6. Functions and Divisors

Up to now we have essentially studied curves for themselves, i. e. no functions on them or maps between them. In fact, as we restrict ourselves to *plane* curves in these notes it does not make too much sense to consider maps between them as these maps would then somehow have to be compatible with the embeddings in the plane, which is quite restrictive and not very natural. But it is still very fruitful to consider functions on plane curves, i. e. maps to the ground field K, as we will see in the following chapters.

It turns out that the theory of such functions on curves is significantly easier from an algebraic point of view if we restrict to smooth and irreducible curves over an algebraically closed field (where irreducibility is automatic for projective curves by Proposition 5.1). So let us make the convention:

From now on, the ground field K is always assumed to be algebraically closed.
Curves are always assumed to be smooth and irreducible.

In particular, by Remarks 1.14 and 3.19 we can then think of a curve as a subset of $\mathbb{A}^2$ resp. $\mathbb{P}^2$.

Let us start by studying polynomial functions on affine curves.

Definition 6.1 (Affine coordinate rings)**.** Let F be a (smooth and irreducible) affine curve (over an algebraically closed field K). We call

$$A(F) := K[x,y]/\langle F \rangle$$

the **coordinate ring** of F.

In order to avoid overly complicated notations, we will not use any special symbols to denote the equivalence classes in $A(F)$, but rather write e. g. $f \in K[x,y]$ or $f \in A(F)$ for a polynomial resp. its equivalence class modulo F.

Remark 6.2 ($A(F)$ as ring of polynomial functions)**.** Clearly, the elements of $A(F)$ determine well-defined polynomial functions on $V(F) \subset \mathbb{A}^2$ to K by evaluation. Conversely, two polynomials f,g in $K[x,y]$ determine the same polynomial function on $V(F)$ if and only if $f-g$ is identically zero on $V(F)$, i. e. $V(F) \subset V(f-g)$. But as F is irreducible (and the ground field is algebraically closed) this is equivalent to $F \mid f-g$ by Corollary 1.13, and thus to $f = g \in A(F)$. In other words, we see that $A(F)$ is exactly the ring of polynomial functions on the curve.

Remark 6.3 (Algebraic properties of $A(F)$)**.**

(a) As the curve F is assumed to be irreducible, the coordinate ring $A(F)$ is an integral domain: If $fg = 0 \in A(F)$ this means that $F \mid fg$, hence $F \mid f$ or $F \mid g$, which means that $f = 0$ or $g = 0$ in $A(F)$.

(b) In contrast to the polynomial ring $K[x,y]$, the coordinate ring $A(F)$ of an affine curve is in general *not* a unique factorization domain as in Fact 1.2. Actually, determining whether a given coordinate ring $A(F)$ is factorial or not is in general a difficult problem. In these notes we will not study this question in detail; we just have to remember that it does not make sense to talk about irreducible decompositions of elements of $A(F)$.

As $A(F)$ is an integral domain we can also construct its quotient field, corresponding to functions on the curve that are given by quotients of polynomials. Just as in Definition 2.1 this gives rise to local rings describing such functions that have a well-defined value at a given point, and thus also on a neighborhood of this point.

Definition 6.4 (Rational functions and local rings)**.** Let F be an affine curve.

(a) The quotient field (see Construction 1.10)

$$K(F) := \operatorname{Quot} A(F) = \left\{ \frac{f}{g} : f,g \in A(F) \text{ with } g \neq 0 \right\}$$

of the coordinate ring is called the field of **rational functions** on F.

(b) A rational function $\varphi \in K(F)$ is called **regular** at a point $P \in F$ if it can be written as $\varphi = \frac{f}{g}$ with $f,g \in A(F)$ and $g(P) \neq 0$. The regular functions at P form a subring of $K(F)$ containing $A(F)$ denoted by

$$\mathcal{O}_{F,P} := \left\{ \frac{f}{g} : f,g \in A(F) \text{ with } g(P) \neq 0 \right\} \quad \subset K(F).$$

This ring of regular functions at P is called the **local ring** of F at P.

(c) There is a well-defined **evaluation map**

$$\mathcal{O}_{F,P} \to K,\ \frac{f}{g} \mapsto \frac{f(P)}{g(P)}$$

which we will simply write as $\varphi \mapsto \varphi(P)$ for $\varphi \in \mathcal{O}_{F,P}$, and whose kernel is

$$I_{F,P} := \left\{ \frac{f}{g} : f,g \in A(F) \text{ with } f(P) = 0 \text{ and } g(P) \neq 0 \right\}.$$

Remark 6.5 (Algebraic interpretation of local rings)**.** As in the case of the ring $\mathcal{O}_{\mathbb{A}^2,P}$ in Remark 2.2, the rings $\mathcal{O}_{F,P}$ are also local rings in the algebraic sense that they contain exactly one maximal ideal, namely $I_{F,P}$. The proof of this statement is the same as before: If I is an ideal in $\mathcal{O}_{F,P}$ which is not a subset of $I_{F,P}$ then there is an element $\frac{f}{g} \in I$ with $f(P) \neq 0$ and $g(P) \neq 0$. But this is then a unit in $\mathcal{O}_{F,P}$, so that $I = \mathcal{O}_{F,P}$.

Alternatively, just as in Remark 2.2 the ring $\mathcal{O}_{F,P}$ is the localization of $A(F)$ at the maximal ideal $\langle x - x_0, y - y_0 \rangle$ with $P = (x_0, y_0)$, and thus a local ring.

It is straightforward to transfer our notion of intersection multiplicity of two curves to a definition of multiplicity of a polynomial or rational function (and hence also of elements of local rings) on a curve. It should be thought of as the order of a zero or pole of such a function as in the introduction to Chapter 2 – an interpretation that will become even more natural in Proposition 6.10 and Remark 6.11.

Construction 6.6 (Multiplicities of rational functions)**.** Let P be a point on an affine curve F.

(a) For a polynomial function $f \in A(F)$ we define its **multiplicity** at P to be

$$\mu_P(f) := \mu_P(F,f) \overset{2.3}{=} \dim \mathcal{O}_{\mathbb{A}^2,P}/\langle F,f \rangle \quad \in \mathbb{N} \cup \{\infty\}.$$

Note that this is well-defined since $f = g \in A(F)$ implies $g = f + hF$ for some polynomial h, and thus $\mu_P(F,f) = \mu_P(F,g)$ by Remark 2.4 (c). By Exercises 2.7 and 2.8 this multiplicity is infinite if and only if f and F have a common component through P, i. e. (since F is irreducible) if and only if $f = 0 \in A(F)$.

The most important property of this multiplicity is that it is additive: By Proposition 2.10 (b) we have

$$\mu_P(fg) = \mu_P(f) + \mu_P(g)$$

for any $f,g \in A(F)$.

(b) For a rational function $\varphi = \frac{f}{g} \in K(F)$ the **multiplicity** at P is defined by

$$\mu_P(\varphi) := \mu_P(f) - \mu_P(g) \quad \in \mathbb{Z} \cup \{\infty\}.$$

Again this is well-defined: As $g \neq 0 \in A(F)$ we have $\mu_P(g) < \infty$ by (a), and if $\frac{f}{g} = \frac{f'}{g'} \in K(F)$ then $fg' = gf' \in A(F)$, so that

$$\mu_P(f) - \mu_P(g) = \mu_P(f') - \mu_P(g')$$

by the additivity of multiplicities of polynomial functions. Moreover, the multiplicity $\mu_P(\varphi)$ is infinite if and only if $\mu_P(f)$ is infinite, i. e. if and only if $f = 0$, and thus $\varphi = 0$.

The additivity of multiplicities immediately extends to rational functions as well: For $\varphi = \frac{f}{g}$ and $\psi = \frac{f'}{g'}$ in $K(F)$ we have

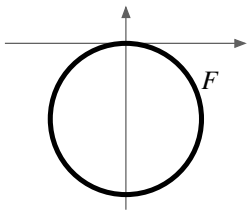
$$\begin{aligned}\mu_P(\varphi\psi) &= \mu_P\left(\frac{ff'}{gg'}\right) = \mu_P(ff') - \mu_P(gg') = \mu_P(f) + \mu_P(f') - \mu_P(g) - \mu_P(g') \\ &= \mu_P(\varphi) + \mu_P(\psi).\end{aligned}$$

In particular, as the multiplicity is finite for elements of $K(F)^*$ this means that μ_P is a group homomorphism from $K(F)^*$ to $\mathbb{Z}$.

If a (polynomial or rational) function has multiplicity $n > 0$ at P we say that it has a **zero** of **order** n at P; if $n < 0$ we say that it has a **pole** of **order** $-n$ at P.

Example 6.7. Consider the rational function $\varphi = \frac{y}{x}$ on the (complex) affine curve $F = y^2 + y + x^2$. A picture of the real points of F is shown in the picture on the right. Using the rules of Chapter 2 for computing intersection multiplicities we obtain at the origin

$$\mu_0(x) = \mu_0(x, y^2 - y - x^2) = \mu_0(x, y^2 - y) = 1$$
$$\text{and} \quad \mu_0(y) = \mu_0(y, y^2 - y - x^2) = \mu_0(y, x^2) = 2,$$



which is also easy to interpret geometrically by Corollary 2.22 as y is the tangent to F there. We conclude that $\mu_0(\varphi) = 2 - 1 = 1$, i. e. that φ has a zero of order 1 at the origin.

Exercise 6.8. Let P be a point on an affine curve F. Check that the local rings of $\mathbb{A}^2$ and F at P are related by $\mathcal{O}_{F,P} \cong \mathcal{O}_{\mathbb{A}^2,P}/\langle F \rangle$, and hence that $\mu_P(f) = \dim \mathcal{O}_{F,P}/\langle f \rangle$ for all $f \in A(F)$.

Remark 6.9 (Multiplicities of regular functions)**.** If $\varphi \in \mathcal{O}_{F,P}$ is an element of the local ring we can write it as $\varphi = \frac{f}{g}$, where $f, g \in A(F)$ with $g(P) \neq 0$. As this means that $\mu_P(g) = 0$, we see that $\mu_P(\varphi) = \mu_P(f) \geq 0$: Elements of the local ring cannot have a pole there. In particular, if φ is a unit then $\mu_P(\varphi^{-1}) = -\mu_P(\varphi) \geq 0$ as well, and we must have $\mu_P(\varphi) = 0$.

One would probably expect that the converse holds as well, i. e. that a rational function without a pole at P is regular at P. Note however that this is not obvious from the definitions, as it might happen just as in Example 6.7 that $\varphi = \frac{f}{g}$ with $\mu_P(f) \geq \mu_P(g) > 0$: In this case we have $\mu_P(\varphi) \geq 0$ but φ is not given as a quotient with non-vanishing denominator, so that it is not visibly regular. Nevertheless this statement turns out to be true as we will show in the next proposition: It is a consequence of the fact that $\mathcal{O}_{F,P}$ is what is called a *discrete valuation ring* in commutative algebra (see also Remark 2.28 (c)).

Proposition 6.10 ($\mathcal{O}_{F,P}$ is a discrete valuation ring)**.** *Let P be a point on an affine curve F.*

(a) *The ideal $I_{F,P}$ is principal, i. e. it can be written as $I_{F,P} = \langle t \rangle$ for some $t \in \mathcal{O}_{F,P}$ (which is unique up to units). We call t a **local coordinate** for F at P.*

(b) *Given a local coordinate t for F at P, every non-zero rational function $\varphi \in K(F)^*$ can be written uniquely as $\varphi = c\,t^n$ for a unit $c \in \mathcal{O}_{F,P}$ and $n \in \mathbb{Z}$, namely for $n = \mu_P(\varphi)$.*

In particular, we have $\varphi \in \mathcal{O}_{F,P}$ if and only if $\mu_P(\varphi) \geq 0$, i. e. if and only if φ does not have a pole at P.

Proof.

(a) Let t be (the class of) a line through P which is not the tangent T_PF, so that $\mu_P(t) = 1$ by Corollary 2.22. As t vanishes at P we have $t \in I_{F,P}$, and thus

$$1 = \mu_P(t) \overset{6.8}{=} \dim \mathcal{O}_{F,P}/\langle t \rangle \geq \dim \mathcal{O}_{F,P}/I_{F,P} \geq 1,$$

where the last inequality holds as the constant function 1 is a non-zero element of $\mathcal{O}_{F,P}/I_{F,P}$. We conclude that we must have equality, and thus $I_{F,P} = \langle t \rangle$. 09

(b) We can write $\varphi = \frac{f}{g}$ with $f, g \in A(F)\backslash\{0\}$. For $m = \mu_P(f)$, i. e. $\mu_P(f) = \mu_P(t^m)$, we then have $\langle F, f \rangle = \langle F, t^m \rangle$ in $\mathcal{O}_{\mathbb{A}^2,P}$ by Proposition 2.26. This means that $\langle f \rangle = \langle t^m \rangle$ in $\mathcal{O}_{F,P}$ by Exercise 6.8, and hence that $f = d\,t^m \in \mathcal{O}_{F,P}$ for a unit d. In the same way we can write $g = e\,t^r$ for a unit e and $r = \mu_P(g)$, and hence $\varphi = c\,t^n$ with $c = \frac{d}{e}$ and $n = m - r$ as desired.

The number n in such a representation is clearly unique: As units have multiplicity 0 by Remark 6.9 we must have $n = \mu_P(c\,t^n) = \mu_P(\varphi)$.

In particular, if $n \geq 0$ then clearly $\varphi = c\,t^n \in \mathcal{O}_{F,P}$; the converse follows again from Remark 6.9. □

Remark 6.11 (Discrete valuation rings in commutative algebra)**.** Over the ground field $K = \mathbb{C}$, the local coordinate t in Proposition 6.10 can be thought of as an analytic local coordinate around P on the 1-dimensional complex manifold $V(F)$ as in Remark 2.28 (b). Consequently, the multiplicity of a function at P just specifies how often this coordinate t can be split off as a linear factor.

As we have mentioned already, in commutative algebra a local ring with the properties of Proposition 6.10 is called a *discrete valuation ring*; the multiplicity $\mu_P(\varphi)$ is therefore also often called the *valuation* of φ. Moreover, Proposition 6.10 (b) means that (in contrast to the ring $A(F)$, see Remark 6.3 (b)), $\mathcal{O}_{F,P}$ is a factorial ring again, with t as the only irreducible element. Despite its more complicated construction, the local ring $\mathcal{O}_{F,P}$ is therefore much simpler than $A(F)$ from an algebraic point of view, and we will often prefer to work with it rather than with polynomials.

The structure of a discrete valuation ring also allows to compute the multiplicity of the sum of two rational functions.

Corollary 6.12. *Let P be a point on a curve F. For any two rational functions $\varphi, \psi \in K(F)$ we have*

$$\mu_P(\varphi + \psi) \geq \min(\mu_P(\varphi), \mu_P(\psi)),$$

with equality holding if $\mu_P(\varphi) \neq \mu_P(\psi)$.

Proof. We may restrict to the case when φ and ψ are non-zero, as the statement is trivial otherwise. By symmetry we may also assume that $n := \mu_P(\varphi) \leq m := \mu_P(\psi)$. Proposition 6.10 then tells us that we can write $\varphi = c\,t^n$ and $\psi = d\,t^m$ for some units c and d and a local coordinate t, and thus

$$\mu_P(\varphi + \psi) = \mu_P\Big(c\,t^n\Big(1 + \frac{d}{c}\,t^{m-n}\Big)\Big) = \mu_P(c\,t^n) + \mu_P\Big(\underbrace{1 + \frac{d}{c}\,t^{m-n}}_{\in \mathcal{O}_{F,P}}\Big) \overset{6.9}{\geq} \mu_P(c\,t^n) = n. \qquad (*)$$

Moreover, if $n \neq m$ then $1 + \frac{d}{c}\,t^{m-n}$ has value 1 at P and hence is a unit in $\mathcal{O}_{F,P}$, which means by Remark 6.9 again that we have equality in $(*)$. □

As a final result on affine curves, we can now show that rational functions that are required to be regular at every point of the curve are exactly the polynomial functions, i. e. the elements of the coordinate ring.

Proposition 6.13 (Global regular functions on affine curves)**.** *Let F be an affine curve. Then*

$$\bigcap_{P \in F} \mathcal{O}_{F,P} = A(F) \quad \subset K(F).$$

Proof. Clearly, all polynomial functions in $A(F)$ are everywhere regular, so it remains to prove the converse. For $\varphi \in \bigcap_{P \in F} \mathcal{O}_{F,P} \subset K(F)$ consider the ideal $I := \{g \in K[x,y] : g\varphi \in A(F)\}$. Then $V(I) = \emptyset$: If we had a point $P \in V(I)$, it would follow first of all that $P \in F$ since $F \in I$. Hence we have $\varphi \in \mathcal{O}_{F,P}$, i. e. we can write $\varphi = \frac{f}{g}$ for polynomials f and g with $g(P) \neq 0$. As $g\varphi = f \in A(F)$ this means that $g \in I$, leading to the contradiction $g(P) = 0$ since $P \in V(I)$.

We conclude that $V(I) = \emptyset$, and hence by the Nullstellensatz of Fact 4.1 that $I = K[x,y]$, which means that $1 \in I$, i. e. $\varphi \in A(F)$. □

Let us now pass on to projective curves, which will be our main objects of interest for the rest of these notes. The constructions of rational functions, local rings, and multiplicities in this case are essentially analogous to the ones considered above, taking care of the fact as in Remark 3.7 that we need to consider homogeneous polynomials resp. quotients of homogeneous polynomials of the same degree.

Definition 6.14 (Homogeneous coordinate rings)**.** Let F be a projective curve.

(a) We call
$$S(F) := K[x,y,z]/\langle F \rangle$$
the **(homogeneous) coordinate ring** of F. As in Remark 6.3 in the affine case, it is an integral domain as F is still assumed to be irreducible.

(b) A non-zero element $f \in S(F)$ is called **homogeneous** of degree d if it can be represented by a homogeneous polynomial of degree d in $K[x,y,z]$. The vector space of these elements, together with 0, will be denoted $S_d(F)$.

Remark 6.15 (Direct sum decomposition of $S(F)$)**.** Even if the representative modulo F of an element in $S(F)$ is not unique, we claim that we still have a direct sum decomposition
$$S(F) = \bigoplus_{d \in \mathbb{N}} S_d(F).$$
In fact, it is obvious that $S(F)$ is the sum of all $S_d(F)$, so let us show that this sum is direct. To do this, assume that $f_0 + \cdots + f_n = 0 \in S(F)$ for some polynomials $f_0, \ldots, f_n$ such that f_d is zero or homogeneous of degree d for all $d = 0, \ldots, n$. This means that $f_0 + \cdots + f_n = gF$ for a polynomial g. Taking the degree-d part of this equation then tells us that $f_d = g_{d-\deg F} F$ (where g_k denotes the degree-k part of g as in Notation 2.16, with $g_k = 0$ for $k < 0$), and thus $f_d = 0 \in S(F)$ for all d.

Construction 6.16 (Rational functions and local rings)**.** Let F be a projective curve.

(a) The field of **rational functions** on F is defined as
$$K(F) := \left\{ \frac{f}{g} : f, g \in S_d(F) \text{ for some } d \in \mathbb{N}, g \neq 0 \right\} \subset \operatorname{Quot} S(F).$$

(b) Analogously to Definition 6.4, we call a rational function $\varphi \in K(F)$ **regular** at a point $P \in F$ if it can be written as $\varphi = \frac{f}{g}$ with $f, g \in S(F)$ homogeneous of the same degree and $g(P) \neq 0$. The regular functions at P form a subring
$$\mathcal{O}_{F,P} := \left\{ \frac{f}{g} \in K(F) : g(P) \neq 0 \right\}$$
of $K(F)$ called the **local ring** of F at P.

(c) The ring of regular functions admits an **evaluation map** $\mathcal{O}_{F,P} \to K$, $\varphi \mapsto \varphi(P)$ with kernel $I_{F,P} := \{\varphi \in \mathcal{O}_{F,P} : \varphi(P) = 0\}$.

Construction 6.17 (Multiplicities of rational functions)**.** Let P be a point on a projective curve F.

(a) For a homogeneous element $f \in S(F)$ we define the **multiplicity** at P as
$$\mu_P(f) := \mu_P(F,f) \overset{3.21}{=} \dim \mathcal{O}_{\mathbb{P}^2,P}/\langle F, f \rangle \in \mathbb{N} \cup \{\infty\}.$$

(b) The **multiplicity** of a rational function $\varphi = \frac{f}{g} \in K(F)$ at P is defined as

$$\mu_P(\varphi) := \mu_P(f) - \mu_P(g).$$

It follows in the same way as in the affine case in Construction 6.6 that these multiplicities are well-defined, infinite exactly for the zero element of $S(F)$ resp. $K(F)$, and additive. The notions of (orders of) zeros and poles are also carried over directly.

Remark 6.18 (Affine and projective local rings)**.** As in Construction 3.20, for a point $P = (x_0 : y_0 : 1)$ on a projective curve F one can check that there is an isomorphism

$$\mathscr{O}_{F,(x_0:y_0:1)} \to \mathscr{O}_{F^{\mathrm{i}},(x_0,y_0)}, \frac{f}{g} \mapsto \frac{f^{\mathrm{i}}}{g^{\mathrm{i}}}$$

sending $I_{F,(x_0:y_0:1)}$ to $I_{F^{\mathrm{i}},(x_0,y_0)}$. Hence the algebraic properties of the local ring as e. g. in Proposition 6.10, Remark 6.11, and Corollary 6.12 carry over directly from the affine to the projective case.

Exercise 6.19. Consider the rational function $\varphi = \frac{x^2}{y^2+yz}$ on the projective curve $F = y^2z + x^3 - xz^2$. Moreover, let $P = (0:0:1) \in F$.

(a) Compute the order $n = \mu_P(\varphi)$.

(b) Determine a local coordinate $t \in \mathscr{O}_{F,P}$.

(c) Give an explicit description of φ in the form $\varphi = ct^n$ for a unit $c \in \mathscr{O}_{F,P}$, where c should be written as $\frac{f}{g}$ for some homogeneous $f, g \in S(F)$ of the same degree with $f(P) \neq 0$ and $g(P) \neq 0$.

Exercise 6.20.

(a) Let P be a point on an affine curve F. Show that there is a rational function $\varphi \in K(F)$ which has exactly one pole which is of order 1 and at P, i. e. such that $\mu_P(\varphi) = -1$ and $\mu_Q(\varphi) \geq 0$ for all $Q \neq P$.

(b) Let P_1 and P_2 be distinct points on a projective conic F. Show that there is a rational function $\varphi \in K(F)$ with $\mu_{P_1}(\varphi) = 1$, $\mu_{P_2}(\varphi) = -1$, and $\mu_P(\varphi) = 0$ at all other points P of F.

Exercise 6.21. Let F be an affine curve. Prove that the affine field of rational functions $K(F)$ is isomorphic to the projective one $K(F^{\mathrm{h}})$.

Before we continue our study of multiplicities of rational functions on projective curves let us introduce the so-called *divisors*, a very convenient piece of notation that allows us to consider the multiplicities at all points of a curve at once. We could have done this already in the affine case, but have chosen not to do so as we will only consider projective curves from now on.

Definition 6.22 (Divisors)**.** Let F be a projective curve.

(a) A **divisor** on F is a formal finite linear combination $a_1P_1 + \cdots + a_nP_n$ of distinct points $P_1, \ldots, P_n \in F$ with integer coefficients $a_1, \ldots, a_n \in \mathbb{Z}$ for some $n \in \mathbb{N}$. Obviously, the divisors on F form an Abelian group under pointwise addition of the coefficients. We will denote it by $\operatorname{Div} F$.

Equivalently, in algebraic terms $\operatorname{Div} F$ is just the *free Abelian group* generated by the points of F (i. e. the group of maps $V(F) \to \mathbb{Z}$ being non-zero at only finitely many points; with a point mapping to its coefficient in the sense above).

(b) A divisor $D = a_1P_1 + \cdots + a_nP_n$ as above is called **effective**, written $D \geq 0$, if $a_i \geq 0$ for all $i = 1, \ldots, n$. If D_1, D_2 are two divisors with $D_2 - D_1$ effective, we also write this as $D_2 \geq D_1$ or $D_1 \leq D_2$. In other words, we have $D_2 \geq D_1$ if and only if the coefficient of any point in D_2 is greater than or equal to the coefficient of this point in D_1. Note that this defines a partial order on $\operatorname{Div} F$.

(c) The **degree** of a divisor $D = a_1P_1 + \cdots + a_nP_n$ is the number $\deg D := a_1 + \cdots + a_n \in \mathbb{Z}$. Obviously, the degree is a group homomorphism $\deg\colon \operatorname{Div} F \to \mathbb{Z}$. Its kernel is denoted by

$$\operatorname{Div}^0 F = \{D \in \operatorname{Div} F : \deg D = 0\}.$$

Note that the name "divisor" in this context is entirely unrelated to the idea of elements of rings dividing one another. Instead, divisors are just given by multiplicities attached to all points on a curve, as appearing naturally in the following situations.

Construction 6.23 (Divisors from polynomials and rational functions)**.** Again, let F be a projective curve. The multiplicities of polynomials and rational functions of Construction 6.17 allow us to define divisors on F as follows.

(a) For a non-zero homogeneous polynomial $f \in S(F)\backslash\{0\}$ the *divisor of* f is defined to be

$$\operatorname{div} f := \sum_{P \in F} \mu_P(f) \cdot P \quad \in \operatorname{Div} F.$$

Hence, the effective divisor $\operatorname{div} f$ contains the data of the zeros of f together with their multiplicities. Note that the sum runs formally over all points of F – but as the number of zeros of f is finite by Remark 3.18, there are only finitely many points in this sum with a non-zero multiplicity, so that we obtain a well-defined divisor.

(b) Similarly, for a non-zero rational function $\varphi \in K(F)^*$ we set

$$\operatorname{div} \varphi := \sum_{P \in F} \mu_P(\varphi) \cdot P \quad \in \operatorname{Div} F.$$

This divisor is not effective; it encodes the zeros and poles of φ together with their multiplicities. By definition, if we write $\varphi = \frac{f}{g}$ as a quotient of two non-zero homogeneous polynomials $f, g \in S(F)\backslash\{0\}$ of the same degree then $\operatorname{div} \varphi = \operatorname{div} f - \operatorname{div} g$.

Example 6.24. Consider the rational function $\varphi = \frac{y}{x}$ on the projective curve $F = y^2 + yz + x^2$ over $\mathbb{C}$, i. e. on the projective closure of the affine curve in Example 6.7. We have seen in this example that φ has a zero of order 1 at $(0:0:1)$. Apart from this point, it is easy to check that the only other point at which y or x vanishes is $(0:-1:1)$, where

$$\mu_{(0:-1:1)}(\varphi) = \mu_{(0:-1:1)}(y) - \mu_{(0:-1:1)}(x) = 0 - 1 = -1.$$

Hence the divisor of φ is

$$\operatorname{div} \varphi = 1 \cdot (0:0:1) - 1 \cdot (0:-1:1).$$

Exercise 6.25. Let $F = y^2z - x^3 + xz^2$. Compute the divisor $\operatorname{div} \frac{y}{z}$ on F.

Remark 6.26 (Additivity of multiplicities for divisors)**.** Let F be a projective curve. The additivity of multiplicities as in Constructions 6.6 and 6.17 translates immediately into the following statements for divisors:

(a) For two homogeneous polynomials $f, g \in S(F)\backslash\{0\}$ we get

$$\begin{aligned}\operatorname{div}(fg) &= \sum_{P \in F} \mu_P(fg) \cdot P = \sum_{P \in F} \mu_P(f) \cdot P + \sum_{P \in F} \mu_P(g) \cdot P \\ &= \operatorname{div} f + \operatorname{div} g.\end{aligned}$$

(b) In the same way we obtain

$$\operatorname{div}(\varphi\psi) = \operatorname{div} \varphi + \operatorname{div} \psi$$

for any two non-zero rational functions $\varphi, \psi \in K(F)^*$. In particular, this means that the map $\operatorname{div}\colon K(F)^* \to \operatorname{Div} F$ is a group homomorphism.

It is also very useful to translate the important theorems of Bézout and Max Noether of Chapter 4 into the language of divisors.

Remark 6.27 (Bézout's Theorem for divisors)**.** For a projective curve F, Bézout's Theorem of Corollary 4.6 implies for the degrees of the divisors of Construction 6.23:

(a) for a non-zero homogeneous polynomial $f \in S(F)\backslash\{0\}$

$$\deg \operatorname{div} f = \sum_{P \in F} \mu_P(f) = \sum_{P \in F} \mu_P(F, f) = \deg F \cdot \deg f;$$

(b) for a non-zero rational function $\varphi \in K(F)^*$ (which we can write as $\varphi = \frac{f}{g}$ with f and g non-zero and homogeneous of the same degree)

$$\deg \operatorname{div} \varphi = \deg \operatorname{div} f - \deg \operatorname{div} g \overset{\text{(a)}}{=} \deg F \cdot \deg f - \deg F \cdot \deg g = 0,$$

i. e. that "a rational function on a projective curve has equally many zeros as poles". In particular, the image of the group homomorphism $\operatorname{div}\colon K(F)^* \to \operatorname{Div} F$ of Remark 6.26 (b) lies in $\operatorname{Div}^0 F$.

Proposition 6.28 (Max Noether's Theorem for divisors)**.** *Let F be a projective curve. Moreover, let $g, h \in S(F)$ be non-zero homogeneous polynomials with $\operatorname{div} g \leq \operatorname{div} h$.*

Then there is a homogeneous polynomial $b \in S(F)$ (of degree $\deg h - \deg g$) with $h = bg$ in $S(F)$, and thus with $\operatorname{div} h = \operatorname{div} b + \operatorname{div} g$.

Proof. As $\operatorname{div} g \leq \operatorname{div} h$ means $\mu_P(g) \leq \mu_P(h)$ for all $P \in F$, Max Noether's Theorem as in Corollary 4.12 (a) implies that there are homogeneous polynomials a and b (of degrees $\deg h - \deg F$ and $\deg h - \deg g$, respectively) such that $h = aF + bg$ in $K[x,y,z]$, and hence $h = bg$ in $S(F)$. The equation $\operatorname{div} h = \operatorname{div} b + \operatorname{div} g$ now follows directly from Remark 6.26 (a) (or Corollary 4.12 (b)). □

As a first consequence of these statements we can identify the rational functions that are regular at every point of the curve. Analogously to Proposition 6.13 we expect such functions to be polynomials – but in the projective case polynomials are only well-defined functions if they are constants:

Corollary 6.29 (Global regular functions on projective curves)**.** *Let F be a projective curve. Then*

$$\bigcap_{P \in F} \mathcal{O}_{F,P} = K \quad \subset K(F),$$

i. e. the only rational functions that are everywhere regular on F are constants.

Proof. Let $\varphi = \frac{f}{g} \in K(F)$ be regular at all points $P \in F$. This means that $0 \leq \mu_P(\varphi) = \mu_P(f) - \mu_P(g)$ for all P, and hence that $\operatorname{div} g \leq \operatorname{div} f$. As f and g have the same degree, Proposition 6.28 then implies that $f = cg$ for a constant c, and hence that $\varphi = \frac{f}{g} = c$ is a constant. □

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Remark 6.30 (Recovering rational functions from their divisors)**.** Corollary 6.29 implies that a rational function $\varphi \in K(F)^*$ on a projective curve F is determined up to scalar multiples by its divisor $\operatorname{div} \varphi$: If ψ is another rational function with $\operatorname{div} \psi = \operatorname{div} \varphi$ then $\operatorname{div} \frac{\psi}{\varphi} = 0$ by Remark 6.26 (b). Proposition 6.10 (b) then implies that $\frac{\psi}{\varphi}$ is a global regular function on F, hence some constant $c \in K^*$ by Corollary 6.29. This means that $\psi = c\varphi$, as we have claimed.

By definition, the group $\operatorname{Div} F$ of divisors on a projective curve F is a very large free Abelian group. As such, it is not very interesting from a group-theoretic point of view. It turns out that we can get a much smaller and more interesting group by considering a certain quotient of $\operatorname{Div} F$ as follows.

Definition 6.31 (Divisor classes and Picard groups)**.** Let F be a projective curve.

(a) A divisor on F is called **principal** if it is the divisor of a non-zero rational function as in Construction 6.23 (b). The set of all principal divisors will be denoted by

$$\operatorname{Prin} F := \{\operatorname{div} \varphi : \varphi \in K(F)^*\}.$$

As the image of the group homomorphism $\operatorname{div}\colon K(F)^* \to \operatorname{Div} F$ of Remark 6.26 (b) it is clearly a subgroup of $\operatorname{Div} F$, and by Remark 6.27 (b) also of $\operatorname{Div}^0 F$.

(b) The quotient group

$$\operatorname{Pic} F := \operatorname{Div} F / \operatorname{Prin} F$$

is called the **Picard group** or group of **divisor classes** on F. Two divisors D_1 and D_2 defining the same element in $\operatorname{Pic} F$, i. e. with $D_1 - D_2 = \operatorname{div} \varphi$ for a rational function $\varphi \in K(F)^*$, are said to be **linearly equivalent**, written $D_1 \sim D_2$. Restricting to divisors of degree 0, we also set

$$\operatorname{Pic}^0 F := \operatorname{Div}^0 F / \operatorname{Prin} F,$$

which is a subgroup of $\operatorname{Pic} F$.

Remark 6.32. By the homomorphism theorem, the degree of divisors induces isomorphisms $\operatorname{Div} F / \operatorname{Div}^0 F \cong \mathbb{Z}$ and $\operatorname{Pic} F / \operatorname{Pic}^0 F \cong \mathbb{Z}$. This means that the Picard group $\operatorname{Pic} F$ and its degree-0 part $\operatorname{Pic}^0 F$ carry essentially the same information. It just depends on the specific application in mind whether it is more convenient to work with $\operatorname{Pic} F$ or $\operatorname{Pic}^0 F$.

Example 6.33 (Picard groups for curves of degree at most 2)**.**

(a) Let F be a projective line. For any point $P \in F$ let l_P be a line through P different from F, so that P is the only intersection point of F and l_P (with multiplicity 1), and hence $\operatorname{div} l_P = P$ on F. For another point $Q \in F$ we then obtain a rational function $\frac{l_P}{l_Q}$ whose divisor is $P - Q$, so that $P - Q \sim 0$ by definition of linear equivalence.

Now any divisor D of degree 0 can be written as $D = P_1 + \cdots + P_n - Q_1 - \cdots - Q_n$ for some points $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ on F, and hence we conclude that

$$D = (P_1 - Q_1) + \cdots + (P_n - Q_n) \sim 0,$$

so that $\operatorname{Pic}^0 F = \{0\}$ is the trivial group.

(b) If F is a projective conic we have seen in Exercise 6.20 (b) that for any two points P and Q on F there is again a rational function with divisor $P - Q$, so that $P \sim Q$. So we conclude again that $\operatorname{Pic}^0 F = \{0\}$ in the same way as in (a).

For curves of bigger degree however, the Picard group is never trivial:

Proposition 6.34. *Let F be a curve of degree $d \geq 3$. Then $P \not\sim Q$ for any two distinct points P and Q on F. In particular, $\operatorname{Pic}^0 F$ is non-trivial.*

Proof. Assume that $P \sim Q$, i. e. that $P - Q = \operatorname{div} \frac{f}{g}$ for some homogeneous polynomials f and g of the same degree. Pick any line l through Q, so that $\operatorname{div} l = E + Q$ for an effective divisor E of degree $\deg E = d - 1 \geq 2$. As

$$\operatorname{div}(fl) = \operatorname{div} g + \operatorname{div} \frac{f}{g} + \operatorname{div} l = \operatorname{div} g + P - Q + E + Q = \operatorname{div} g + E + P \geq \operatorname{div} g$$

it follows from Max Noether's Theorem in Proposition 6.28 that there is a line l' with $\operatorname{div} l' = E + P$.

But $\deg E \geq 2$ means that E contains at least two points (or one point with multiplicity at least 2). Hence l and l' have to pass through them (resp. be tangent to F at the one point with multiplicity at least 2). As this fixes the line uniquely, it follows that $l = l'$, and thus that $P = Q$.

We conclude that $P \not\sim Q$ for $P \neq Q$, and thus that $P - Q \neq 0 \in \operatorname{Pic}^0 F$. □

Corollary 6.35 (Embedding of a curve in its Picard group)**.** *Let P_0 be a fixed base point on a projective curve F of degree at least 3. Then the map*

$$\Phi \colon V(F) \to \operatorname{Pic}^0 F,\ P \mapsto P - P_0$$

is injective.

Proof. If $\Phi(P) = \Phi(Q)$ then $P - P_0 \sim Q - P_0$, hence $P \sim Q$, and thus $P = Q$ as points in $V(F)$ by Proposition 6.34. □

Remark 6.36. For a projective curve F of degree $\deg F \geq 3$, Corollary 6.35 gives us a natural embedding (after choosing a base point) of the curve F into its degree-0 Picard group $\mathrm{Pic}^0 F$. This is a very interesting statement, as it gives us a natural map between mathematical objects of totally different types (namely a variety and a group).

In the next chapter we will see that this map is even a bijection if $\deg F = 3$, making this correspondence between varieties and groups even more surprising and useful.

Exercise 6.37.

(a) Let F be a projective curve, and let f be a homogeneous polynomial with $\operatorname{div} f = D + E$ for two divisors D and E on F. Show: If D' is linearly equivalent to D and $D' + E$ is effective then there is a homogeneous polynomial g with $\operatorname{div} g = D' + E$.

(b) Let P, Q, R, S be four distinct points on a cubic curve F with $P + Q \sim R + S$. Show that the intersection point of the lines $\overline{PQ}$ and $\overline{RS}$ lies on F.