tropical Gromov-Witten theory in a satisfactory way. Hereby, the subsections 1.1, 1.2, 1.3, 1.4 and 1.5 recall the important definitions and results from [AR07] and [AR08] (however, note that our notations will sometimes slightly differ from the original ones). Parts of this summary already appeared in [MR08].

1.1. **Cycles.** A *cycle* X is a balanced (weighted, pure-dimensional, rational and polyhedral) complex (resp. fan) in a finite-dimensional vector space $V = \Lambda \otimes \mathbb{R}$ with underlying lattice Λ (the most common case is $V = \mathbb{R}^r$, whose underlying lattice, if not specified otherwise, is $\mathbb{Z}^r$). The top-dimensional polyhedra (resp. cones) in X are called *facets*, the codimension one polyhedra (resp. cones) are called *ridges*. *Balanced* means that for each ridge $\tau \in X$ the following *balancing condition at* τ is satisfied: The weighted sum of the primitive vectors of the facets σ around τ

$$\sum_{\substack{\sigma \in X^{(\dim(X))} \\ \tau < \sigma}} \omega(\sigma) v_{\sigma/\tau}$$

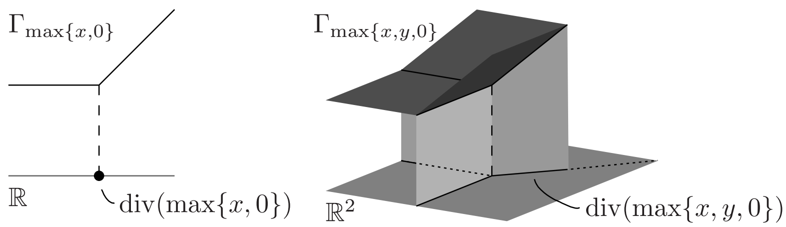
vanishes "modulo τ", or, precisely, lies in the linear vector space spanned by τ, denoted by V_τ. Here, a *primitive vector* $v_{\sigma/\tau}$ *of* σ *modulo* τ is a vector in Λ that points from τ towards σ and fulfills the primitive condition: The lattice $\mathbb{Z}v_{\sigma/\tau} + (V_\tau \cap \Lambda)$ must be equal to the lattice $V_\sigma \cap \Lambda$. Slightly differently, in [AR07] the class of $v_{\sigma/\tau}$ modulo V_τ is called primitive vector and $v_{\sigma/\tau}$ is just a representative of it. We will abbreviate the lattice $V_\sigma \cap \Lambda$ by Λ_σ.
The *support of* X, denoted by X, is the union of all facets in X with non-zero weight. We call X *irreducible* if for any cycle Y of the same dimension with $|Y| \subseteq |X|$ there exists an integer $\mu \in \mathbb{Z}$ such that $Y = \mu \cdot X$. The *positive part of* X, denoted by X^+, is the set of all faces contained in a facet with positive weight. A *general element* x *of* X is an element $x \in |X|$ that lies in the interior of a facet.

1.2. **Cycles modulo refinement.** By abuse of notation, a *cycle* is also a class of balanced fans with common refinement and agreeing weights. A *rational function* φ on such a class is just a rational function on a fan X contained in the class. We can generalize our intersection product to such classes of fans $[X]$ by defining $\varphi \cdot [X] := [\varphi \cdot X]$. In the following, we try to avoid these technical aspects whenever possible. We will also omit the brackets distinguishing between fans and their classes, hoping that no confusion arises.

1.3. **The divisor of a rational function and intersection products.** A *(non-zero) rational function on* X is a function $\varphi : |X| \to \mathbb{R}$ that is integer affine (resp. linear) on each polyhedron (resp. cone). Here, *integer linear* means that it maps lattice elements to integers and *integer affine* means that it is a sum of an integer linear function (called the *linear part*) and a real constant. The *divisor of* φ, denoted by $\operatorname{div}(\varphi) = \varphi \cdot X$, is the balanced subcomplex (resp. subfan) of X constructed in [AR07, 3.3], namely the codimension one skeleton $X \setminus X^{(\dim X)}$ together with the weights $\omega_{\varphi \cdot X}(\tau)$ for each ridge $\tau \in X$. These weights are given by the formula

$$\omega_{\varphi \cdot X}(\tau) = \sum_{\substack{\sigma \in X^{(k)} \\ \tau < \sigma}} \omega(\sigma)\varphi_\sigma(v_{\sigma/\tau}) - \varphi_\tau\Big(\sum_{\substack{\sigma \in X^{(k)} \\ \tau < \sigma}} \omega(\sigma) v_{\sigma/\tau}\Big),$$

where $\varphi_\sigma : V_\sigma \to \mathbb{R}$ denotes the linear part of the affine function $\varphi|_\sigma$. Note that the balancing condition of X around τ ensures that the argument of φ_τ is indeed an element of V_τ. The motivation for this definition is illustrated in the following picture.



The graph Γ_φ of φ in $X \times \mathbb{R}$ is a polyhedral complex whose polyhedra are in one-to-one correspondence with those of X, but in general Γ_φ is not balanced. However, it can be completed to a cycle by adding facets in $(0,-1)$-direction at each ridge of Γ_φ, equipped with the above weights. Now, if we (imaginary) intersect this tropically completed graph of φ with $X \times \{-\infty\}$ (i.e. compute the tropical zero locus of φ), we obtain the cycle $\operatorname{div}(\varphi) = \varphi \cdot X$ of our definition.
If φ is globally affine (resp. linear), all weights are zero, which we denote by $\varphi \cdot X = 0$. Let the *support of* φ, denoted by $|\varphi|$, be the subcomplex of X containing the points $x \in |X|$ where φ is not locally affine. Then we have $|\varphi \cdot X| \subseteq |\varphi|$. Furthermore, the intersection product is bilinear (see [AR07, 3.6]). As the restriction of a rational function to a subcycle is again a rational function, we can also form multiple intersection products $\varphi_1 \cdot \ldots \cdot \varphi_l \cdot X$. In this case we will sometimes omit "$\cdot X$" to keep formulas shorter. Note that multiple intersection products are commutative (see [AR07, 3.7]).

By [AR07, definition 9.3] it is also possible to form the intersection product of two cycles X, Y in $V = \Lambda \otimes \mathbb{R}$: We choose coordinates $x_1, \ldots, x_r$ on Λ (and denote the same coordinates on the second factor of $V \times V$ by $y_1, \ldots, y_r$). Then the diagonal Δ in $V \times V$ is given by $\Delta = \max\{x_1, y_1\} \cdots \max\{x_r, y_r\} \cdot (V \times V)$. Furthermore we consider the function $\pi : \Delta \to V, (x,x) \mapsto x$. Then the intersection product of X and Y in V is given by

$$X \cdot Y := \pi_*\big(\max\{x_1, y_1\} \cdots \max\{x_r, y_r\} \cdot (X \times Y)\big).$$

This intersection product is independent of the chosen coordinates, commutative, associative, bilinear, admits the identity element V and satisfy $(\varphi \cdot X) \cdot Y = \varphi \cdot (X \cdot Y)$, where φ is a rational function on X.

1.4. **Morphisms and projection formula.** A *morphism of cycles* $X \subseteq V = \Lambda \otimes \mathbb{R}$ *and* $Y \subseteq V' = \Lambda' \otimes \mathbb{R}$ is a map $f : |X| \to |Y|$ that is induced from a linear map from Λ to Λ' and that maps each polyhedron (resp. cone) of X into a one of Y. We call f an *isomorphism* and write $X \cong Y$, if there exists an inverse morphism and if for all facets $\sigma \in X$ we have $\omega_X(\sigma) = \omega_Y(f(\sigma))$.
Such a morphism *pulls back rational functions* φ on Y to rational functions $f^*(\varphi) = \varphi \circ f$ on X. Note that the second condition of a morphism makes sure that we do not have to refine X further. $f^*(\varphi)$ is already affine (resp. linear) on each cone. The inclusion $|f^*(\varphi)| \subseteq f^{-1}(|\varphi|)$ holds, as the composition

of an affine and a linear function is again affine.
Furthermore, we can *push forward subcycles* Z of X to subcycles $f_*(Z)$ of Y of same dimension. This is due [GKM07, 2.24 and 2.25] in the case of fans and can be generalized to complexes (see [AR07, 7.3]). We can omit further refinements here if we assume that $f(\sigma) \in Y$ for all $\sigma \in X$. Then $f_*(Z)$ is defined by assigning the following weights to the $\dim(Z)$-dimensional polyhedra $\sigma' \in Y$:

$$\omega_{f_*(Z)}(\sigma') = \sum_{\substack{\sigma \in X \\ f(\sigma)=\sigma'}} |\Lambda_{\sigma'}/f(\Lambda_\sigma)| \cdot \omega_Z(\sigma)$$

By definition we have $|f_*(Z)| \subseteq f(|Z|)$.
The *projection formula* (see [AR07, 4.8]) connects all the above constructions via

$$f_*(f^*(\varphi) \cdot X) = \varphi \cdot f_*(X).$$

1.5. **Rational equivalence.** Here we summarize the definitions and results of [AR08].
Let X be a zero-dimensional cycle. Then *degree* $\deg(X)$ *of* X denotes the sum of the weights of all points in Y.
Now let X be an arbitrary cycle and let $\varphi, \widetilde{\varphi}$ be two rational functions on X. We call them *(rationally) equivalent* if $\varphi - \widetilde{\varphi}$ is the sum of a bounded and a globally linear function. Obviously, this property is preserved when pulled back. Furthermore, if Y is an one-dimensional subcycle of X, then $\deg(\varphi \cdot X) = \deg(\widetilde{\varphi} \cdot Y)$ holds (see [AR07, lemma 8.3]).
Let X be a cycle and let Y be a subcycle. We call Y *rationally equivalent to zero*, denoted by $Y \sim 0$, if there exists a morphism $f : X' \to X$ and a bounded rational function ϕ on X' such that

$$f_*(\phi \cdot X') = Y.$$

This property commutes with taking cartesian products, intersection products (of functions as well as of cycles) and with pushing forward. Moreover, if Y is zero-dimensional, then $Y \sim 0$ implies $\deg(Y) = 0$.
Let $\widetilde{Y}$ be another subcycle of X. Then we call Y and $\widetilde{Y}$ *rationally equivalent* if $Y - \widetilde{Y}$ is rationally equivalent to zero.
If X, Y live in $V = \Lambda \otimes \mathbb{R}$, we call them *numerically equivalent* if for any cycle Z in V of complementary dimension the equation

$$\deg(X \cdot Z) = \deg(Y \cdot Z)$$

holds.

The easiest example of rationally equivalent cycles are translations: Let X be a cycle in $V = \Lambda \otimes \mathbb{R}$ and let us denote by $X + v$ denote the translation of X by an arbitrary vector $v \in V$. Then

$$X \sim X + v$$

holds.

Let X be a cycle in $V = \Lambda \otimes \mathbb{R}$. We define the *degree* or *recession fan* of X, denoted by $\delta(X)$, as follows: $\delta(X)$ is is supported on the purely $\dim(X)$-dimensional part of the polyhedral set

$$\bigcup_{\sigma \in X} \operatorname{rc}(\sigma).$$

Here, the *recession cone* $\operatorname{rc}(\sigma)$ of a polyhedron σ is defined to be the cone containing all vectors $v \in V$ such that, starting at an arbitrary point $x \in \sigma$, the ray $x + \mathbb{R}v$ is contained in σ. Now, for a fine enough fan structure on this polyhedral set, the weights are given by

$$\omega_{\delta(X)}(\sigma') := \sum_{\substack{\sigma \in X \\ \sigma' \subseteq \operatorname{rc}(\sigma)}} \omega_X(\sigma).$$

In particular, if X is a curve, then $\delta(X)$ is just the union of all unbounded rays in X and the weights are the sums of the weights of the rays in X of given direction. Geometrically, we simply shrink all bounded edges of X to a point and move the final single vertex to the origin.

The main result of [AR08] is that for cycles X in $V = \Lambda \otimes \mathbb{R}$, rational equivalence, numerical equivalence and "having the same degree" coincides. To prove this, an important substep is to show that X is always rationally equivalent to its degree,

$$X \sim \delta(X).$$

1.6. **Local computation of intersection products.** Let X be a cycle and let $\tau \in X$ be a polyhedron in X. We define the *star of X at τ* to be the fan

$$\text{Star}_X(\tau) := \{\bar{\sigma} | \tau < \sigma \in X\},$$

where $\bar{\sigma}$ denotes the cone in V/V_τ spanned by the image of $\sigma - \tau$ under the quotient map $q : V \to V/V_\tau$. We make it into a cycle by defining $\omega_{\text{Star}_X(\tau)}(\bar{\sigma}) = \omega_X(\sigma)$ for all facets $\bar{\sigma}$ of $\text{Star}_X(\tau)$ (note that q preserves the codimension of the polyhedra). This fan contains all the local information of X around τ and can be considered as the tropical version of a small neighborhood of an interior point of τ (divided by the "trivial part" V_τ). Its dimension equals the codimension of τ in X. As an example for our way of speaking, we call a cycle X *locally irreducible* if for all $\tau \in X$ the cycle $\text{Star}_X(\tau)$ is irreducible. Note that locally irreducible implies irreducible, but not the other way around.

Let furthermore φ be a rational function on X. Choose an arbitrary affine function ψ with $\varphi|_\tau = \psi|_\tau$. Then $\varphi - \psi$ induces a rational function on $\text{Star}_X(\tau)$ which we denote by φ^τ (and call it a *germ of φ at τ*). This function is only unique up to adding a linear function, which suffices for our intersection-theoretic purposes.

In the following proposition we will express the locality of our intersection product in terms of these notions.

Proposition 1.1. *Let X be a cycle with polyhedra $\tau < \sigma \in X$. Let $\varphi, \varphi_1, \ldots \varphi_l$ be rational functions on X. Then the following statements are true:*

(a) $\text{Star}_{\text{Star}_X(\tau)}(\sigma) = \text{Star}_X(\sigma)$
(b) $(\varphi^\tau)^\sigma = \varphi^\sigma$ *(up to adding a linear function)*
(c) $\text{Star}_{\varphi \cdot X}(\tau) = \varphi^\tau \cdot \text{Star}_X(\tau)$
(d) $\text{Star}_{\varphi_1 \cdot \ldots \cdot \varphi_l \cdot X}(\tau) = \varphi_1^\tau \cdot \ldots \cdot \varphi_l^\tau \cdot \text{Star}_X(\tau)$
(e) *If $l = \dim(X) - \dim(\tau)$, then $\omega_{\varphi_1 \cdot \ldots \cdot \varphi_l \cdot X}(\tau) = \omega_{\varphi_1^\tau \cdot \ldots \cdot \varphi_l^\tau \cdot \text{Star}_X(\tau)}(\{0\})$, i.e. we can compute the weight of τ in $\varphi_1 \cdot \ldots \cdot \varphi_l \cdot X$ "locally" in $\text{Star}_X(\tau)$.*

Proof. (a) and (b) are immediate consequences of the definitions. (d) follows from (c) by induction and (e) is just a special case of (d), namely when $\varphi_1^\tau \cdot \ldots \cdot \varphi_l^\tau \cdot \text{Star}_X(\tau)$ is zero-dimensional. Hence we are left with (c).

Let $r := \dim(X) - \dim(\tau)$ be the codimension of τ in X. The statement is trivial when $r = 0$: Both sides are 0. Assume $r = 1$. In this case, we only have to check

$$\omega_{\varphi \cdot X}(\tau) = \omega_{\varphi^\tau \cdot \text{Star}_X(\tau)}(\{0\}).$$

By adding an affine function we can assume that $\varphi|_\tau = 0$ without changing the intersection product and in particular the weight of τ in $\varphi \cdot X$. But then we can replace both weights according to their definition and observe that

$$\omega_{\varphi \cdot X}(\tau) = \sum_{\substack{\sigma \in X^{(\dim(X))} \\ \tau < \sigma}} \omega(\sigma) \varphi_\sigma(v_{\sigma/\tau}) = \sum_{\bar{\sigma} \in \text{Star}_X(\tau)^{(1)}} \omega(\bar{\sigma}) \varphi^\tau(v_{\bar{\sigma}/\{0\}}) = \omega_{\varphi^\tau \cdot \text{Star}_X(\tau)}(\{0\})$$

holds true, as $[v_{\sigma/\tau}] = v_{\bar{\sigma}/\{0\}} \in V/V_\tau$.

Now let us assume $r > 1$ and let τ' be a ridge in X. Then we can use the previous case as well as (a) and (b) and obtain

$$\omega_{\varphi \cdot X}(\tau) \overset{r=1}{=} \omega_{\varphi^{\tau'} \cdot \text{Star}_X(\tau')}(\{0\}) \overset{\text{(a), (b)}}{=} \omega_{(\varphi^\tau)^{\tau'} \cdot \text{Star}_{\text{Star}_X(\tau)}(\tau')}(\{0\}) \overset{r=1}{=} \omega_{\varphi^\tau \cdot \text{Star}_X(\tau)}(\bar{\tau}'),$$

which proves the claim. ∎

We can extend this to the case of the intersection product of two cycles.

Lemma 1.2. *Let X, Y be two cycles in $V = \mathbb{R} \otimes \Lambda$. Then the equation*

$$\operatorname{Star}_{X \cdot Y}(\tau) = \operatorname{Star}_X(\tau) \cdot \operatorname{Star}_Y(\tau).$$

holds for all polyhedra $\tau \in X \cdot Y$.

Proof. First, we fix some notation. Let $x_1, \ldots, x_r$ be a lattice basis of $\Lambda^\vee$ such that the first $d := \operatorname{codim}_V(\tau)$ elements generate $V_\tau^\perp$. When we consider the product $\Lambda \times \Lambda$, the same coordinates on the second factor will be denoted by $y_1, \ldots, y_r$. Furthermore, let $\Delta : V \to V \times V, x \mapsto (x, x)$ denote the diagonal map. By definition of the intersection product of cycles and using 1.1 (d) we have to compute

$$\operatorname{Star}_{\max\{x_1,y_1\} \cdots \max\{x_r,y_r\} \cdot (X \times Y)}(\Delta(\tau)) = \max\{x_1, y_1\} \cdots \max\{x_r, y_r\} \cdot \operatorname{Star}_{X \times Y}(\Delta(\tau))$$

and

$$\max\{x_1, y_1\} \cdots \max\{x_d, y_d\} \cdot (\operatorname{Star}_X(\tau) \times \operatorname{Star}_Y(\tau))$$

respectively. Thus the statement follows from the fact that

$$\begin{array}{rcl} \max\{x_{d+1}, y_{d+1}\} \cdots \max\{x_r, y_r\} \cdot (V \times V/\Delta(V_\tau)) & \to & V/V_\tau \times V/V_\tau, \\ (x, y) & \mapsto & (x, y) \end{array}$$

is an isomorphism and can be restricted to an isomorphism of $\max\{x_{d+1}, y_{d+1}\} \cdots \max\{x_r, y_r\} \cdot \operatorname{Star}_{X \times Y}(\Delta(\tau))$ and $\operatorname{Star}_X(\tau) \times \operatorname{Star}_Y(\tau)$. □

1.7. **Transversal Intersections.** If we intersect two cycles X, Y the generic case is the following:

Definition 1.3. Let X, Y be two cycles in $V = \Lambda \otimes \mathbb{R}$ of codimension c resp. d. We say X *and* Y *intersect transversally* if $X \cap Y$ is of pure codimension $c + d$ and if for each facet τ in $X \cap Y$ the corresponding neighbourhoods $\operatorname{Star}_X(\tau)$ and $\operatorname{Star}_Y(\tau)$ are (transversal) affine subspaces of V.

In this case, by locality of the intersection product, the computation of $X \cdot Y$ can be reduced to the intersection of vector spaces. This motivates the following study of intersections of linear functions and spaces.

Lemma 1.4. *Let $h_1, \ldots, h_l$ be integer linear functions on V ($l \leq \dim(V) =: r$) and define the rational functions $\varphi_i := \max\{h_i, 0\}$ on V. Let $H : V \to \mathbb{R}^l$ be the linear function with $H(x) = (h_1(x), \ldots, h_l(x))$ and let us assume that H has full rank. Then $\varphi_1 \cdot \ldots \cdot \varphi_l \cdot V$ is equal to the subspace $\ker(H)$ with weight $\operatorname{ind}(H) := |\mathbb{Z}^l / H(\Lambda)|$. Here we give V the fan structure consisting of all cones where each of the h_i is either positive or zero or negative, with all weights being 1.*

Proof. Let us assume $l = 1$ first (i.e. $H = h_1$) In this case we have to compute the weight of the only ridge in V which is $h_1^\perp = \ker(H)$. This ridge is contained in the two facets corresponding to $h_i \geq 0$ and $h_i \leq 0$. Let $v_\geq = -v_\leq$ be corresponding primitive vectors. This implies that for example $v_\geq$ generates the one-dimensional lattice $\Lambda / h_1^\perp \cong h_1(\Lambda)$ and therefore $|\mathbb{Z}/h_1(\Lambda)| = h_1(v_\geq)$. On the other hand we can compute the weight of $h_1^\perp$ in $h_1 \cdot V$ to be

$$\omega_{h_1 \cdot V}(h_1^\perp) = \varphi_1(v_\geq) + \varphi_1(v_\leq) = h_1(v_\geq) + 0 = |\mathbb{Z}/h_1(\Lambda)|.$$

Now we make induction for $l > 1$. The induction hypothesis says that $\varphi_2 \cdot \ldots \varphi_l \cdot V$ is equal to the subspace $\ker(H')$ with weight $\operatorname{ind}(H')$, where $H' = h_2 \times \ldots \times h_l$. By applying the case $l = 1$ to the vector space $\ker(H') = (\ker(H') \cap \mathbb{Z}^r) \otimes \mathbb{R}$, we obtain that $\varphi_1 \cdot \ldots \varphi_l \cdot V$ is equal to the subspace $h_1^\perp \cap \ker(H') = \ker(H)$ with weight $\operatorname{ind}(h_1|_{\ker(H')}) \cdot \operatorname{ind}(H')$. We have to show that this weight coincides with $\operatorname{ind}(H)$. This follows from the exact sequence

$$\begin{array}{ccccccccc} 0 & \to & h_1(\ker(H') \cap \Lambda) & \to & H(\Lambda) & \to & H'(\Lambda) & \to & 0 \\ & & h_1(x) & \mapsto & H(x) = (h_1(x), 0) & & & & \\ & & & & H(x) & \mapsto & H'(x) & & \end{array}$$

and its induced quotient sequence

$$0 \quad \to \quad \mathbb{Z}^{l-1}/H'(\Lambda) \quad \to \quad \mathbb{Z}^l/H(\Lambda) \quad \to \quad \mathbb{Z}/h_1(\ker(H') \cap \Lambda) \quad \to \quad 0 \ .$$

□

Remark 1.5. In the special case $l = r$ the weight of $\{0\}$ in the intersection product $\varphi_1 \cdot \ldots \cdot \varphi_r \cdot V$ is $|\mathbb{Z}^r/H(\Lambda)|$, which equals $|\det(M)|$ where M is a matrix representation of H with respect to a lattice basis of Λ and the standard basis of $\mathbb{Z}^r$. This version of the statement is contained in [MR08]. Note that it can be extended to the case where H has not full rank, as then the intersection product as well as the determinant $\det(M)$ are zero.

Now we use this lemma to compute the intersection of two linear subspaces.

Lemma 1.6. *Let U, W be two subspaces of $V = \mathbb{R} \otimes \Lambda$ (with rational slope) such that $U + W = V$. If we consider U, W as cycles with weight 1, their intersection product can be computed to be*

$$U \cdot W = |\Lambda/\Lambda_U + \Lambda_W| \cdot (U \cap W).$$

Proof. By definition we have to compute

$$\max\{x_1, y_1\} \cdots \max\{x_r, y_r\} \cdot (U \times W),$$

(where we chose arbitrary coordinates on Λ). Instead of $\max\{x_i, y_i\}$, we can as well substract the linear function y_i and use the functions $\max\{x_i - y_i, 0\}$. Now we can apply 1.4. In our case, the function H is just

$$\begin{aligned} H : \Lambda \times \Lambda \quad &\to \quad \Lambda, \\ (x, y) \quad &\mapsto \quad x - y. \end{aligned}$$

Restricted to $U \times W$, this provides

$$U \cdot W = |\Lambda/H(\Lambda_U \times \Lambda_W)| \cdot \pi_*(\ker(H)) = |\Lambda/\Lambda_U \mp \Lambda_W| \cdot (U \cap W).$$

□

Now, as a combination of 1.2 and 1.6, we obtain the following result.

Corollary 1.7. *Let X, Y be two cycles in $V = \mathbb{R} \otimes \Lambda$ that intersect transversally. Then $X \cdot Y = (X \cap Y, \omega_{X \cap Y})$ with the following weight function: Any facet τ in $X \cap Y$ is the intersection of two facets σ, σ' in X resp. Y. Then the weight of $\tau = \sigma \cap \sigma'$ is*

$$\omega_{X \cap Y}(\sigma \cap \sigma') = \omega_X(\sigma)\omega_Y(\sigma')|\Lambda/\Lambda_\sigma + \Lambda_{\sigma'}|.$$

1.8. **Comparison to the "fan displacement rule".** In [FS94] the authors introduce Minkowski weights to describe the Chow cohomology groups of a toric variety combinatorially. In particular, they compute the cup-product of these cohomology groups in terms of Minkowski weights. In this subsection we show explicitly that, when we interpret Minkowski weights as tropical cycles, this cup-product coincides with our product of tropical cycles. Another approach to this topic is given in [Katz06, section 9]

Let Θ be a complete fan in a vector space $V = \mathbb{R} \otimes \Lambda$ of dimension r (in [FS94], the fan is called Δ and the lattice is called N). Let $\Theta^{(k)}$ denote the set of k-dimensional cones in Θ (in [FS94], the exponent indicates the codimension, i.e. $\Delta^{(k)}$ means $\Theta^{(r-k)}$).

Definition 1.8 (cf. [FS94], section 2)**.** A *Minkowski weight c of codimension k* is an integer-valued function on $\Theta^{(r-k)}$ that satisfies for any $\tau \in \Theta^{(r-k-1)}$

$$\sum_{\substack{\sigma \in \Theta^{(r-k)} \\ \tau \subseteq \sigma}} c(\sigma) v_{\sigma/\tau} \in \Lambda_\tau$$

(in [FS94], primitive vectors are denoted by $n_{\sigma,\tau}$).

Let c be a Minkowski weight of codimension k. Of course, if we set $X(c)$ to be the fan $\bigcup_{0\leq i\leq r-k}\Theta^{(i)}$ with weight function c, the Minkowski weight condition precisely coincides with our balancing condition, i.e. $X(c)$ is a tropical cycle of codimension k.

In [FS94] it is shown that Minkowski weights are in one-to-one correspondence with the operational Chow cohomology classes of the toric variety associated to the fan Θ and therefore admit a cup-product with the following properties. Let c, c' be Minkowski weights of codimension k, k'. Then the cup-product $c \cup c'$ is a Minkowski weight of codimension $k + k'$ given by

$$(c\cup c')(\tau) = \sum_{\substack{\sigma\in\Theta^{r-k}\\ \sigma'\in\Theta^{r-k'}\\ \tau\subseteq\sigma,\sigma'}} m^{\tau}_{\sigma,\sigma'}\cdot c(\sigma)\cdot c'(\sigma').$$

Here, the coefficients are not unique but depend on the choice of a generic vector $v \in V$. If we fix such a vector v, then

$$m^{\tau}_{\sigma,\sigma'} = \begin{cases} |\Lambda/\Lambda_\sigma + \Lambda_{\sigma'}| & \text{if } (\sigma+v)\cap\sigma'\neq\emptyset,\\ 0 & \text{otherwise}\end{cases}$$

(cf. [FS94, introduction]).

The tools introduced in the previous sections make it quite easy to prove rigorously that the cup-product of Minkowski weights coincides with our intersection product of tropical cycles in V.

Theorem 1.9. *Let c, c' be Minkowski weights of codimension k, k'. Then the following equation holds:*

$$X(c)\cdot X(c') = X(c\cup c')$$

Proof. For each facet τ in $X(c \cup c')$ we have to show

$$\omega_{X(c)\cdot X(c')}(\tau) = (c\cup c')(\tau).$$

First, note that we can compute both sides locally on $\mathrm{Star}_\Theta(\tau)$, where we of course define the "local" Minkowski weights by $\bar{c}(\bar{\sigma}) := c(\sigma)$ and $\bar{c}'(\bar{\sigma}') := c'(\sigma')$. For the left hand side this follows from 1.2 and for the right hand side it follows from $|\Lambda/\Lambda_\sigma + \Lambda_{\sigma'}| = |(\Lambda/\Lambda_\tau)/((\Lambda_\sigma + \Lambda_{\sigma'})/\Lambda_\tau)|$.
Therefore we can assume $k + k' = r$ and $\tau = \{0\}$. In this case, by plugging in the definition on the right hand side and choosing a generic vector $v \in V$, it remains to show

$$\deg(X(c)\cdot X(c')) = \sum_{\substack{\sigma\in\Theta^{r-k}\\ \sigma'\in\Theta^{r-k'}\\ (\sigma+v)\cap\sigma'\neq\emptyset}} |\Lambda/\Lambda_\sigma + \Lambda_{\sigma'}|\cdot c(\sigma)\cdot c'(\sigma').$$

Now, for a generic vector $v \in V$ we can assume that $X(c) + v$ and $X(c')$ intersect transversally (in fact, this is what the authors of [FS94] mean by a generic vector). Note that, in fact, the sum on the right hand side runs through all points in the intersection of $X(c) + v$ and $X(c')$. Therefore, by 1.7 it equals $\deg((X(c) + v) \cdot X(c'))$. But as $X(c) + v$ and $X(c)$ are rationally equivalent, the equation $\deg(X(c) \cdot X(c')) = \deg((X(c) + v) \cdot X(c'))$ holds and the statement follows. □

1.9. **Convexity and Positivity.** A non-zero cycle X is called *positive*, denoted $X > 0$, if all weights are non-negative. By throwing away the facets with weight 0 (and all polyhedra contained in only such facets) we can assume all weights to be positive. A rational function φ on X is called *convex* if it is locally the restriction of a convex function on V. The pull-back $f^*(\varphi)$ of a convex function is again convex, as the composition of a convex function and a linear map is again convex. Moreover, if Z is a subcycle of X, then $\varphi|_{|Z|}$ is also convex on Z. Combining positivity and convexity we get the following result.

Lemma 1.10. *Let X be a positive cycle and let φ be a convex function on X. Then*

(a) $\varphi \cdot X$ *is positive and*
(b) $|\varphi| = |\varphi \cdot X|$.

Proof. First of all note that we can assume that X is a one-dimensional fan, as all intersection weights can be computed locally modulo the ridge (cf. 1.1 (c)) and convexity is preserved when adding linear functions or when considering the function induced on the quotient. Thus we assume that $X = \{\{0\}, \rho_1, \ldots, \rho_r\}$ is a one-dimensional fan with positive weights $\omega(\rho_i) > 0$. The statements of the lemma translate to

(a) φ convex $\Rightarrow \varphi \cdot X > 0$,
(b) φ convex, $\varphi \cdot X = 0 \Rightarrow \varphi$ linear.

We use the following criteria for linearity and convexity. Let φ be a rational function on X and let us abbreviate the primitive vector of the ray ρ_i by v_i. Then

i) φ is linear if and only if for all $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ with $\sum_i \lambda_i v_i = 0$ it holds
$$\sum_i \lambda_i \varphi(v_i) = 0,$$
ii) φ is convex if and only if for all positive $\lambda_1, \ldots, \lambda_r \geq 0$ with $\sum_i \lambda_i v_i = 0$ it holds
$$\sum_i \lambda_i \varphi(v_i) \geq 0.$$

Now let φ be convex. We can apply criterion ii) to the coefficients $\omega(\rho_i)$, which are positive and satisfy $\sum_i \omega(\rho_i) v_i = 0$. This provides
$$\omega_{\varphi \cdot X}(\{0\}) = \sum_i \omega(\rho_i)\varphi(v_i) \geq 0,$$
which proves (a).
For (b), let us assume that $\sum_i \omega(\rho_i)\varphi(v_i) = 0$ (i.e. $\varphi \cdot X = 0$) but φ is not linear. Then by i) there exist $\lambda_1, \ldots, \lambda_r$ with $\sum_i \lambda_i v_i = 0$ but $\sum_i \lambda_i \varphi(v_i) \neq 0$. W.l.o.g. we can assume $\sum_i \lambda_i \varphi(v_i) < 0$ (otherwise we replace λ_i by $-\lambda_i$). For large enough $C \in \mathbb{R}$ the coefficients $\lambda'_i := \lambda_i + C\omega(\rho_i)$ are all positive and still satisfy $\sum_i \lambda'_i v_i = 0$ and $\sum_i \lambda'_i \varphi(v_i) < 0$, which contradicts ii). Therefore φ is linear, which proves (b). □

An easy but useful application of this lemma is the following one:

Lemma 1.11. *Let $f : X \to Y$ be a morphism of cycles and let us assume that Y is positive. Let furthermore $\varphi_1, \ldots, \varphi_l$ denote convex functions on Y. Then the following equation of sets holds:*
$$|f^*(\varphi_1) \cdots f^*(\varphi_l) \cdot X| \subseteq f^{-1}(|\varphi_1 \cdots \varphi_l \cdot Y|)$$

Proof. This can be proven by an easy induction: If $l = 1$ we have
$$|f^*(\varphi_1) \cdot X| = |f^*(\varphi_1)| \subseteq f^{-1}(|\varphi_1|) = f^{-1}(|\varphi_1 \cdot Y|),$$
where the equalities follow from 1.10 (a). Now for arbitrary l we can apply the case of a single function to φ_l, obtaining
$$|f^*(\varphi_l) \cdot X| \subseteq f^{-1}(|\varphi_l \cdot Y|).$$
This shows that we can restrict the morphism f to $f : f^*(\varphi_l) \cdot X \to \varphi_l \cdot Y$. As $\varphi_l \cdot Y$ is still positive by 1.10 (b), we can apply the induction hypothesis to this restriction, which yields the result. □

1.10. **Complete intersections.** We define the set of m-dimensional *complete intersections* $Z_m^{\text{c.i.}}(X) \subset Z_m(X)$ to be the set of m-dimensional cycles in X obtained as a intersection product $\varphi_1 \cdots \varphi_l \cdot X$ (where $l = \dim(X) - m$).
Let $C, C' \in Z_*^{\text{c.i.}}(X)$ be complete intersections given by $C = \varphi_1 \cdots \varphi_l \cdot X$ and $C' = \varphi'_1 \cdots \varphi'_{l'} \cdot X$. Then we define
$$C \cdot C' := \varphi_1 \cdots \varphi_l \cdot \varphi'_1 \cdots \varphi'_{l'} \cdot X.$$
Using commutativity of the intersection product of functions this multiplication is independent of the chosen functions, commutative and satisfies $|C \cdot C'| = |C| \cap |C'|$. Note that, if $X = V = \Lambda \otimes \mathbb{R}$, it

follows from [AR07, corollary 9.8] that this definition coincides with the usual intersection product of cycles.
Let $C \in Z_m^{\text{c.i.}}(X)$ be given by $C = \varphi_1 \cdots \varphi_l \cdot X$ and let $f : Y \to X$ be a tropical morphism. Then we would like to define the pull-back of C along f to be the complete intersection

$$f^*(C) := f^*(\varphi_1) \cdots f^*(\varphi_l) \cdot Y.$$

However, in general this definition is not independent of the chosen functions $\varphi_1, \ldots, \varphi_l$. But it works in the following case:

Corollary 1.12. *Let X, Y be two cycles and let $\pi : X \times Y \to X$ be the projection onto the first factor. Moreover, let Z be a complete intersection of $X \times Y$ and consider the map $f = \pi|_Z : Z \to X$. Now, if $C = \varphi_1 \cdots \varphi_l \cdot X$ is a complete intersection in X, then the pull-back*

$$f^*(C) := f^*(\varphi_1) \cdots f^*(\varphi_l) \cdot Z$$

is well-defined and the equation

$$|f^*(C)| \subseteq f^{-1}(|C|)$$

holds.

Proof. First, we apply [AR07, 9.6], which yields

$$\pi^*(\varphi_1) \cdots \pi^*(\varphi_l) \cdot (X \times Y)(\varphi_1 \cdots \varphi_l \cdot X) \times Y = C \times Y.$$

Therefore $f^*(\varphi_1) \cdots f^*(\varphi_l) \cdot Z$ is just the product of the complete intersections $C \times Y$ and Z, which does not depend on any choices. Moreover, its support is contained in $|C \times Y|$ and the equation of sets follows. $\square$

Remark 1.13 (Pulling back preserves numerical equivalence). Let C, C' be complete intersections in $\mathbb{R}^r$ and let $f : Y \to \mathbb{R}^r$ be a tropical morphism. Then, if C and C' are numerically equivalent, also $f^*(C)$ and $f^*(C')$ are numerically equivalent in the following sense: If Z is an arbitrary complete intersection in Y of complementary dimension, then

$$\deg(f^*(C) \cdot Z) = \deg(f^*(C') \cdot Z)$$

holds. This follows from the projection formula:

$$\deg(f^*(C) \cdot Z) = \deg(f_*(f^*(C) \cdot Z)) = \deg(C \cdot f_*(Z))$$

In particular, if we move around C in V, the numerical properties of the pull-backs of the original and the translated cycle coincide. This motivates the following subsection about general position.

1.11. **General position.** We now investigate what we can say about the set-theoretic pre-image of a general translation of a cycle under a morphism f.

Lemma 1.14. *Let X be a pure-dimensional polyhedral complex and let $f : X \to \mathbb{R}^r$ be a morphism of polyhedral complexes (i.e. f is linear on every polyhedron of X). Furthermore, let C be a polyhedral complex in $\mathbb{R}^r$ and consider the subcomplex $f^{-1}(C)$ of X consisting of all polyhedra $\tau \cap f^{-1}(\gamma), \tau \in X, \gamma \in C$. Then for a general translation $C' = C + v$ (i.e. $v \in \mathbb{R}^r$ can be chosen from an open dense subset of $\mathbb{R}^r$) the codimension of each non-empty polyhedron $\tau \cap f^{-1}(\gamma)$ of X is equal to*

$$\operatorname{codim}_X(\tau \cap f^{-1}(\gamma)) = \operatorname{codim}_X(\tau) + \operatorname{codim}_{\mathbb{R}^r}(\gamma).$$

Proof. For each τ in X and γ in C we consider the map

$$f_\tau : \text{AffSpan}(\tau) \to \mathbb{R}^r,$$

induced by $f|_\tau$. Now we are interested in $\tau \cap f^{-1}(\gamma') = \tau \cap f_\tau^{-1}(\gamma')$ for general translations γ' of γ. We have to distinguish the cases $\text{Im}(f_\tau) + V_\gamma = \mathbb{R}^r$ and $\text{Im}(f_\tau) + V_\gamma \neq \mathbb{R}^r$. In the latter case, $f_\tau^{-1}(\gamma')$ is empty for general γ'. In the former case, $f_\tau^{-1}(\gamma')$ is a polyhedron of dimension $\dim(\tau) + \dim(\gamma) - r$, and for general γ' it is disjoint from τ or intersects the interior of τ, in which case $\tau \cap f_\tau^{-1}(\gamma')$ has the same dimension $\dim(\tau) - \operatorname{codim}_{\mathbb{R}^r}(\gamma)$, which is the expected dimension.

As there are only finitely many pairs τ, γ, this holds simultanously for all pairs for general enough translations of C. $\square$

This technical statement has the following more applicable consequences:

Corollary 1.15 (Preimages of general translations)**.** *Let $f_k : X \to \mathbb{R}^r, k = 1, \ldots, n$ be morphisms of pure-dimensional polyhedral complexes and let $C_k, k = 1, \ldots, n$ be cycles in $\mathbb{R}^r$. Then for a general translation $C'_k = C_k + v_k, v_k \in \mathbb{R}^r$ the following holds: Either $Z := f_1^{-1}(C'_1) \cap \ldots \cap f_n^{-1}(C'_n)$ is empty or*

(a) *the codimension of Z in X equals the sum*

$$\operatorname{codim}_X(Z) = \sum_{k=1}^{n} \operatorname{codim}_{\mathbb{R}^r}(C_k),$$

(b) *Z is pure-dimensional,*

(c) *if a polyhedron α of Z is contained in a polyhedron τ of X, the codimensions satisfy $\operatorname{codim}_X(\tau) \leq \operatorname{codim}_Z(\alpha)$ (in particular, the interior of a facet of Z is contained in the interior of a facet of X),*

(d) *if the images $f_k(\alpha)$ of a polyhedron α of Z are contained in polyhedra γ_k of C_k, the codimensions satisfy $\sum_{k=1}^{n} \operatorname{codim}_{C_k}(\gamma_k) \leq \operatorname{codim}_Z(\alpha)$.*

Proof. It is easy to prove the statement in the case $n = 1$: (a), (b) and (c) are immediate consequences of 1.14 and (d) follows from applying 1.14 to the $(r - \operatorname{codim}_Z(\alpha) - 1)$-dimensional skeleton of C_1 (if γ_1 belonged to this skeleton, α would be contained in its preimage, which (for general translations) contradicts (a)). Now the statement follows if we apply the case of a single morphism to $f_1 \times \ldots \times f_n : X \to (\mathbb{R}^r)^n$ and $C := C_1 \times \ldots \times C_n$. $\square$

Remark 1.16. Sticking to the notation of the previous statement, let us assume that X is a cycle and that the maps f_k are tropical morphisms. Moreover, we assume that the maps f_k are projections (at least after composing with an isomorphism) and that the complexes C_k are complete intersections. Then $f_1^*(C_1) \cdots f_n^*(C_n)$ is also a pure-dimensional object of the same dimension as $f_1^{-1}(C'_1) \cap \ldots \cap f_n^{-1}(C'_n)$. Indeed, 1.12 shows that

$$|f_1^*(C_1) \cdots f_n^*(C_n)| \subseteq f_1^{-1}(C'_1) \cap \ldots \cap f_n^{-1}(C'_n)$$

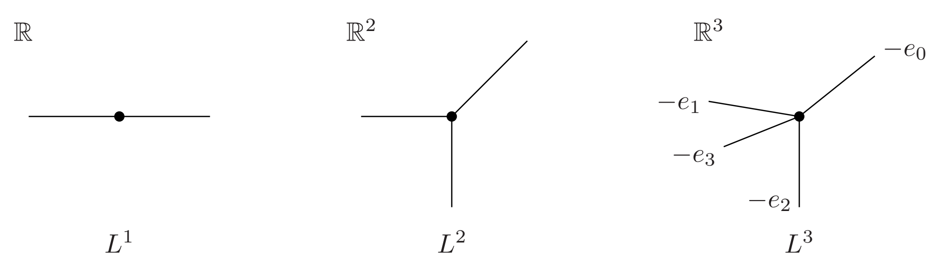
holds. Hence we should think of $f_1^*(C_1) \cdots f_n^*(C_n)$ as the polyhedral set $f_1^{-1}(C'_1) \cap \ldots \cap f_n^{-1}(C'_n)$ with the additional data of weights.

2. Intersections on the space of abstract curves

Let us start with a definition of smooth abstract curves. As a local model of a curve we will use the following fan. Let $e_1, \ldots, e_r$ be the standard basis in $\mathbb{R}^r$ and set $e_0 := -e_1 - \ldots - e_r$. We define the one-dimensional fan

$$L^r := \{\{0\}, \mathbb{R}_{\geq}(-e_0), \ldots, \mathbb{R}_{\geq}(-e_r)\},$$

with weights $\omega(\mathbb{R}_{\geq}(-e_i)) = 1$ for all i. This fan is balanced because of $e_0 + \ldots + e_r = 0$.



Note that this fan is also irreducible, as $e_0 + \ldots + e_r = 0$ is the only equation that the generating vectors fulfill.

Definition 2.1. A *smooth abstract curve* C is a one-dimensional connected cycle that is locally isomorphic to L^r for suitable r (i.e. for each vertex V in C we have $\text{Star}_C(V) \cong L^{val(V)}$). The *genus of* C is the first Betti number of $|C|$. An *n-marked smooth abstract curve* $(C, x_1, \ldots, x_n)$ is a smooth abstract curve C with n unbounded rays (called *leaves*), which are labelled by $x_1, \ldots x_n$. If we instead label the leaves by elements of some finite set I, we will call it an I-marked curve.

Remark 2.2. As no other abstract curves will be considered we will often omit "smooth". Note that by definition C is (locally) irreducible. We will always consider abstract curves up to isomorphisms. Still, our definition provides C with the structure of a connected graph equipped with a metric, which is essentially the definition of an abstract curve in existing literature, in particular in [GKM07]. This "old" definition has the disadvantage, that, when embedding these graphs, the balancing condition must be included in the definition of the embedding morphism (see [GKM07, 4.1]).
A way out of this is to use the glueing techniques developed in [AR07] and start with a genuine abstract connected cycle of dimension one. Beyond the structure of a metric graph, these objects contain the data of local pictures of the balancing condition around each vertex V. But this picture is uniquely fixed to be $L^{val(V)}$ by the requirement of smoothness. Using this more complicated definition, an embedding morphism is just a morphism of cycles $f : C \to \mathbb{R}^r$.
Our definition here requires additionally that a global embedding of our curve exists (which we then forget as we identify isomorphic curves). Basically we do this to avoid glueing. However, when investigating universal families it will turn out that our definition is not a restriction (at least for rational curves), but that each rational curve in the "old" sense has a canonical embedding as a smooth curve in some big $\mathbb{R}^r$.

We now give a criterion to decide whether a one-dimensional fan with $r+1$ rays is isomorphic to L^r or not. Following our way of speaking, we could also call it a smoothness criterion.

Lemma 2.3 (Smoothness criterion)**.** *Let X be a one dimensional fan in $V = \Lambda \otimes \mathbb{R}$ with $r+1$ rays, all with weight 1 and generated by the primitive vectors $v_0, \ldots, v_r$. Then the following are equivalent:*

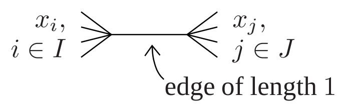
(a) *X is isomorphic to L^r.*
(b) *For arbitrary real coefficients $\lambda_0, \ldots, \lambda_r \in \mathbb{R}$ we have*
   i)
$$\sum_{i=0}^{r} \lambda_i v_i = 0 \quad \Leftrightarrow \quad \lambda_0 = \ldots = \lambda_r \quad \Leftrightarrow \quad \lambda_i - \lambda_j = 0 \text{ for all } i, j,$$
   ii)
$$\sum_{i=0}^{r} \lambda_i v_i \in \Lambda \quad \Leftrightarrow \quad \lambda_i - \lambda_j \in \mathbb{Z} \text{ for all } i, j.$$

Proof. As (b) holds for L^r, the direction (a) $\Rightarrow$ (b) follows.
On the other hand, if (b) holds, an isomorphism is given by
$$\begin{aligned} \Omega : \mathbb{R}^r &\to V_X = \langle |X| \rangle_{\mathbb{R}}, \\ e_i &\mapsto v_i. \end{aligned}$$
This linear map is well-defined and bijective by (b) i). Moreover by ii) (and also i)) it follows that the vectors v_i generate the lattice $\Lambda \cap V_X$. Therefore $\Omega(\mathbb{Z}^r) = \Lambda \cap V_X$, which implies that Ω as well as its inverse map are integer maps, hence tropical morphisms. □

The tropical analogue $\mathcal{M}_n := \mathcal{M}_{0,n}, n \geq 3$ of the space of stable n-marked abstract curves is (a quotient of) the space of trees, or the tropical Grassmanian (see [GKM07, section 3], and also [SS04], [Mi07]). The fan $\mathcal{M}_n$ is stratified by cones corresponding to combinatorial types of trees. A general curve (i.e. an element in the interior of a facet) is a 3-valent metric tree. All facets are equipped with weight 1.

The important thing is that, since $\mathcal{M}_n \subseteq \mathbb{R}^{\binom{n}{2}}/\mathrm{Im}(\Phi_n)$ fulfills the balancing condition and therefore is a tropical cycle, our intersection-theoretic constructions from above are available on this moduli space. The coordinates in $\mathbb{R}^{\binom{n}{2}}$ describe the metric of the tree, i.e. the $\{i,j\}$-entry measures the distance between the vertices adjacent to the leaves x_i and x_j. If we work with $\mathcal{M}_{n+1}$, the extra leaf is labelled by x_0. In the following we assume $n \geq 4$ ($\mathcal{M}_3$ is just a point). Whenever we write $I|J$, we mean that I and J form a non-trivial partition of $[n] = \{1,\ldots,n\}$ (or of $\{0\} \cup [n]$ if we work with $\mathcal{M}_{n+1}$). If $|I| \neq 1 \neq |J|$, such a partition describes a ray in $\mathcal{M}_n$ generated by the abstract curve with only one bounded edge

$V_{I|J} :=$



$\in \mathcal{M}_n.$

A cone of $\mathcal{M}_n$ contains the ray generated by $V_{I|J}$ if and only if the corresponding combinatorial type contains a bounded edge that subdivides the leaves into I and J. In this sense, we can regard the set of partitions as "global" labels of the edges of a curve, where $I|J$ labels the leaf x_i if $I = \{i\}$ or $J = \{i\}$ and a bounded edge otherwise.

We will sometimes also think of $V_{I|J}$ as a vector in $\mathbb{R}^{\binom{n}{2}}$, in which case we also allow $|I| = 1$ or $|J| = 1$ to get easier formulas. However, as $V_{\{k\}|[n]\setminus\{k\}} = \Phi(0,\ldots,0,1,0,\ldots,0)$, these vectors vanish modulo $\mathrm{Im}(\Phi)$. Note that the underlying lattice of $\mathbb{R}^{\binom{n}{2}}$ is *not* $\mathbb{Z}^{\binom{n}{2}}$, but is the lattice generated by these vectors $V_{I|J}$, denoted by Λ_n (see [GKM07, 3.3]).

We will now define divisors respectively rational functions that play the role of "boundary" divisors in our moduli space. They all lie in the codimension one skeleton of $\mathcal{M}_n$, therefore represent higher-valent curves. Note that our nomenclature is a bit confusing here. Even if we call all curves parametrized by $\mathcal{M}_n$ smooth, we consider the codimension one skeleton of $\mathcal{M}_n$ to be (part of) the boundary of $\mathcal{M}_n$ which classically consists of singular curves.

As $\mathcal{M}_n$ is simplicial, we can define a rational function on $\mathcal{M}_n$ by assigning an integer to each $I|J$: The integers are the values of the function at $V_{I|J}$ and on each cone, we extend the function by linearity.

Definition 2.4. We define the rational function $\varphi_{I|J}$ by

$$\varphi_{I|J}(V_{I'|J'}) := \begin{cases} 1 & \text{if } I = I', \\ 0 & \text{otherwise}, \end{cases}$$

Furthermore, we use the notation

$$\varphi_{k,l} := \varphi_{\{k,l\}|[n]\setminus\{k,l\}}$$

for $k \neq l$.

The ridges of $\mathcal{M}_n$ are combinatorial types of curves with one 4-valent vertex, which we will draw like this:

$${}^{A}_{D}\!\times\!{}^{B}_{C}$$

Here A, B, C and D denote the four parts of the combinatorial type adjacent to the 4-valent vertex and by abuse of notations also the sets of leaves belonging to this part (as, in most cases, this is the only information needed).

When we want to compute the weight of a ridge ${}^{A}_{D}\!\times\!{}^{B}_{C}$ in the divisor of a rational function on $\mathcal{M}_n$, we need to know how $\mathcal{M}_n$ looks like locally around ${}^{A}_{D}\!\times\!{}^{B}_{C}$. In fact, it is easy to see that $\mathrm{Star}_{\mathcal{M}_n}({}^{A}_{D}\!\times\!{}^{B}_{C})$ contains three facets corresponding to the three types of removing the 4-valent vertex by inserting a new bounded edge. The (representatives of the) primitive vectors are $V_{A\cup B|C\cup D}$, $V_{A\cup C|B\cup D}$ and $V_{A\cup D|B\cup C}$. For the balancing condition around ${}^{A}_{D}\!\times\!{}^{B}_{C}$, it suffices to show the equation

$$V_{A\cup B|C\cup D} + V_{A\cup C|B\cup D} + V_{A\cup D|B\cup C} = V_{A|B\cup C\cup D} + V_{B|A\cup C\cup D} + V_{C|A\cup B\cup D} + V_{D|A\cup B\cup C},$$

as all vectors on the right hand side lie in the vector space spanned by the ridge ${}^A_D\times{}^B_C$, as required. But the equation follows from the fact that, on the level of metric trees, the distance between two marked leaves is identical on both sides: If both leaves belong to the same set A, B, C, D, the distance is 0, if not, it is 2.
This discussion also shows that $\mathcal{M}_n$ is locally irreducible: As $\mathrm{Star}_{\mathcal{M}_n}({}^A_D\times{}^B_C)$ contains the minimal number of three facets, all with weight 1, it is necessarily irreducible for all ${}^A_D\times{}^B_C$.

Let us now compute the divisors of the functions $\varphi_{I|J}$.

Lemma 2.5. *The weight of a facet in* $\mathrm{div}(\varphi_{I|J})$ *(which is a ridge in* $\mathcal{M}_n$*) is (up to permuting the names of* I*,* J *and of* A*,* B*,* C*,* D *respectively, which we always allow in the sequel)*

$$\omega_{\varphi_{I|J}}({}^A_D\times{}^B_C) = \begin{cases} 1 & \text{if } I = A \cup B, \\ -1 & \text{if } I = A, \\ 0 & \text{otherwise} \end{cases}$$

Proof. Following from the previous discussion, the weight of ${}^A_D\times{}^B_C$ in $\mathrm{div}(\varphi_{I|J})$ is by definition

$$\begin{aligned} \omega_{\varphi_{I|J}}({}^A_D\times{}^B_C) &= \varphi_{I|J}(V_{A\cup B|C\cup D}) + \varphi_{I|J}(V_{A\cup C|B\cup D}) + \varphi_{I|J}(V_{A\cup D|B\cup C}) \\ &\quad -\varphi_{I|J}(V_{A|B\cup C\cup D}) - \varphi_{I|J}(V_{B|A\cup C\cup D}) - \varphi_{I|J}(V_{C|A\cup B\cup D}) - \varphi_{I|J}(V_{D|A\cup B\cup C}). \end{aligned}$$

Hence, this weight is 1 if I is the union of two of the sets A, B, C, D and is -1 if I equals one of the four sets. Otherwise, it is 0. □

Remark 2.6. These divisors were computed before by Matthias Herold (see [H]).

Remark 2.7. We will regard the divisors $\mathrm{div}(\varphi_{I|J})$ as the tropical analogue of the irreducible components of the boundary of the moduli space of stable curves $\overline{M}_{0,n}$. The positive part $\mathrm{div}(\varphi_{I|J})^+$ can be regarded as the set of curves with bounded edge $I|J$ whose length has shrunk to zero. One might think of such curves as reducible curves having two components with leaves I and J respectively and glued together at the 4-valent vertex. The negative part can be considered as a correction term due to the non-compactness of the tropical moduli space.
This point of view is justified by the fact that in the following we will reprove many of the algebro-geometric statements concerning the intersection-theoretic behaviour of boundary divisors. As a general reference for the algebro-geometric theory we use the unpublished paper "Notes on psi classes" [K] by Joachim Kock. It is very useful for our purposes as it contains all the statements we are interested in in down-to-earth terms.

Lemma 2.8 (cf. [K] 1.2.5)**.** *The equation*

$$\varphi_{i,j} \cdot \varphi_{i,k} \cdot \mathcal{M}_n = 0$$

holds for $n \geq 4$ *and pairwise different* $k, l, i \in [n]$.

Proof. An abstract curve C cannot simultanously have bounded edges with partitions $\{i,j\}|\{i,j\}^c$ and $\{i,k\}|\{i,k\}^c$ (as for example the first partition forces i and k to be adjacent to the same 3-valent vertex). Let C be a curve in $|\varphi_{i,k}|$. At least after resolving a 4-valent vertex, it contains an edge with partition $\{i,k\}|\{i,k\}^c$ and can therefore *not* contain an edge with partition $\{i,j\}|\{i,j\}^c$. But $\varphi_{i,j}$ just measures the length of such an edge if present. Thus, $\varphi_{i,j}|_{|\varphi_{i,k}|} \equiv 0$. □

Analogues of Psi-classes on tropical $\mathcal{M}_n$ have been defined recently by G. Mikhalkin ([Mi07]). How such Psi-classes intersect is discussed in [KM07]. We use the notion Psi-divisor instead of Psi-class to emphasize that, in contrast to the algebro-geometric case, tropically Psi-divisors are *not* defined up to rational equivalence. In fact, 2.24 suggests that we should think of a tropical Psi-divisor as a boundary representation of the corresponding Psi-class. Let us recall the important definitions and results of [KM07] here.

Definition 2.9. We define the *k-th Psi-function* ψ_k by

$$\psi_k(V_{I|J}) := \frac{|I|(|I|-1)}{(n-1)(n-2)}$$

for all partitions $I|J$ with $|I|, |J| \geq 2$ and $k \in J$.

Remark 2.10. Our function ψ_k equals the function $\frac{1}{\binom{n-1}{2}} f_k$ defined in [KM07] (follows from [KM07, Lemma 2.6]). In particular, ψ_k is a convex function (cf. [KM07, Remark 2.5]). Note that in this paper, ψ_k and $\varphi_{I|J}$ denote functions and *not* their corresponding divisors. On the other hand, as mentioned in subsection 1.10, this is only a matter of notation. For intersection-theoretic purposes, the actual choice of a function defining the same divisor does not matter.

Remark 2.11. Obviously the numbers $\psi_k(V_{I|J})$ are only rational. A generalization of intersection theory to rational numbers is straightforward, but nearly unnecessary: The weights of the divisor of ψ_k turn out to be integers (see the following proposition) and there exist integer rational functions producing the same divisor (see 2.24). This particular function ψ_k was chosen in [KM07] because of its symmetry.

Proposition 2.12 (see [KM07] 3.5)**.** *The divisor* $\operatorname{div}(\psi_k)$ *consists of the cones corresponding to trees where the marked leaf* k *is at a* 4*-valent vertex, i.e. the weight of a facet in* $\operatorname{div}(\psi_k)$ *(which is a ridge in* $\mathcal{M}_n$*) is*

$$\omega_{\psi_k}\left({}^{A}_{D}\times{}^{B}_{C}\right) = \begin{cases} 1 & \textit{if } \{k\} = A, \\ 0 & \textit{otherwise} \end{cases}$$

Notation 2.13. As in the conventional case we will introduce the following τ-notation that makes formulas shorter and hides "unimportant" data such as the number of marked leaves. For any positive integers $a_1, \ldots, a_n$ we define

$$(\tau_{a_1} \cdot \ldots \cdot \tau_{a_n}) := \psi_1^{a_1} \cdot \ldots \cdot \psi_n^{a_n} \cdot \mathcal{M}_n.$$

Every factor τ_{a_k} stands for a marked leaf and the index a_k serves as the exponent with which the corresponding Psi-function appears in the intersection product. If $\sum a_k = \dim(\mathcal{M}_n) = n-3$, the above cycle is zero-dimensional (in fact, its only point corresponds to the curve without bounded edges where all leaves are adjacent to one single vertex) and we define

$$\langle \tau_{a_1} \cdot \ldots \cdot \tau_{a_n} \rangle := \deg\left(\psi_1^{a_1} \cdot \ldots \cdot \psi_n^{a_n} \cdot \mathcal{M}_n\right).$$

The main theorem of [KM07] computes these intersection products of Psi-divisors:

Theorem 2.14 (Intersections of Psi-divisors for abstract curves, see [KM07] 4.1)**.** *The intersection product* $(\tau_{a_1} \cdot \ldots \cdot \tau_{a_n})$ *is the subfan of* $\mathcal{M}_n$ *consisting of the closure of the cones of dimension* $n - 3 - \sum_{i=1}^n a_i$ *whose interior curves* C *have the property:*
Let $k_1, \ldots, k_q \subseteq N$ *be the marked leaves adjacent to a vertex* V *of* C*. Then the valence of* V *is*

$$\operatorname{val}(V) = a_{k_1} + \ldots + a_{k_q} + 3.$$

Let us define the multiplicity of this vertex to be $\operatorname{mult}(V) := \binom{\operatorname{val}(V)-3}{a_{k_1},\ldots,a_{k_q}}$*. Then the weight of such a cone* σ *in* X *is*

$$\omega_X(\sigma) = \prod_V \operatorname{mult}(V),$$

where the product runs through all vertices V *of an interior curve of* σ*.*

In this section we will reprove the zero-dimensional case of this theorem (see 2.22). To do this, we first have to analyze how Psi- and boundary divisors intersect and how they behave when pulled back or pushed forward along forgetful morphisms.

Lemma 2.15 (cf. [K] 1.2.7)**.** *It holds*

$$\varphi_{i,j} \cdot \psi_i \cdot \mathcal{M}_n = 0$$

for $n \geq 4$ *and* $k \neq l \in [n]$*.*

Proof. Curves in $|\psi_i|$ cannot contain a bounded edge with partition $\{i,j\}|\{i,j\}^c$, as the leaf i does not lie at a 3-valent vertex. Thus $\varphi_{i,j}$ vanishes on $|\psi_i|$. □

The forgetful map $\mathcal{M}_{n+1} \to \mathcal{M}_n$ that forgets the extra leaf x_0 is denoted by ft_0 (cf. [GKM07, 3.8]). By [GKM07, 3.9] this map is a tropical morphism. Therefore we can ask how Psi-functions behave when pulled back along ft_0.

Proposition 2.16 (Pull-back of Psi-functions, cf. [K] 1.3.1)**.** *Let $n \geq 4$ and let $\mathrm{ft}_0 : \mathcal{M}_{n+1} \to \mathcal{M}_n$ be the morphism that forgets the leaf x_0. For $k \in [n]$ it holds*

$$\mathrm{div}(\psi_k) = \mathrm{div}(\mathrm{ft}_0^* \psi_k) + \mathrm{div}(\varphi_{0,k}).$$

Proof. This can be proven by explicitly computing the weights of the codimension one faces of the three divisors. We distinguish four cases (up to renaming A, B, C and D):

$\omega_f(\tfrac{A}{D}\times\tfrac{B}{C})$	$f=\psi_k$	$f=\mathrm{ft}_0^*\psi_k$	$f=\varphi_{0,k}$
$A=\{0,k\}$	0	1	-1
$A=\{0\}, B=\{k\}$	1	0	1
$A=\{0,\ldots\}, B=\{k\}$	1	1	0
otherwise	0	0	0

□

Corollary 2.17 (cf. [K] 1.3.2 and 1.3.3)**.** *Let $n \geq 4$ and let $\mathrm{ft}_0 : \mathcal{M}_{n+1} \to \mathcal{M}_n$ be the morphism that forgets the leaf x_0. Then for $k \in [n]$ the following formulas hold:*

(a)
$$\varphi_{0,k}^2 = -\,\mathrm{ft}_0^*(\psi_k)\cdot\varphi_{0,k}$$

(b)
$$\psi_k^a = \mathrm{ft}_0^*(\psi_k)^a + \mathrm{ft}_0^*(\psi_k)^{a-1}\cdot\varphi_{0,k}$$

(c)
$$\psi_k^a = \mathrm{ft}_0^*(\psi_k)^a + (-1)^{a-1}\varphi_{0,k}^a$$

Proof. All the formulas are easy applications of 2.15 and 2.16. □

Lemma 2.18. *Let $n \geq 4$ and let $\mathrm{ft}_0 : \mathcal{M}_{n+1} \to \mathcal{M}_n$ be the morphism that forgets the leaf x_0 and choose $k \in [n]$. Then*

$$\mathrm{ft}_{0*}(\mathrm{div}(\varphi_{0,k})) = \mathrm{ft}_{0*}(\mathrm{div}(\psi_k)) = \mathcal{M}_n.$$

Proof. We show $\mathrm{ft}_{0*}(\mathrm{div}(\varphi_{0,k})) = \mathcal{M}_n$ by direct computation: Let σ' be a facet of $\mathcal{M}_n$ corresponding to a 3-valent combinatorial type. Let V be the vertex adjacent to k. Then there exists precisely one cone σ in $\mathrm{div}(\varphi_{0,k})$ whose image under ft_0 is σ', namely the cone obtained by attaching the additional leaf x_0 to the vertex V. Moreover, on such a cone, the length of the bounded edges remain unchanged under ft_0 and therefore $\mathrm{ft}_0(\Lambda_\sigma) = \Lambda_{\sigma'}$. On the other hand, cones in $\mathrm{div}(\varphi_{0,k})$ with negative weight are not mapped injectively, as in this case x_0 is adjacent to a 3-valent vertex and stabilization is needed. This shows that $\mathrm{ft}_{0*}(\mathrm{div}(\varphi_{0,k})) = \mathcal{M}_n$.
The equation $\mathrm{ft}_{0*}(\mathrm{div}(\psi_k)) = \mathcal{M}_n$ follows from the same argument or by using 2.16, the projection formula and $\mathrm{ft}_{0*}(\mathcal{M}_{n+1}) = 0$ (because the dimension is too big). □

Proposition 2.19 (Univeral family ft_0 for abstract curves)**.** *Let p be a point in $\mathcal{M}_n$ and let $C_p = \mathrm{ft}_0^{-1}(p)$ be the fibre of p under the forgetful morphism $\mathrm{ft}_0 : \mathcal{M}_{n+1} \to \mathcal{M}_n$. Then the following holds:*

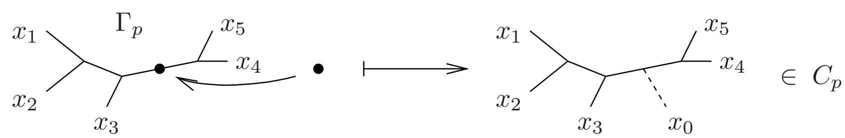
(a) *C_p has the canonical structure of a one-dimensional polyhedral complex.*
(b) *The leaves of C_p (as graph itself) are the facets where x_0 and another leaf x_i lie at the same 3-valent vertex (i.e. the leaves are given by $L_i := \overline{\{y \in C_p | \varphi_{0,i}(y) > 0\}}$). Moreover $p \in \mathcal{M}_n$ represents the n-marked metric graph $(C_p, L_1, \ldots, L_n)$.*
(c) *When we equip all its facets with weight 1, C_p is a smooth abstract curve (in the sense of 2.1).*

(d) *Let* $\sum_k \mu_k p_k = \varphi_1 \cdot \ldots \cdot \varphi_{n-3} \cdot \mathcal{M}_n$ *be a zero-dimensional cycle in* $\mathcal{M}_n$ *obtained as the intersection product of convex functions* φ_j*. Then*

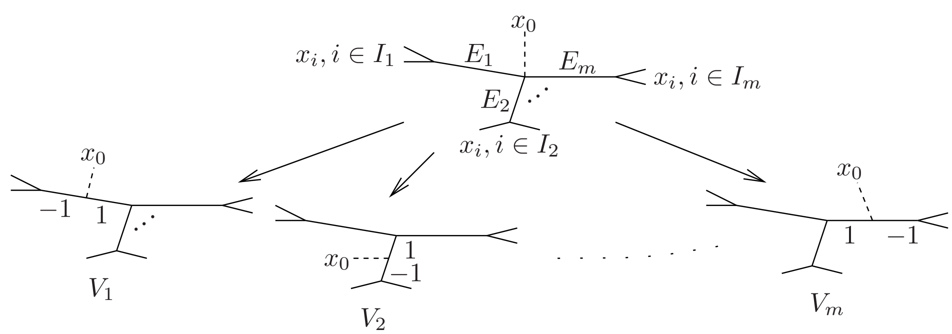
$$\mathrm{ft}_0^*(\varphi_1) \cdot \ldots \cdot \mathrm{ft}_0^*(\varphi_{n-3}) \cdot \mathcal{M}_{n+1} = \sum_k \mu_k C_{p_k}.$$

We write this as $\mathrm{ft}_0^*(\sum_k \mu_k p_k) = \sum_k \mu_k C_{p_k}$.

Proof. (a): As polyhedral complex, C_p consists of the polyhedra $\mathrm{ft}_0|_\sigma^{-1}(p)$ for each cone σ of $\mathcal{M}_{n+1}$. The dimension of these polyhedra can be at most one as $\dim(f_0(\sigma)) \geq \dim(\sigma) - 1$ (it depends on whether x_0 is adjacent to a 3-valent or higher-valent vertex).
(b): Let Γ_p denote the n-marked metric graph represented by p. The bijective map $\Gamma_p \to C_p$ indicated in the picture identifies the two graphs.



(c): Let V be a vertex of C_p. It corresponds to the metric graph Γ_p with the extra leaf x_0 adjacent to one of the vertices. Let us label the other edges adjacent to this vertex by $1, \ldots, m$ and let us divide the other leaves $[n] = I_1 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} I_m$ according to via which edge one reaches x_i from x_0. There are m facets in C_p adjacent to V corresponding to moving x_0 on one of the edges. Hereby on has to shorten the edge $I_k|I_k^c$ as much as the length of $I_k \cup \{x_0\}|(I_k \cup \{x_0\})^c$ increases.



Thus the primitive integer vector of the corresponding facet with respect to V is given by

$$V_k := V_{I_k \cup \{x_0\}} - V_{I_k}.$$

Note that this formula as well as the following ones also holds in the case that I_k consists only of a single leaf x_i (which means x_i is adjacent to the same vertex as x_0), as $V_{\{x_i\}} = 0 \in \mathbb{R}^{\binom{n+1}{2}}/\mathrm{Im}(\Phi_{n+1})$. To prove the statement we now use 2.3 and verify the conditions i) and ii), which can be done by applying some formulas of [KM07]. Let $\mathcal{S}$ be the set of two-element subsets of $[n]$ (i.e. not containing 0). It follows from [KM07, 2.3, 2.4, 2.6] that the vectors $V_S, S \in \mathcal{S}$ fulfill i) and ii) (with $V = \mathbb{R}^{\binom{n+1}{2}}/\mathrm{Im}(\Phi_{n+1})$ and $\Lambda = \Lambda_n$). Furthermore [KM07, 2.6] gives us a representation of our vectors in terms of the vectors V_S, namely

$$V_{I_k} = \sum_{\substack{S \in \mathcal{S} \\ S \subseteq I_k}} V_S$$

$$V_{I_k \cup \{x_0\}} = \sum_{\substack{S \in \mathcal{S} \\ S \cap I_k = \emptyset}} V_S, = -\Big(\sum_{\substack{S \in \mathcal{S} \\ S \cap I_k \neq \emptyset}} V_S \Big),$$

and therefore

$$V_k = -\Big(\sum_{S\in\mathcal{S}} |S \cap I_k| \cdot V_S\Big).$$

Now let $\lambda_1, \dots, \lambda_m$ be arbitrary real coefficients. Then we obtain the formula

$$\sum_{k=1}^{m} \lambda_k V_k = -\Big(\sum_{\substack{\{i,j\}\in\mathcal{S} \\ i\in I_k, j\in I_{k'}}} (\lambda_k + \lambda_{k'}) \cdot V_{\{i,j\}}\Big).$$

Now all differences of two coefficients on the left hand side $\lambda_k - \lambda'_k$ can be obtained as differences of two coefficients on the right hand side (choose elements $i \in I_k, j \in I_{k'}, l \in I_{k''}$; then the coefficients of $V_{\{i,l\}}$ and $V_{\{j,l\}}$ differ by $\lambda_k + \lambda_{k''} - \lambda_{k'} - \lambda_{k''} = \lambda_k - \lambda_{k'}$). Conversely, a right hand side difference of coefficients equals the sum of two left hand side differences. (The coefficients of $V_{\{i_1,i_2\}}$ and $V_{\{j_1,j_2\}}$ differ by $(\lambda_{k_1} - \lambda_{l_1}) + (\lambda_{k_2} - \lambda_{l_2})$, where $i_1 \in I_{k_1}, i_2 \in I_{k_2}, j_1 \in I_{l_1}, j_2 \in I_{l_2}$.) Hence, as conditions 2.3 i) and ii) hold for the vectors V_S, they also hold for the vectors V_k.
(d): First of all, the set-theoretic equation

$$|\operatorname{ft}_0^*(\varphi_1) \cdot \ldots \cdot \operatorname{ft}_0^*(\varphi_{n-3}) \cdot \mathcal{M}_{n+1}| \subseteq \operatorname{ft}_0^{-1}(|\varphi_1 \cdot \ldots \cdot \varphi_{n-3} \cdot \mathcal{M}_n|) = \bigcup_k |C_{p_k}|.$$

follows from 1.11. But the sets $|C_{p_k}|$ are pairwise disjoint (as they are fibres of pairwise different points) and belong to irreducible cycles (as the curves C_{p_k} are smooth abstract curves). Thus any one-dimensional cycle whose support lies in $\bigcup_i |C_{p_k}|$ is actually a sum $\sum_k \lambda_k C_{p_k}, \lambda_k \in \mathbb{Z}$. So it remains to check that in our case these coefficients λ_k coincide with μ_k. To do this, we choose an arbitrary leaf $x_i \neq x_0$ and consider the function $\varphi_{0,i}$ on C_{p_k}. On the leaf L_i of C_{p_k}, where x_0 and x_i are adjacent to the same 3-valent vertex, it measures the length of the third edge, elsewhere it is constantly zero. Thus $\varphi_{0,i} \cdot C_{p_k} = V_{p_k}$, where V_{p_k} is the vertex of C_{p_k} adjacent to L_i (where x_0 and x_i lie together at a higher-valent vertex). Thus we get

$$\operatorname{ft}_{0*}\big(\varphi_{0,i} \cdot (\sum_k \lambda_k C_{p_k})\big) = \operatorname{ft}_{0*}\big(\sum_k \lambda_k V_{p_k}\big) = \sum_k \lambda_k p_k.$$

On the other hand we can use projection formula and 2.18 and compute

$$\operatorname{ft}_{0*}\big(\varphi_{0,i} \cdot \operatorname{ft}_0^*(\varphi_1) \cdot \ldots \cdot \operatorname{ft}_0^*(\varphi_{n-3}) \cdot \mathcal{M}_{n+1}\big) = \varphi_1 \cdot \ldots \cdot \varphi_{n-3} \cdot \operatorname{ft}_{0*}(\varphi_{0,i} \cdot \mathcal{M}_{n+1}) = \sum_k \mu_k p_k.$$

Comparing the coefficients proves the statement. □

Remark 2.20. Hence there is a one-to-one correspondence between curves according to the "old" definition (i.e. as metric graphs) and definition 2.1. In particular, $\mathcal{M}_n$ parametrizes smooth abstract curves in our sense.

Theorem 2.21 (String equation for abstract curves, cf. [K] 1.4.2)**.** *For zero-dimensional intersection products of Psi-divisors the following holds:*

$$\langle \tau_0 \prod_{k=1}^{n} \tau_{a_k} \rangle_d = \sum_{i=1}^{n} \langle \tau_{a_i - 1} \prod_{k\neq i} \tau_{a_k} \rangle_d$$

Proof. The proof is identical to the algebro-geometric one: We have to compute degree of the intersection product $\prod_{k=1}^n \psi_k^{a_k} \cdot \mathcal{M}_{n+1}$. First we replace each term $\psi_k^{a_k}$ $(k \neq 0)$ by $\operatorname{ft}_0^*(\psi_k)^{a_k} + \operatorname{ft}_0^*(\psi_k)^{a_k - 1} \cdot \varphi_{0,k}$ using 2.17 (b) and multiply the product out. As $\varphi_{0,k} \cdot \varphi_{0,k'} = 0$ for $k \neq k'$ (see 2.8), we only get the following $n+1$ terms:

$$\prod_{k=1}^{n} \operatorname{ft}_0^*(\psi_k)^{a_k} \cdot \mathcal{M}_{n+1} + \sum_{i=1}^{n} \operatorname{ft}_0^*(\psi_i)^{a_i - 1} \cdot \prod_{k\neq i} \operatorname{ft}_0^*(\psi_k)^{a_k} \cdot \varphi_{0,i} \cdot \mathcal{M}_{n+1}$$

Now we push this cycle forward along ft_0 and use projection formula. The first term vanishes for dimension reasons and, as $\varphi_{0,i}$ pushes forward to $\mathcal{M}_n$ by 2.18, the other terms provide the desired result. □

Remark 2.22. As in the classical case, the string equation suffices to compute all intersection numbers of Psi-divisors of abstract curves (see [K, 1.5.1]). Namely, if $\sum a_i = n - 3$, the equation

$$\langle \tau_{a_1} \cdot \ldots \cdot \tau_{a_n} \rangle = \frac{(n-3)!}{a_1! \cdot \ldots \cdot a_n!}$$

holds. This was proven in [KM07, 4.2] using the paper's main theorem [KM07, 4.1] (cited here in 2.14). Note, however, that in order to prove the string equation it was not necessary to use [KM07, 4.1].

Lemma 2.23. *Let $n > 4$ and let $\mathrm{ft}_0 : \mathcal{M}_{n+1} \to \mathcal{M}_n$ be the morphism that forgets the last leaf. Then*

$$\mathrm{ft}_{0*}(\mathrm{div}(\varphi_{I|J})) = \begin{cases} \mathcal{M}_n & \textit{if } I = \{0,k\} \textit{ or } J = \{0,k\} \textit{ for some } k \in [n], \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. The first part is shown in 2.18. So let us consider the second part: There exist $i \in I$ and $j \in J$, both different from 0. Consider a facet σ' in $\mathcal{M}_n$ corresponding to a combinatorial type where x_i and x_j are adjacent to the same 3-valent vertex V. All ridges in $\mathcal{M}_{n+1}$ mapping onto σ', are obtained by attaching x_0 to any of the vertices. If not attached to V, the induced partition A, B, C, D cannot separate i and j. If attached to V, the induced partition is $\{0\}, \{i\}, \{j\}, D$. It follows from $\{0, i\} \neq I$ and $\{0, j\} \neq J$ that D intersects both I and J and therefore none of these types is contained in $\mathrm{div}(\varphi_{I|J})$. Hence σ' is not contained in the push-forward of $\mathrm{div}(\varphi_{I|J})$. But $\mathcal{M}_n$ is irreducible, thus $\mathrm{ft}_{0*}(\mathrm{div}(\varphi_{I|J})) = 0$. □

Lemma 2.24 (cf. [K] 1.5.2)**.** *For $n \geq 4$ we define*

$$(x_1|x_2, x_3) := \sum_{\substack{I|J \\ 1 \in I;\, 2,3 \in J}} \mathrm{div}(\varphi_{I|J}).$$

Then

$$\mathrm{div}(\psi_1) = (x_1|x_2, x_3).$$

Proof. We use induction on the number of leaves n. For $n = 4$, only the partition $\{1,4\}|\{2,3\}$ contributes to the sum. But $\mathrm{div}(\psi_1)$ as well as $\mathrm{div}(\varphi_{1,4|2,3})$ is just the single vertex in $\mathcal{M}_4$ parametrizing the curve $\frac{1}{4}\!\times\!\frac{2}{3}$ with weight 1. For the induction step, assume $n \geq 4$ and consider the morphism $\mathrm{ft}_0 : \mathcal{M}_{n+1} \to \mathcal{M}_n$ that forgets the leaf x_0 and let $I'|J'$ be a partition of $[n]$. Then $\mathrm{ft}_0^*(\varphi_{I'|J'})$ measures the sum of the lengths of the edges separating I' and J' if present. Hence we obtain

$$\mathrm{ft}_0^*(\varphi_{I'|J'}) = \varphi_{I' \cup \{0\}|J'} + \varphi_{I'|J' \cup \{0\}}.$$

Using the induction hypothesis, we conclude that $\mathrm{ft}_0^*(\psi_1)$ equals the sum on the right hand side except for the partition $\{0,1\}|\{0,1\}^c$. This missing summand is provided by 2.16. □

Lemma 2.25 (cf. [K] 1.6.1)**.** *Let $n \geq 4$ and let $\mathrm{ft}_0 : \mathcal{M}_{n+1} \to \mathcal{M}_n$ be the morphism that forgets the leaf x_0. Then*

$$\mathrm{ft}_{0*}(\mathrm{div}(\psi_0)) = (n-2)\mathcal{M}_n.$$

Proof. 1. version: We express ψ_0 as $(x_0|x_1, x_2)$ by 2.24 and use linearity of the push-forward. Lemma 2.23 says that we get *one* $\mathcal{M}_n$ for each $\varphi_{\{0,k\}|\{0,k\}^c}$ and zero for each other $\varphi_{I|J}$. As k runs through $\{3, \ldots, n\}$, the statement follows.
2. version: Alternatively we can obtain the result by directly computing the number of facets in $\mathrm{div}(\psi_0)$ mapping onto a fixed facet in $\mathcal{M}_n$ as in 2.18: The ridges in $\mathcal{M}_{n+1}$ mapping onto a fixed facet in $\mathcal{M}_n$ are obtained by attaching the extra leaf n at an arbitrary vertex and in each case the corresponding lattice index is 1. Now each of these ridges has a 4-valent vertex adjacent to x_0, hence is contained in $\mathrm{div}(\psi_0)$. So it remains to count the number of vertices of a 3-valent rational graph with n leaves, which is $n-2$. (Let v be the number of vertices, let e be the number of bounded edges. As there are no loops, we have $e = v - 1$. On the other hand, count the number of flags in the given graph. This is $3v$ (each vertex is 3-valent, hence provides 3 flags) as well as $2e + n$ (each bounded edge provides two, each leaf provides one flag). Plugging in gives $v = n - 2$.) □

Proposition 2.26 (Dilaton equation for abstract curves, cf. [K] 1.6.2)**.** *Let $\langle \prod_{k=1}^{n} \tau_{a_k} \rangle$ be a zero-dimensional intersection product. Then*

$$\langle \tau_1 \cdot \prod_{k=1}^{n} \tau_{a_k} \rangle = (n-2) \langle \prod_{k=1}^{n} \tau_{a_k} \rangle.$$

Proof. The proof is identical to the algebro-geometric one, using 2.17, 2.15, 2.18, 2.25 and the projection formula.
As degree is preserved, we push forward $(\tau_1 \cdot \prod_{k=1}^{n} \tau_{a_k})$ along the forgetful morphism ft_0 forgetting the extra leaf x_0 corresponding to the factor τ_1. To see what happens, we use 2.17 (b) and replace each term $\psi_k^{a_k}$ by $\mathrm{ft}_0^*(\psi_k)^{a_k} + \mathrm{ft}_0^*(\psi_k)^{a_k-1} \cdot \varphi_{0,k}$. When we multiply the whole product out, all summands containing a factor $\varphi_{0,k}$ vanish when multiplied with ψ_0 (see 2.15). It follows

$$\psi_0 \cdot \prod_{k=1}^{n} \psi_k^{a_k} = \psi_0 \cdot \prod_{k=1}^{n} \mathrm{ft}_0^*(\psi_k)^{a_k}$$

and the projection formula together with $\mathrm{ft}_{0*}(\mathrm{div}(\psi_0)) = (n-2)\mathcal{M}_n$ from 2.25 gives the desired result. □

3. Intersections on the space of parametrized curves

A *(labelled) degree* Δ *in* $\mathbb{R}^r$ is a finite set of labels together with a map $\Delta \to \mathbb{Z}^r \setminus \{0\}$ to the set of non-zero integer vectors. Furthermore the images of this map, denoted by $v(x_i), i \in \Delta$ as they will later play the role of the directions of the leaves x_i, sum up to zero, i.e. $\sum_{i \in \Delta} v(x_i) = 0$. The number of elements in Δ is denoted by $\#\Delta$ (to distingiush it from the support of a cycle). As an example, we define the *projective degree* d *(in dimension* r*)* to be the set $[(r+1)d]$ with the map

$$\begin{array}{rcl} [(r+1)d] & \to & \mathbb{Z}^r \setminus \{0\}, \\ 1, \ldots, d & \mapsto & -e_0, \\ d+1, \ldots, 2d & \mapsto & -e_1, \\ \vdots & & \vdots \\ rd+1, \ldots, (r+1)d & \mapsto & -e_r, \end{array}$$

where, as usual, $e_1, \ldots, e_r$ denote the standard basis vectors and $e_0 := e_1 + \ldots + e_r$.

Definition 3.1. An n*-marked (labelled) parametrized curve of degree* Δ *in* $\mathbb{R}^r$ is a tuple (C, h), where C is an $[n] \cup \Delta$-marked smooth abstract curve and $h : C \to \mathbb{R}^r$ is a tropical morphism such that for all leaves x_i the ray $h(x_i) \subseteq \mathbb{R}^r$ has direction $v(x_i)$. Here $v(x_i)$ is set to be zero if $i \in [n]$. The genus of (C, h) is defined to be the genus of C.

Remark 3.2. The leaves $x_i, i \in [n]$ are called *marked leaves*, as they correspond to the marked points of stable maps classically. Marked leaves are contracted by h. In contrast to that we call the leaves $x_i, i \in \Delta$ *non-contracted leaves*. Our curves are called "labelled" as also the non-contracted leaves are labelled.
Two parametrized curves (C, h) and (C', h') are called isomorphic (and therefore identified in the following) if there exists an isomorphism $\Phi : C \to C'$ identifying the labels and satisfying $h = h' \circ \Phi$.
Let us compare our definition to [GKM07, definition 4.1]. Conditions (a) and (b) in that definition make sure that h is a tropical morphism in our sense (at least locally; but again, considering the universal family of $\mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta)$ we will see that a global integer affine map h always exists). Condition (c) is also contained in our definition.

Let $\mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta)$ be the moduli space of rational n-marked labelled parametrized curves of degree Δ in $\mathbb{R}^r$. Its construction as a tropical cycle can be found in [GKM07, 4.7]. After fixing one of the marked leaves x_i as *anchor leaf* (we avoid "root leaf" as, from the botanic point of view, this is nonsense), we

can identify $\mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta)$ with $\mathcal{M}_{[n]\cup\Delta} \times \mathbb{R}^r$, where the first factor parametrizes the abstract curve C and the second factor contains the coordinates of the image point of the anchor leaf x_i. So again, cones in $\mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta)$ correspond to combinatorial types of the underlying abstract curves, but this time the minimal cone is not zero- but r-dimensional because we can move the curve in $\mathbb{R}^r$.
For enumerative purposes, we would like to identify curves whose only difference is the labelling of the non-contracted leaves. Let $\mathcal{M}_n(\mathbb{R}^r, \Delta)$ denote the set of these *unlabelled* curves. Then the number of elements in a general fibre of the map $\mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta) \to \mathcal{M}_n(\mathbb{R}^r, \Delta)$ forgetting the labelling of the non-contracted leaves equals the number of possibilities to label a general unlabelled curve, which is

$$\Delta! := \prod_{v \in \mathbb{Z}^r \setminus \{0\}} n(v)!,$$

where $n(v)$ denotes the number of times v occurs as $v(x_i), i \in \Delta$. Therefore each enumerative invariant computed on $\mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta)$ must simply be divided by $\Delta!$ to get the corresponding one in $\mathcal{M}_n(\mathbb{R}^r, \Delta)$.
From now on, $I|J$ denotes a (non-empty) partition of $[n] \cup \Delta$ (or $\{0\} \cup [n] \cup \Delta$ if we work with $\mathcal{M}_{n+1}^{\text{lab}}(\mathbb{R}^r, \Delta)$). Again such partitions can be used as global labels of the edges of our curves. The direction of the image of the corresponding edge under h is given by

$$v_{I|J} := \sum_{i \in I} v(x_i) = -(\sum_{j \in J} v(x_j))$$

(as an exception, the ordering of I and J plays a little role here, namely $v_{I|J} = -v_{J|I}$). We call $I|J$ *reducible* if $v_{I|J} = 0$ (i.e. if the corresponding edge is contracted). This is equivalent to requiring that the corresponding splitted sets $\Delta_I = I \cap \Delta$ and $\Delta_J = J \cap \Delta$ fulfill the balancing condition, i.e. are degrees on its own. Also the marked leaves split up into $[n] = \{i \in I | v(x_i) = 0\} \cup \{j \in I | v(x_j) = 0\}$. In this sense, the partition corresponds (nearly) to a conventional partition $(S', \beta' | S'', \beta'')$ of the marked points $S = S' \cup S''$ and the degree $\beta = \beta' + \beta''$, occuring for example in the splitting lemma [K] 5.2.1. However, note that in the tropical setting it is possible to permute non-contracted leaves with the same direction vector between I and J without changing the corresponding conventional partition, hence in general several tropical reducible partitions correspond to the same conventional partition.
In contrast to that, the irreducible partitions $I|J$ do not have a counterpart in the algebro-geometric moduli space. A way to explain this is the following. If we let grow the length of the edge $I|J$ towards infinity, the image of our curve under h remains unchanged if $I|J$ is reducible. If not, some part of the curve (depending on where we picked our anchor leaf) moves towards the "boundary" of $\mathbb{R}^r$. In our yet uncompactified tropical moduli space, we have no limit point for this movement. Hence, these partitions stand for the difference between the boundary structures of the tropical resp. algebro-geometric moduli spaces. As a consequence, to recover classical enumerative results tropically, one principally can follow two strategies. One could try to compactify the tropical moduli spaces and extend intersection theory to the new boundary; up to now, no rigorous attempts in this direction have been made. Or one must check that a in particular count the irreducible partitions $I|J$ do not contribute (which in existing literature is contained in proving the existence of an contracted edge when the $\mathcal{M}_4$-coordinate is arbitrarily big, see [GM05, 5.1] and [MR08, 4.4]).

Note that, independently of the choice of a anchor leaf, there exists a forgetful map $\text{ft}' : \mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta) \to \mathcal{M}_{[n]\cup\Delta}$ forgetting just the position of a curve in $\mathbb{R}^r$. This forgetful map $\text{ft}' : \mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta) \to \mathcal{M}_{[n]\cup\Delta}$ is a morphism of tropical varieties, as after choosing a anchor leaf and identifying $\mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta)$ with $\mathcal{M}_{[n]\cup\Delta} \times \mathbb{R}^r$, ft' is just the projection onto the first factor. We use this to define Psi-functions on $\mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta)$.

Definition 3.3 (Psi-functions for parametrized curves)**.** For a partition $I|J$ of $[n] \cup \Delta$ we define the function $\varphi_{I|J}$ on $\mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta)$ to be $\text{ft}'^*(\varphi_{I|J}^{\text{abstr}})$, where $\varphi_{I|J}^{\text{abstr}}$ is the corresponding function on $\mathcal{M}_{[n]\cup\Delta}$. For $i = 1, \ldots, n$ we define *the k-th Psi-function on* $\mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta)$ to be $\psi_k := \text{ft}'^*(\psi_k^{\text{abstr}})$, where the ψ_k^{abstr} is the k-th Psi-function on $\mathcal{M}_{[n]\cup\Delta}$.

Remark 3.4. Again, in spite of defining functions we are actually interested in its divisors. Note that by 1.12 the pull-backs of the respective divisors do not depend on the particular functions.

We can immediately generalize statement 2.14 to parametrized curves.

Lemma 3.5 (Intersections of Psi-divisors for parametrized curves)**.** *Let $a_1, \dots, a_n$ be positive integers and let $X = \prod_{k=1}^n \psi_k^{a_k} \cdot \mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta)$ be a product of Psi-divisors. Then X is the subfan of $\mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta)$ consisting of the closure of the cones of dimension $n + \#\Delta - 3 - \sum_{i=1}^n a_i$ whose interior curves C have the property:*
Let $k_1, \dots, k_q \subseteq [n]$ be the marked leaves adjacent to a vertex V of C. Then the valence of V is

$$\operatorname{val}(V) = a_{k_1} + \ldots + a_{k_q} + 3.$$

Let us define the multiplicity of this vertex to be $\operatorname{mult}(V) := \binom{\operatorname{val}(V)-3}{a_{k_1},\dots,a_{k_q}}$. *Then the weight of such a cone σ in X is*

$$\omega_X(\sigma) = \prod_V \operatorname{mult}(V),$$

where the product runs through all vertices V of an interior curve of σ.

Proof. Choose an anchor leaf and identify $\mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta)$ with $\mathcal{M}_{[n]\cup\Delta} \times \mathbb{R}^r$. Then ft' is just the projection on the first factor and we can apply [AR07, 9.6], i.e. instead of intersecting the pull-backs of the f_k on the product, we can just intersect the f_k on the first factor and then multiply with $\mathbb{R}^2$. Thus,

$$X = (\prod_{k=1}^n (\psi_k^{\text{abstr}})^{a_k} \cdot \mathcal{M}_{[n]\cup\Delta}) \times \mathbb{R}^r,$$

where here ψ_k^{abstr} denotes a Psi-function on $\mathcal{M}_{[n]\cup\Delta}$. Now, as the weight of $\mathbb{R}^r$ is one and the combinatorics of a curve do not change under ft', the statements follows from 2.14. □

Lemma 3.6. *Let ft_0 be the map $\mathcal{M}_{n+1}^{\text{lab}}(\mathbb{R}^r, \Delta) \to \mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta)$ that forgets the extra leaf x_0 and assume $n + \#\Delta \geq 4$ (and $n \geq 1$). Furthermore, let x_i, x_j, x_k be pairwise different leaves. Then the following equations hold (where all the occurring intersection products are computed in $\mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta)$ or $\mathcal{M}_{n+1}^{\text{lab}}(\mathbb{R}^r, \Delta)$ respectively):*

(a) (cf. 2.8)

$$\varphi_{i,j} \cdot \varphi_{i,k} = 0$$

(b) (cf. 2.15)

$$\varphi_{i,j} \cdot \psi_i = 0$$

(c) (cf. 2.16)

$$\operatorname{div}(\psi_k) = \operatorname{div}(\operatorname{ft}_0^* \psi_k) + \operatorname{div}(\varphi_{0,k})$$

(d) (cf. 2.17 (a))

$$\varphi_{0,k}^2 = -\operatorname{ft}_0^*(\psi_k) \cdot \varphi_{0,k}$$

(e) (cf. 2.17 (b))

$$\psi_k^a = \operatorname{ft}_0^*(\psi_k)^a + \operatorname{ft}_0^*(\psi_k)^{a-1} \cdot \varphi_{0,k}$$

(f) (cf. 2.17 (c))

$$\psi_k^a = \operatorname{ft}_0^*(\psi_k)^a + (-1)^{a-1} \varphi_{0,k}^a$$

(g) (cf. 2.18)

$$\operatorname{ft}_{0*}(\operatorname{div}(\varphi_{0,k})) = \operatorname{ft}_{0*}(\operatorname{div}(\psi_k)) = \mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta)$$

(h) (cf. 2.23)

$$\operatorname{ft}_{0*}(\operatorname{div}(\varphi_{I|J})) = \begin{cases} \mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta) & \textit{if } I = \{0,k\} \textit{ or } J = \{0,k\} \textit{ for some } k \in [n], \\ 0 & \textit{otherwise}. \end{cases}$$

(i) (cf. 2.24)

$$\operatorname{div}(\psi_i) = (x_i | x_j, x_k) := \sum_{\substack{I|J \\ i \in I;\, j,k \in J}} \operatorname{div}(\varphi_{I|J}),$$

where the sum runs also through non-reducible *partitions.*

(j) (cf. 2.25)

$$\mathrm{ft}_{0*}(\mathrm{div}(\psi_0)) = (n + \#\Delta - 2)\mathcal{M}_n^{\mathrm{lab}}(\mathbb{R}^r, \Delta),$$

(which is different *to the algebro-geometric factor $n-2$ that equals the abstract case).*

Proof. As in the proof of 3.5, we apply [AR07, 9.6] to the morphism $\mathrm{ft}' : \mathcal{M}_n^{\mathrm{lab}}(\mathbb{R}^r, \Delta) = \mathcal{M}_{[n]\cup\Delta} \times \mathbb{R}^r \to \mathcal{M}_{[n]\cup\Delta}$ forgetting the position in $\mathbb{R}^r$. This means that instead of computing the intersection product on $\mathcal{M}_n^{\mathrm{lab}}(\mathbb{R}^r, \Delta)$ we can compute them on $\mathcal{M}_{[n]\cup\Delta}$ and therefore use the corresponding statements for abstract curves. For statements (c) – (h) and (j) we also use $\mathrm{ft}_0 = \mathrm{ft}_0^{\mathrm{abstr}} \times \mathrm{id}$. □

Definition 3.7 (Evaluation maps and its pull-backs)**.** The *evaluation map* $\mathrm{ev}_k : \mathcal{M}_n^{\mathrm{lab}}(\mathbb{R}^r, \Delta) \to \mathbb{R}^r$, for $k \in [n]$, maps each parametrized curve (C, h) to the position of its k-th leaf $h(x_k)$ (see [GKM07, 4.2]). If we choose one of the marked leaves, say x_a, as anchor leaf, then the evaluation maps are morphisms from $\mathcal{M}_{[n]\cup\Delta} \times \mathbb{R}^r$ to $\mathbb{R}^r$ obeying the following mapping rule:

$$(C^{\mathrm{abstr}}, P) \mapsto P + \sum_{\substack{I|J \\ a\in I, k\in J}} \varphi_{I|J}(C^{\mathrm{abstr}})\, v_{I|J}$$

In particular, if our anchor leaf is chosen to be x_k, then ev_k is just the projection onto the second factor. Let $C \in Z_m^{\mathrm{c.i.}}(\mathbb{R}^r)$ be given by $C = h_1 \cdot \ldots \cdot h_l \cdot X$. Then we can apply 1.12 which states that there is a well-defined *pull-back of C along* ev_k

$$\mathrm{ev}_k^*(C) := \mathrm{ev}_k^*(h_1) \cdot \ldots \cdot \mathrm{ev}_k^*(h_l).$$

Proposition 3.8 (Univeral family $\mathrm{ft}_0, \mathrm{ev}_0$ for parametrized curves)**.** *Let p be a point in $\mathcal{M}_n^{\mathrm{lab}}(\mathbb{R}^r, \Delta)$ and let $C_p = \mathrm{ft}_0^{-1}(p)$ be the fibre of p under the forgetful morphism $\mathrm{ft}_0 : \mathcal{M}_{n+1}^{\mathrm{lab}}(\mathbb{R}^r, \Delta) \to \mathcal{M}_n^{\mathrm{lab}}(\mathbb{R}^r, \Delta)$. Then the following holds:*

(a) *When we equip all its facets with weight 1, C_p is a rational smooth abstract curve. Its leaves are naturally $[n] \cup \Delta$-marked by $L_i := \overline{\{y \in C_p | \varphi_{0,i}(y) > 0\}}$.*
(b) *The tuple $(C_p, \mathrm{ev}_0 |_{|C_p|})$ is an n-marked parametrized curve of degree Δ. Moreover, p represents $(C_p, \mathrm{ev}_0 |_{|C_p|})$.*
(c) *Let $\sum_k \mu_k p_k = \varphi_1 \cdot \ldots \cdot \varphi_{n+\#\Delta-3} \cdot \mathcal{M}_n^{\mathrm{lab}}(\mathbb{R}^r, \Delta)$ be a zero-dimensional cycle in $\mathcal{M}_n^{\mathrm{lab}}(\mathbb{R}^r, \Delta)$ obtained as the intersection product of convex functions φ_j. Then*

$$\mathrm{ft}_0^*(\varphi_1) \cdot \ldots \cdot \mathrm{ft}_0^*(\varphi_{n+\#\Delta-3}) \cdot \mathcal{M}_{n+1}^{\mathrm{lab}}(\mathbb{R}^r, \Delta) = \sum_k \mu_k C_{p_k}.$$

We write this as $\mathrm{ft}_0^*(\sum_k \mu_k p_k) = \sum_k \mu_k C_{p_k}$.

Proof. (a): First of all, let us fix an anchor leaf $x_a, a \in [n]$ in order to identify $\mathcal{M}_{n+1}^{\mathrm{lab}}(\mathbb{R}^r, \Delta) = \mathcal{M}_{n+\#\Delta+1} \times \mathbb{R}^r$ and $\mathcal{M}_n^{\mathrm{lab}}(\mathbb{R}^r, \Delta) = \mathcal{M}_{[n]\cup\Delta} \times \mathbb{R}^r$. We use again $\mathrm{ft}_0 = \mathrm{ft}_0^{\mathrm{abstr}} \times \mathrm{id}$, where $\mathrm{ft}_0^{\mathrm{abstr}}$ is the corresponding forgetful map on the abstract spaces. Then the fibre of $p = (p', P)$ equals $C_{p'} \times \{P\}$, where $C_{p'}$ is the $[n] \cup \Delta$-marked rational smooth abstract curve considered in 2.19 (a)–(c).
(b): We have to check that the direction of the rays $\mathrm{ev}_0(L_i)$ are correct. For curves in L_i, the only length that varies is that of the third edge adjacent to the same 3-valent vertex as x_i and x_0. Hence we can use the description of ev_0 in 3.7 and obtain for all $y \in L_i$

$$\mathrm{ev}_0 |_{L_i}(y) = Q + \varphi_{0,i}(y) \cdot v_{\{0,i\}|\{0,i\}^c},$$

where $Q \in \mathbb{R}^r$ is some constant vector. But $v_{\{0,i\}|\{0,i\}^c} = v(x_i) + v(x_0) = v(x_i)$ is the expected direction.
To show that $p = (p', P)$ represents $(C_p, \mathrm{ev}_0 |_{|C_p|})$ it actually suffices to prove that the anchor leaf L_a of C_p is mapped to the point P under ev_0, which is obviously the case as $\mathrm{ev}_0 |_{L_a} = \mathrm{ev}_a |_{L_a}$ and ev_a is just the projection on the second factor of $C_{p'} \times \{P\}$.
(c): We can use literally the same proof as in the abstract case 2.19 (d) using 3.6 (g). □

Notation 3.9 (Tropical Gromov-Witten invariants)**.** Let us now extend our τ-notation to the case of parametrized curves. For any positive integers $a_1, \ldots, a_n$ and complete intersection cycles $C_1, \ldots, C_n \in Z_*^{\text{c.i.}}(\mathbb{R}^r)$ we define

$$(\tau_{a_1}(C_1) \cdot \ldots \cdot \tau_{a_n}(C_n))_\Delta^{\mathbb{R}^r} := \psi_1^{a_1} \cdot \text{ev}_1^*(C_1) \cdot \ldots \cdot \psi_n^{a_n} \cdot \text{ev}_n^*(C_n) \cdot \mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta).$$

Once again, each factor $\tau_{a_k}(C_k)$ stands for a marked leaf subject to a_k Psi-conditions and to the condition that it must meet C_k. Let c_k be the codimension of C_k in $\mathbb{R}^r$. If $\sum(a_k + c_k) = \dim(\mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta)) = n + \#\Delta + r - 3$, the above cycle is zero-dimensional and we denote its degree by

$$\langle \tau_{a_1}(C_k) \cdot \ldots \cdot \tau_{a_n}(C_k) \rangle_\Delta^{\mathbb{R}^r}.$$

These numbers are called *tropical descendant Gromov-Witten invariants*.

Remark 3.10 (Enumerative relevance of tropical Gromov-Witten invariants). Let $(\tau_{a_1}(C_1) \cdot \ldots \cdot \tau_{a_n}(C_n))$ be an intersection product as defined above. If we set $X = \prod_{k=1}^n \psi_k^{a_k} \cdot \mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta)$ and apply 1.15 to the morphisms $\text{ev}_k : X \to \mathbb{R}^r$, we can conclude the following (as discussed in 1.16): After replacing all the cycles C_k by general translations (called *general conditions* in the following), $Z := \tau_{a_1}(C_1) \cdot \ldots \cdot \tau_{a_n}(C_n))$ is the set of curves C such that

- every vertex $V \in C$ with adjacent marked leaves $k_1, \ldots, k_q$ fulfills
$$\text{val}(V) \geq a_{k_1} + \ldots + a_{k_q} + 3,$$
- for all $k = 1, \ldots, n$ it holds
$$\text{ev}_k(C) \in C_k.$$

Additionally, the facets of Z (i.e. general curves) are equipped with (possibly zero) weights.
Moreover, assume that all the cycles C_k can be described by convex functions $h_1 \cdots h_l \cdot \mathbb{R}^r$. Then by 1.10, all these weights are positive (in particular, $|Z|$ really *is* the set of such curves).
Thus, if Z is zero-dimensional, $\deg(Z) = \langle \tau_{a_1}(C_k) \cdot \ldots \cdot \tau_{a_n}(C_k) \rangle$ is the number of curves satisfying the above properties, counted with a certain integer multiplicity/weight. Now again, if all C_k can be described by convex functions, all these multiplicities and in particular $\langle \tau_{a_1}(C_k) \cdot \ldots \cdot \tau_{a_n}(C_k) \rangle$ are positive.

Remark 3.11. Let $\text{ft}_0 : \mathcal{M}_{n+1}^{\text{lab}}(\mathbb{R}^r, \Delta) \to \mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta)$ be the morphism that forgets the leaf x_0. Then by abuse of notation the equation

$$\text{ft}_0^*(\text{ev}_k) = \text{ev}_k$$

holds for all $k \in [n]$.

Theorem 3.12 (String equation for parametrized curves, cf. [K] 4.3.1)**.** *Let* $(\tau_0(\mathbb{R}^r) \cdot \prod_{k=1}^n \tau_{a_k}(C_k))_\Delta$ *be a zero-dimensional cycle. Then*

$$\langle \tau_0(\mathbb{R}^r) \cdot \prod_{k=1}^n \tau_{a_k}(C_k) \rangle_\Delta = \sum_{k=1}^n \langle \tau_{a_k - 1}(C_k) \cdot \prod_{l \neq k} \tau_{a_l}(C_l) \rangle_\Delta.$$

Theorem 3.13 (Dilaton equation for parametrized curves, cf. [K] 4.3.1)**.** *The following equation holds:*

$$\langle \tau_1(\mathbb{R}^r) \cdot \prod_{k=1}^n \tau_{a_k}(C_k) \rangle_\Delta = (n + \#\Delta - 2) \langle \prod_{k=1}^n \tau_{a_k}(C_k) \rangle_\Delta.$$

Proofs. In both cases, the proofs are completely analogous to the abstract case using 3.6 and 3.11. □

Remark 3.14. Note that the factor appearing in the dilaton equation is different from the algebro-geometric one, due to $\text{ft}_{0*}(\psi_0) = (n + \#\Delta - 2) \cdot \mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta)$ (cf. 3.6 (j)).

Lemma 3.15 (cf. [K] 5.1.6)**.** *Let* h *be a rational function. Then*

$$\text{ev}_k^*(h) \cdot \varphi_{k,l} \cdot \mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta) = \text{ev}_l^*(h) \cdot \varphi_{k,l} \cdot \mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta)$$

Proof. In all curves corresponding to points in $\operatorname{div}(\varphi_{k,l})$, the leaves k and l lie at a common vertex. Therefore their coordinates in $\mathbb{R}^r$ must agree, which means $\operatorname{ev}_k|_{|\operatorname{div}(\varphi_{k,l})|} = \operatorname{ev}_l|_{|\operatorname{div}(\varphi_{k,l})|}$. The result follows. □

For a given labelled degree Δ, we define $\delta(\Delta)$ to be the associated unlabelled degree in the sense of subsection 1.5: $\delta(\Delta)$ is the one-dimensional balanced fan in $\mathbb{R}^r$ consisting of all the rays generated by the direction vectors $v_k, k \in \Delta$ appearing in Δ. The weight of such a ray $\mathbb{R}_{\geq}v$, where v is primitive, is given by

$$\sum_{\substack{k\in\Delta \\ v_k \in \mathbb{Z}_{>0}v}} |\mathbb{Z}v/\mathbb{Z}v_k|.$$

Obviously, let $(C,h) \in \mathcal{M}_n^{\text{lab}}(\mathbb{R}^r,\Delta)$ be an arbitrary n-marked parametrized curve of degree Δ, then by definition $\delta(h(C)) = \delta(\Delta)$ holds.
For a given rational function h on $\mathbb{R}^r$ we define $h \cdot \Delta$ to be $\deg(h \cdot \delta(\Delta))$.

Proposition 3.16 (cf. [K] 5.1.5)**.** *Let h be a rational function on $\mathbb{R}^r$ and define $Y := \operatorname{ev}_0^*(h) \cdot \mathcal{M}_{n+1}^{\text{lab}}(\mathbb{R}^r,\Delta)$. Then*

$$\operatorname{ft}_{0*}(Y) = (h\cdot\Delta)\mathcal{M}_n^{\text{lab}}(\mathbb{R}^r,\Delta).$$

Proof. As our moduli space $\mathcal{M}_n^{\text{lab}}(\mathbb{R}^r,\Delta)$ is irreducible, we know that $\operatorname{ft}_{0*}(Y) = \alpha \cdot \mathcal{M}_n^{\text{lab}}(\mathbb{R}^r,\Delta)$ for an integer α. To compute this number, we set $m := n + \#\Delta + r - 3$ and consider the zero-dimensional intersection product $Z = \varphi_1 \cdots \varphi_m \cdot \mathcal{M}_n^{\text{lab}}(\mathbb{R}^r,\Delta)$ of arbitrary convex functions $\varphi_1,\ldots,\varphi_m$ such that $\deg(Z) \neq 0$ (e.g. $Z = \psi_1^{m-r} \cdot \operatorname{ev}_1(P)$ for some point $P \in \mathbb{R}^r$). If we pull back Z along ft_0, we know by the projection formula

$$\deg(\operatorname{ev}_0(h) \cdot \operatorname{ft}_0^*(Z)) = \alpha \cdot \deg(Z).$$

On the other hand, by the universal family property of ft_0 we know that Z is the union of the curves represented by the points in Z (with according weights) and therefore the push-forward $\operatorname{ev}_{0*}(\operatorname{ft}_0^*(Z))$ is rationally equivalent to its degree

$$\delta(\operatorname{ev}_{0*}(\operatorname{ft}_0^*(Z))) = \deg(Z) \cdot \delta(\Delta).$$

So, applying the projection formula to ev_0, we obtain

$$\deg(\operatorname{ev}_0(h) \cdot \operatorname{ft}_0^*(Z)) = \deg(Z) \cdot (h \cdot \Delta).$$

But this implies $h \cdot \Delta = \alpha$, which proves the claim. □

Theorem 3.17 (Divisor equation, cf. [K] 4.3.2)**.** *Let h be a rational function on $\mathbb{R}^r$ and let $(\prod_{k=1}^n \tau_{a_k}(C_k))_\Delta$ be a one-dimensional cycle. Then*

$$\langle \tau_0(h) \cdot \prod_{k=1}^n \tau_{a_k}(C_k)\rangle_\Delta = (h\cdot\Delta)\langle \prod_{k=1}^n \tau_{a_k}(C_k)\rangle_\Delta + \sum_{k=1}^n \langle \tau_{a_k-1}(h\cdot C_k) \prod_{l\neq k} \tau_{a_l}(C_l)\rangle_\Delta.$$

Proof. First we use 3.6 (e) and (a): We replace each factor $\psi_k^{a_k}$ by $\operatorname{ft}_0^*(\psi_k)^{a_k} + \operatorname{ft}_0^*(\psi_k)^{a_k-1} \cdot \varphi_{0,k}$ and multiply out. All terms containing two φ-factors vanish. In terms with only one factor $\varphi_{0,k}$, we replace $\operatorname{ev}_0(h)$ by $\operatorname{ev}_k(h)$ using 3.15. Now we push forward along ft_0 and produce the desired equation by applying 3.16 and $\operatorname{ft}_{0*}(\operatorname{div}(\varphi_{0,k})) = \mathcal{M}_n^{\text{lab}}(\mathbb{R}^r,\Delta)$. □

4. Splitting Curves

The basic fact used to compute intersection invariants of $\overline{M}_{g,n}(X,\beta)$ is the recursive structure of its boundary: Its irreducible components correspond to reducible curves with a certain partition of the combinatoric data and therefore are (nearly) a product of two "smaller" moduli spaces. In this section we will investigate how far this principle can be carried over to the tropical world.

4.1. **The case of abstract curves.**

Definition 4.1. Let S be a finite set. By $\mathcal{M}_S$ we denote the moduli space of $|S|$-marked tropical curves $\mathcal{M}_{|S|}$ where we label the leaves by elements in S. For each partition $I|J$ of $[n]$ we construct the map $\rho_{I|J} : \mathcal{M}_{I\cup\{x\}} \times \mathcal{M}_{J\cup\{y\}} \to \varphi_{I|J} \cdot \mathcal{M}_n$ by the following rule: Given two curves $(p_I, p_J) \in \mathcal{M}_{I\cup\{x\}} \times \mathcal{M}_{J\cup\{y\}}$, we remove the extra leaves x and y and glue the curves together at the two vertices to which these leaves have been adjacent. We could also say, we glue x and y together by creating a bounded edge whose length we define to be 0. In the coordinates of the space of tree metrics, this map is given by the linear map

$$\begin{aligned} \rho_{I|J} : \mathbb{R}^{\binom{I}{2}} \times \mathbb{R}^{\binom{J}{2}} &\to \mathbb{R}^{\binom{n}{2}}, \\ (p_I, p_J) &\mapsto p, \end{aligned}$$

where

$$p_{k,l} := \begin{cases} p_{I\,k,l} & \text{if } k,l \in I, \\ p_{J\,k,l} & \text{if } k,l \in J, \\ p_{I\,k,x} + p_{J\,y,l} & \text{if } k \in I, l \in J. \end{cases}$$

Attention: This map does *not* induce a linear map on the corresponding quotients in which our moduli spaces are balanced and therefore $\rho_{I|J}$ is *not* a tropical morphism of our moduli spaces. Even more, $\rho_{I|J}$ is not even locally linear around ridges of our moduli spaces considered as balanced complexes in the quotients. On the other hand, $\rho_{I|J}$ is at least piecewise linear (i.e. it is linear on all cones of $\mathcal{M}_{I\cup\{x\}} \times \mathcal{M}_{J\cup\{y\}}$). Its image is a polyhedral complex, namely the positive part of $\varphi_{I|J} \cdot \mathcal{M}_n$ (i.e. it consists of all (faces of) facets ${}^{A}_{D}\!\times^{B}_{C}$ with $A \cup B = I$).

Definition 4.2 (Morphisms of rational polyhedral complexes)**.** Let X and Y be (rational) polyhedral complexes. Then a *morphism of polyhedral complexes* is a map $\rho : |X| \to |Y|$ that satisfies for each polyhedron $\sigma \in X$

(a) $\rho(\sigma) \in Y$,
(b) $\rho|_\sigma$ is affine linear,
(c) $\rho(\Lambda_\sigma) \subseteq \Lambda_{\rho(\sigma)}$.

We call ρ an *isomorphism of polyhedral complexes* if there exists an inverse morphism. It other words, an isomorphism is a bijection between $|X|$ and $|Y|$ (as well as between X and Y) and $\rho(\Lambda_\sigma) = \Lambda_{\rho(\sigma)}$ for all $\sigma \in X$.

Lemma 4.3 (Intersections of Psi-functions with the boundary)**.** *The facets of the fan $\varphi_{I|J} \cdot \psi_1^{a_1} \cdot \ldots \cdot \psi_n^{a_n} \cdot \mathcal{M}_n$ with positive weight are precisely the cones σ in $\mathcal{M}_n$ with the following properties: Consider a curve in the interior of σ. Let $E(V) \in [n]$ be the set of leaves adjacent to a vertex V and let $P(V)$ be the $val(V)$-fold partition of $[n]$ obtained by removing V. Then the following holds:*

(a) *There exists one special vertex V_{spec} whose partition $P(V_{spec})$ is a subpartition of $I|J$ and whose valence is $(\sum_{k\in E(V)} a_k) + 4$.*
(b) *Let m_I be the number of sets in $P(V_{spec})$ contained in I. Then $m_I + 1 = (\sum_{k\in E(V)\cap I} a_k) + 3$ (together with (a), the analogue $m_J + 1 = (\sum_{k\in E(V)\cap J} a_k) + 3$ follows). In particular, $m_I, m_J > 1$.*
(c) *The valence of all other vertices V equals $(\sum_{k\in E(V)} a_k) + 3$.*

Furthermore, the facets of $\varphi_{I|J} \cdot \psi_1^{a_1} \cdot \ldots \cdot \psi_n^{a_n} \cdot \mathcal{M}_n$ with negative weight fulfill the same properties (a) and (c) and the property

(b') *Let m_I (resp. m_J) be the number of sets in $P(V_{spec})$ contained in I (resp. J). Then $m_I = 1$ or $m_J = 1$, i.e. $I \in P(V_{spec})$ or $J \in P(V_{spec})$.*

Proof. We know how $X := \psi_1^{a_1} \cdot \ldots \cdot \psi_n^{a_n} \cdot \mathcal{M}_n$ looks like by 2.14. In the combinatorial type of a facet of X the valence of each vertex is $(\sum_{k\in E(V)} a_k) + 3$; in the combinatorial type of a ridge, there is one special vertex V_{spec} with valence $(\sum_{k \in E(V)} a_k) + 4$. The balancing condition of a ridge is given by the equation

$$\sum_{I'|J'} \omega_{I'|J'} V_{I'|J'} = \sum_{\substack{I'|J' \\ I' \in P(V_{\text{spec}})}} \lambda_{I'|J'} V_{I'|J'},$$

where the left hand sum runs through all superpartitions $I'|J'$ of $P(V_{\text{spec}})$ not appearing in the right hand sum, $\omega_{I'|J'}$ denotes the weight of the facet obtained by inserting an edge $I'|J'$ and $\lambda_{I'|J'}$ is some (rational) coefficient. Therefore the weight ω that this ridge obtains when intersecting X with $\varphi_{I|J}$ is given by

$$\omega = \begin{cases} 0 & \text{if } I|J \text{ is } \textit{not} \text{ a superpartition of } P(V_{\text{spec}}), \\ \lambda_{I|J} & \text{if } I \in P(V_{\text{spec}}) \text{ or } J \in P(V_{\text{spec}}), \\ \omega_{I|J} & \text{otherwise.} \end{cases}$$

This already shows two implications: As all weights $\omega_{I'|J'}$ are at least non-negative, a ridge can only obtain a negative weight if it fulfills conditions (a), (b') and (c). On the other hand, if a ridge of X satisfies properties (a), (b) and (c), then $\omega_{I|J}$ and hence the ridge obtains a positive weight. It remains to show the converse, which can be done by proving that all $\lambda_{I'|J'}$ are non-negative. To see this, we consider the balancing equation in $\mathbb{R}^{\binom{r}{2}}$ and compare some coordinate entries.
Let K be an arbitrary element of $P(V_{\text{spec}})$; we want to show that $\lambda_K := \lambda_{K|K^c}$ is non-negative. We choose two more arbitrary elemtnts L_1, L_2 in $P(V_{\text{spec}})$ and fix some leaves $k \in K$, $l_i \in L_i$. Now the k, l_i-entry of the right hand side equals $\lambda_K + \lambda_{L_i}$ and analogously the l_1, l_2-entry equals $\lambda_{L_1} + \lambda_{L_2}$. Therefore, by adding the two k, l_i-entries and substracting the l_1, l_2-entry we get $2\lambda_K$. Meanwhile, on the left hand side we get

$$\begin{aligned} 2\lambda_K &= \sum_{\substack{I'|J' \\ k \in I' \\ l_1 \in J'}} \omega_{I'|J'} + \sum_{\substack{I'|J' \\ k \in I' \\ l_2 \in J'}} \omega_{I'|J'} - \sum_{\substack{I'|J' \\ l_1 \in I' \\ l_2 \in J'}} \omega_{I'|J'} \\ &= \sum_{I'|J'} \alpha_{I'|J'} \omega_{I'|J'}, \end{aligned}$$

where

$$\alpha_{I'|J'} = \begin{cases} 2 & \text{if } k \in I', \ l_1, l_2 \in J' \\ 0 & \text{if } k, l_1 \in I', \ l_2 \in J' \\ 0 & \text{if } k, l_2 \in I', \ l_1 \in J' \\ 0 & \text{if } k, l_1, l_2 \in I'. \end{cases}$$

But as all the weights $\omega_{I'|J'}$ are non-negative, it follows that λ_K is non-negative. □

Lemma 4.4. *The map*

$$\rho_{I|J} : \Big(\prod_{k \in I} \psi_k^{a_k} \cdot \mathcal{M}_{I \cup \{x\}}\Big) \times \Big(\prod_{k \in J} \psi_k^{a_k} \cdot \mathcal{M}_{J \cup \{y\}}\Big) \to (\varphi_{I|J} \cdot \psi_1^{a_1} \cdot \ldots \cdot \psi_n^{a_n} \cdot \mathcal{M}_n)^+$$

is a well-defined isomorphism of polyhedral complexes.

Proof. We have to check the conditions of 4.2. Using the lengths of the bounded edges as local coordinates on the cones, this is straightforward. The inverse map is given by splitting a given curve at its special vertex V_{spec}. □

4.2. **The case of parametrized curves.**

Definition 4.5. Let $I|J$ be a reducible partition and let Δ_I, Δ_J be the corresponding splitting of the tropical degree Δ. Let $Z = \max(x_1, y_1) \cdot \ldots \cdot \max(x_r, y_r) \cdot \mathbb{R}^r \times \mathbb{R}^r$ denote the diagonal in $\mathbb{R}^r \times \mathbb{R}^r$ and consider the map

$$\text{ev}_x \times \text{ev}_y : \mathcal{M}^{\text{lab}}_{I \cup \{x\}}(\mathbb{R}^r, \Delta_I) \times \mathcal{M}^{\text{lab}}_{J \cup \{y\}}(\mathbb{R}^r, \Delta_J) \to \mathbb{R}^r \times \mathbb{R}^r.$$

We define

$$Z_{I|J} := (\mathrm{ev}_x \times \mathrm{ev}_y)^*(Z)$$

We furthermore define $\pi_{I|J} : Z_{I|J} \to \mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta)$ by

$$\begin{array}{rcl} \mathcal{M}_{I\cup\{x\}} \times \mathbb{R}^r \times \mathcal{M}_{J\cup\{y\}} \times \mathbb{R}^r & \to & \mathcal{M}_{[n]\cup\Delta} \times \mathbb{R}^r \\ \big((p_I, P), (p_J, Q)\big) & \mapsto & (\rho(p_I, p_J), P), \end{array}$$

where we choose the same anchor leaf for $\mathcal{M}_{I\cup\{x\}}^{\text{lab}}(\mathbb{R}^r, \Delta_I)$ and $\mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta)$.

Lemma 4.6. *The map*

$$\pi_{I|J} : \psi_1^{a_1} \cdot \ldots \cdot \psi_n^{a_n} \cdot Z_{I|J} \to (\varphi_{I|J} \cdot \psi_1^{a_1} \cdot \ldots \cdot \psi_n^{a_n} \cdot \mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta))^+$$

is a well-defined isomorphism of polyhedral complexes.

Proof. This follows from 4.4 and from $\mathrm{ev}_x|_{Z_{I|J}} = \mathrm{ev}_y|_{Z_{I|J}}$ (which follows from both 1.11 (Z is described by convex functions) as well as from 1.12 ($\mathrm{ev}_x \times \mathrm{ev}_y$ can be considered as a projection)). □

Remark 4.7. Obviously the positions of the marked leaves are preserved under $\pi_{I|J}$, i.e. (by abuse of notation) for $i \in I$ (resp. $j \in J$) it holds $\mathrm{ev}_i \circ \pi_{I|J} = \mathrm{ev}_i$ (resp. $\mathrm{ev}_j \circ \pi_{I|J} = \mathrm{ev}_j$).

Lemma 4.8. *Let $E = (\varphi_{I|J} \cdot \tau_{a_1}(C_1) \cdot \ldots \cdot \tau_{a_n}(C_n))_\Delta$ be a zero-dimensional cycle. Then all points of E lie in $(\varphi_{I|J} \cdot \psi_1^{a_1} \cdot \ldots \cdot \psi_n^{a_n} \cdot \mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta))^+$.*

Proof. By 1.1 we can compute the weight of a point $p \in E$ locally around p in $X := \varphi_{I|J} \cdot \psi_1^{a_1} \cdot \ldots \cdot \psi_n^{a_n} \cdot \mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta)$, namely we can focus on $\mathrm{Star}_X(p)$. Assume $p \notin (\varphi_{I|J} \cdot \psi_1^{a_1} \cdot \ldots \cdot \psi_n^{a_n} \cdot \mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta))^+$. Then curves corresponding to points in $\mathrm{Star}_X(p)$ contain a bounded edge corresponding to the partition $I|J$ (see 4.3). But as $I|J$ is chosen to be reducible, this edge is a contracted bounded edge whose length does not change the positions of the marked leaves in $\mathbb{R}^r$. Therefore, if we denote by $\mathrm{ev} = \mathrm{ev}_1 \times \ldots \times \mathrm{ev}_n$ the product of all evaluation maps, the image of $\mathrm{Star}_X(p)$ under ev has smaller dimension which implies $\mathrm{ev}_*(\mathrm{Star}_X(p)) = 0$. Hence, by projection formula, the weight of p in E must zero. □

We now simplify the situation by choosing general incidence conditions. The following statement combines 1.15, in particular item (c), and the preceding result.

Corollary 4.9. *Let $E = (\varphi_{I|J} \cdot \tau_{a_1}(C_1) \cdot \ldots \cdot \tau_{a_n}(C_n))_\Delta$ be a zero-dimensional cycle. If we substitute the cycles C_i by general translation, we can assume that all points of E lie in the interior of a facet of $(\varphi_{I|J} \cdot \psi_1^{a_1} \cdot \ldots \cdot \psi_n^{a_n} \cdot \mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta))^+$. This operation does not change the degree of E by remark 1.13.*

Proposition 4.10. *Let $E = (\varphi_{I|J} \cdot \tau_{a_1}(C_1) \cdot \ldots \cdot \tau_{a_n}(C_n))_\Delta$ be a zero-dimensional cycle. Then the equation*

$$\langle \varphi_{I|J} \cdot \tau_{a_1}(C_1) \cdot \ldots \cdot \tau_{a_n}(C_n) \rangle_\Delta = \langle \tau_{a_1}(C_1) \cdot \ldots \cdot \tau_{a_n}(C_n) \cdot Z_{I|J} \rangle_{\Delta_I, \Delta_J}$$

holds.

Proof. We denote $X := \psi_1^{a_1} \cdot \ldots \cdot \psi_n^{a_n} \cdot Z_{I|J}$ and $Y := \varphi_{I|J} \cdot \psi_1^{a_1} \cdot \ldots \cdot \psi_n^{a_n} \cdot \mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, \Delta)$ and assume that the conditions C_i are general. Then 4.9 implies that, for each point $p \in E$, we have an isomorphism of cycles $\pi_{I|J} : \mathrm{Star}_X(\pi_{I|J}^{-1}(p)) \to \mathrm{Star}_Y(p)$. By 1.1 this suffices to show that the weights of p and $\pi_{I|J}^{-1}(p)$ in their respective intersection products coincide. □

4.3. **Splitting the diagonal.** Up to now, we have seen that intersecting with a "boundary" function $\varphi_{I|J}$ leads to intersection products in two smaller moduli spaces $\mathcal{M}^{\text{lab}}_{I\cup\{x\}}(\mathbb{R}^r, \Delta_I)$ and $\mathcal{M}^{\text{lab}}_{J\cup\{y\}}(\mathbb{R}^r, \Delta_J)$. However, the factor $(\text{ev}_x \times \text{ev}_y)^*(Z)$ still connects these two smaller spaces. In order to finally arrive at recursive equations of Gromov-Witten invariants, it is desirable to distribute this diagonal factor onto the two moduli spaces and to obtain independent intersection products there. In the algebro-geometric case, this can be easily done as the *class* of the diagonal Z in e.g. $\mathbb{P}^r \times \mathbb{P}^r$ can be written as the sum of products of classes in the factors

$$[Z] = [L^0 \times L^r] + [L^1 \times L^{r-1}] + \ldots + [L^r \times L^0],$$

where L^i denotes an i-dimensional linear space in $\mathbb{P}^r$. But this can *not* copied tropically (see below). For the first time, we face a serious problem which is connected to the non-compactness of our moduli space $\mathcal{M}^{\text{lab}}_n(\mathbb{R}^r, \Delta)$: Our notion of rational equivalence is "too strong" for this application, as it is inspired by the idea that two rational equivalent objects should be rational equivalent in *any* compactification. However, we will discuss here how far the conventional plan can be carried out anyways.

The general plan is the following: Set

$$X_I := (\tau_0(\mathbb{R}^r) \cdot \prod_{k\in I} \tau_{a_k}(C_k))_{\Delta_I} \qquad \text{in } \mathcal{M}^{\text{lab}}_{I\cup\{x\}}(\mathbb{R}^r, \Delta_I)$$

and

$$X_J := (\tau_0(\mathbb{R}^r) \cdot \prod_{k\in J} \tau_{a_k}(C_k))_{\Delta_J} \qquad \text{in } \mathcal{M}^{\text{lab}}_{J\cup\{y\}}(\mathbb{R}^r, \Delta_J).$$

We want to compute the degree of

$$(\tau_{a_1}(C_1) \cdot \ldots \cdot \tau_{a_n}(C_n) \cdot Z_{I|J})_{\Delta_I,\Delta_J} = (\text{ev}_x \times \text{ev}_y)^*(Z) \cdot (X_I \times X_J),$$

or, by the projection formula,

$$\deg(Z \cdot (\text{ev}_x(X_I) \times \text{ev}_y(X_J))).$$

Now we would like to replace the diagonal Z by something like

$$S := \sum_\alpha (M_\alpha \times N_\alpha),$$

where M_α, N_α are cycles in $\mathbb{R}^r$ such that S intersects $\text{ev}_x(X_I) \times \text{ev}_y(X_J)$ like Z. But note that S cannot be rationally equivalent to Z (in the sense of [AR08]), as this would imply that both cycles must have the same recession fan, i.e. must have the same directions towards infinity. To come out of this, we need more information about how the push-forwards $\text{ev}_x(X_I)$ and $\text{ev}_y(X_J)$ look like; in particular, we would like to know how their degrees/recession fans can look like. Let us formalize this first.

Let Θ be a complete simplicial fan in $\mathbb{R}^r$ and let $Z_k(\Theta)$ be the group of k-dimensinional cycles X whose support lies in the k-dimesnional skeleton of Θ, i.e. $|X| \subseteq |\Theta^{(k)}|$. Fix a basis of $Z_*(\Theta) := \oplus_{k=0}^r Z_k(\Theta)$ denoted by $B_0, \ldots, B_m$ (where we may assume $B_0 = \{0\}$ and $B_m = \mathbb{R}^r$). If the degree $\delta(X)$ of an arbitrary cycle is contained in $Z_*(\Theta)$, we say X is Θ-*directional*. For such a cycle there exist integer coefficients λ_e such that $X \sim \delta(X) = \sum_{e=1}^m \lambda_e B_e$.
For each ray $\rho \in \Theta^{(1)}$ with primitive vector v_ρ let φ_ρ be the rational function on Θ uniquely defined by

$$\varphi_\rho(v_{\rho'}) = \begin{cases} 1 & \text{if } \rho' = \rho, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.11. *The linear map*

$$\begin{array}{rcl} Z_*(\Theta) & \to & \mathbb{Z}^{m+1}, \\ X & \mapsto & (\deg(B_0 \cdot X), \ldots, \deg(B_m \cdot X)), \end{array}$$

(where $\deg(.)$ *is set to be zero if the dimension of the argument is non-zero) is injective.*

Proof. Let $X \in Z_k(\Theta)$ be an element of the kernel, which implies that $\deg(X \cdot Y) = 0$ for all $Y \in Z_{r-k}(\Theta)$. Now, in fact the remaining is identical to the proof of [AR08, Lemma 6]: Assume that X is non-zero and therefore there exists a cone $\sigma \in \Theta^{(k)}$ such that $\omega_X(\sigma) \neq 0$. As Θ is simplicial, this cone is generated by k rays $\rho_1, \ldots, \rho_k$. Let us consider $\varphi_{\rho_k} \cdot X$ and in particular the weight of $\tau := \langle \rho_1, \ldots, \rho_{k-1} \rangle$ in this intersection product: As primitive vector $v_{\sigma/\tau}$ we can use $\frac{1}{|\Lambda_\sigma / \Lambda_\tau + \Lambda_{\rho_k}|} v_{\rho_k}$ (it might not be an integer vector, but modulo V_τ, it is a primitive generator of σ). Analogously, we can get any primitive vector around τ as a multiple of an appropriate v_ρ. But as φ_{ρ_k} is zero on all of these vectors but v_{ρ_k}, we get

$$\omega_{\varphi_{\rho_k} \cdot X}(\tau) = \frac{\omega_X(\sigma)}{|\Lambda_\sigma / \Lambda_\tau + \Lambda_{\rho_k}|} \neq 0.$$

Now induction shows

$$\deg(\varphi_{\rho_1} \cdots \varphi_{\rho_k} \cdot X) = \omega_{\varphi_{\rho_1} \cdots \varphi_{\rho_k} \cdot X}(\{0\}) = \frac{\omega_X(\sigma)}{|\Lambda_\sigma / \Lambda_{\rho_1} + \ldots + \Lambda_{\rho_k}|} \neq 0.$$

This means we have found a Θ-directional cycle $Y := \varphi_{\rho_1} \cdots \varphi_{\rho_k} \cdot \mathbb{R}^r \in Z_{r-k}(\Theta)$ with $\deg(X \cdot Y) \neq 0$, which contradicts the assumption that X is an element of the kernel. □

With respect to the basis $B_0, \ldots, B_m$, the map defined in the previous lemma has the matrix representation $\alpha := (\deg(B_e \cdot B_f))_{ef}$. Obviously α is a symmetric matrix. The lemma implies that this matrix is invertible over $\mathbb{Q}$, and we denote the inverse by $(\beta_{ef})_{ef}$. The coefficients of this matrix can be used to replace the diagonal Z of $\mathbb{R}^r \times \mathbb{R}^r$ by a sum of products of cycles in the two factors (namely $\sum_{e,f} \beta_{ef}(B_e \times B_f)$) — at least with respect to Θ-directional cycles.

Lemma 4.12. *Let $X \sim \sum_e \lambda_e B_e, Y \sim \sum_f \mu_e B_e$ be two Θ-directional cycles in $\mathbb{R}^r$ with complementary dimension. Then*

$$\deg(Z \cdot (X \times Y)) = \deg(X \cdot Y) = \sum_{e,f} \deg(X \cdot B_e) \beta_{ef} \deg(Y \cdot B_f).$$

Proof. Denote $\lambda := (\lambda_1, \ldots, \lambda_m), \mu := (\mu_1, \ldots, \mu_m)$. We get

$$\begin{aligned} \sum_{e,f} \deg(X \cdot B_e) \beta_{ef} \deg(Y \cdot B_f) &= (\alpha \cdot \lambda)^T \cdot \beta \cdot (\alpha \cdot \mu) \\ &= \lambda^T \cdot \alpha^T \cdot \beta \cdot \alpha \cdot \mu \\ &= \lambda^T \cdot \alpha \cdot \beta \cdot \alpha \cdot \mu \\ &= \lambda^T \cdot \alpha \cdot \mu = \deg(X \cdot Y). \end{aligned}$$

□

Using this, our original goal of deriving a tropical splitting lemma can be formulated as follows.

Theorem 4.13 (Splitting Lemma, cf. [K] 5.2.1)**.** *Let $E = (\varphi_{I|J} \cdot \prod_{k=1}^n \tau_{a_k}(C_k))_\Delta^{\mathbb{R}^r}$ be a zero-dimensional cycle, where $I|J$ is a reducible partition. Moreover, let us assume that Θ is a complete simplicial fan such that (with the notations from above) $\mathrm{ev}_x(X_I)$ and $\mathrm{ev}_y(X_J)$ are Θ-directional. Let $B_0, \ldots, B_m$ be a basis of $Z_*(\Theta)$ and let $(\beta_{ef})_{ef}$ be the inverse matrix (over $\mathbb{Q}$) of $(\deg(B_e \cdot B_f))_{ef}$. Then the following equation holds:*

$$\langle \varphi_{I|J} \cdot \prod_{k=1}^n \tau_{a_k}(C_k) \rangle_\Delta = \sum_{e,f} \langle \prod_{k \in I} \tau_{a_k}(C_k) \cdot \tau_0(B_e) \rangle_{\Delta_I} \ \beta_{ef} \ \langle \tau_0(B_f) \cdot \prod_{k \in J} \tau_{a_k}(C_k) \rangle_{\Delta_J}$$

Proof. Follows from the general plan above and 4.10. □

Remark 4.14. The toric version of this subsection could be formulated as follows: We fix a complete toric variety X corresponding to the simplicial fan Θ. The geometric meaning of the fact that Θ is simplicial is that every Weil-divisor of X is also Cartier. The group $Z_k(\Theta)$ equals the group of Minkowski weights of codimension $r-k$ and therefore is isomorphic to the Chow cohomology group $A^{r-k}(X)$ of X (cf. [FS94]). Consequently, the basis $B_0, \dots, B_m$ corresponds to a basis of the cohomology classes. Constructing the functions φ_ρ by determining its values on the rays is analogous to constructing the Cartier-divisor whose corresponding Weil-divisor is the torus-invariant divisor associated to ρ. It was shown in subsection 1.8 that our tropical intersection product coincides with the fan displacement rule in [FS94] and therefore compatible with the cup-product of the corresponding cohomology classes. In particular, this shows that the matrix $(\deg(B_e \cdot B_f))_{ef}$ and its algebro-geometric counterpart coincide (in particular, 4.11 also follows from the corresponding algebro-geometric statement). Moreover, this implies that the coefficients β_{ef} appearing in the splitting lemma really are the same as in the associated algebro-geometric version.

4.4. **The directions of families of curves.** In order to recursively determine invariants, the above splitting lemma is only useful if, at least for a certain class of invariants, the fan of directions Θ is fixed and well-known. This is one of the main problems when transferring the algebro-geometric theory to the tropical set-up. However, in this subsection we will show that in some cases the problem can be solved.

Remark 4.15. In the easiest case, namely if $r = 1$, the situation is trivial: There is one unique complete simplicial fan $\Theta = \{\mathbb{R}_{\leq 0}, \{0\}, \mathbb{R}_{\geq 0}\}$ and any subcycle is Θ-directional. Also, it is obvious that 4.12 holds (with $B_0 = \{0\}, B_1 = \mathbb{R}$).

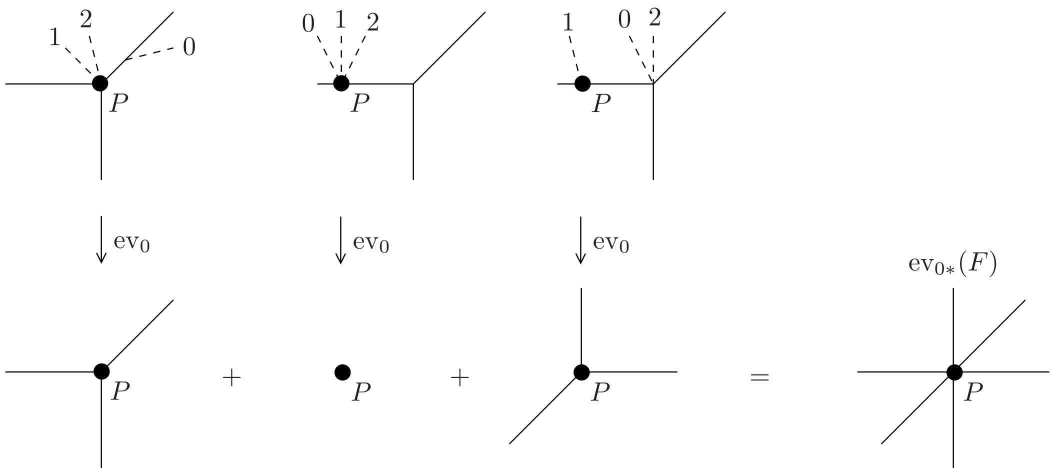
Let us now consider curves in the plane. Let $F = (\tau_0(\mathbb{R}^2) \cdot \prod_{k=1}^n \tau_{a_k}(C_k))_\Delta^{\mathbb{R}^2}$ be a one-dimensional family of plane curves (with unrestricted leaf x_0). We define $\Theta(F)$ to the complete fan in $\mathbb{R}^2$ which contains the following rays: All directions appearing in Δ and furthermore all rays in $\delta(C_k)$ if $\dim(C_k) = 1$ and $a_k > 0$.

Lemma 4.16. *Let $F = (\tau_0(\mathbb{R}^2) \cdot \prod_{k=1}^n \tau_{a_k}(C_k))_\Delta^{\mathbb{R}^2}$ be a one-dimensional family of plane curves (with unrestricted leaf x_0). Let us furthermore assume that $a_k \leq 1$ if $\dim(C_k) = 2$ (i.e. if a leaf is not restricted by ev-conditions, only one Psi-condition is allowed). Then $\mathrm{ev}_{0*}(F)$ is $\Theta(F)$-directional.*

Proof. As before, we replace each factor $\psi_k^{a_k}$ by $\mathrm{ft}_0^*(\psi_k)^{a_k} + \mathrm{ft}_0^*(\psi_k)^{a_k-1} \cdot \varphi_{0,k}$ and multiply out. Consider the term without φ-factors: It is the fiber of $(\prod_{k=1}^n \tau_{a_k}(C_k))_\Delta$ (which is finite) under ft_0 (see univeral family property 3.8) and moreover the push-forward of the fibre along ev_0 is just the sum/union of the images in $\mathbb{R}^r$ of the parametrized curves corresponding to the points in $\prod_{k=1}^n \tau_{a_k}(C_k))_\Delta$. But these curves have degree Δ, thus by definition their images are $\Theta(F)$-directional.
So let us consider the term with the factor $\varphi_{0,k}$. Here, ev_0 and ev_k coincide (see 3.15), so we can in fact compute the push-forward along ev_k. As $\mathrm{ev}_k = \mathrm{ev}_k \circ \mathrm{ft}_0$ (by abuse of notation), we can first push-forward along ft_0 and get the term $(\tau_{a_k-1}(C_k) \cdot \prod_{l \neq k} \tau_{a_l}(C_l))$.
Now, if $\dim(C_k) = 2$, by our assumptions $a_k - 1 = 0$ – in which case we can use induction to prove the statement – or this term does not appear at all.
On the other hand, if $\dim(C_k) = 0, 1$, we can use the fact that the push-forward is certainly contained in C_k – therefore, $\dim(C_k) = 0$ is trivial and $\dim(C_k) = 1$ works as we added the directions of C_k to $\Theta(F)$ if $a_k > 0$.
This finishes the proof, as all terms with more φ-factors vanish. □

Remark 4.17. A weaker version of this lemma can be obtained by directly investigating on how the image under ev_0 of an unbounded ray in F looks like, using general conditions (see [MR08, 3.7]).

Remark 4.18. Consider the family $F = (\tau_0(\mathbb{R}^2)\tau_0(P)\tau_2(\mathbb{R}^2))_1^{\mathbb{R}^2} = \mathrm{ev}_1^*(P) \cdot \psi_2^2 \cdot \mathcal{M}_3^{\mathrm{lab}}(\mathbb{R}^2, 1)$ of curves of projective degree 1. It consists of the following types of curves:



Its push-forward along ev_0 also contains the inverted standard directions $(1,0)$, $(0,1)$ and $(-1,-1)$. Therefore this family is a counterexample of our statement if we drop the condition on the number of Psi-conditions allowed at leaves not restricted by incidence conditions.

Remark 4.19. For higher dimensions ($r \geq 2$), only few cases are explored. If we restrict to projective degree d and banish all Psi-conditions, i.e. for a family $F = (\tau_0(\mathbb{R}^r) \cdot \prod_{k=1}^n \tau_0(C_k))_d$ of arbitrary dimension, it is proven in [GZ] that $\mathrm{ev}_{0*}(F)$ is Θ-directional, where Θ is the complete simplicial fan in $\mathbb{R}^r$ consisting of all cones generated by at most r of the vectors $-e_0, -e_1, \ldots, -e_r$. We conjecture that a similar proof also works for Psi-conditions at point-conditions. Beyond this, the behaviour of the push-forwards is mainly unknown up to now.

5. WDVV EQUATIONS AND TOPOLOGICAL RECURSION

5.1. **WDVV equations.** Let x_i, x_j, x_k, x_l be pairwise different marked leaves and consider the forgetful map $\mathrm{ft} : \mathcal{M}_n^{\mathrm{lab}}(\mathbb{R}^r, \Delta) \to \mathcal{M}_{\{i,j,k,l\}}$.

Lemma 5.1. *The equation*

$$\mathrm{ft}^*(\varphi_{\{i,j\}|\{k,l\}}) = \sum_{\substack{I|J \\ i,j \in I, k,l \in J}} \varphi_{I|J}$$

holds, where the sum on the right side runs through all *(also non-reducible) partitions with* $i, j \in I$ *and* $k, l \in J$.

Proof. Note that $\mathrm{ft}(V_{I|J}) = V_{I \cap \{i,j,k,l\} | J \cap \{i,j,k,l\}}$. Therefore $\varphi(\mathrm{ft}(V_{I|J})) = 1$ if $i, j \in I, k, l \in J$ and zero otherwise. □

Now we face the crucial difference to the conventional setting: The right sum also runs over non-reducible partitions, which do not correspond to something in the algebro-geometric case. Let us add up only those $\varphi_{I|J}$ with $I|J$ non-reducible and denote the sum by ϕ, i.e.

$$\phi_{i,j|k,l} := \sum_{\substack{I|J \text{ non-red.} \\ i,j \in I, k,l \in J}} \varphi_{I|J}$$

We would like to show that $\phi_{i,j|k,l}$ is bounded, as then it would not change the degree of a zero-dimensional intersection product and could derive the same formulas as in the conventional case. So let us investigate what this fucntion measures:
Let $F = (\prod_{k=1}^n \tau_{a_k}(C_k))_\Delta$ be a one-dimensional family of curves with general conditions. Consider a facet σ of F representing curves *with* contracted bounded edge E (called *reducible curves*). Then we can change the length of E while keeping all other lenghts and our curve will still match the incidence

conditions. As our conditions are general, the set of curves fulfilling the incidence conditions set-theoretically is also 1-dimensional. Hence, all curves in σ just differ by the length of E, whereas all other lengths are fixed. But this means that $\phi_{i,j|k,l}$ is constant on σ.
Now, let σ be a facet of F representing curves *without* contracted bounded edge E (called *non-reducible curves*). This means, for all non-reducible partitions $I|J$, the respective function $\varphi_{I|J}$ is identically zero on σ. Therefore, on σ, $\phi_{i,j|k,l}$ coincides with $\mathrm{ft}^*(\varphi_{\{i,j\}|\{k,l\}})$.

Lemma 5.2. *Let $F = (\prod_{k=1}^n \tau_{a_k}(C_k))_\Delta$ be a one-dimensional family of curves with general conditions. Let σ be a facet of F. Then*

$$\phi_{i,j|k,l}|_\sigma = \begin{cases} \varphi_{\{i,j\}|\{k,l\}} \circ \mathrm{ft} & \textit{if interior curves of } \sigma \textit{ are non-reducible} \\ \textit{const} & \textit{otherwise}. \end{cases}$$

In other words: Proving that $\phi_{i,j|k,l}$ is bounded on a family one-dimensional family F is the same as *proving that curves in F with large $\mathcal{M}_{i,j,k,l}$-coordinate must contain a contracted bounded edge*. This is the way of speaking in existing literature (e.g. [GM05, proposition 5.1], [KM06, proposition 6.1], [MR08, section 4]). We will address this difficult problem in its own subsection and first state the desired results here.

Lemma 5.3 (cf. [K] 5.3.2)**.** *Let $F = (\prod_{k=1}^n \tau_{a_k}(C_k))_\Delta$ be a one-dimensional family of curves. Furthermore assume that $\phi_{i,j|k,l}$ is bounded. Then the equation*

$$\langle \mathrm{ft}^*(\varphi_{\{i,j\}|\{k,l\}}) \cdot \prod_{k=1}^n \tau_{a_k}(C_k)\rangle_\Delta = \sum_{\substack{I|J \text{ reducible} \\ i,j\in I, k,l\in J}} \langle \varphi_{I|J} \cdot \prod_{k=1}^n \tau_{a_k}(C_k)\rangle_\Delta$$

holds.

Proof. This follows from 5.1 and the fact that the degree of a bounded function intersected with a one-dimensional cycle is zero. Therefore, if $\phi_{i,j|k,l}$ is bounded, the degree of

$$\langle \phi_{i,j|k,l} \cdot \prod_{k=1}^n \tau_{a_k}(C_k)\rangle_\Delta$$

is zero and hence this term can be omitted. □

Finally, we can state the following version of the WDVV equations:

As before, we fix a complete simplicial fan Θ and a basis $B_0, \ldots, B_m$ of $Z_*(\Theta)$. Furthermore, let $(\beta_{ef})_{ef}$ be the inverse matrix (over $\mathbb{Q}$) of the matrix $(\deg(B_e \cdot B_f))_{ef}$.

Theorem 5.4 (WDVV equations, cf. [K] 5.3.3)**.** *Let $F = (\prod_{k=1}^n \tau_{a_k}(C_k))_\Delta$ be a one-dimensional family of curves and fix four pairwise different marked leaves x_i, x_j, x_k, x_l. Moreover, we assume that the following conditions hold:*

(a) *For any reducible partition $I|J$ with $i, j \in I; k, l \in J$ or $i, k \in I; j, l \in J$ the push-forwards $\mathrm{ev}_x(X_I)$ and $\mathrm{ev}_y(X_J)$ are Θ-directional (with notations from section 4).*
(b) *The functions $\phi_{i,j|k,l}$ and $\phi_{i,k|j,l}$ are bounded on F.*

Then the WDVV equation

$$\sum_{\substack{I|J \text{ reducible} \\ i,j\in I,k,l\in J}} \sum_{e,f} \langle \prod_{k\in I} \tau_{a_k}(C_k) \cdot \tau_0(B_e)\rangle_{\Delta_I} \ \beta_{ef} \ \langle \tau_0(B_f) \cdot \prod_{k\in J} \tau_{a_k}(C_k)\rangle_{\Delta_J}$$
$$= \sum_{\substack{I|J \text{ reducible} \\ i,k\in I,j,l\in J}} \sum_{e,f} \langle \prod_{k\in I} \tau_{a_k}(C_k) \cdot \tau_0(B_e)\rangle_{\Delta_I} \ \beta_{ef} \ \langle \tau_0(B_f) \cdot \prod_{k\in J} \tau_{a_k}(C_k)\rangle_{\Delta_J}$$

holds, where the sums run through reducible partitions only.

Proof. The statement follows from 5.3 and the fact that on $\mathcal{M}_{\{i,j,k,l\}}$ the functions $\varphi_{\{i,j\}|\{k,l\}}$ and $\varphi_{\{i,k\}|\{j,l\}}$ are rationally equivalent. In fact, they only differ by a linear function and therefore have the same divisor, namely the single vertex in $\mathcal{M}_{\{i,j,k,l\}}$. □

Remark 5.5. In the algebro-geometric version of these equations (cf. [K, 5.3.3] or (with proofs) [FP, equation (54) and (55)]) the big sum(s) usually run like $\sum_{\beta_1,\beta_2}\sum_{A,B}$, where β_1, β_2 are cohomology classes such that $\beta_1+\beta_2=\beta$ and $A \cup B = [n]$ is a partition of the marks. We can proceed accordingly and let our sum run through unlabelled instead of labelled degrees, as unlabelled degrees correspond via Minkowski weights to cohomology classes. If we collect all reducible partitions $I \cup J = \Delta \cup [n]$, such that the unlabelled degrees $\delta(\Delta_I), \delta(\Delta_J)$ coincide, we obtain a class of $\frac{\Delta!}{\Delta_I!\cdot\Delta_J!}$ elements. On the other hand, as mentioned at the beginning of section 3, counting curves with labelled non-contracted leaves leads to an overcounting by the factor $\Delta!$, i.e. if $\delta := \delta(\Delta)$ is an unlabelled degree, we should define

$$\langle\prod_{k=1}^n \tau_{a_k}(C_k)\rangle_\delta := \frac{1}{\Delta!}\langle\prod_{k=1}^n \tau_{a_k}(C_k)\rangle_\Delta.$$

So by switching to "unlabelled" invariants, the above factor $\frac{\Delta!}{\Delta_I!\cdot\Delta_J!}$ cancels and we obtain

$$\begin{aligned}&\sum_{\substack{\delta_I,\delta_J\\ \delta_I+\delta_J=\delta}}\ \sum_{\substack{A\cup B=[n]\\ i,j\in A,k,l\in B}}\ \sum_{e,f}\langle\prod_{k\in A}\tau_{a_k}(C_k)\cdot\tau_0(B_e)\rangle_{\delta_I}\ \beta_{ef}\ \langle\tau_0(B_f)\cdot\prod_{k\in B}\tau_{a_k}(C_k)\rangle_{\delta_J}\\ =&\sum_{\substack{\delta_I,\delta_J\\ \delta_I+\delta_J=\delta}}\ \sum_{\substack{A\cup B=[n]\\ i,k\in A,j,l\in B}}\ \sum_{e,f}\langle\prod_{k\in A}\tau_{a_k}(C_k)\cdot\tau_0(B_e)\rangle_{\delta_I}\ \beta_{ef}\ \langle\tau_0(B_f)\cdot\prod_{k\in B}\tau_{a_k}(C_k)\rangle_{\delta_J},\end{aligned}$$

which is now combinatorially identical to the algebro-geometric version.

5.2. **Topological recursion.** In the same flavour as in the previous subsection, we will also formulate a tropical version of the equations known as "topological recursion".

Let x_i, x_k, x_l be pairwise different marked leaves. We know from 2.24 that we can express the Psi-divisor ψ_i in terms of "boundary" divisors, namely

$$\operatorname{div}(\psi_i) = \sum_{\substack{I|J\\ i\in I,k,l\in J}} \operatorname{div}(\varphi_{I|J}).$$

Now again we give a name to the term that has no algebro-geometric counterpart

$$\phi_{i|k,l} = \sum_{\substack{I|J \text{ non-red.}\\ i\in I;k,l\in J}} \varphi_{I|J}.$$

As in the previous subsection, we can describe this function as follows.

Lemma 5.6. *Let $F = (\prod_{k=1}^n \tau_{a_k}(C_k))_\Delta$ be a one-dimensional family of curves with general conditions. Let σ be a facet of F. Then*

$$\phi_{i|k,l}|_\sigma = \begin{cases} \sum \textit{length of edge that seperates } i \textit{ from } k,l & \textit{if interior curves of } \sigma \textit{ are non-reducible}\\ \textit{const} & \textit{otherwise}.\end{cases}$$

Again, we fix a complete simplicial fan Θ and a basis $B_0,\dots,B_m$ of $Z_*(\Theta)$. Furthermore, let $(\beta_{ef})_{ef}$ be the inverse matrix (over $\mathbb{Q}$) of the matrix $(\deg(B_e\cdot B_f))_{ef}$.

Theorem 5.7 (Topological recursion, cf. [K] 5.4.1)**.** *Let $F = (\prod_{k=1}^n \tau_{a_k}(C_k))_\Delta$ be a one-dimensional family of curves and fix three pairwise different marked leaves x_i, x_k, x_l. Moreover, we assume that the following conditions hold:*

(a) *For any reducible partition $I|J$ with $i\in I; k,l\in J$ the push-forwards $\operatorname{ev}_x(X_I)$ and $\operatorname{ev}_y(X_J)$ are Θ-directional (with notations from section 4).*

(b) *The function $\phi_{i|k,l}$ is bounded on F.*

Then the topological recursion

$$\langle \psi_i \cdot \prod_{k=1}^{n} \tau_{a_k}(C_k) \rangle_\Delta = \sum_{\substack{I|J \text{ reducible} \\ i \in I, k,l \in J}} \sum_{e,f} \langle \prod_{k \in I} \tau_{a_k}(C_k) \cdot \tau_0(B_e) \rangle_{\Delta_I} \ \beta_{ef} \ \langle \tau_0(B_f) \cdot \prod_{k \in J} \tau_{a_k}(C_k) \rangle_{\Delta_J}$$

holds, where the sum runs through reducible partitions only.

Remark 5.8. In the same way as in 5.5 we obtain the "unlabelled" version

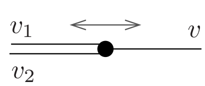
$$\langle \psi_i \cdot \prod_{k=1}^{n} \tau_{a_k}(C_k) \rangle_\delta = \sum_{\substack{\delta_I, \delta_J \\ \delta_I + \delta_J = \delta}} \sum_{\substack{A \,\dot\cup\, B = [n] \\ i \in A, k,l \in B}} \sum_{e,f} \langle \prod_{k \in A} \tau_{a_k}(C_k) \cdot \tau_0(B_e) \rangle_{\delta_I} \ \beta_{ef} \ \langle \tau_0(B_f) \cdot \prod_{k \in B} \tau_{a_k}(C_k) \rangle_{\delta_J},$$

which coincides combinatorially with the algebro-geometric version of this equation.

5.3. **Contracted bounded edges.** As a preparation for the more difficult case of plane curves, we first assume $r = 1$.

Lemma 5.9. *Let $P_1, \ldots, P_n$ be points in general position in $\mathbb{R}^1$ and let $F = (\prod_{k=1}^{n} \tau_{a_k}(P_k))_d^{\mathbb{R}^1}$ be a one-dimensional family in $\mathcal{M}_n^{\text{lab}}(\mathbb{R}^1, d)$. Then for any choice of marked leaves x_i, x_j, x_k, x_l, the functions $\phi_{i,j|k,l}$ and $\phi_{i|k,l}$ are bounded on F.*

Proof. For general conditions, F set-theoretically coincides with the set of curves satisfying the given incidence and valence conditions. Consider a general curve $C \in F$. Then C is also a general curve in the Psi-product $X := \prod_{k=1}^{n} \psi_k^{a_k}$. As we cut down X by n point conditions and $\dim(F) = 1$, the dimension of X must be $n+1$, hence C contains n bounded edges. As C is a rational curve, this implies $n+1$ vertices. Therefore there exists a vertex V not adjacent to a marked leaf $x_k, k \in [n]$. Now, either one of the three adjacent edges is a contracted bounded edge. Then the deformation of C in F is given by changing the length of this edge, but this does not affect $\phi_{i,j|k,l}$ or $\phi_{i|k,l}$ by definition. Or, if all of the adjacent edges are non-contracted, the deformation of C in F is given by moving V (and changing the lengths accordingly).



Note that the edge v cannot be unbounded as its direction "vector" is not primitive. Therefore, if this deformation is supposed to be unbounded, v_1, v_2 must be unbounded. But in this case only the length of v grows infinitely. But as v does not separate any marked leaves, this does not change $\phi_{i,j|k,l}$ and $\phi_{i|k,l}$. □

Now let us consider the case of plane curves, i.e. $r = 2$. We fix the following notation: Let $F = (\prod_{k=1}^{n} \tau_{a_k}(C_k))_\Delta^{\mathbb{R}^2}$ be a one-dimensional family of plane curves with general conditions and and let $L \,\dot\cup\, M \,\dot\cup\, N = [n]$ be the partition of the labels such that

$$\operatorname{codim}(C_k) = \begin{cases} 0 & \text{if } k \in L, \\ 1 & \text{if } k \in M, \\ 2 & \text{if } k \in N. \end{cases}$$

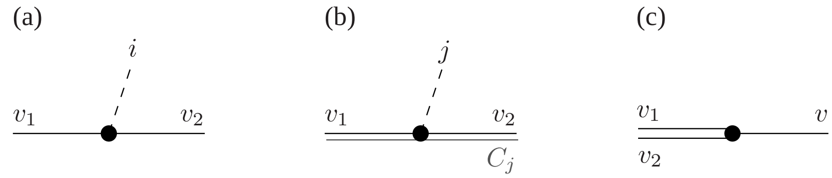
First we study how the deformation of a general curve C in F can look like.

Lemma 5.10 (Variation of [MR08] 4.4)**.** *Let us assume*

i) $a_k = 0$ *for all* $k \in L \cup M$*, i.e. Psi-conditions are only allowed together with point conditions.*

Then the following holds:
Let σ be a facet of F and let $C \in \sigma$ be a general curve. Then the deformation of C inside σ is described by one of the following cases:

- **(I)** *C contains a* contracted bounded edge. *Then the deformation inside σ is given by changing the length of this edge arbitrarily.*
- **(II)** *C has a three-valent* degenerated vertex *V of one of the following three types:*
  - (a) *One of the adjacent edges is a marked leaf $i \in L$.*
  - (b) *One of the adjacent edges is a marked leaf $j \in M$ and the linear spans of the corresponding line C_j at $\operatorname{ev}_j(C)$ and of the other two edges adjacent to V coincide (i.e. the curves C and the C_j do* not *intersect transversally at $\operatorname{ev}_j(C)$).*
  - (c) *All edges adjacent to V are non-contracted, but their span near V is still only one-dimensional; w.l.o.g. we denote the edge alone on one side of V by v and the two edges on the other side by v_1, v_2.*



In all these cases the deformation inside σ is given by moving V.
- **(III)** *C contains a* movable string *S, i.e. a two-valent subgraph of C homeomorphic to $\mathbb{R}$ such that all edges are non-contracted and all vertices of S are three-valent in C and not degenerated in the sense of case (II). Then the deformation of C is given by moving S while all vertices not contained in S remain fixed (in particular, only edges in or adjacent to S change their lengths).*

Proof. Again, for general conditions, F set-theoretically coincides with the set of curves satisfying the given incidence and valence conditions. Thus finding the deformation of C inside σ is the same as finding a way of changing the position and the length of the bounded edges of C such that the resulting curve still meets the incidence conditions C_k.
It is obvious that in the cases (I) and (II) changing the length of the contracted bounded edge respectively moving the degenerated vertex V leads to such deformations.
In case (III) the non-degeneracy of the vertices makes sure that both ends of S consist of non-contracted ends and that a small movement of one of these ends leads to a well-defined movement of the whole string (a more detailed description can be found in the proof of [MR08, 4.4]).
Finally, this list of cases is really complete, as C always contains a string whose vertices are three-valent in C and whose ends are either non-contracted leaves or marked leaves in L. This follows from the same calculation as in [MR08, 4.3], with the only difference that we have to replace the number $3d$ by $\#\Delta$. □

We have now seen how a general curve $C \in F$ can be deformed. In a second step, we will now focus on unbounded deformations.

Definition 5.11. A fan Θ in $\mathbb{R}^2$ is called *strongly unimodular* if *any* two independent primitive vectors generating rays of Θ form a basis of $\mathbb{Z}^2$.
For a given degree Δ let $\Theta(\Delta)$ be the fan consisting of all rays generated by a direction vector appearing in Δ (i.e. $\Theta(\Delta)$ is the fan supporting $\delta(\Delta)$). A degree Δ in $\mathbb{R}^2$ is called *strongly unimodular* if $\Theta(\Delta)$ is strongly unimodular and if all direction vectors appearing in Δ are primitive. This ensures that for every pair of independent vectors v_1, v_2 appearing in Δ, the dual triangle to the fan spanned by v_1, v_2 and $-(v_1 + v_2)$ does not contain interior lattice points.

Remark 5.12. Let us investigate which (one-dimensional) fans in $\mathbb{R}^2$ (up to isomorphism) are strongly unimodular. This discussion is also contained in [Fra, 5.3].
Let Θ be a fan in $\mathbb{R}^2$ and let PG be the set of all primitive generators of rays of Θ. W.l.o.g. we can

assume that $(-1,0)$ is contained in PG. Then being strongly unimodular requires that the y-coordinate of all vectors in PG is either 1, 0 or -1. But note that two vectors $(\alpha,1)$ and $(\beta,1)$ form a lattice basis if and only if $|\alpha-\beta|=1$ (i.e. the vectors must be neighbours in $\mathbb{Z}\times\{1\}$). In particular, at most two such vectors can appear in PG (and analogously, PG can contain at most two vectors with y-coordinate -1). Therefore let $r=0,1,2$ (resp. $s=0,1,2$) be the number of vectors in PG with positive (resp. negative) y-coordinate and let $t=1,2$ be the number of vectors in PG with y-coordinate 0. Moreover, for any two vectors $(\alpha,1)$ and $(\beta,-1)$ we must have $|\alpha+\beta|=0$ or 1 (i.e. $(\beta,-1)$ must coincide with or must be a neighbour of $-(\alpha,1)$). Keeping this in mind, we can distinguish the following cases:

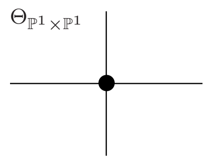
(a) $r=s=0, t=2$
We obtain the degenerated fan $\Theta_{\mathbb{P}^1\times\mathbb{K}^*}$.

$\Theta_{\mathbb{P}^1\times\mathbb{K}^*}$

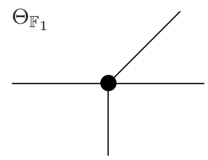
(b) $r=s=0, t=1$ or $r=0, s\neq 0$ or $r\neq 0, s=0$
In this case, the fan Θ can not appear as $\Theta(\Delta)$, as no assignment of positive weights can make Θ balanced.

(c) $r=s=1, t=2$
We obtain the two non-isomorphic fans $\Theta_{\mathbb{P}^1\times\mathbb{P}^1}$ and $\Theta_{\mathbb{F}_1}$.

$\Theta_{\mathbb{P}^1\times\mathbb{P}^1}$

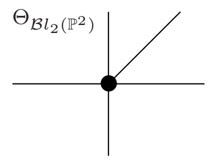


$\Theta_{\mathbb{F}_1}$



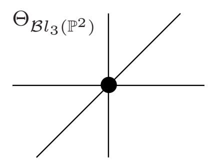
(d) $r=1, s=t=2$ or $s=1, r=t=2$
Up to isomorphisms we obtain the fan $\Theta_{\mathcal{B}l_2(\mathbb{P}^2)}$.

$\Theta_{\mathcal{B}l_2(\mathbb{P}^2)}$



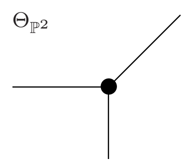
(e) $r=s=t=2$
We obtain the fan $\Theta_{\mathcal{B}l_3(\mathbb{P}^2)}$.

$\Theta_{\mathcal{B}l_3(\mathbb{P}^2)}$



(f) $r=s=t=1$
We obtain the fan $\Theta_{\mathbb{P}^2}$.

$\Theta_{\mathbb{P}^2}$



(g) $r=2, s\geq 1, t=1$ or $r\geq 1, s=2, t=1$
In this cases $|\Theta|$ must contain a one-dimensional subspace. Therefore, after applying an automorphism of $\mathbb{Z}^2$, we can assume $t=2$, which was dealt with in the other cases.

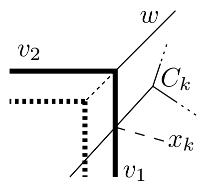
Thus Δ is strongly unimodular if and only if all direction vectors are primitive and $\Theta(\Delta)$ corresponds to one of the following toric varieties $\mathbb{P}^1\times\mathbb{K}^*$, $\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$, $\mathbb{F}_1$, $\mathcal{B}l_2(\mathbb{P}^2)$, $\mathcal{B}l_3(\mathbb{P}^2)$.

Lemma 5.13 (Variation of [MR08] 4.4)**.** *We assume*

i) $a_k=0$ *for all* $k\in L\cup M$*,*
ii) Δ *is strongly unimodular.*

Then the following holds:
Let σ be a unbounded *facet of F and let $C \in \sigma$ be a general curve. Then the deformation of C in σ is described by one of the following cases:*

- **(I)** *C contains a contracted bounded edge whose length can be changed arbitrarily.*
- **(II)** *C has a three-valent degenerated vertex V of one the three types described above. Furthermore, in the cases (a) and (b) (of 5.10 (II)) one of the edges v_1, v_2 is bounded, the other one unbounded, whereas in case (c) the edge v is bounded and v_1, v_2 are unbounded.*
- **(III)** *C contains a movable string S with two non-contracted leaves v_1, v_2 and* only one adjacent bounded edge w. *The deformation of C is given by increasing the length of w.*



Furthermore, if $x_k, k \in M$ is a marked leaf adjacent to S, then $h(x_k)$ is a general point in an unbounded facet of C_k whose outgoing direction vector v lies in the interior of the cone spanned by v_1, v_2.

Proof. Nothing happens in the cases (I), (II) (a) and (b). In case (II) (c), the edge v cannot be unbounded as $v = -v_1 - v_2$ is not primitive. Therefore the two edges on the other side of V must be unbounded. In case (III), the proof of the first statement is fully contained in the last part of the proof of [MR08, 4.4] (using the fact that Δ is strongly unimodular in the last step). The second statement concerning adjacent marked leaves $x_k, k \in M$ is obvious as the deformation is supposed to be unbounded. □

Theorem 5.14. *Let x_i, x_j, x_k, x_l be pairwise different marked leaves and let us assume*

- i) *$a_k = 0$ for all $k \in L \cup M$,*
- ii) *Δ is strongly unimodular,*
- iii) *if $i, j \in M$ (resp. $k, l \in M$), then for any pair of independent direction vectors v_1, v_2 appearing in Δ, the interior of the cone spanned by v_1, v_2 does not intersect both degrees $\delta(C_i)$ and $\delta(C_j)$ (resp. $\delta(C_k)$ and $\delta(C_l)$).*

Then $\phi_{i,j|k,l}$ is bounded.
If we additionally require

- iv) *$i \in N$,*

then also $\phi_{i|k,l}$ is bounded.

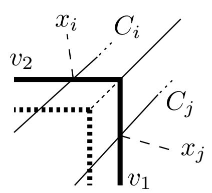
Proof. As conditions i) and ii) hold, we can apply 5.13, which describes the unbounded facets of F. We have to show that $\phi_{i,j|k,l}$ (resp. $\phi_{i|k,l}$) is bounded on these facets. In case (I), the only changing length is that of a contracted edge and therefore not measured by both $\phi_{i,j|k,l}$ and $\phi_{i|k,l}$. In case (II), the edge whose length is growing infinitely cannot separate more then one marked leaf $x_k, k \in L \cup M$ from the others. Therefore this length cannot contribute to $\phi_{i,j|k,l}$ and — by condition iv) — to $\phi_{i|k,l}$. Finally, condition iii) (and also condition iv)) is made such that $\phi_{i,j|k,l}$ and $\phi_{i|k,l}$ are also bounded in case (III). □

Remark 5.15. The conditions i) – iv) appearing in the above statements are not only sufficient but, in most cases, also necessary for the statements to hold:

- iv) If condition iv) in 5.14 is not satisfied, we can get the following things:
  - If $i \in L$, then the degenerated vertex of type (a) leads to an unbounded $\phi_{i|k,l}$.

- If $i \in M$ and ρ is a ray in C_i whose direction vector v_ρ also appears in Δ, then in general we will find curves in F with a degenerated vertex of type (b), whose unbounded movement will make $\phi_{i|k,l}$ unbounded.
- If $i \in M$ and ρ is a ray in C_i whose direction vector v_ρ lies between two direction vectors v_1, v_2 appearing in Δ, this will in general lead to curves in F with unbounded deformations of case (III) such that the outward directions are v_1, v_2 and such that x_i is adjacent to the moved string. So again, $\phi_{i|k,l}$ is in general unbounded.

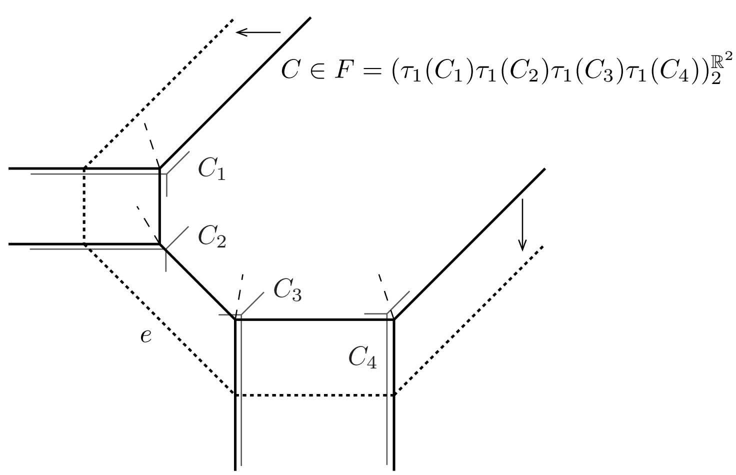
iii) If condition iii) is not satisfied, we will in general get unbounded deformations of the following type:



In this case we have $i, j \in M$ and the interior of the cone spanned by v_1, v_2 contains direction vectors of both C_i and C_j. As in general x_k, x_l will lie on the other side of the growing edge w, $\phi_{i,j|k,l}$ will be unbounded.

ii) If we drop condition ii), i.e. if we allow non-unimodular degrees Δ, two things can happen: If we allow non-primitive direction vectors, then we get deformations of type (II) (c) with unbounded edge v. Therefore the lengths of v_1 and v_2, which can in general separate arbitrary marked leaves, grow infinitely. If $\Theta(\Delta)$ is not supposed to be strongly unimodular, then the description of unbounded deformations of case (III) in 5.13 becomes incorrect, as there will appear more complicated strings with more adjacent bounded edges than just one. The example of $\mathbb{F}_2$ is analyzed in detail in [Fra] and [FM, e.g. 2.10].

i) If we drop condition i), i.e. if we allow Psi-conditions also at marked leaves which are not fixed by points, we end up with more complicated kinds of deformations of general curves in F. The following picture shows an example of an unbounded deformation in a one-dimensional family of plane curves of projective degree 2.



Here, C has to meet all the four tropical lines $C_1, \ldots, C_4$ with one Psi-condition. Note that the indicated deformation of C is indeed unbounded and that the length of the $(1, -1)$-edge e grows infinitely. This example can be extended in the following way: One can glue arbitrary (fixed) curves to the non-contracted leaves of C in direction $(1, 1)$, obtaining more families admitting such a deformation. In particular, the edge e can separate arbitrary kinds of points, showing that in general $\phi_{i,j|k,l}$ and $\phi_{i|k,l}$ can be unbounded for any choice of i, j, k, l.

In higher dimensions ($r \geq 3$), up to now only the following case is studied:

Theorem 5.16 ([Zim] 4.86). *Let $F = (\prod_{k=1}^{n} \tau_0(V_k))_d^{\mathbb{R}^r}$ be a one-dimensional family of curves of projective degree d in $\mathbb{R}^r$ which do* not *satisfy Psi-conditions, but incidence conditions given by conventional linear spaces $V_k \subseteq \mathbb{R}^r$. Then for any choice of $\{i, j, k, l\} \in [n]$ the function $\phi_{i,j|k,l}$ is bounded on F.*

5.4. **Comparison to the algebro-geometric invariants.** In the special case of an empty degree, denoted by $\Delta = 0$, the situation is analogous to the algebro-geometric one.

Lemma 5.17. *Let $Z = (\prod_{k=1}^{n} \tau_{a_k}(C_k))_0$ be a zero-dimensional intersection product in $\mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, 0)$. Then $\deg(Z)$ is non-zero if and only if $\sum_{k=1}^{n} \operatorname{codim}(C_k) = r$ (or equivalently $\sum_{k=1}^{n} a_k = n-3$). In this case,*

$$\deg(Z) = \binom{n-3}{a_1, \ldots, a_n} \deg(C_1 \cdots C_k)$$

holds.

Proof. By definition $\mathcal{M}_n^{\text{lab}}(\mathbb{R}^r, 0)$ is isomorphic to $\mathcal{M}_n \times \mathbb{R}^r$. Moreover, as $\Delta = 0$, all evaluation maps ev_i coincide with the projection onto the second factor, which we therefore denote by ev. Now let $X := \prod_{k=1}^{n} \psi_k^{a_k} = (\prod_{k=1}^{n} (\psi_k^{\text{abstr}})^{a_k}) \times \mathbb{R}^r$ be the intersection of all Psi-divisors. Then the projection formula applied to ev yields

$$\deg(Z) = \deg(C_1 \cdots C_n \cdot \operatorname{ev}_*(X)).$$

But $\operatorname{ev}_*(X)$ is non-zero if and only if $\sum_{k=1}^{n} a_k = n-3$. If so, by 2.22 we know $\operatorname{ev}_*(X) = \binom{n-3}{a_1,\ldots,a_n} \cdot \mathbb{R}^r$, which proves the statement. □

Now we are finally ready to compare the tropical invariants for plane tropical curves to the algebro-geometric ones using the equations proven in the previous subsections.

Theorem 5.18. *Let*

- *Θ be a complete (not necessarily strongly) unimodular fan in $\mathbb{R}^2$ and let $X := X(\Theta)$ denote the corresponding smooth toric variety,*
- *$\gamma_1, \ldots, \gamma_n \in A^*(X)$ be cohomology classes of X, which correspond to Minkowski weights by* [FS94, theorem 2.1], *and denote by $C_1, \ldots, C_n$ the corresponding tropical Θ-directional cycles,*
- *Δ be a strongly unimodular degree whose unlabelled degree $\delta(\Delta)$ is Θ-directional, and let $\beta \in A^1(X)$ be the corresponding cohomology class.*
- *$a_1, \ldots, a_n$ be non-negative integers such that $a_k = 0$ if $\dim(C_k) > 0$.*

Then the tropical and algebro-geometric descendant Gromov-Witten invariants satisfy

$$\frac{1}{\Delta!} \langle \tau_{a_1}(C_1) \cdots \tau_{a_n}(C_n) \rangle_{\Delta}^{\mathbb{R}^2} = \langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle_{\beta}^{X}.$$

Proof. First we choose a basis $B_0, \ldots, B_m$ of $Z_*(\Theta)$. Via [FS94, theorem 2.1] this also describes a basis $\eta_0, \ldots, \eta_m$ of $A^*(X)$, and [FS94, proposition 3.1] together with 1.9 prove that

$$\deg(B_e \cdot B_f) = \deg(\eta_e \cdot \eta_f)$$

holds. This implies that, if we use WDVV equations or topological recursion with respect to these bases, then the diagonal coefficients $\beta_e f$ appearing in the tropical and in the algebro-geometric setting coincide. Thus, as discussed in the remarks 5.5 and 5.8 the numbers $\frac{1}{\Delta!} \langle \tau_{a_1}(C_1) \cdots \tau_{a_n}(C_n) \rangle_{\Delta} = \langle \tau_{a_1}(C_1) \cdots \ \tau_{a_n}(C_n) \rangle_{\delta(\Delta)}$ and $\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle_{\beta}^{X}$ satisfy a certain set of identical WDVV and topological recursion equations (where on the tropical side we have to be slightly more careful about i, j, k, l satisfying condition iii) and iv) of 5.14) as well as the string and divisor equation. Therefore we can finish the proof by showing that the numbers can be computed recursively, using these equations, from some initial numbers and that these initial numbers coincide.

We separate the labels of the marked leaves into the sets $L \cup M \cup N = [n]$ according to the (co)dimension of C_k as in subsection 5.3. First we use topological recursion to reduce the number of Psi-conditions: We pick a marked leaf x_i with $a_i > 0$ (and therefore $i \in N$) and an arbitrary pair of marked leaves x_k, x_l satisfying condition iii) of 5.14. If such x_k, x_l do not exist, we can add them using the divisor equation backwards with appropriate rational functions h_k, h_l. Namely, if $X = \mathbb{P}^1 \times \mathbb{P}^1$ we can use $h_k = h_l = \max\{0, x, y, x+y\}$, otherwise we can use $h_k = h_l = \max\{0, x, y\}$. Note also that this choice ensures that $h_k \cdot \Delta = h_l \cdot \Delta$ is non-zero for every possible degree, so we do not divide by zero. After eliminating all Psi-conditions in this way, we can assume $a_k = 0$ for all $k \in [n]$, i.e. we are back in the case of usual (primary) Gromov-Witten invariants. After applying string and divisor equation we can assume that $L = M = \emptyset$ and it remains to compute invariants of the form $\langle \prod_{k=1}^n \tau_0(P_k) \rangle_\Delta$ for points $P_1, \ldots, P_n \in \mathbb{R}^2$. Comparing dimension shows $\#\Delta = n+1$. Let us first consider the general case $n \geq 3$. Here we consider the one-dimensional family $F = (\tau_0(C_i)\tau_0(C_j) \prod_{k=1}^{n-1} \tau_0(P_k))_\Delta$ with arbitrary Θ-directional curves C_i, C_j such that $C_i \cdot C_j$ is non-zero and such that condition iii) of 5.14 is satisfied (e.g. we can choose the divisors of the functions we chose above). We let x_i, x_j be the first two marked leaves as indicated, and choose $k, l \in [n-1]$ arbitrarily. In the corresponding WDVV equation only one extremal partition $I|J$ with $\Delta_I = 0, \Delta_J = \Delta$ does not vanish. This follows from 5.17 and $\operatorname{codim}(P_k) + \operatorname{codim}(P_l), \operatorname{codim}(C_i) + \operatorname{codim}(P_k), \operatorname{codim}(C_j) + \operatorname{codim}(P_l) > 2$. Moreover, the only remaining extremal partition $I = \{i, j\}, J = \Delta \cup [n-1]$ provides the term

$$\langle \tau_0(C_i)\tau_0(C_j)\tau_0(\mathbb{R}^2) \rangle_0 \cdot \langle \tau_0(P) \prod_{k=1}^{n-1} \tau_0(P_k) \rangle_\Delta = \deg(C_i \cdot C_j) \cdot \langle \prod_{k=1}^{n} \tau_0(P_k) \rangle_\Delta .$$

Hence, we can lead back the computation of $\langle \prod_{k=1}^n \tau_0(P_k) \rangle_\Delta$ to invariants of smaller degree. We can repeat this until we arrive at the initial invariants with $n = 1$ or $n = 2$. In these cases $\#\Delta = 2$ or $\#\Delta = 3$ and therefore the only possible degrees (up to identification via linear isomorphisms of $\mathbb{Z}^r$) are $\Delta = \{-e_1, e_1\}$ and $\Delta = \{-e_1, -e_2, e_1 + e_2\}$. In both cases, it is easy to show by direct computation that $\langle \tau_0(P_1) \rangle_\Delta = 1$ and $\langle \tau_0(P_1)\tau_0(P_2) \rangle_\Delta = 1$ hold. □

Remark 5.19. Note that the left hand tropical side of the equation

$$\frac{1}{\Delta!} \langle \tau_{a_1}(C_1) \cdots \tau_{a_n}(C_n) \rangle_\Delta^{\mathbb{R}^2} = \langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle_\beta^X$$

is in fact independent of the fan Θ (provided that Θ is fine enough), and therefore the right hand algebro-geometric side does also not depend on X. This implies that, for two complete smooth toric surfaces and for "common" cohomology classes, their descendant Gromov-Witten invariants appearing in the theorem coincide. This seems to be a new result for *descendant* invariants. Without Psi-classes, overlapping results were proven in the course of studying Gromov-Witten invariants of blow-ups (cf. [GP], [G], [Hu]).

Remark 5.20. Similarly we can deal with the case $r = 1$, i.e. we can prove

$$\frac{1}{d!^2} \langle \tau_0(\mathbb{R}^1)^l \prod_{k=1}^{n} \tau_{a_k}(P_k) \rangle_d^{\mathbb{R}^1} = \langle \tau_0([\mathbb{P}^1])^l \prod_{k=1}^{n} \tau_{a_k}([pt]) \rangle_d^{\mathbb{P}^1},$$

where the left hand side is a tropical, the right hand side a conventional invariant and $[pt]$ denotes the class of a point $pt \in \mathbb{P}^1$. In fact, after applying the string equation, we are left with the case where $l = 0$. Now we use 5.9 and topological recursion to reduce the number of Psi-conditions (where, if $n < 3$, we first add more marked leaves using the divisor equation). Finally, when $a_k = 0$ for all $k \in [n]$, it follows $d = 1$ and we can compute directly $\langle \tau_0(P) \rangle_1^{\mathbb{R}^1} = 1$. For the case of the rational Hurwitz numbers $H_d^0 := \langle \tau_1([pt])^{2d-2} \rangle_d^{\mathbb{P}^1}$, this result was basically known before (cf. [CJM08, lemma 9.7]), but the proof is based on the different point of view. In [CJM08] the result is a specialization of considerations for higher genus, not for higher dimension r as it is the case here.

Remark 5.21. The discussion in 5.15 and the factor $n + \#\Delta - 2$ appearing in the tropical dilaton equation 3.13, instead of $n - 2$ in the algebro-geometric version, show that for more difficult degrees

Δ (if $r = 2$) and for Psi-conditions at marked leaves x_k with $\dim(C_k) > 0$, the corresponding tropical and conventional invariants are in general different. For example, if we add a marked leaf that has to satisfy only a Psi-condition, the different factors in the dilaton equations immediately lead to different invariants.

Remark 5.22. Based on 4.19 and 5.16, we can extend the above equalities of tropical and conventional primary Gromov-Witten invariants to higher dimensions. We postpone this to [GZ].

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