TROPICAL ENUMERATIVE INVARIANTS OF $\mathbb{F}_0$ AND $\mathbb{F}_2$

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Abstract. There is an equation relating numbers of curves on $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ satisfying incidence conditions and numbers of curves on $\mathbb{F}_2$ satisfying incidence conditions. The purpose of this paper is to give a tropical proof of this equation (for some cases). We use two tropical methods. The first method proves the formula for rational curves. We use induction on the degree and two Kontsevich-type formulas for curves on $\mathbb{F}_0$ and on $\mathbb{F}_2$. The formula for $\mathbb{F}_2$ was not known before and is proved using tropical geometry. The second method proves the formula for small degree and any positive genus and uses lattice paths.

1. Introduction

In tropical geometry, algebraic curves are replaced by certain balanced piece-wise linear graphs called tropical curves. Tropical geometry has gained lots of attention recently. One of the interesting results is that we can determine numbers of algebraic curves on toric surfaces satisfying incidence conditions by counting the corresponding tropical curves instead (Mikhalkin's Correspondence Theorem, see [8]). This is true in particular for the toric surfaces $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_2$. The tropical numbers can be determined using certain lattice paths in the polygon dual to the toric surface (see [8], theorem 2).

Gromov-Witten invariants can be thought of as "virtual" solutions to enumerative problems. They are deformation invariants, thus they coincide for the two surfaces $\mathbb{F}_0$ and $\mathbb{F}_2$. For $\mathbb{F}_0$, Gromov-Witten invariants are enumerative, i.e. they count curves on $\mathbb{F}_0$ satisfying incidence conditions. For $\mathbb{F}_2$, they are not, but it is known how they are related to enumerative numbers. Therefore there is an equation relating the enumerative numbers of $\mathbb{F}_0$ and $\mathbb{F}_2$.

The purpose of this paper is to give a tropical proof of this equation (for some cases), using Mikhalkin's Correspondence Theorem.

Let us introduce this equation in more details. Let C denote the class of a general section of $\mathbb{F}_2$ and F the class of the fiber of ruling. Then the Picard group of $\mathbb{F}_2$ is generated by C and F. The exceptional curve is linearly equivalent to $C - 2F$. The Picard group of $\mathbb{F}_0$ is generated by two fibers of ruling which we will denote by C and F as well. We can degenerate $\mathbb{F}_2$ to $\mathbb{F}_0$ such that the class $aC + bF$ on $\mathbb{F}_2$ degenerates to the class $aC + (a + b)F$ on $\mathbb{F}_0$. Then for nonnegative a, b with $a + b \geq 1$ we have

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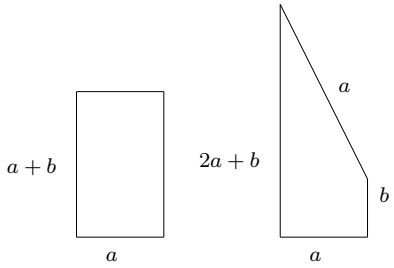
$$N^g_{\mathbb{F}_0}(a,a+b)=\sum_{k=0}^{a-1}\binom{b+2k}{k}N^g_{\mathbb{F}_2}(a-k,b+2k). \tag{1}$$

where $N^g_{\mathbb{F}_0}(a,a+b)$ and $N^g_{\mathbb{F}_2}(a,b)$ denote the numbers of nodal irreducible curves of genus g of class $aC+(a+b)F$ in $\mathbb{F}_0$ (resp. of class $aC+bF$ in $\mathbb{F}_2$) through $4a+2b+g-1$ points in general position. (See [1], theorem 3.1.1, for rational curves and [12], section 8.3, for arbitrary genus.)

Let us now introduce the analogous tropical numbers. The polygon corresponding to the divisor class $aC+(a+b)F$ on $\mathbb{F}_0$ is a rectangle with vertices $(0,0)$, $(a,0)$, $(a,a+b)$ and $(0,a+b)$, the polygon corresponding to the divisor class $aC+bF$ on $\mathbb{F}_2$ is a quadrangle with vertices $(0,0)$, $(a,0)$, (a,b) and $(0,2a+b)$.



We consider plane tropical curves dual to these polygons. Thus, we consider plane tropical curves of degree $\Delta_{\mathbb{F}_0}(a,a+b)$ and $\Delta_{\mathbb{F}_2}(a,b)$, where $\Delta_{\mathbb{F}_0}(a,a+b)$ denotes the multiset of the vectors $(-1,0)$ and $(1,0)$ each $a+b$ times and $(0,-1)$ and $(0,1)$ each a times and $\Delta_{\mathbb{F}_2}(a,b)$ denotes the multiset of the vectors $(-1,0)$ $2a+b$ times, $(0,-1)$ a times, $(1,0)$ b times and $(2,1)$ a times. We denote by $\mathcal{N}^g_{\mathbb{F}_0}(a,a+b)$ (resp. $\mathcal{N}^g_{\mathbb{F}_2}(a,b)$) the number of irreducible plane *tropical* curves of degree $\Delta_{\mathbb{F}_0}(a,a+b)$ (resp. $\Delta_{\mathbb{F}_2}(a,b)$) and genus g through $4a+2b+g-1$ points in general position (see [8]).

Our central result is the following theorem:

Theorem 1.1
The following equation holds for

- *nonnegative integers a, b with $a+b\geq 1$ and $g=0$, and for*
- *$0\leq a\leq 2$, $b\geq 0$ with $a+b\geq 1$ and any $g\geq 0$:*

$$\mathcal{N}^g_{\mathbb{F}_0}(a,a+b)=\sum_{k=0}^{a-1}\binom{b+2k}{k}\mathcal{N}^g_{\mathbb{F}_2}(a-k,b+2k). \tag{2}$$

Of course this theorem (and even the more general case) is an immediate consequence of equation 1 and Mikhalkin's Correspondence theorem which states that $N^g_{\mathbb{F}_0}(a,a+b)=\mathcal{N}^g_{\mathbb{F}_0}(a,a+b)$ and $N^g_{\mathbb{F}_2}(a,b)=\mathcal{N}^g_{\mathbb{F}_2}(a,b)$. However, we want to give a proof within tropical geometry.

We use two different tropical methods to prove theorem 1.1.

To prove the statement for nonnegative a, b with $a+b\geq 1$ and $g=0$, we use induction on the degree and generalizations of Kontsevich's formula for enumerative

numbers on $\mathbb{F}_0$ and $\mathbb{F}_2$. While Kontsevich's formula for $\mathbb{F}_0$ (see theorem 2.13) was known and can be proved without tropical geometry ([4], section 9), our formula for $\mathbb{F}_2$ (see theorem 2.12) is new and was derived using tropical geometry. To derive a Kontsevich-type formula tropically, we compute numbers of curves satisfying point and line conditions and mapping to a special point in tropical $\mathcal{M}_{0,4}$ under the forgetful map. To prove Kontsevich's formula for $\mathbb{P}_2$, one can show that all such curves have a contracted bounded edge and can thus be interpreted as reducible tropical curves ([6]). For $\mathbb{F}_2$, this statement is no longer true. Instead, we get a correction-term corresponding to curves that do not have a contracted bounded edge. We show that these curves can also be interpreted as reducible curves in a different way.

To prove theorem 1.1 for $0 \leq a \leq 2$, $b \geq 0$ with $a + b \geq 1$ and any $g \geq 0$, we use Mikhalkin's lattice path algorithm to count tropical curves (see theorem 2 of [8]) and observations about those lattice paths from [7].

Unfortunately, it seems that none of the above methods can be generalized to other cases easily.

The paper is organized as follows. In section 2, we prove our tropical Kontsevich formulas for $\mathbb{F}_0$ and $\mathbb{F}_2$. In section 3, we use those formulas to prove theorem 1.1 for a, b with $a + b \geq 1$ and $g = 0$ using induction. In section 4, we prove theorem 1.1 for $0 \leq a \leq 2$, $b \geq 0$ with $a + b \geq 1$ and any $g \geq 0$ using lattice paths.

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2. Tropical Kontsevich formulas for $\mathbb{F}_0$ and $\mathbb{F}_2$

To derive tropical Kontsevich formulas for $\mathbb{F}_0$ and $\mathbb{F}_2$, we generalize the ideas of [6]. Let us start by recalling some notations we will use.

Notation 2.1
Let $\Delta = \Delta_{\mathbb{F}_2}(a, b)$ and let $\mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^2, \Delta)$ denote the space of rational parametrized tropical curves in $\mathbb{R}^2$ of degree Δ, with $\#\Delta + n$ ends all of which are labelled, and n of which are contracted ends (see definition 4.1 of [5]). Let

$$\text{ft} : \mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^2, \Delta) \to \mathcal{M}_{0,4}$$

denote the forgetful map which forgets all ends but the first 4 contracted ends (see definition 4.1 of [6]), and

$$\text{ev}_i : \mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^2, \Delta) \to \mathbb{R}^2$$

the evaluation at the contracted end labelled i (see definition 3.3 of [6]). Pick two rational functions φ_A and φ_B on $\mathcal{M}_{0,4}$ (in the sense of [2], definition 3.1) that correspond to abstract tropical curves λ_A (resp. λ_B) where the ends x_1 and x_2 come together at a vertex (resp. where x_1 and x_3 come together) and where the length parameter of the bounded edge is very large.

Remark 2.2
Note that we use the space of parametrized tropical curves with labelled ends here. The reason is that one can show that this space is a tropical fan (proposition 4.7 of [5]) and that we can thus use the tropical intersection theory from [2]. Since

we want to count tropical curves without the labels of the non-contracted ends, we have to divide by a factor of $|G|$, where G is the group of possible permutations of the labels. In the Kontsevich formula we want to prove (theorem 2.12), we sum over all possibilities to split the degree $\Delta = \Delta_{\mathbb{F}_2}(a,b)$ into two smaller degrees. To be precise, we would have to sum over all possibilities to pick a *labelled* subset of non-contracted ends forming the smaller degrees. This factor together with the factors for labelling the ends in the small degrees exactly cancel with the total factor of $|G|$. In the following, we will therefore neglect the fact that non-contracted ends are labelled.

The difference of φ_A and φ_B is globally given by a bounded rational function on $\mathcal{M}_{0,4}$. Therefore, the tropical Cartier divisors $[\varphi_A]$ and $[\varphi_B]$ are rationally equivalent and by lemma 8.5 of [2] their pull-backs $[\mathrm{ft}^*\,\varphi_A]\cdot\mathcal{M}^{\mathrm{lab}}_{0,n}(\mathbb{R}^2,\Delta)$ and $[\mathrm{ft}^*\,\varphi_B]\cdot\mathcal{M}^{\mathrm{lab}}_{0,n}(\mathbb{R}^2,\Delta)$ are rationally equivalent as well.

Set $n=\#\Delta$ and choose rational functions $\varphi_1,\ \varphi_2,\ \varphi_{3_1},\varphi_{3_2}\ \ldots,\ \varphi_{n_1},\varphi_{n_2}$ on $\mathbb{R}^2$ that correspond to tropical curves L_1 and L_2 of degree $\Delta_{\mathbb{F}_2}(1,0)$ and to points $p_3,\ldots,p_n\in\mathbb{R}^2$ in general position. We can set $\varphi_1=\max\{x-p_{1_1},2(y-p_{1_2}),0\}$ and $\varphi_2=\max\{x-p_{2_1},2(y-p_{2_2}),0\}$ to get L_1 and L_2 in this case.

Because of the above, we have

$$\begin{aligned}&\deg([\mathrm{ev}_1^*\,\varphi_1\cdot\mathrm{ev}_2^*\,\varphi_2\cdot\prod_{i=3}^{n}(\mathrm{ev}_i^*\,\varphi_{i1}\cdot\mathrm{ev}_i^*\,\varphi_{i2})\cdot\mathrm{ft}^*\,\varphi_A\cdot\mathcal{M}^{\mathrm{lab}}_{0,n}(\mathbb{R}^2,\Delta)])\\ =&\deg([\mathrm{ev}_1^*\,\varphi_1\cdot\mathrm{ev}_2^*\,\varphi_2\cdot\prod_{i=3}^{n}(\mathrm{ev}_i^*\,\varphi_{i1}\cdot\mathrm{ev}_i^*\,\varphi_{i2})\cdot\mathrm{ft}^*\,\varphi_B\cdot\mathcal{M}^{\mathrm{lab}}_{0,n}(\mathbb{R}^2,\Delta))].\end{aligned}\tag{3}$$

Remark 2.3
Both above expressions are 0-dimensional tropical intersection products as defined in [2], even if the set-theoretical intersection is higher-dimensional. If we pick the conditions to be general however, the set-theoretical intersection equals the support of the intersection product. That means that the intersection products above equal the sums of tropical curves in $\mathcal{M}^{\mathrm{lab}}_{0,n}(\mathbb{R}^2,\Delta)$ that satisfy the conditions, i.e. that pass through L_1, L_2, $p_3,\ldots,p_n$ and map to λ_A resp. $\lambda_B\in\mathcal{M}_{0,4}$ under ft, counted with multiplicity. This can be shown analogously to [9], lemma 3.1.

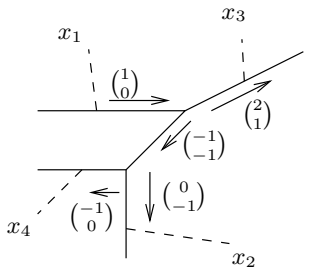
The following lemma will enable us to compute the multiplicity with which we have to count each curve satisfying the conditions in the intersection product:

Lemma 2.4
Let X be an abstract tropical variety (in the sense of [2]*, definition 5.12) of dimension k and $\varphi_1,\ \ldots,\ \varphi_k$ rational functions on X. Moreover, let $P\in X$ be a point in the interior of a cone σ of maximal dimension in X. Assume that φ_i is of the form $\varphi=\max\{\psi_i,\chi_i\}$ locally around P, where $\psi_i,\chi_i : X\to\mathbb{R}$ denote $\mathbb{Z}$-affine functions with $\psi_i(P)=\chi_i(P)$. Let $(\psi_i-\chi_i)_L$ denote the linear part of the affine function $(\psi_i-\chi_i)$ and let A be the $(k\times k)$-matrix with entries $((\psi_i-\chi_i)_L(u_j))_{i,j}$ for a basis $u_1,\ldots,u_k$ of the lattice underlying X at σ. Then the coefficient of P in the intersection product $\varphi_1\cdot\ldots\cdot\varphi_k\cdot X$ is equal to $\omega(\sigma)\cdot|\det(A)|$.*

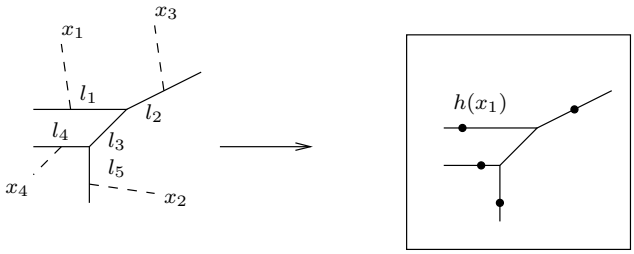
Proof:
The computation of the coefficient of P in the intersection product is local around P. Thus, we may assume that X is a tropical fan in some vector space V and extend σ to the affine vector space V_σ spanned by σ. Furthermore, we may consider the rational functions $\varphi_i = \max\{\psi_i, \chi_i\}$ on the whole space V_σ. Moreover, we may replace the rational functions $\max\{\psi_i, \chi_i\}$ by $\max\{\psi_i, \chi_i\} - \chi_i = \max\{\psi_i - \chi_i, 0\}$ as changing a rational function by a linear function does not affect the intersection product. We define the morphism $g = (\psi_1 - \chi_1, \ldots, \psi_k - \chi_k) : X \to \mathbb{R}^k$. Then we have $\varphi_i = g^*\mu_i$ for $\mu_i : \mathbb{R}^k \to \mathbb{R}; (a_1, \ldots, a_k) \mapsto \max\{a_i, 0\}$ and for all $1 \leq i \leq k$. By the projection formula ([2], proposition 4.8) the multiplicity of $P \in X$ in the intersection product $\varphi_1 \cdot \ldots \cdot \varphi_k \cdot \sigma = g^*\mu_1 \cdot \ldots \cdot g^*\mu_k \cdot \sigma$ is equal to the multiplicity of 0 in $\mathbb{R}^k$ in the intersection product $g_*(g^*\mu_1 \cdot \ldots \cdot g^*\mu_k \cdot \sigma) = \mu_1 \cdot \ldots \mu_k \cdot g_*\sigma$. For dimensional reasons the cycle $g_*\sigma$ is the whole target space $\mathbb{R}^k$ with some weight. But this weight is $\omega(\sigma) \cdot |\det(A)|$. Note $\mu_1 \cdot \ldots \cdot \mu_k \cdot \mathbb{R}^k$ is the origin with weight 1. This finishes the proof. □

Remark 2.5
If σ is a cone in $\mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^2, \Delta)$, it corresponds to a combinatorial type, i.e. a homeomorphism class of a graph plus direction vectors for all edges (see [6], 2.9). We can deform a parametrized tropical curve (Γ, h, x_i) within σ by changing the length of the bounded edges or translating the image $h(\Gamma)$. Thus a basis for the lattice underlying $\mathcal{M}^{\text{lab}}_{0,n}(\mathbb{R}^2, \Delta)$ at σ is given by the position of a root vertex $h(V)$ and the length of all bounded edges. By remark 3.2 of [6], the absolute value of the determinant of the matrix A from lemma 2.4 above is independent from the choice of a root vertex and an order of the bounded edges.

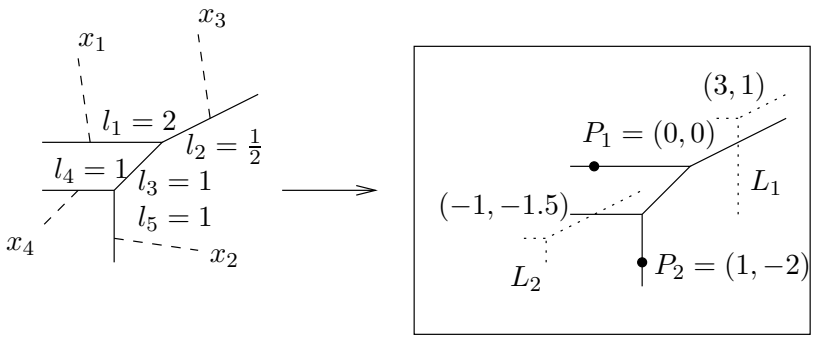
Example 2.6
Assume σ is the cone in $\mathcal{M}^{\text{lab}}_{0,4}(\mathbb{R}^2, \Delta_{\mathbb{F}_2}(1,0))$ corresponding to the combinatorial type pictured below.



Following remark 2.5, we choose the position of $h(x_1)$ and the lengths $l_1, \ldots, l_5$ of the bounded edges as coordinates for σ.



The following curve C inside σ (where $h(x_1) = (0,0)$, $l_1 = 2$, $l_2 = \frac{1}{2}$ and $l_3 = l_4 = l_5 = 1$) goes through the points P_1 (which is cut out by $\max\{x,0\}$ and $\max\{y,0\}$) and P_2 (cut out by $\max\{x,1\}$ and $\max\{y,-2\}$) and through L_1 (cut out by $\max\{x-3,2y-2,0\}$) and L_2 (cut out by $\max\{x+1,2y+3,0\}$) and maps to the abstract tropical curve with x_1 and x_3 at one vertex and length parameter 1 under ft. Denote by λ a rational function on $\mathcal{M}_{0,4}$ that cuts out this curve.



Because C satisfies the conditions, it contributes to the intersection product

$$\begin{aligned}&\mathrm{ev}_1^*(\max\{x,0\})\cdot \mathrm{ev}_1^*(\max\{y,0\})\,\mathrm{ev}_2^*(\max\{x,1\})\cdot \mathrm{ev}_2^*(\max\{y,-2\})\\ &\cdot \mathrm{ev}_3^*(\max\{x-3,2y-2,0\})\cdot \mathrm{ev}_4^*(\max\{x+1,2y+3,0\})\\ &\cdot \mathrm{ft}^*(\lambda)\cdot \mathcal{M}_{0,4}^{\mathrm{lab}}(\mathbb{R}^2,\Delta_{\mathbb{F}_2}(1,0)).\end{aligned}$$

Let us compute the multiplicity with which it contributes using lemma 2.4. Locally at $h(x_3)$, the function $\max\{x-3,2y-2,0\}$ equals $\max\{2y-2,0\}$ and locally at $h(x_4)$, the function $\max\{x+1,2y+3,0\}$ equals $\max\{x+1,2y+3\}$. Hence locally we have

$$\begin{aligned}&\mathrm{ev}_3^*(\max\{x-3,2y-2,0\}) = \max\{2h(x_3)_y-2,0\} \text{ and}\\ &\mathrm{ev}_4^*(\max\{x+1,2y+3,0\}) = \max\{h(x_4)_x+1,2h(x_4)_y+3\}.\end{aligned}$$

We can rewrite the pullbacks along ev_1 and ev_2 analogously. Locally, ft equals the map that sends a curve in σ with coordinates $(h(x_1),l_1,\ldots,l_5)$ to l_3. We also have to write the linear part of the evaluation pullbacks in the basis of σ, i.e. in

$h(x_1), l_1, \dots, l_5$. For this, note first that

$$h(x_2) = h(x_1) + l_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + l_3 \cdot \begin{pmatrix} -1 \\ -1 \end{pmatrix} + l_5 \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$
$$h(x_3) = h(x_1) + l_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + l_2 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and}$$
$$h(x_4) = h(x_1) + l_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + l_3 \cdot \begin{pmatrix} -1 \\ -1 \end{pmatrix} + l_4 \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Thus, the linear part of $2h(x_3)_y - 2$ equals

$$2h(x_1)_y + 2l_2$$

and the linear part of $h(x_4)_x + 1 - 2h(x_4)_y - 3$ equals

$$h(x_1)_x + l_1 - l_3 - l_4 - 2(h(x_1)_y - l_3) = h(x_1)_x - 2h(x_1)_y + l_1 + l_3 - l_4.$$

So, if we plug for example the vector which has a 1 at l_2 and 0 everywhere else into the linear part of $2h(x_3)_y - 2$, we get 2. If we plug the vector which has a 1 at l_4 and 0 everywhere else into the linear part of $h(x_4)_x + 1 - 2h(x_4)_y - 3$, we get -1. Continuing like this, we can see that the matrix A equals:

	$h(x_1)_x$	$h(x_1)_y$	l_1	l_2	l_3	l_4	l_5
$(P_1)_x$	1	0	0	0	0	0	0
$(P_1)_y$	0	1	0	0	0	0	0
$(P_2)_x$	1	0	1	0	-1	0	0
$(P_2)_y$	0	1	0	0	-1	0	-1
L_1	0	2	0	2	0	0	0
L_2	1	-2	1	0	1	-1	0
ft	0	0	0	0	1	0	0

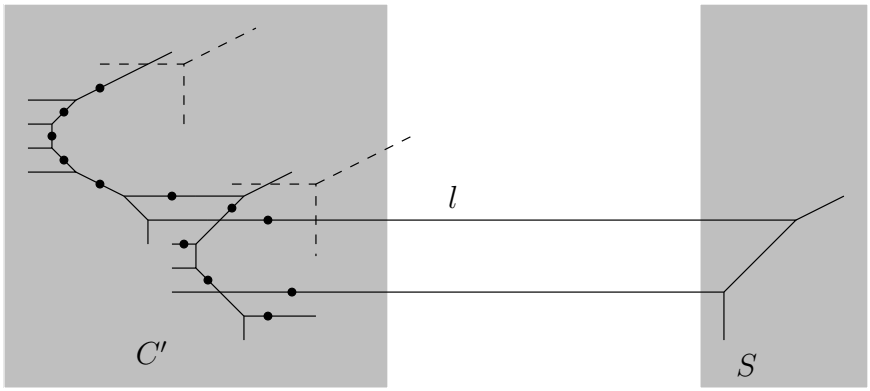
Since $|\det(A)| = 2$, the curve C contributes with multiplicity 2 to the intersection product above.

Notation 2.7
Let C be a curve contributing to a 0-dimensional intersection product as in example 2.6 consisting of evaluation pullbacks and the pullback of a curve in $\mathcal{M}_{0,4}$ under ft (resp. only evaluation pullbacks). Then we denote by $\mathrm{mult}_{\mathrm{ev}\times\mathrm{ft}}(C)$ (resp. $\mathrm{mult}_{\mathrm{ev}}(C)$) the multiplicity with which C contributes to the intersection product, which equals the absolute value of the determinant of the linear part of the combined evaluation and forgetful maps, as we have seen in 2.4.

Remark 2.8
Note that by [6], proposition 3.8, $\mathrm{mult}_{\mathrm{ev}}(C)$ equals the usual multiplicity of a tropical curve as defined in [8], 4.15, i.e. the multiplicity with which it contributes to the count of $\mathcal{N}^g_{\mathbb{F}_2}(a,b)$.

In the following, we want to describe both sides of equation 3 in detail. We want to study the set of curves that satisfy the conditions, and their multiplicity. We will see that we can interpret the curves as reducible curves, and count the contributions from each component separately. This will lead to the formula of theorem 2.12 we want to prove.

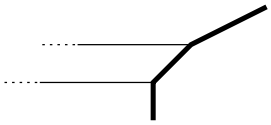
Remark 2.9
Using the notations from 2.1, let C be a tropical curve in $\mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^2, \Delta)$ passing through L_1, L_2, $p_3, \ldots, p_n$ and mapping to λ_A under ft (hence a curve C that contributes to the left hand side of equation 3 with multiplicity $\text{mult}_{\text{ev} \times \text{ft}}(C)$). We would like to generalize proposition 5.1 of [6], which states that C has a contracted bounded edge. However, this is not true in the case of $\mathbb{F}_2$. We can have curves like the one shown in the following picture (where the length l is very large) which do allow a very large $\mathcal{M}_{0,4}$-coordinate.



Even though those curves fail to have a contracted bounded edge, we can still interpret them as reducible curves by cutting off the part which is far away to the right (in the picture denoted by S). The remaining part (in the picture denoted by C') is a reducible curve of degree $\Delta_{\mathbb{F}_2}(a-1, b+2)$. The existence of such curves with a very large $\mathcal{M}_{0,4}$-coordinate leads to the second part of the sum in the recursion formula of theorem 2.12. The part S which is far away to the right is called a *string* following [6], definition 3.5.

Lemma 2.10
Using the notations from 2.1, let $C \in \mathcal{M}_{0,n}^{lab}(\mathbb{R}^2, \Delta)$ be a tropical curve that passes through L_1, L_2, p_3, …, p_n, maps to λ_A under ft *and has a non-zero multiplicity* $\text{mult}_{\text{ev} \times \text{ft}}(C)$. *Then* ***either***

(1) *C has a contracted bounded edge* ***or***
(2) *C contains a string*



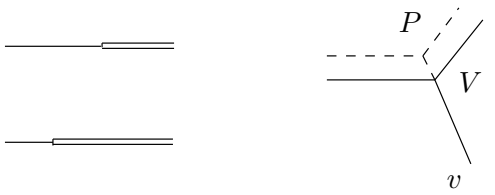
(see remark 2.9) that can be moved to the right.

Proof:
The beginning of the proof is similar to proposition 5.1 of [6].

We will show that the set of all points $\text{ft}(C)$ is bounded in $\mathcal{M}_{0,4}$ where C runs over all curves $C \in \mathcal{M}_{0,n}^{\text{lab}}(\mathbb{R}^2, \Delta)$ with non-zero multiplicities $\text{mult}_{\text{ev} \times \text{ft}}(C)$ that satisfy the conditions but have no contracted bounded edge and no string moving to the right as in the picture. By proposition 2.11 of [10] there are only finitely many

combinatorial types in $\mathcal{M}^{\mathrm{lab}}_{0,n}(\mathbb{R}^2, \Delta)$. Thus, we may restrict ourselves to tropical curves C of a fixed combinatorial type α. Furthermore, we may assume the curves corresponding to α are 3-valent.

Let C be such a curve and let C' be the curve obtained from C by forgetting the first and the second marked point. Then C' has a string Γ' which follows analogously to remark 3.7 of [6]. We claim that using the string, we can deform C' in a 1-parameter family within its combinatorial type without changing the images of the marked points. To see this, assume first that there is a vertex V contained in the string such that the directions of the adjacent edges do not span $\mathbb{R}^2$ (case (A) in the picture below). Then we can change the lengths of the adjacent edges without changing the image inside $\mathbb{R}^2$, in particular without changing the image of any marked point. Next assume that there is no such vertex contained in the string (case (B)). Then we can take one of the ends of the string (which is necessarily non-contracted) and move it slightly in a non-zero direction modulo its linear span. Consider the next vertex V and let v be the adjacent edge not contained in the string. Then v is non-contracted and our moved end will meet the affine span of v at some point P. So we change the length of v such that it ends at P (while keeping the position of its second vertex fixed). Then we also move the second edge of the string to P and go on to the next vertex. Continuing like this, we produce a 1-dimensional deformation of C' that keeps the images of the marked points fixed.

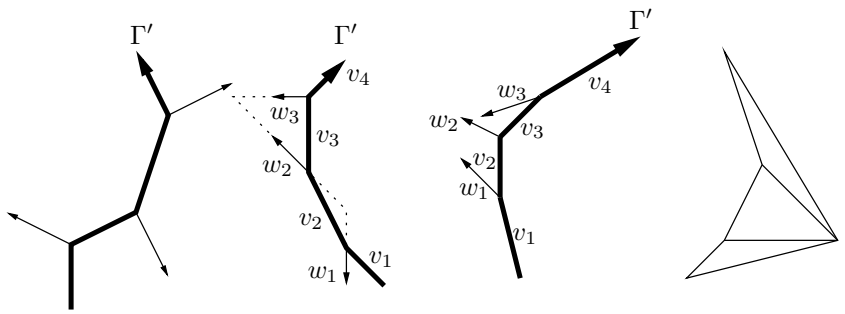


(A) (B)

Assume we could deform C' in a more than 1-dimensional family while keeping the images of the marked points fixed. Then C' moves in an at least 1-dimensional family with the image point under all evaluations and the forgetful map fixed. Then $\mathrm{mult}_{\mathrm{ev}\times\mathrm{ft}}(C) = 0$, which is a contradiction to our assumption. In particular, we can see that we cannot have more than one string. Note that the edges adjacent to Γ' must be bounded since otherwise we would have two strings.

Now we show that the 1-dimensional deformation of C' is either bounded itself or does not affect the image under ft. From this, the statement follows.

First assume that there are bounded edges adjacent to Γ' to both sides of Γ' as shown in (i). Then the deformation of C' with the combinatorial type and the conditions fixed are bounded to both sides. This means that the lengths of all inner edges are bounded except possibly the edges adjacent to x_1 and x_2. This is sufficient to ensure that the image of these curves under ft is bounded in $\mathcal{M}_{0,4}$, too.

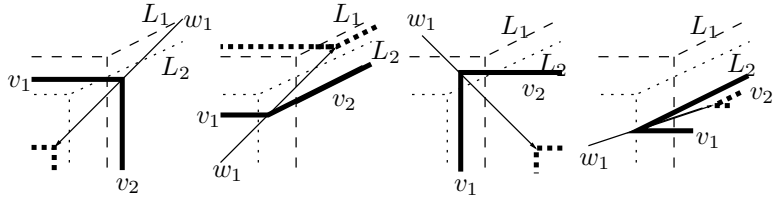


(i) (ii) (iii) (iv)

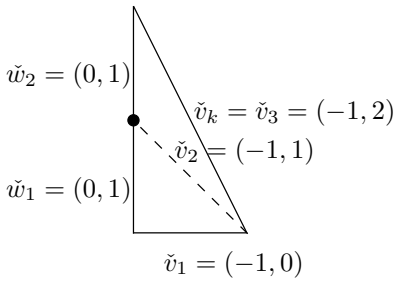
Now assume that all bounded edges adjacent to Γ' are on one side of Γ' (say after picking an orientation of Γ' on the left side). Denote the direction vectors of the edges of Γ' by $v_1, \dots, v_k$ and the direction vectors of the adjacent bounded edges by $w_1, \dots, w_{k-1}$. As above, the movement of Γ' to the left with the combinatorial type and the conditions fixed is bounded. If one of the directions w_{i+1} is obtained from w_i by a right turn, then the edges corresponding to w_i and w_{i+1} meet to the right of Γ' as shown in (ii). This restricts the movement of Γ' to the right with the combinatorial type and the conditions fixed, too, since the edge corresponding to v_{i+1} then receives length 0. Hence, as above, the image of these plane tropical curves under ft is bounded in $\mathcal{M}_{0,4}$ as well. Thus, we may assume that for all i, $1 \leq i \leq k-2$ the direction w_{i+1} is either the same as w_i or obtained from w_i by a left turn as shown in (iii). The balancing condition then ensures that for all i both the directions v_{i+1} and $-w_{i+1}$ lie in the angle between v_i and $-w_i$. Therefore, all directions v_i and $-w_i$ lie in the angle between v_1 and $-w_1$. In particular, the string Γ' cannot have any self-intersections in $\mathbb{R}^2$. We can therefore pass to the local dual picture where the edges dual to w_i correspond to a concave side of a polygon whose other two edges are dual to v_1 and v_k as shown in (iv). But note that both v_1 and v_k are outer directions of a plane tropical curve of degree Δ. Thus, v_1 and v_k must be $\binom{-1}{0}$, $\binom{0}{-1}$, $\binom{1}{0}$ or $\binom{2}{1}$. Consequently, their dual edges have direction vectors $\pm\binom{0}{-1}$, $\pm\binom{1}{0}$, $\pm\binom{0}{1}$ or $\pm\binom{-1}{2}$. We have to distinguish two cases

(a) v_1 and v_k are $\binom{0}{-1}$ and $\binom{2}{1}$, i.e. their dual edges have direction vectors $\pm(-1,0)$ and $\pm(-1,2)$
(b) v_1 and v_k are not $\binom{0}{-1}$ and $\binom{2}{1}$.

In case (b) the triangles spanned by two of those vectors do not admit any further integer points. Therefore we have $k = 2$ and the string consist just of the two unbounded edges corresponding to v_1 and v_2 that are connected to the rest of the plane tropical curve by exactly one internal edge corresponding to w_1. It remains to show that for all possibilities for v_1 and v_2 in case (b) the union of the corresponding edges finally becomes disjoint from at least one of the chosen curves L_1 and L_2 as the length of the edge corresponding to w_1 increases. This can be proved by a case-by-case analysis as shown in the following picture:



In case (a) the triangle spanned by the two vectors $\binom{-1}{0}$ and $\binom{-1}{2}$ admits exactly one further integer point.



In the picture, we denote the duals of the vectors v_i and w_i by $\check{v}_i$ and $\check{w}_i$. Thus, in case (a) we may have $k = 3$ and the string Γ' may consist of the two unbounded edges corresponding to v_1 and v_3 and the bounded edge corresponding to v_2 that is connected to the rest of the plane tropical curve by the two edges corresponding to w_1 and w_2. In this case, the movement of the string is indeed not bounded to the right. Then we are in case (2) of lemma 2.10. This finishes the proof of the lemma. □

Lemma 2.11
Let C be a curve of type (2) of lemma 2.10 then

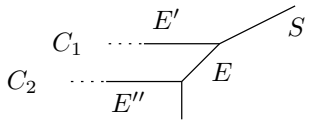
$$\mathrm{mult}_{\mathrm{ev}\times\mathrm{ft}}(C) = \mathrm{mult}_{\mathrm{ev}_1}(C_1)\cdot\mathrm{mult}_{\mathrm{ev}_2}(C_2)\cdot 2\cdot(C_1\cdot L_1)_{x_1}\cdot(C_1\cdot L_2)_{x_2}$$

where $\mathrm{mult}_{\mathrm{ev}_i}(C_i)$ denotes the multiplicity of the evaluation map at the $\#\Delta_i - 1$ points of $x_3,\dots,x_n$ that lie on C_i for $i \in \{1,2\}$ and $(C'\cdot C'')_p$ denotes the intersection multiplicity of the plane tropical curves C' and C'' at the point $p \in C'\cap C''$. Here, C_1 and C_2 denote the two irreducible components of the part C' of C that we get when cutting off the string S as in remark 2.9.

Proof:
Since $\mathrm{mult}_{\mathrm{ev}\times\mathrm{ft}}(C)$ equals the absolute value of the determinant of the map $\mathrm{ev}\times\mathrm{ft}$ in local coordinates, we set up the matrix A for $\mathrm{ev}\times\mathrm{ft}$ as in lemma 2.4 and compute its determinant. The local coordinates are the position of a root vertex and the length of all bounded edges, respectively the coordinates of the images of the contracted edges and the length coordinate of the bounded edge of the image under ft in $\mathcal{M}_{0,4}$. Because of remark 2.5, the absolute value of the determinant does not depend on the special choice of such coordinates.

There are exactly two bounded edges that connect the string S with the rest of the curve. We denote these bounded edges by E' and E'' and the unique bounded

edge that is contained in the string by E. Their lengths are denoted by l', l'' and l, respectively.



As the length of the $\mathcal{M}_{0,4}$-coordinate is very large and there is no contracted bounded edge, the lengths l', l and l'' must count towards the length of the $\mathcal{M}_{0,4}$-coordinate. That is, x_1 and x_2 have to be on one side of those three edges and x_3 and x_4 on the other. Let us call the part with x_1 and x_2 C_1, and assume without restriction that E' belongs to C_1. Put the root vertex on the E'-side. Then the columns of the matrix A corresponding to the lengths l' and l'' read:

	l'	l''
evaluation at a point behind E'	0	0
	0	0
evaluation at a point behind E''	1	-1
	0	0
$\mathcal{M}_{0,4}$-coordinate	1	1

If we add the column corresponding to the length l' to the column corresponding to the length l'', then the column corresponding to the length $l'+l''$ has only one entry 2 and all other entries 0. Thus, we get a factor of 2 and to compute the determinant of the matrix A we may drop both the $\mathcal{M}_{0,4}$-row and the column corresponding to the edge E''.

Now, we consider the first marked point x_1. We require that the plane tropical curve C passes through L_1 at this point. Let E_1 and E_2 be the two adjacent edges of x_1. We denote their common direction vector by $v = \binom{v_1}{v_2}$ and their lengths by l_1 and l_2, respectively. We may assume that the root vertex is on the E_1-side of x_1. Assume that both E_1 and E_2 are bounded. If x_1 is contracted to a point on an unbounded edge of L_1 with direction vector $\binom{u_1}{u_2}$, then the columns of the matrix corresponding to l_1 and l_2, respectively, read

evaluation at ...	l_1	l_2
... x_1	$\lvert u_2v_1 - u_1v_2\rvert$	0
... a point reached via E_1 from x_1	0	0
... a point reached via E_2 from x_1	v	v

Note that there are two rows for each marked point $x_3, \ldots, x_n$ that is reached via E_1 or E_2 from x_1 and there is one row for the marked point x_2. If we subtract the column corresponding to l_2 from the column corresponding to l_1, then we obtain a column with only one non-zero entry. So for the determinant we get $(C_1 \cdot L_1)_{x_1}$ as a factor and may drop both the row corresponding to x_1 and the column corresponding to l_1. Now assume that one of the edges E_1 and E_2 is unbounded. Assume that E_1 is bounded and E_2 is unbounded. Then there is a column corresponding to l_1 but no column corresponding to l_2. The column corresponding to l_1 has only one non-zero entry and the same argument as above

holds. Note that it is not possible that both E_1 and E_2 are unbounded. Taking the factor $(C_1 \cdot L_1)_{x_1}$ into account, we can now forget the marked point x_1 and straighten the 2-valent vertex to produce only one bounded edge out of E_1 and E_2.

If we forget the marked point x_2, we obtain a factor of $(C_1 \cdot L_2)_{x_2}$ in the same way.

Now, we consider again the string S. Remember that we split up the plane tropical curve C at this string into the two parts C_1 and C_2. We choose the boundary vertex of the bounded edge E' at the C_1-side as root vertex and denote it by V. Then the matrix reads

evaluation at a ...	root	lengths in C_1	l'	l	lengths in C_2
... point behind E'	I_2	$*$	0	0	0
... point behind E''	I_2	0	$\binom{1}{0}$	$\binom{-1}{-1}$	$*$

where I_2 denotes the 2 by 2 unit matrix. Assume there are n_1 marked points besides x_1 and x_2 on C_1 and n_2 marked points besides x_3 and x_4 on C_2, where $n_1 + n_2 = n - 4 = \#\Delta - 4 = 4a + 2b - 4$. Assume the degree of C_1 is $\Delta_{\mathbb{F}_2}(a_1, b_1)$ and the degree of C_2 is $\Delta_{\mathbb{F}_2}(a_2, b_2)$. Then $a_1 + a_2 = a - 1$ and $b_1 + b_2 = b + 2$ as we observed in remark 2.9. Since $\mathrm{mult}_{\mathrm{ev} \times \mathrm{ft}}(C) \neq 0$ we must have $n_1 = 4a_1 + 2b_1 - 1$ and $n_2 + 2 = 4a_2 + 2b_2 - 1$ (because then the curves will be fixed by the points). Thus (after forgetting x_1 and x_2) C_1 has $n_1 + 4a_1 + 2b_1 = 2n_1 + 1$ unbounded edges and thus $2n_1 - 2$ bounded edges. Hence $2n_1 - 2$ length coordinates belong to bounded edges in C_1. C_2 has $n_2 + 2 + 4a_2 + 2b_2 = 2n_2 + 5$ unbounded edges and thus $2n_2 + 2$ length coordinates belong to C_2. Furthermore, there are n_1 points behind E' and there are $n_2 + 2$ points behind E''.

If we add the l'-column to the l-column and then multiply the l-column by -1, then we obtain the following matrix whose determinant has the same absolute value as the determinant that we are looking for.

evaluation at a ...	root	lengths in C_1	l'	l	lengths in C_2
... point behind E'	I_2	$*$	0	0	0
... point behind E''	I_2	0	$\binom{1}{0}$	$\binom{0}{1}$	$*$

Note that this matrix has a block form. The block at the top right is a zero block. We denote the top left block of size $2n_1$ by A_1 and the bottom right block of size $2n_2+4$ by A_2. Then, the determinant that we are looking for is $|\det(A_1)| \cdot |\det(A_2)|$.

But the matrix A_1 is the matrix of evaluation at marked points in C_1 and A_2 is the matrix of evaluation at marked points in C_2. Thus, we have $|\det(A_1)| = \mathrm{mult}_{\mathrm{ev}_1}(C_1)$ and $|\det(A_2)| = \mathrm{mult}_{\mathrm{ev}_2}(C_2)$.

Together, we have $\mathrm{mult}_{\mathrm{ev} \times \mathrm{ft}}(C) = \mathrm{mult}_{\mathrm{ev}_1}(C_1) \cdot \mathrm{mult}_{\mathrm{ev}_2}(C_2) \cdot 2 \cdot (C_1 \cdot L_1)_{x_1} \cdot (C_1 \cdot L_2)_{x_2}$. □

We are now ready to prove the main result of this section:

Theorem 2.12 (tropical Kontsevich formula for $\mathbb{F}_2$)
Let a and b be non-negative integers with $a+b \geq 1$. Then

$$\begin{aligned}\mathcal{N}^0_{\mathbb{F}_2}(a,b) = &\frac{1}{2}\sum \phi_{2_1}(a_1,b_1)\mathcal{N}^0_{\mathbb{F}_2}(a_1,b_1)\mathcal{N}^0_{\mathbb{F}_2}(a_2,b_2)\\ &+\frac{1}{2}\sum \phi_{2_2}(a_1,b_1)\mathcal{N}^0_{\mathbb{F}_2}(a_1,b_1)\mathcal{N}^0_{\mathbb{F}_2}(a_2,b_2)\end{aligned}$$

where the first sum goes over all (a_1,b_1) and (a_2,b_2) satisfying

$$(a_1,b_1)+(a_2,b_2)=(a,b),$$

$0 \leq a_1 \leq a$, $0 \leq b_1 \leq b$ and $(0,0) \neq (a_1,b_1) \neq (a,b)$, and the second sum goes over all (a_1,b_1) and (a_2,b_2) satisfying

$$(a_1,b_1)+(a_2,b_2)=(a-1,b+2),$$

$0 \leq a_1 \leq a-1$ and $0 < b_1 < b+2$. We use the shortcuts $\phi_{2_1}(a_1,b_1)$ for

$$\begin{aligned}\phi_{2_1}(a_1,b_1) = &(2a_1+b_1)(2a_2+b_2)(a_1b_2+a_2b_1+2a_1a_2)\binom{4a+2b-4}{4a_1+2b_1-2}\\ &-(2a_1+b_1)(2a_1+b_1)(a_1b_2+a_2b_1+2a_1a_2)\binom{4a+2b-4}{4a_1+2b_1-1}\end{aligned} \tag{4}$$

and $\phi_{2_2}(a_1,b_1)$ for

$$\begin{aligned}\phi_{2_2}(a_1,b_1) = &2(2a_1+b_1)(2a_2+b_2)(b_1b_2)\binom{4a+2b-4}{4a_1+2b_1-2}\\ &-2(2a_1+b_1)(2a_1+b_1)(b_1b_2)\binom{4a+2b-4}{4a_1+2b_1-1}.\end{aligned} \tag{5}$$

Proof:
Let C be a curve passing through L_1, L_2, $p_3,\ldots,p_n$ and mapping to λ_A under ft, i.e. a curve that contributes to the left hand side of equation 3 because of remark 2.3. By lemma 2.4 and notation 2.7, it has to be counted with multiplicity $\mathrm{mult}_{\mathrm{ev}\times\mathrm{ft}}(C)$. We will show that C can be interpreted as a reducible curve, and that its multiplicity $\mathrm{mult}_{\mathrm{ev}\times\mathrm{ft}}(C)$ can be split into factors corresponding to the irreducible components.

Because of lemma 2.10 we know that C either has a contracted bounded edge or a string that can be moved to the right as in remark 2.9.

If it has a contracted bounded edge, then it is possible that this edge is adjacent to the marked ends x_1 and x_2. Then the two marked ends are contracted to the same point in $\mathbb{R}^2$, which has to be an intersection point of L_1 and L_2. Let us call this point p. Let C' denote the curve that arises after forgetting the marked point x_1. Analogously to 5.5.a) of [6] we can show that $\mathrm{mult}_{\mathrm{ev}\times\mathrm{ft}}(C) = \mathrm{mult}_{\mathrm{ev}'}(C')\cdot(L_1.L_2)_p$, where ev' now denotes the evaluation of x_2 at a point combined with all other point evaluations and $(L_1.L_2)_p$ denotes the intersection product of L_1 and L_2 at p. Instead of counting those curves C, we can hence count curves C' meeting the points $p_3,\ldots,p_n$ and an intersection point of L_1 and L_2. Since $(L_1.L_2)=2$ by the tropical Bézout's theorem (4.2 of [11]), we can conclude that those curves C contribute $2\mathcal{N}^0_{\mathbb{F}_2}(a,b)$ to the left hand side of equation 3.

If x_1 and x_2 are not adjacent to the contracted bounded edge, then there have to be bounded edges to both sides of the contracted bounded edge, since all other marked

points have to meet different points. If there are bounded edges on both sides of the contracted bounded edge, we can cut the bounded edge thus producing a reducible curve with two new contracted ends z_1 and z_2. Let us call the two components C_1 and C_2. Since C maps to λ_A under ft, x_1 and x_2 have to be on C_1. Let us call the degree of C_1 $\Delta_{\mathbb{F}_2}(a_1, b_1)$ and the degree of C_2 $\Delta_{\mathbb{F}_2}(a_2, b_2)$, then we must have $(a_1, b_1) + (a_2, b_2) = (a, b)$, $0 \leq a_1 \leq a$, $0 \leq b_1 \leq b$ and $(0, 0) \neq (a_1, b_1) \neq (a, b)$. Analogously to 5.5.b) of [6], we can forget x_1 and x_2 thus producing a factor of $(C_1.L_1)_{x_1} \cdot (C_1.L_2)_{x_2}$. (By abuse of notation, we use the x_i here for the point in $\mathbb{R}^2$ to which the end x_i is contracted to.) With the same arguments as in 5.5.b) of [6], we can see that

$$\mathrm{mult}_{\mathrm{ev}\,\times\,\mathrm{ft}}(C) = \mathrm{mult}_{\mathrm{ev}_1}(C_1)\,\mathrm{mult}_{\mathrm{ev}_2}(C_2)(C_1 \cdot C_2)_{z_1=z_2}(C_1.L_1)_{x_1} \cdot (C_1.L_2)_{x_2},$$

where $\mathrm{mult}_{\mathrm{ev}_1}(C_1)$ denotes the multiplicity of the evaluation at the points on C_1. Note that $4a_1+2b_1-1$ of the other marked points have to be on C_1. Now instead of counting the curves C with a contracted bounded edge and bounded edges on both sides, we can pick $4a_1+2b_1-1$ of the points $p_5, \ldots, p_n$ ($\binom{4a+2b-4}{4a_1+2b_1-1}$ possibilities) and count curves C_1 through those points, and C_2 through the remaining points. Again by tropical Bézout's theorem we have $(C_1.L_1) = (2a_1+b_1)$ choices to attach x_1 and $(C_1.L_2) = (2a_1 + b_1)$ choices to attach x_2, and we have $(C_1.C_2) = (a_1b_2 + a_2b_1 + 2a_1a_2)$ choices to glue C_1 and C_2 to a possible C. Thus those curves contribute

$$\sum (2a_1+b_1)(2a_1+b_1)(a_1b_2+a_2b_1+2a_1a_2)\binom{4a+2b-4}{4a_1+2b_1-1}\mathcal{N}^0_{\mathbb{F}_2}(a_1, b_1)\mathcal{N}^0_{\mathbb{F}_2}(a_2, b_2),$$

where the sum goes over all $(a_1, b_1) + (a_2, b_2) = (a, b)$, $0 \leq a_1 \leq a$, $0 \leq b_1 \leq b$ and $(0, 0) \neq (a_1, b_1) \neq (a, b)$. In the formula we want to prove, we can see this contribution negatively on the right hand side.

Finally, if C has a string as in remark 2.9, then by lemma 2.11 we can conclude that C contributes

$$\mathrm{mult}_{\mathrm{ev}\,\times\,\mathrm{ft}}(C) = \mathrm{mult}_{\mathrm{ev}_1}(C_1) \cdot \mathrm{mult}_{\mathrm{ev}_2}(C_2) \cdot 2 \cdot (C_1 \cdot L_1)_{x_1} \cdot (C_1 \cdot L_2)_{x_2}.$$

Instead of counting such curves C, we can pick $4a_1+2b_1-1$ of the points $p_5, \ldots, p_n$ ($\binom{4a+2b-4}{4a_1+2b_1-1}$ possibilities) and count curves C_1 of degree $\Delta_{\mathbb{F}_2}(a_1, b_1)$ through those points, and C_2 of degree $\Delta_{\mathbb{F}_2}(a_2, b_2)$ through the remaining points, where now $(a_1, b_1) + (a_2, b_2) = (a-1, b+2)$. There are again $(2a_1 + b_1)$ possibilities to attach x_1 and also $(2a_1 + b_1)$ possibilities to attach x_2 to C_1. There are b_1b_2 choices to pick the edges E' and E'' to which we can attach the string S. Hence those curves contribute

$$\sum 2(2a_1+b_1)(2a_1+b_1)(b_1b_2)\binom{4a+2b-4}{4a_1+2b_1-1}\mathcal{N}^0_{\mathbb{F}_2}(a_1, b_1)\mathcal{N}^0_{\mathbb{F}_2}(a_2, b_2),$$

where now the sum goes over all $(a_1, b_1) + (a_2, b_2) = (a-1, b+2)$, $0 \leq a_1 \leq a-1$, $0 < b_1 < b+2$ and $(0, 0) \neq (a_1, b_1) \neq (a, b)$. In the formula we want to prove, this contribution appears negatively on the right hand side.

Performing the same analysis for the right hand side of equation 3 and collecting the terms to the different sides, the statement follows.

□

The following formula for $\mathbb{F}_0$ can be proved analogously. The proof is easier in fact, since all curves have a contracted bounded edge and the special case of curves having a string that can be moved to the right as in remark 2.9 does not occur here. We therefore skip the proof. For more details, see [3].

Theorem 2.13 (tropical Kontsevich formula for $\mathbb{F}_0$)
Let a and b be non-negative integers with $a+b \geq 1$. Then

$$\mathcal{N}^0_{\mathbb{F}_0}(a, a+b) = \frac{1}{2} \sum \phi_0(a_1, a_1+b_1)\mathcal{N}^0_{\mathbb{F}_0}(a_1, a_1+b_1)\mathcal{N}^0_{\mathbb{F}_0}(a_2, a_2+b_2)$$

where the sum goes over all (a_1, b_1) and (a_2, b_2) satisfying

$$(a_1, a_1+b_1) + (a_2, a_2+b_2) = (a, a+b),$$

$0 \leq a_1 \leq a$, $-a_1 \leq b_1 \leq b+a_2$ *and* $(0,0) \neq (a_1, a_1+b_1) \neq (a, a+b)$. *We use the shortcut $\phi_0(a_1, a_1+b_1)$ for*

$$\begin{aligned}\phi_0(a_1, a_1+b_1) = {} & (2a_1+b_1)(2a_2+b_2)(a_1b_2+a_2b_1+2a_1a_2)\binom{4a+2b-4}{4a_1+2b_1-2} \\ & -(2a_1+b_1)(2a_1+b_1)(a_1b_2+a_2b_1+2a_1a_2)\binom{4a+2b-4}{4a_1+2b_1-1}.\end{aligned} \tag{6}$$

3. The proof for nonnegative a, b with $a+b \geq 1$ and $g = 0$

Recall the equation from theorem 1.1 we want to prove. Note that for $k = a$ we would obtain a summand $\binom{b+2a}{a}\mathcal{N}^0_{\mathbb{F}_2}(0, (b+2a))$ which is 0 for all a, $b \in \mathbb{Z}_{\geq 0}$ with $a+b \geq 1$ except for $a = 0$ and $b = 1$. As in this special case the statement still holds, we may add this summand for $k = a$ and deal with the slightly modified equation

$$\mathcal{N}^0_{\mathbb{F}_0}(a, a+b) = \sum_{k=0}^{a} \binom{b+2k}{k} \mathcal{N}^0_{\mathbb{F}_2}(a-k, b+2k).$$

We will see later that this is useful.

Example 3.1
Let us consider the formula for small a in more detail. For $a = 0$ and $a = 1$ we have $\mathcal{N}^0_{\mathbb{F}_0}(a, a+b) = \mathcal{N}^0_{\mathbb{F}_2}(a, b)$ for all $b \in \mathbb{Z}_{\geq 0}$. Hence, in these cases the Gromov-Witten invariants are enumerative for $\mathbb{F}_2$ as well. For $a = 2$ and $b = 0$ we have $\mathcal{N}^0_{\mathbb{F}_2}(2, 0) = 10$ while the associated Gromov-Witten invariant is $\mathcal{N}^0_{\mathbb{F}_0}(2, 2) = 12$. This is the first interesting case. The formula gives a interpretation of the difference 2 in terms of a deformation of $\mathbb{F}_2$:

$$\mathcal{N}^0_{\mathbb{F}_0}(2,2) = 12 = 10 + 2 \times 1 = \mathcal{N}^0_{\mathbb{F}_2}(2,0) + \binom{2}{1}\mathcal{N}^0_{\mathbb{F}_2}(1,2) + \underbrace{\binom{4}{2}\mathcal{N}^0_{\mathbb{F}_2}(0,4)}_{=0}.$$

We need the following combinatorial indentity involving binomial coefficients for our proof.

Lemma 3.2
Let n, m, $k \in \mathbb{N}$. Then

$$\sum_{i=0}^{k}(mi+n(k-i)-2i(k-i))\binom{n}{i}\binom{m}{k-i} = 2\cdot n\cdot m\cdot \binom{n+m-2}{k-1}.$$

Proof:

The equation is essentially a consequence of Vandermonde's identity which states that

$$\sum_{i=0}^{k}\binom{n}{i}\binom{m}{k-i} = \binom{n+m}{k}.$$

Using this we have

$$\sum_{i=0}^{k} mi\binom{n}{i}\binom{m}{k-i} = \sum_{i=0}^{k} nm\binom{n-1}{i-1}\binom{m}{k-i} = nm\binom{n+m-1}{k-1}$$

and

$$\sum_{i=0}^{k} n(k-i)\binom{n}{i}\binom{m}{k-i} = \sum_{i=0}^{k} nm\binom{n}{i}\binom{m-1}{k-i-1} = nm\binom{n+m-1}{k-1}$$

and

$$\begin{aligned}\sum_{i=0}^{k}(-2i(k-i))\binom{n}{i}\binom{m}{k-i} &= -2\sum_{i=0}^{k} nm\binom{n-1}{i-1}\binom{m-1}{k-i-1}\\ &= -2nm\binom{n+m-2}{k-2}\end{aligned}$$

as $i\binom{n}{i} = n\binom{n-1}{i-1}$ and $(k-i)\binom{m}{k-i} = m\binom{m-1}{k-i-1}$ and thus

$$\begin{aligned}&\sum_{i=0}^{k}(mi+n(k-i)-2i(k-i))\binom{n}{i}\binom{m}{k-i}\\ &\quad= 2nm\left(\binom{n+m-1}{k-1}-\binom{n+m-2}{k-2}\right)\\ &\quad= 2nm\binom{n+m-2}{k-1}\end{aligned}$$

where the last equality follows by Pascal's rule.

□

Proof of theorem 1.1 for non-negative a, b with $a+b\geq 1$ and $g=0$:
We will prove that the statement holds for all integers $a \geq 0$ and $b \geq -a$ with $2a+b \geq 1$ by induction on $2a+b$. Note that b may be negative. But as $b \geq -a$ and $2a+b \geq 1$ we have $a+b \geq 0$ and hence the left hand side is well defined. On the right hand side we may have negative entries. We define $\mathcal{N}^0_{\mathbb{F}_2}(a,b)$ to be 0 for all $b<0$, $a \geq 0$. In particular, we conclude that the statement holds for all non-negative integers a and b with $a+b \geq 1$.

The induction beginning for $a = 0$ and $b = 1$ resp. $a = 1$ and $b = -1$ is straight forward, we need to use the extra summand with $k = a$ however.

Now let $a \geq 0$ and $b \geq -a$ be integers such that $2a + b \geq 1$.

We can assume that the statement holds for all integers $a_i \geq 0$ and $b_i \geq -a_i$ with $1 \leq 2a_i + b_i < 2a + b$.

First let us consider the left hand side. We know by theorem 2.13 that

$$2\mathcal{N}^0_{\mathbb{F}_0}(a, a+b) = \sum \phi_0(a_1, a_1+b_1)\mathcal{N}^0_{\mathbb{F}_0}(a_1, a_1+b_1)\mathcal{N}^0_{\mathbb{F}_0}(a_2, a_2+b_2)$$

where the sum goes over all (a_1, b_1) and (a_2, b_2) satisfying $(a_1, a_1+b_1)+(a_2, a_2+b_2) = (a, a+b)$, $0 \leq a_1 \leq a$, $-a_1 \leq b_1 \leq b+a_2$ and $(0,0) \neq (a_1, a_1+b_1) \neq (a, a+b)$, and $\phi_0(a_1, a_1+b_1)$ is defined by equation 6.

As $2a_1 + b_1 < 2a + b$ and $2a_2 + b_2 < 2a + b$ for all a_1, a_2, b_1 and b_2 we have by the induction hypothesis that $\mathcal{N}^0_{\mathbb{F}_0}(a_1, a_1+b_1) = \sum_{i=0}^{a_1} \binom{b_1+2i}{i}\mathcal{N}^0_{\mathbb{F}_2}(a_1-i, b_1+2i)$ and $\mathcal{N}^0_{\mathbb{F}_0}(a_2, a_2+b_2) = \sum_{j=0}^{a_2} \binom{b_2+2j}{j}\mathcal{N}^0_{\mathbb{F}_2}(a_2-j, b_2+2j)$. Hence we have

$$\begin{aligned}
2\mathcal{N}^0_{\mathbb{F}_0}(a, a+b) = \sum \phi_0(a_1, a_1+b_1) \cdot &\left(\sum_{i=0}^{a_1} \binom{b_1+2i}{i}\mathcal{N}^0_{\mathbb{F}_2}(a_1-i, b_1+2i)\right) \\
&\cdot \left(\sum_{j=0}^{a_2} \binom{b_2+2j}{j}\mathcal{N}^0_{\mathbb{F}_2}(a_2-j, b_2+2j)\right) \\
= \sum_{k=0}^{a} \sum_{\substack{i=0 \\ j=k-i}}^{k} \sum &\left(\binom{b_1+2i}{i}\binom{b_2+2j}{j}\phi_0(a_1, a_1+b_1)\right. \\
&\left. \cdot \mathcal{N}^0_{\mathbb{F}_2}(a_1-i, b_1+2i)\mathcal{N}^0_{\mathbb{F}_2}(a_2-j, b_2+2j)\right)
\end{aligned}$$

Let us consider the range of a_1 and b_1 in the third sum for a fixed k and i. As $\mathcal{N}^0_{\mathbb{F}_2}(a_1-i, b_1+2i) = 0$ for all $0 \leq a_1 < i$ and $\mathcal{N}^0_{\mathbb{F}_2}(a_2-j, b_2+2j) = 0$ for all $0 \leq a_2 < j$ (i.e. for all $a - 0 = a \geq a - a_2 = a_1 > a - j = a - k + i$) we can forget about the summands where $0 \leq a_1 < i$ or $a - k + i < a_1 \leq a$. As $\binom{b_1+2i}{i} = 0$ for all $b_1 < -i$ and $\binom{b_2+2j}{j} = 0$ for all $b_2 < -j$ (i.e. for all $b - b_2 = b_1 > b + j$) the range of those b_1 which give a contribution is $-i \leq b_1 \leq b + j$. We may add summands for $-2i \leq b_1 < -i$ and $b + j < b_1 \leq b + 2j$ since they are 0 anyway. Hence we may restrict our attention to those $(a_1, a_1+b_1)+(a_2, a_2+b_2) = (a, a+b)$ with $i \leq a_1 \leq a-k+i$ and $-2i \leq b_1 \leq b+2j$ such that $(0,0) \neq (a_1, a_1+b_1) \neq (a, a+b)$. With the definitions

$$\begin{array}{ll} a_1' := a_1 - i & a_2' := (a-k) - a_1' = a_2 - j \\ b_1' := b_1 + 2i & b_2' := (b+2k) - b_1' = b_2 + 2j \end{array}$$

this is equivalent to considering all pairs

$$(a_1', b_1') + (a_2', b_2') = (a-k, b+2k)$$

with

$$0 \leq a_1' \leq a-k \text{ and } 0 \leq b_1' \leq b+2k \text{ such that } (0,0) \neq (a_1', b_1') \neq (a, b).$$

Let the sums in the following equation go over those pairs now. Then

$$\begin{aligned}2\mathcal{N}^0_{\mathbb{F}_0}(a,a+b) &= \sum_{k=0}^{a}\sum_{\substack{i=0\\ j=k-i}}^{k}\sum\Bigg(\binom{b_1'}{i}\binom{b_2'}{j}\phi_0(a_1'+i,a_1'+b_1'-i)\\ &\qquad\cdot\mathcal{N}^0_{\mathbb{F}_2}(a_1',b_1')\mathcal{N}^0_{\mathbb{F}_2}(a_2',b_2')\Bigg)\\ &= \sum_{k=0}^{a}\sum\Bigg(\sum_{\substack{i=0\\ j=k-i}}^{k}\binom{b_1'}{i}\binom{b_2'}{j}\phi_0(a_1'+i,a_1'+b_1'-i)\\ &\qquad\cdot\mathcal{N}^0_{\mathbb{F}_2}(a_1',b_1')\mathcal{N}^0_{\mathbb{F}_2}(a_2',b_2')\Bigg)\end{aligned}$$

where

$$\begin{aligned}&\phi_0(a_1'+i,a_1'+b_1'-i)\\ &\quad=(2a_1'+b_1')(2a_2'+b_2')(a_1'b_2'+a_2'b_1'+2a_1'a_2'\\ &\qquad\qquad+b_2'i+b_1'j-2ij)\binom{4a+2b-4}{4a_1'+2b_1'-2}\\ &\quad-(2a_1'+b_1')(2a_1'+b_1')(a_1'b_2'+a_2'b_1'+2a_1'a_2'\\ &\qquad\qquad+b_2'i+b_1'j-2ij)\binom{4a+2b-4}{4a_1'+2b_1'-1}\\ &\quad=(2a_1'+b_1')(2a_2'+b_2')(a_1'b_2'+a_2'b_1'+2a_1'a_2')\binom{4a+2b-4}{4a_1'+2b_1'-2}\\ &\qquad\qquad-(2a_1'+b_1')(2a_1'+b_1')(a_1'b_2'+a_2'b_1'+2a_1'a_2')\binom{4a+2b-4}{4a_1'+2b_1'-1}\\ &\quad+(2a_1'+b_1')(2a_2'+b_2')(b_2'i+b_1'j-2ij)\binom{4a+2b-4}{4a_1'+2b_1'-2}\\ &\qquad\qquad-(2a_1'+b_1')(2a_1'+b_1')(b_2'i+b_1'j-2ij)\binom{4a+2b-4}{4a_1'+2b_1'-1}\end{aligned}$$

by equation (6) in theorem 2.13.

Let us stop here and consider the right hand side of the equation we want to prove. We have by Theorem 2.12

$$\begin{aligned}&\sum_{k=0}^{a}\binom{b+2k}{k}2\mathcal{N}^0_{\mathbb{F}_2}(a-k,b+2k)\\ &=\sum_{k=0}^{a}\binom{b+2k}{k}\cdot\Big(\sum\phi_{2_1}(a_1,b_1)\mathcal{N}^0_{\mathbb{F}_2}(a_1,b_1)\mathcal{N}^0_{\mathbb{F}_2}(a_2,b_2)\\ &\qquad\qquad+\sum\phi_{2_2}(a_1,b_1)\mathcal{N}^0_{\mathbb{F}_2}(a_1,b_1)\mathcal{N}^0_{\mathbb{F}_2}(a_2,b_2)\Big)\end{aligned}$$

where the first sum goes over all pairs such that $(a_1,b_1)+(a_2,b_2)=(a-k,b+2k)$ and the second sum goes over all pairs such that $(a_1,b_1)+(a_2,b_2)=(a-(k+1),b+2(k+1))$. We use the shortcuts $\phi_{2_1}(a_1,b_1)$ and $\phi_{2_2}(a_1,b_1)$ as defined in equation 4 and 5.

Since for $k=0$ the binomial coefficient $\binom{b+2(k-1)}{k-1}$ is 0 and for $k=a$ there are no a_1 and b_1 which satisfy $(a_1,b_1)+(a_2,b_2)=(a-(k+1),b+2(k+1))$ we can merge the two sums and get

$$\sum_{k=0}^{a}\sum\Bigg(\Bigg(\binom{b+2k}{k}\phi_{2_1}(a_1,b_1)+\binom{b+2(k-1)}{k-1}\phi_{2_2}(a_1,b_1)\Bigg)\cdot\mathcal{N}^0_{\mathbb{F}_2}(a_1,b_1)\mathcal{N}^0_{\mathbb{F}_2}(a_2,b_2)\Bigg)$$

where the sum now goes over all pairs such that $(a_1,b_1)+(a_2,b_2)=(a-k,b+2k)$.

Thus it remains to show that

$$\begin{aligned}&\sum_{k=0}^{a}\sum\Bigg(\sum_{\substack{i=0\\ j=k-i}}^{k}\binom{b_1}{i}\binom{b_2}{j}\phi_0(a_1+i,a_1+b_1-i)\Bigg)\cdot\mathcal{N}^0_{\mathbb{F}_2}(a_1,b_1)\mathcal{N}^0_{\mathbb{F}_2}(a_2,b_2)\\ =&\sum_{k=0}^{a}\sum\Bigg(\binom{b+2k}{k}\phi_{2_1}(a_1,b_1)+\binom{b+2(k-1)}{k-1}\phi_{2_2}(a_1,b_1)\Bigg)\\ &\qquad\cdot\mathcal{N}^0_{\mathbb{F}_2}(a_1,b_1)\mathcal{N}^0_{\mathbb{F}_2}(a_2,b_2).\end{aligned}$$

Therefore we will show that

$$\begin{aligned}&\sum_{\substack{i=0\\ j=k-i}}^{k}\binom{b_1}{i}\binom{b_2}{j}\phi_0(a_1+i,a_1+b_1-i)\\ =&\binom{b+2k}{k}\phi_{2_1}(a_1,b_1)+\binom{b+2(k-1)}{k-1}\phi_{2_2}(a_1,b_1)\end{aligned}$$

for all $k\in\{0,...,a\}$ and for all integers $0\le a_1,\ a_2\le a-k$, $0\le b_1,\ b_2\le b+2k$ with $a_1+a_2=a-k$, $b_1+b_2=b+2k$ and $(0,0)\neq(a_1,b_1)\neq(a-k,b+2k)$.

We use the identity from Lemma 3.2.

It is

$$\begin{aligned}&\sum_{\substack{i=0\\ j=k-i}}^{k}\binom{b_1}{i}\binom{b_2}{j}\phi_0(a_1+i,a_1+b_1-i)\\ \overset{(6)}{=}&\sum_{\substack{i=0\\ j=k-i}}^{k}\Bigg[\binom{b_1}{i}\binom{b_2}{j}\\ &\cdot\Bigg((2a_1+b_1)(2a_2+b_2)(a_1b_2+a_2b_1+2a_1a_2)\binom{4a+2b-4}{4a_1+2b_1-2}\\ &-(2a_1+b_1)(2a_1+b_1)(a_1b_2+a_2b_1+2a_1a_2)\binom{4a+2b-4}{4a_1+2b_1-1}\\ &+(2a_1+b_1)(2a_2+b_2)(b_2i+b_1j-2ij)\binom{4a+2b-4}{4a_1+2b_1-2}\\ &-(2a_1+b_1)(2a_1+b_1)(b_2i+b_1j-2ij)\binom{4a+2b-4}{4a_1+2b_1-1}\Bigg)\Bigg]\end{aligned}$$

$$
\begin{aligned}
&= \Bigg(\sum_{\substack{i=0 \\ j=k-i}}^{k} \binom{b_1}{i}\binom{b_2}{j} \Bigg) \\
&\qquad \cdot \Bigg((2a_1+b_1)(2a_2+b_2)(a_1b_2+a_2b_1+2a_1a_2)\binom{4a+2b-4}{4a_1+2b_1-2} \\
&\qquad\quad - (2a_1+b_1)(2a_1+b_1)(a_1b_2+a_2b_1+2a_1a_2)\binom{4a+2b-4}{4a_1+2b_1-1} \Bigg) \\
&\quad + \Bigg(\sum_{\substack{i=0 \\ j=k-i}}^{k} \binom{b_1}{i}\binom{b_2}{j}(b_2i+b_1j-2ij) \Bigg) \\
&\qquad \cdot \Bigg((2a_1+b_1)(2a_2+b_2)\binom{4a+2b-4}{4a_1+2b_1-2} \\
&\qquad\quad - (2a_1+b_1)(2a_1+b_1)\binom{4a+2b-4}{4a_1+2b_1-1} \Bigg) \\
&= \binom{b_1+b_2}{i+j} \Bigg((2a_1+b_1)(2a_2+b_2)(a_1b_2+a_2b_1+2a_1a_2)\binom{4a+2b-4}{4a_1+2b_1-2} \\
&\qquad\quad - (2a_1+b_1)(2a_1+b_1)(a_1b_2+a_2b_1+2a_1a_2)\binom{4a+2b-4}{4a_1+2b_1-1} \Bigg) \\
&\quad + \Bigg(2b_1b_2\binom{b_1+b_2-2}{i+j-1} \Bigg) \Bigg((2a_1+b_1)(2a_2+b_2)\binom{4a+2b-4}{4a_1+2b_1-2} \\
&\qquad\quad - (2a_1+b_1)(2a_1+b_1)\binom{4a+2b-4}{4a_1+2b_1-1} \Bigg) \\
&= \binom{b_1+b_2}{i+j} \Bigg((2a_1+b_1)(2a_2+b_2)(a_1b_2+a_2b_1+2a_1a_2)\binom{4a+2b-4}{4a_1+2b_1-2} \\
&\qquad\quad - (2a_1+b_1)(2a_1+b_1)(a_1b_2+a_2b_1+2a_1a_2)\binom{4a+2b-4}{4a_1+2b_1-1} \Bigg) \\
&\quad + \binom{b_1+b_2-2}{i+j-1} \Bigg(2(2a_1+b_1)(2a_2+b_2)b_1b_2\binom{4a+2b-4}{4a_1+2b_1-2} \\
&\qquad\quad - 2(2a_1+b_1)(2a_1+b_1)b_1b_2\binom{4a+2b-4}{4a_1+2b_1-1} \Bigg) \\
&= \binom{b+2k}{k}\phi_{2_1}(a_1,b_1) + \binom{b+2(k-1)}{k-1}\phi_{2_2}(a_1,b_1)
\end{aligned}
$$

where the last equality follows from equation (4) and (5). This completes the proof. □

4. The proof for $0 \le a \le 2$, $b \ge 0$ with $a+b \ge 1$ and any $g \ge 0$

First, we have to introduce another enumerative invariant, namely the numbers $\tilde{\mathcal{N}}^g_{\mathbb{F}_0}(a,a+b)$ and $\tilde{\mathcal{N}}^g_{\mathbb{F}_2}(a,b)$ of *not necessarily irreducible* plane tropical curves of degree $\Delta_{\mathbb{F}_0}(a,a+b)$ (resp. $\Delta_{\mathbb{F}_2}(a,b)$) and genus g through $4a+2b+g-1$ points in general position (see [8]). Obviously, $\mathcal{N}^g_{\mathbb{F}_0}(a,a+b)$ equals $\tilde{\mathcal{N}}^g_{\mathbb{F}_0}(a,a+b)$ minus the number of *reducible* curves satisfying the conditions.

By [8], theorem 2 we can determine the tropical enumerative numbers $\tilde{\mathcal{N}}^g_{\mathbb{F}_0}(a, a+b)$ and $\tilde{\mathcal{N}}^g_{\mathbb{F}_2}(a,b)$ both of $\mathbb{F}_0$ and $\mathbb{F}_2$ by counting λ-increasing lattice paths of length $4a+2b+g-1$ in the polygon corresponding to the toric surface $\mathbb{F}_0$ respectively $\mathbb{F}_2$ and the divisor class $aC+(a+b)F$ respectively $aC+bF$. (Each path has to be counted with a certain multiplicity. For more information, see [8] or [7].) Here, we fix λ to be of the form $\lambda : \mathbb{R}^2 \to \mathbb{R}$, $\lambda(x,y) = x - \varepsilon y$, where ε is a small irrational number.

We will first show the following modified version of theorem 1.1 and use this later to prove the theorem for $0 \leq a \leq 2$, $b \geq 0$ with $a+b \geq 1$ and any $g \geq 0$.

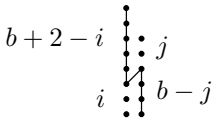
Lemma 4.1
The following equation holds for

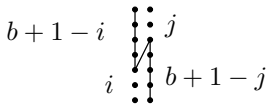
- $0 \leq a \leq 1$, $b \geq 0$ *with* $a+b \geq 1$ *and any* $g \in \mathbb{Z}$, *and for*
- $a = 2$, $b \geq 0$ *with* $a+b \geq 1$ *and any* $g \geq 0$*:*

$$\tilde{\mathcal{N}}^g_{\mathbb{F}_0}(a, a+b) = \sum_{k=0}^{a-1} \binom{b+2k}{k} \tilde{\mathcal{N}}^g_{\mathbb{F}_2}(a-k, b+2k). \tag{7}$$

Proof:
If $a = 0$, then the polygon corresponding to $\mathbb{F}_0$ and bF equals the polygon corresponding to $\mathbb{F}_2$ and bF. It is just a vertical line of integer length b. Hence the number of lattice paths agrees, since the polygon in which we count agrees.

If $a = 1$, then a path with $2b+3-g$ steps has to miss g lattice points of the polygons dual to $\Delta_{\mathbb{F}_2}(1,b)$ resp. $\Delta_{\mathbb{F}_0}(1,b+1)$. Since it cannot have any steps of integer length bigger 1 in the boundary, it looks like the paths in the following picture, where $i+j=g$.





Thus the left hand side of the equation equals

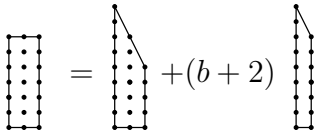
$$\sum_{i+j=g} \binom{b+2-i}{j} \binom{b-j}{i}$$

and the right hand side of the equation equals

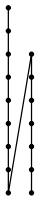
$$\sum_{i+j=g} \binom{b+1-i}{j} \binom{b+1-j}{i}$$

which are both equal to $\binom{2b+2-g}{g}$ because of Vandermonde's identity.

Let $a = 2$ and $g \geq 0$. Since $a = 2$, the sum on the right hand has only two summands, as indicated in the following picture.

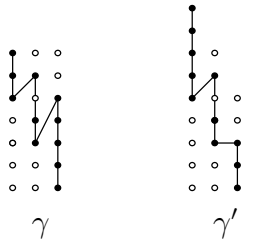


If $g > 0$, then no path of length $4a + 2b + g - 1 = 2b + g + 7$ fits into the second polygon on the right hand side. Hence that summand is 0 in this case. If $g = 0$ then there is exactly one path of length $2b + 7$ which fits into the second polygon on the right hand side, and it counts with multiplicity 1:



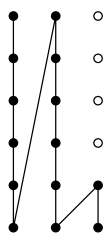
Thus we have to show $\tilde{\mathcal{N}}^g_{\mathbb{F}_0}(2, 2+b) = \tilde{\mathcal{N}}^g_{\mathbb{F}_2}(2, b)$ if $g > 0$, and $\tilde{\mathcal{N}}^0_{\mathbb{F}_0}(2, 2+b) = \tilde{\mathcal{N}}^0_{\mathbb{F}_0}(2, b) + b + 2$.

Let γ be any path in the polygon dual to $\Delta_{\mathbb{F}_0}(2, b+2)$. We want to associate a path γ' in the polygon dual to $\Delta_{\mathbb{F}_2}(2, b)$ to it. First note that by lemma 3.6 and remark 3.7 of [7], each path that does not count 0 has two sort of steps: some that go down vertically and others that move exactly one column to the right (with a simultaneous move up or down) — see the picture below. (Although stated for a triangle in [7], this statement is with the same arguments true the polygons dual to $\Delta_{\mathbb{F}_0}(a, a+b)$ and $\Delta_{\mathbb{F}_2}(a, b)$.) We define γ' in the following way: γ' has two extra steps in the first column, γ' coincides with γ in the second column, and it has two steps less in the last column:



More precisely, $\gamma'(0) := (0, b+4)$, $\gamma'(1) := (0, b+3)$, $\gamma'(i) := \gamma(i-2)$ for all $i \geq 2$ such that the x -coordinate of $\gamma(i)$ is less than or equal to 1 and $\gamma'(i) := (2, \gamma(i-2)_y - 2)$ for all i such that the x-coordinate of $\gamma(i)$ is 2. Note that in the above picture the lattice points which are not images of γ respectively γ' are drawn white.

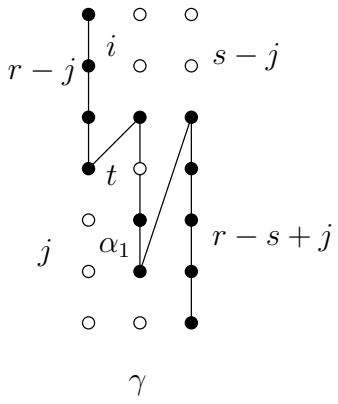
It is possible to associate γ' to γ if $\gamma(2b + g + 7 - 2)$ is in the $x = 2$-column (recall that γ has $2b + g + 7$ steps). Then $\gamma(2b + g + 7 - 2)_y - 2 \geq 0$. This holds if $g > 0$ for any path, and if $g = 0$ for any path except the one which takes every step in the first two columns and only one in the last column:



γ_0

We denote this path by γ_0, and we always assume $\gamma \neq \gamma_0$ in the following. Note that the multiplicity of γ_0 is equal to $\text{mult}(\gamma_0) = \binom{b+2}{b+1} = b+2$.

It is too much to hope that the multiplicity of γ and γ' coincides. Let us compute the multiplicity of both paths. Note first that if γ has a step of lattice length bigger 1 in the $x=0$- or the $x=2$-column, then its multiplicity is 0 (and the same holds for γ'), so we do not need to consider it. Let us assume γ has α_i steps of lattice length i in the column $x=1$, and let $t := \sum_{i\geq 2} \alpha_i \cdot i$. Denote by j the number of free lattice points on $x=0$, by i the number of free lattice points above the ones taken by γ, and by $s-j$ the number of free lattice points on $x=2$. Let $r := b+2$. Then there are $r-\alpha_1-t-i$ free lattice points on $x=1$ below γ.



γ

In the picture, $r=6$, $j=3$, $i=2$, $t=2$, $\alpha_1=1$ and $s-j=2$.

By [7], proposition 3.8, the multiplicity of γ is equal to

$$\text{mult}(\gamma) = (I^\alpha)^2 \cdot \binom{r-i-t}{j}\binom{r-j}{i}\binom{r-s+j}{r-\alpha_1-t-i}\binom{\alpha_1+i}{s-j},$$

where I^α is a shortcut for $\prod_i i^{\alpha_i}$.

To see this, note that no step of lattice length bigger 1 on the column $x=1$ can be part of a parallelogram, since on $x=0$ and $x=2$, only steps of lattice length 1 are allowed. Thus the binomial factors above count the numbers of ways to arrange parallelograms with edge length 1 (both above and below γ), and the factor in front corresponds to the double areas of triangles involving the steps of higher lattice length on $x=1$ (see remark 3.9 of [7]).

Analogously,

$$\operatorname{mult}(\gamma') = (I^\alpha)^2 \cdot \binom{r-i-t}{j}\binom{r-j+2}{i}\binom{r-s+j-2}{r-\alpha_1-t-i}\binom{\alpha_1+i}{s-j}.$$

Proposition 3.8 of [7] is only stated for a triangles, but taking remark 3.10 of [7] into account, it can be generalized to polygons dual to $\Delta_{\mathbb{F}_0}(a, a+b)$ and $\Delta_{\mathbb{F}_2}(a, b)$ with the same arguments.

As already said, in general $\operatorname{mult}(\gamma)$ will not be equal to $\operatorname{mult}(\gamma')$. However, we can take a set of paths γ in the rectangle such that the sum of the multiplicities coincides with the sum of the multiplicities of the corresponding paths γ' in the dual of $\Delta_{\mathbb{F}_2}(2, b)$: We take all paths γ such that their values for α_i (for all i) and s coincides. That is, we let i vary from 0 to $r - t - \alpha_i$ and j vary from 0 to s. We denote the set of all those paths by $\Gamma(s, (\alpha_i)_i)$. The sum of the multiplicities of all paths γ in the rectangle in $\Gamma(s, (\alpha_i)_i)$ is then equal to

$$(I^\alpha)^2 \cdot \sum_{j=0}^{s} \sum_{i=0}^{r-\alpha_1-t} \binom{r-i-t}{j}\binom{r-j}{i}\binom{r-s+j}{r-\alpha_1-t-i}\binom{\alpha_1+i}{s-j}. \tag{8}$$

The sum of the corresponding paths γ' in $\Delta(2, b)$ is equal to

$$(I^\alpha)^2 \cdot \sum_{j=0}^{s} \sum_{i=0}^{r-\alpha_1-t} \binom{r-i-t}{j}\binom{r-j+2}{i}\binom{r-s+j-2}{r-\alpha_1-t-i}\binom{\alpha_1+i}{s-j}. \tag{9}$$

Using the Mathematica package MultiSum (see [14], respectively [13] for more information), we can show that the sum in (8) (neglecting the factor $(I^\alpha)^2$ which coincides for both expressions anyway) — that we will denote by $F(r, (\alpha_i)_i, s)$ from now on — fulfills the following recurrence:

$$(2r-s+2)(\alpha_1+r-t+2)\cdot F(r,(\alpha_i)_i,s) - 2(r^2+\alpha_1 r - tr + 4r - \alpha_1 s - 2t + 4)\cdot$$
$$F(r+1,(\alpha_i)_i,s+1) - (s+2)(\alpha_1 - r + t - 2)\cdot F(r+2,(\alpha_i)_i,s+2) = 0.$$

The sum in (9) — denoted by $G(r, (\alpha_i)_i, s)$ — satisfies the same recurrence. As $(\alpha_1 - r + t - 2) \neq 0$ we only need to check the initial values $r = \alpha_1 + t$, $r = \alpha_1 + t + 1$, $s = 0$ and $s = 1$ in order to show that the two sums are equal. If $r = \alpha_1 + t$, $F(r, (\alpha_i)_i, s) = G(r, (\alpha_i)_i, s)$ is easy to see. If $r = \alpha_1 + t + 1$, then $F(r, (\alpha_i)_i, s) = G(r, (\alpha_i)_i, s)$ is equivalent to

$$\sum_{j=0}^{s} \binom{\alpha_1+1}{s-j}\binom{\alpha_1}{s} = \sum_{j=0}^{s} \binom{\alpha_1}{s-j}\binom{\alpha_1+1}{s},$$

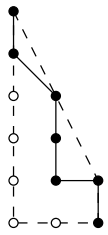
which is true by Vandermonde's identity. If $s = 0$, $F(r, (\alpha_i)_i, s) = G(r, (\alpha_i)_i, s)$ is again easy. If $s = 1$, both sides can be seen to be equal to

$$(\alpha_1 + r - t)\binom{2r-1}{r-\alpha_1-t} + \binom{2r-2}{r-\alpha_1-t-1}$$

using Vandermonde's identity.

Thus $F(r, (\alpha_i)_i, s) = G(r, (\alpha_i)_i, s)$ holds and therefore the sum of the multiplicities of all paths γ in $\Gamma(s, (\alpha_i)_i)$ is equal to the sum of multiplicities of the corresponding paths γ'.

At last, note that any path in the dual of $\Delta_{\mathbb{F}_2}(2,b)$ which is not equal to γ' for some γ in the rectangle counts with multiplicity 0, since we can not arrange parallelograms as needed (see again proposition 3.8 and remark 3.9 of [7]):



To sum up, if $g > 0$ then the number of lattice paths in the rectangle is equal to

$$\sum_{s=0}^{b-g+1} \sum_{(\alpha_i)_i} \sum_{\gamma\in\Gamma(s,(\alpha_i)_i)} \text{mult}(\gamma) = \sum_{s=0}^{b-g+1} \sum_{(\alpha_i)_i} \sum_{\gamma\in\Gamma(s,(\alpha_i)_i)} \text{mult}(\gamma')$$

and the right hand side covers all paths in $\Delta(2,b)$ which do not count zero. Thus

$$\tilde{\mathcal{N}}^g_{\mathbb{F}_0}(2,2+b) = \sum_\gamma \text{mult}(\gamma) = \sum_\gamma \text{mult}(\gamma') = \tilde{\mathcal{N}}^g_{\mathbb{F}_2}(2,b).$$

If $g = 0$ then the number of lattice paths in the rectangle is equal to

$$\begin{aligned}\text{mult}(\gamma_0) + &\sum_{s=0}^{b+1} \sum_{(\alpha_i)_i} \sum_{\gamma\in\Gamma(s,(\alpha_i)_i)} \text{mult}(\gamma) \\ &= (b+2) + \sum_{s=0}^{b} \sum_{(\alpha_i)_i} \sum_{\gamma\in\Gamma(s,(\alpha_i)_i)} \text{mult}(\gamma')\end{aligned}$$

and thus

$$\tilde{\mathcal{N}}^0_{\mathbb{F}_0}(2,2+b) = \tilde{\mathcal{N}}^0_{\mathbb{F}_2}(2,b) + (b+2)\tilde{\mathcal{N}}^0_{\mathbb{F}_2}(1,b-2).$$

□

Proof of theorem 1.1 for $0 \le a \le 2$, $b \ge 0$ with $a + b \ge 1$ and any $g \ge 0$:
Let $a = 0$. Since we have to fit paths with $2b + g - 1$ steps inside a line of integer length b, we get $g = -b + 1$. Hence $g \ge 0$ if and only if $b = 1$. This is the only case in which we have an irreducible curve. For $a = 0$ and $b = 1$, the equation trivially holds.

Let $a = 1$. Since we have to fit $2b + g - 1$ steps inside the polygons corresponding to $C + bF$ on $\mathbb{F}_2$ resp. $C + (b + 1)F$ on $\mathbb{F}_0$, we have $g \le 0$. Again, there is only one case in which we get an irreducible curve, namely $g = 0$. For $g = 0$, there is only one path in both polygons, it counts with multiplicity one on both sides and corresponds to an irreducible curve. Hence also in this case the equation is true.

Let $a = 2$. We know that the equation of lemma 4.1 is true, and we want to use it in order to deduce the equation of theorem 1.1 by showing that the number of reducible curves on both sides agrees. We use induction on b. For $b = 0$, we can see easily that there are no reducible curves for both sides, so the equation follows. Now we can assume that the equation is true for any $c < b$. Since $g \ge 0$, there are

no reducible curves of degree $\Delta_{\mathbb{F}_2}(1, b+2)$ that contribute to the right hand side. How many reducible curves of degree $\Delta_{\mathbb{F}_2}(2, b)$ are there? A reducible curve C could either be equal to $(\bigcup_{j=1}^{i} C_j) \cup C'$, $i \geq 1$, where each C_j is of degree $\Delta_{\mathbb{F}_2}(0,1)$ and C' is of degree $\Delta_{\mathbb{F}_2}(2, b')$, where $i + b' = b$, or it could be $(\bigcup_{j=1}^{i} C_j) \cup \tilde{C}_1 \cup \tilde{C}_2$, $i \geq 0$, where again each C_j is of degree $\Delta_{\mathbb{F}_2}(0,1)$, $\tilde{C}_1$ is of degree $\Delta_{\mathbb{F}_2}(1, b_1)$ and $\tilde{C}_2$ is of degree $\Delta_{\mathbb{F}_2}(1, b_2)$, with $b_1 + b_2 + i = b$. In the first case, we have $i \cdot 0 + g' - (i+1) + 1 = g$ if g' is the genus of C', so $g' = g + i$. Since g' is less than or equal to the number of interior points in the polygon dual to $\Delta_{\mathbb{F}_2}(2, b')$, it is less than or equal to $b' + 1 = b - i + 1$. Thus $2i \leq b + 1 - g$ or $i \leq \lfloor \frac{b+1-g}{2} \rfloor =: h$. Hence from this first case we get a contribution of $\sum_{i=1}^{h} \mathcal{N}_{\mathbb{F}_2}^{g+i}(2, b-i)$ of reducible curves. In the second case, we can show analogously that $g_1 + g_2 = g + i + 1$. But since $\tilde{C}_1$ and $\tilde{C}_2$ are of degree $\Delta_{\mathbb{F}_2}(1, b_1)$ resp. $\Delta_{\mathbb{F}_2}(1, b_2)$, we have $g_1 = g_2 = 0$, hence $g + i + 1 = 0$ which is not possible since $g \geq 0$. Thus by the induction assumption we know that the total contribution of reducible curves on the right hand side equals $\sum_{i=1}^{h} \mathcal{N}_{\mathbb{F}_0}^{g+i}(2, b-i+2)$. It is easy to see following the same arguments that we have the same contribution on the left hand side. Thus theorem 1.1 follows. □

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