

Plane Algebraic Curves – Problem Set 5

due Tuesday, June 23 at 10:15

- (1) Consider the rational function $\varphi = \frac{x^2}{y^2+yz}$ on the projective curve $F = y^2z + x^3 - xz^2$. Moreover, let $P = (0:0:1) \in F$.
 - (a) Compute the order $n = \mu_P(\varphi)$.
 - (b) Determine a local parameter $t \in \mathcal{O}_{F,P}$.
 - (c) Give an explicit description of φ in the form $\varphi = ct^n$ for some $c \in \mathcal{O}_{F,P}^*$, where c should be written as $\frac{f}{g}$ for some homogeneous $f, g \in S(F)$ of the same degree with $f(P) \neq 0$ and $g(P) \neq 0$.
- (2)
 - (a) Let P be a point on an affine curve F . Show that there is a rational function $\varphi \in K(F)$ which has exactly one pole which is simple and at P , i. e. such that $\mu_P(\varphi) = -1$ and $\mu_Q(\varphi) \geq 0$ for all $Q \neq P$.
 - (b) Let P_1 and P_2 be distinct points on a projective conic F . Show that there is a rational function $\varphi \in K(F)$ with $\mu_{P_1}(\varphi) = 1$, $\mu_{P_2}(\varphi) = -1$, and $\mu_P(\varphi) = 0$ at all other points P of F .
- (3) Let P be a point on an affine curve F . Show that there are ring isomorphisms
 - (a) $\mathcal{O}_{F,P} \cong \mathcal{O}_{\mathbb{A}^2,P} / \langle F \rangle$;
 - (b) $K(F) \cong K(F^h)$.
- (4)
 - (a) Let F be a projective curve, and let f be a homogeneous polynomial with $\text{div } f = D + E$ for two divisors D and E on F . Show: If $D' - D$ is the divisor of a non-zero rational function on F and $D' + E$ is effective then there is a homogeneous polynomial g with $\text{div } g = D' + E$.
 - (b) Let P, Q, R, S be four distinct points on a cubic curve F . Show that $(P + Q) - (R + S)$ is the divisor of a non-zero rational function on F if and only if the intersection point of the lines $\overline{PQ}$ and $\overline{RS}$ lies on F .