Enumerative Geometry

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Notes for a class

taught at the University of Kaiserslautern 2003/2004

— preliminary version —

Contents

0. INTRODUCTION

Enumerative geometry is the branch of algebraic geometry concerned with counting curves in varieties that satisfy some given conditions. We give some (classical and modern) examples of enumerative problems and sketch how they can often be reduced to the computation of intersection products on suitable moduli spaces of curves.

0.1. **What is enumerative geometry?** Let X be a complex variety, usually assumed to be smooth and projective. The goal of enumerative geometry is simply to count curves in X that satisfy some given conditions. These conditions can be of various types: we can require that the curves have specified genus, specified degree, intersect given subvarieties of X, are tangent to a given subvariety of X, have certain singularities, and so on. The only requirement is that the conditions are chosen so that we expect a *finite* number of curves satisfying them. We are then asking for this finite number.

Let us illustrate these ideas by some examples.

Example 0.1.1. Probably the easiest enumerative question that one can ask is: how many lines are there in the projective plane $\mathbb{P}^2$ through two given (distinct) points? The answer here is obviously 1.

Example 0.1.2. Let us extend example 0.1.1 to conics, i. e. plane curves of degree 2. Note that a conic is uniquely given by its equation

$$a_0x_0^2 + a_1x_0x_1 + a_2x_0x_2 + a_3x_1^2 + a_4x_1x_2 + a_5x_2^2 = 0,$$

and conversely the a_i are determined up to a common scalar by the conic. So we can think of the projective space $\mathbb{P}^5$ with homogeneous coordinates a_i as the space of all conics. We say that $\mathbb{P}^5$ is a *moduli space* for plane conics.

Now let $P \in \mathbb{P}^2$ be a given point. Then a conic (determined by the a_i) passes through P if and only if the above equation is satisfied if we set x_0, x_1, x_2 to be the coordinates of P. This is obviously one linear condition in the coordinates a_i of the moduli space $\mathbb{P}^5$. As the moduli space is 5-dimensional we see that we get a finite number of conics if we require them to pass through 5 given points in the plane. In fact, we get exactly one such conic as the intersection of the 5 linear conditions in $\mathbb{P}^5$ is a single point.

There are two potential problems here that we should mention though:

(i) We should check that the linear equations in $\mathbb{P}^5$ given by the 5 incidence conditions are in fact independent, so that their intersection is really just a point (and not a higher-dimensional subspace).

(ii) Not all points in the moduli space $\mathbb{P}^5$ describe smooth conics. Some of them correspond to unions of two lines or even double lines. In other words, the "true" moduli space of smooth conics is not $\mathbb{P}^5$ itself but rather an open subset $U \subset \mathbb{P}^5$. The complement $\mathbb{P}^5 \backslash U$ is usually called the *boundary* of the moduli space. We cannot know a priori whether the point in the moduli space that is the intersection of the 5 linear conditions lies in U or not. In other words, it may be that there is no *smooth* conic through the 5 given points.

Both problems can actually arise for some special choices of the 5 points. For example, if we choose all 5 points to lie on a line then there is no smooth conic through these points at all, but on the other hand there is an infinite family of reducible conics through them (namely the line through the points together with any other line).

We will show now however that this cannot happen if we pick the 5 points *in general position*. This means: there is a dense open subset $V \subset (\mathbb{P}^2)^5$ such that for any $(P_1, \dots, P_5) \in V$

there is precisely one smooth conic through $P_1, \dots, P_5$. In fact, in this case we can say explicitly what this open subset V looks like: all we have to require is that no three of the marked points lie on a line. It is then obvious that there is no reducible conic through the five points. Moreover, if we had two distinct smooth conics through the points then these two conics would meet in five points, which is a contradiction to Bézout's theorem.

One would probably expect in general that the above problems (intersection products of too big dimension and components of the result in the boundary of the moduli space) do not occur if we pick the conditions on the curves in a general way. This is not true however; we will see a counterexample in example 0.1.6.

Example 0.1.3. Example 0.1.2 obviously extends to curves of higher degree: plane curves of degree d are parametrized by the projectivization of the vector space of homogeneous degree-d polynomials in 3 variables, which has dimension $\binom{d+2}{2} - 1$. Arguing as above we see that there is exactly 1 curve in $\mathbb{P}^2$ of degree d that passes through $\binom{d+2}{2} - 1$ general given points (see exercise 0.2.1).

The above examples were very easy because the moduli spaces and conditions were all linear. We usually express this by saying that the curves form a *linear system*. It is obvious then that the answer to our enumerative problem must be 1 since a zero-dimensional linear space is necessarily a single point. But in general of course neither the moduli space nor the conditions need be linear, and consequently the answer to an enumerative problem need not always be 1. Let us give some examples of this.

Example 0.1.4. In this example we want to answer the following question: how many *singular* plane cubic curves are there through 8 given points?

We have seen in example 0.1.3 that plane cubics are parametrized by a projective space $\mathbb{P}^9$. The 8 point conditions are again linear conditions in this $\mathbb{P}^9$, so what we have to analyze is the new condition that the curves be singular.

To do so define the function

$$F = a_0 x_0^3 + a_1 x_0^2 x_1 + a_2 x_0 x_1^2 + \cdots + a_9 x_2^3$$

describing a general cubic curve in $\mathbb{P}^2$ with coefficients a_i. Consider the variety

$$S = Z\left(\frac{\partial F}{\partial x_0}, \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}\right) \subset \mathbb{P}^9 \times \mathbb{P}^2$$

and its projection $\pi : S \to \mathbb{P}^9$. By the projective Jacobi criterion of [G] proposition 4.4.8 (ii) the fiber of S over a point $P \in \mathbb{P}^9$ is precisely the set of singular points of the cubic curve determined by the point P in the moduli space $\mathbb{P}^9$. So the image $\pi(S) \subset \mathbb{P}^9$ is the locus of singular cubic curves. Its class is easily determined: every equation $\frac{\partial F}{\partial x_i}$ is homogeneous of degree 1 in the coordinates of $\mathbb{P}^9$ and homogeneous of degree 2 in the coordinates of $\mathbb{P}^2$. So if we denote by $H \in A_*(\mathbb{P}^9)$ and $L \in A_*(\mathbb{P}^2)$ the class of a hyperplane in $\mathbb{P}^9$ and $\mathbb{P}^2$ respectively, the class of each zero locus of $\frac{\partial F}{\partial x_i}$ is $H + 2L$ (where we use the same letters H and L to denote the pull-back classes on $\mathbb{P}^9 \times \mathbb{P}^2$. The class of S is therefore

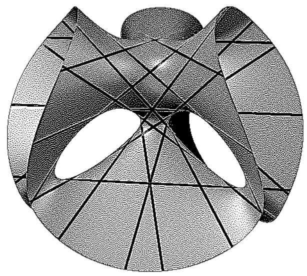
$$[S] = (H + 2L)^3 = H^3 + 6H^2L + 12HL^2 + 8L^3 \in A_8(\mathbb{P}^9 \times \mathbb{P}^2),$$

and so we conclude that

$$\pi_*[S] = 12H \in A_8(\mathbb{P}^9)$$

by the description of the push-forward of [G] construction 9.2.9. The condition of being a singular curve is therefore a hypersurface of degree 12 in the moduli space $\mathbb{P}^9$. As the 8 point conditions are again linear we see that there are 12 singular plane cubic curves through 8 given points.

Example 0.1.5. Let X be a smooth cubic surface in $\mathbb{P}^3$. We have seen in [G] section 4.5 and example 10.3.15 that there are exactly 27 lines in X. Let us briefly recall the intersection-theoretic computation that leads to this number.



Our moduli space is the 4-dimensional Grassmannian variety $G(1,3)$ of lines in $\mathbb{P}^3$. There is a tautological rank-2 subbundle F of the trivial bundle $\mathbb{C}^4$ on $G(1,3)$ whose fiber over a point $[L] \in G(1,3)$ (where $L \subset \mathbb{P}^3$ is a line) is precisely the 2-dimensional subspace of $\mathbb{C}^4$ whose projectivization is L. Dualizing, we get a surjective morphism of vector bundles $(\mathbb{C}^4)^\vee \to F^\vee$ that corresponds to restricting a linear function on $\mathbb{C}^4$ (or $\mathbb{P}^3$) to the line L. Taking the d-th symmetric power of this morphism we arrive at a surjective morphism $S^d(\mathbb{C}^4)^\vee \to S^d F^\vee$ that corresponds to restricting a homogeneous polynomial of degree d on $\mathbb{P}^3$ to L.

Now let $f = 0$ be the equation of X. By what we have just said the polynomial f determines a section of $S^3 F^\vee$ whose set of zeros in $G(1,3)$ is precisely the set of lines that lie in X (i. e. the set of lines on which f vanishes). As $S^3 F^\vee$ is a vector bundle of rank 4 we expect finitely many zeros of this section. Their number is given by [G] proposition 10.3.12 as the degree of the top Chern class $c_4(S^3 F^\vee)$ on $G(1,3)$. This degree can be computed explicitly to be 27; see [G] example 10.3.15 for details.

It should be noted that — in the same way as in example 0.1.2 (i) — this computation does *not* show that the number of lines in X is actually finite; we have to prove this in some other way (see [G] section 4.5). Only then do we know that the Chern class computation above gives the correct answer.

Example 0.1.6. Let us now give an example of an enumerative problem where the naïve intersection-theoretic computation does not give the right answer. Consider again conics in $\mathbb{P}^2$ with associated moduli space $\mathbb{P}^5$. We have seen in example 0.1.2 that there is exactly one conic through 5 points (in general position) since incidence conditions with points are linear conditions in the moduli space $\mathbb{P}^5$.

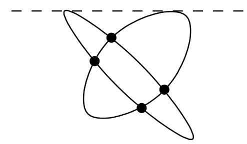
We will now replace some of the incidence conditions by tangency conditions. Let us first replace only one condition and ask: how many conics in $\mathbb{P}^2$ are tangent to a given line and moreover pass through 4 given points? Let us analyze the tangency condition. By a change of coordinates we may assume that the line is given by the equation $x_0 = 0$. Then a conic with equation

$$a_0x_0^2 + a_1x_0x_1 + a_2x_0x_2 + a_3x_1^2 + a_4x_1x_2 + a_5x_2^2 = 0$$

is tangent to this line if and only if the restriction of this equation to the line

$$a_3x_1^2 + a_4x_1x_2 + a_5x_2^2 = 0$$

has a double zero somewhere, i. e. if and only if the discriminant $a_4^2 - 2a_3a_5$ of this quadratic equation is zero. So we see that the tangency condition is a quadratic equation in the a_i. Intersecting this with 4 (linear) incidence conditions we conclude that there are exactly 2 conics that are tangent to a line and pass through 4 points.



Let us now replace all incidence conditions by tangencies and ask: how many plane conics are tangent to 5 given lines in general position? The naïve answer would be $2^5 = 32$ as the intersection of 5 quadratic conditions in $\mathbb{P}^5$. This is not true however as one can see from the theory of dual curves (see [G] exercise 4.6.11): the dual curve of a smooth conic is again a conic, and in the dual picture tangency conditions are translated into incidence conditions. So the number of conics tangent to 5 lines must be the same as the number of conics through 5 points, namely 1.

Why did the intersection-theoretic computation give the wrong answer? The problems arise from the points in $\mathbb{P}^5$ corresponding to double lines. Note that every double line intersects any other (distinct) line in one point with multiplicity two, so it counts as a tangent according to our definition above. The space of all conics tangent to the 5 given lines therefore includes the complete 2-dimensional space of double lines in $\mathbb{P}^5$. Hence the intersection-theoretic number 32 cannot be interpreted as the number of solutions to our enumerative problem.

Example 0.1.7. It happens frequently that a very simple enumerative question has a very complicated solution or is even still unsolved. For example, we can extend example 0.1.6 to higher degree and ask (similarly to example 0.1.3: how many plane curves are there that are tangent to $\binom{d+2}{2} - 1$ lines in general position? Although this question seems to be very similar to the ones that we have studied above its answer is still unknown.

Remark 0.1.8. After having studied a series of examples let us now summarize the general strategy to solve enumerative problems:

(1) Set up a moduli space that describes the curves one wants to study. The moduli space has to be compact (see below). It will therefore usually have "boundary points" that do not correspond to curves that one wants to count.
(2) Imposing the given conditions on the curve corresponds to an intersection product on the moduli space. These conditions have to be chosen so that the resulting intersection product is a cycle of dimension 0. As the moduli space is compact the degree of this 0-cycle is well-defined. It can be considered to be the "expected solution" of the enumerative problem.
(3) Finally we have to find out whether the geometric intersection of the conditions in (2) really has dimension 0 (i. e. the conditions are independent) and does not contain any points in the boundary of the moduli space (maybe for general choice of the conditions). If this is not the case then the "expected result" of (2) has to be corrected based on an explicit analysis of the geometry.

The biggest problem is usually that of finding a suitable moduli space. Note that the moduli space is certainly not uniquely defined by the problem we want to study:

- There are many ways to compactify a non-compact (moduli) space.
- We can parametrize curves in a variety X either by describing them as embedded subvarieties of X or as pairs (C, f) where C is an abstract curve and $f : C \to X$ a morphism. The resulting moduli spaces are usually different.
- If we want to count curves in a projective variety $X \subset \mathbb{P}^N$ we can either start with the moduli space of curves in $\mathbb{P}^N$ and later impose the condition that the curves actually be in X, or we can start with a moduli space of curves in X in the first place.

Of course the final answer to the enumerative problem should not depend on these choices. Different choices of moduli spaces will however lead to completely different computations: some moduli spaces may be easy to describe as a variety so that intersection products can be computed without much effort, but their boundary may be so complicated that the step from (2) to (3) above cannot be carried out. If one tries to solve this problem by picking a more sophisticated moduli space that does not give rise to complicated boundary contributions then the moduli space may become intractable as a variety, so that the intersection product (2) cannot even be computed any more.

This is in fact the point where classical enumerative geometry was stuck for a long time. For every enumerative question one had to construct and study a moduli space whose only purpose was to solve this one single problem. The situation changed only about 10 years ago with the invention of the theory of so-called stable maps that we will present in these notes.

Example 0.1.9. The transition from "classical" to "modern" enumerative geometry was in fact inspired by theoretical physicists. In 1989 the string theorists Candelas et al. claimed that they can compute the numbers n_d of genus-zero curves of degree d in a general hypersurface of degree 5 in $\mathbb{P}^4$. It is expected by a simple dimension count that these numbers are indeed finite (see exercise 0.2.4). The prediction of the physicists reads as follows. Set

$$\sum_{d\geq 0} \frac{\prod_{i=1}^{5d}(5H+i)}{\prod_{i=1}^{d}(H+i)^5}\, q^d =: F_0 + F_1 H + F_2 H^2 + \cdots \qquad (F_i \in \mathbb{Q}[[q]]).$$

Then define rational numbers N_d recursively by the equation of formal power series in q

$$F_2 = \frac{1}{2}\frac{F_1^2}{F_0} + \frac{1}{5}\sum_{d>0} d\, N_d\, q^d\, F_0 \exp\left(d\,\frac{F_1}{F_0}\right).$$

Then the enumerative numbers n_d are given by the recursion

$$N_d = \sum_{k|d} \frac{n_{d/k}}{k^3}.$$

The first few numbers are as follows.

d	n_d
1	2875
2	609250
3	317206375
4	242467530000
5	229305888887625
6	248249742118022000
7	295091050570845659250
8	375632160937476603550000
9	503840510416985243645106250
10	704288164978454686113488249750

The computation of the physicists was based on arguments from topological quantum field theory that are not understandable to mathematicians. At the time this result was published it was even a surprise to mathematicians that there is a reasonably simple generating function that computes all numbers n_d in one go. In fact, it is not even obvious that the numbers n_d defined by the recursion above are positive integers. To verify these results in classical enumerative geometry people had to find a suitable moduli space separately for every degree d. They checked the results up to degree 4, but the moduli spaces for degrees 3 and 4 were already so complicated that it was clear that no universal formula like the one above could be proven that way.

About five years ago the above formula has been proven mathematically using the theory of stable maps.

0.2. **Exercises.** Note: As we have not developed any theory yet, you are not expected to be able to solve the following problems in a mathematically precise way. Rather, they are just meant as some "food for thought" if you want to think a little further about the examples considered in this section.

Exercise 0.2.1. In example 0.1.3 check that there is in fact a unique *smooth* plane curve of degree d through $\binom{d+2}{2}-1$ given points if these points are in general position. What does "general position" mean here? Is it sufficient — as in example 0.1.2 — that no three of the points lie on a line?

Exercise 0.2.2. Generalize the statement of example 0.1.4 to plane curves of higher degree. More precisely, consider singular plane curves of degree d that pass through $n(d)$ given points. How big must $n(d)$ be so that we get a finite number of curves, and what is this number then?

Exercise 0.2.3. Solve the remaining cases of example 0.1.6: for any $a,b \geq 0$ with $a+b=5$ determine the number of plane conics that are tangent to a given lines and pass through b given points in general position. (Hint: For $a \leq 2$ show that the naïve computation gives the right answer. Then use the theory of dual curves for the cases $b \leq 2$.)

Exercise 0.2.4. Let $X \subset \mathbb{P}^N$ be a smooth hypersurface of degree e for some $e \geq 1$. Compute the expected dimension of the space of morphisms $\mathbb{P}^1 \to X$ of degree $d \geq 1$. For the case $N=4, e=5$ conclude that for every degree d one expects a finite number of curves of genus zero of degree d on a hypersurface of degree 5 in $\mathbb{P}^4$. (Hint: you may want to use corollary 1.1.5 and lemma 1.1.7.)

1. MODULI SPACES OF RATIONAL CURVES

1.1. **Set-theoretic description of the smooth case.** We have motivated in the introduction that our main goal will be the study of moduli spaces of curves in varieties. For simplicity let us first drop the variety from the picture and study just moduli spaces of curves. On the other hand we have to make the picture slightly more complicated by adding marked points to the curves. These are just distinguished points on the curves that will later become the points where "the specified conditions happen", e. g. the points of intersections with given subvarieties, the singular points, and so on.

Throughout these notes we will work over the field of complex numbers; all schemes and morphisms are assumed to be over $\mathbb{C}$ without further notice. A *curve* will always be reduced, connected, and projective (but not necessarily smooth or irreducible) unless stated otherwise.

Definition 1.1.1. Let $n \geq 0$ be an integer. A **smooth n-pointed curve** is a tuple $\mathcal{C} = (C, x_1, \dots, x_n)$, where C is a smooth curve and the x_i are distinct points on C. The points x_i are called the **marked points** of C. The **genus** of $\mathcal{C}$ is defined to be the genus of C. A smooth n-pointed curve is said to be **rational** (resp. **elliptic**) if its genus is 0 (resp. 1).

A morphism $(C, x_1, \dots, x_n) \to (C', x'_1, \dots, x'_n)$ of smooth n-pointed curves is a morphism $f : C \to C'$ such that $f(x_i) = x'_i$ for all i. For any $g \geq 0$ the set of all smooth n-pointed curves of genus g modulo isomorphism is denoted $M_{g,n}$. It will be called the moduli space of smooth n-pointed curves of genus g.

In this first section we will mainly be concerned with rational curves. So let us figure out first what smooth rational curves look like.

Lemma 1.1.2. *Let C be a smooth curve, and let X be a projective variety. Assume that we are given a non-empty open subset $U \subset C$ and a morphism $f : U \to X$. Then f extends uniquely to a morphism $\tilde{f} : C \to X$.*

Proof. Let $f : U \to X$ be a morphism. As X is assumed to be projective we may replace X by some $\mathbb{P}^N$. By [G] lemma 7.5.14 the morphism $f : U \to \mathbb{P}^N$ is then given by $f(x) = (s_0(x) : \cdots : s_N(x))$, where the s_i are global sections of some line bundle $\mathcal{L}$ on U.

Now let $P \in C \backslash U$ be a point. As the question is local around P we can assume that $\mathcal{L}$ is trivial, i. e. that the s_i are just regular functions on U and thus rational functions on $U \cup \{P\}$. For all i let $m_i \in \mathbb{Z}$ be the order of s_i at P. Denote the minimum of all m_i by m. By possibly shrinking U again we can assume by [G] lemma 7.5.6 that there is a regular function φ_P on U that vanishes at P with multiplicity 1 and has no further zeros or poles on U. Then f can be rewritten as

$$f(x) = \left(\frac{s_0(x)}{\varphi_P(x)^m} : \cdots : \frac{s_N(x)}{\varphi_P(x)^m} \right).$$

But by the choice of m all entries are now regular at P, and at least one of them is non-zero. So f has a unique extension to P. □

Corollary 1.1.3. *Let C and C' be smooth curves. The following are equivalent:*

(i) $C \cong C'$.
(ii) *C and C' are **birational**, i. e. they have isomorphic non-empty open subsets.*
(iii) *The fields of rational functions $K(C)$ and $K(C')$ are isomorphic.*

Proof. (i) ⇒ (ii) is trivial.

(ii) ⇒ (i): Let $U \subset C$ and $U' \subset C'$ be isomorphic non-empty open subsets. The isomorphism $f : U \to U'$ extends to a morphism $\tilde{f} : C \to C'$ by lemma 1.1.2. In the same way the inverse $f^{-1} : U' \to U$ extends to a morphism $\widetilde{f^{-1}} : C' \to C$. These two morphisms must be inverse to each other since $\tilde{f} \circ \widetilde{f^{-1}}$ and $\widetilde{f^{-1}} \circ \tilde{f}$ are the identity on a non-empty open subset and hence on the whole curve.

(ii) $\Rightarrow$ (iii) is clear as $K(C) = K(U)$ for any non-empty open subset $U \subset C$ by [G] exercise 2.6.9 (iv).

(iii) $\Rightarrow$ (ii): Let $U \subset C$ be an affine open subset, and let $f_1, \dots, f_k \in A(U)$ be generators of its coordinate ring. As $A(U) \subset K(C) = K(C')$ the f_i are rational functions on C'. Hence we can pick an open subset $U' \subset C'$ on which the f_i are regular. We are thus getting a $\mathbb{C}$-algebra homomorphism $A(U) \to A(U')$ that corresponds to a morphism $U' \to U$.

We can now apply the same construction to U' to arrive at a morphism $V \to U'$ for some non-empty open subset $V \subset C$. By construction the two maps are inverse to each other where defined (the two $\mathbb{C}$-algebra homomorphisms are just restrictions of the identity $K(C) = K(C')$ after all). So C and C' are birational. □

Remark 1.1.4. The equivalence (ii) $\Leftrightarrow$ (iii) of corollary 1.1.3 works in fact for varieties of any dimension (with the same proof that we have given). Only the equivalence (i) $\Leftrightarrow$ (ii) is special to smooth curves (with blow-ups as the standard counterexample).

Corollary 1.1.5. *Any smooth rational curve is isomorphic to* $\mathbb{P}^1$.

Proof. Let C be a smooth curve of genus 0. Pick a point $P \in C$. By the Riemann-Roch theorem [G] 7.7.3 we have

$$h^0(\mathcal{O}_C(P)) - h^0(\omega_C \otimes \mathcal{O}_C(-P)) = 1 + 1 - 0 = 2.$$

But the line bundle $\omega_C \otimes \mathcal{O}_C(-P)$ does not have global sections since its degree is -3 by [G] corollary 7.6.6. So there are two linearly independent sections s_0, s_1 of $\mathcal{O}_C(P)$. They define a rational map $(s_0 : s_1) : C \dashrightarrow \mathbb{P}^1$ that must in fact be a morphism by lemma 1.1.2. The degree of this morphism is $\deg \mathcal{O}_C(P) = 1$, so $[K(C) : K(\mathbb{P}^1)]$ is a field extension of degree 1 by [G] proposition 9.2.8. Hence $K(C) \cong K(\mathbb{P}^1)$. The statement of the corollary now follows from the equivalence (i) $\Leftrightarrow$ (iii) of corollary 1.1.3. □

Remark 1.1.6. The name *rational curve* for a curve of genus 0 actually comes from the above corollaries. In general, a variety is called rational if it is birational to some projective space (hence in the case of curves to $\mathbb{P}^1$). By corollary 1.1.5 every smooth curve of genus 0 is isomorphic (hence birational) to $\mathbb{P}^1$. On the other hand, every smooth curve that is birational to $\mathbb{P}^1$ is in fact isomorphic to $\mathbb{P}^1$ by corollary 1.1.3, hence has genus 0.

We have just seen that every smooth rational curve C admits an isomorphism to $\mathbb{P}^1$. This isomorphism is however not unique. In fact, we will see in the following lemma that we can require in addition that three marked points of C are mapped to some given points in $\mathbb{P}^1$.

Lemma 1.1.7. *Let x_1, x_2, x_3 and x'_1, x'_2, x'_3 be two sets of three distinct points in $\mathbb{P}^1$. Then there is a unique isomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that $f(x_i) = x'_i$ for $i = 1, 2, 3$.*

Proof. First of all note that any isomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ is of the form $f(s : t) = (as + bt : cs + dt)$ for some a, b, c, d. Passing to an affine coordinate $x = \frac{s}{t}$ on $\mathbb{P}^1$ (that is allowed to take on the value ∞) we can thus write the isomorphism f as

$$f : \mathbb{P}^1 \to \mathbb{P}^1, \quad x \mapsto \frac{ax + b}{cx + d}.$$

For simplicity let us first assume that the three image points are $x'_1 = 0, x'_2 = 1, x'_3 = \infty$. Then the conditions $f(x_i) = x'_i$ mean that

(i) $f(x_1) = 0 \Rightarrow ax_1 + b = 0 \Rightarrow b = -ax_1$;
(ii) $f(x_3) = \infty \Rightarrow cx_3 + d = 0 \Rightarrow d = -cx_3$;
(iii) $f(x_2) = 1 \Rightarrow ax_2 + b = cx_2 + d$, so by (i) and (ii) $a(x_2 - x_1) = c(x_2 - x_3)$.

As $x_2 \neq x_1$ and $x_2 \neq x_3$ equation (iii) fixes a and c uniquely up to a common scalar. Equations (i) and (ii) then fix b and d as well. So altogether the above equations fix a,b,c,d up to a common scalar and hence f uniquely. It is given by

$$f(x) = \frac{x-x_1}{x-x_3} \cdot \frac{x_2-x_3}{x_2-x_1} =: c(x;x_1,x_2,x_3).$$

This function is commonly called the **cross ratio**. It is the unique isomorphism of $\mathbb{P}^1$ that takes the three points x_1,x_2,x_3 to $0,1,\infty$, respectively.

In the general case when x'_1,x'_2,x'_3 are also arbitrary distinct given points an isomorphism of the required type is obviously given by $c^{-1}(\,\cdot\,;x'_1,x'_2,x'_3) \circ c(\,\cdot\,;x_1,x_2,x_3)$. It is unique since otherwise we could compose two different isomorphisms with $c(\,\cdot\,;x'_1,x'_2,x'_3)$ to get two different isomorphisms that map x_1,x_2,x_3 to $0,1,\infty$, respectively, in contradiction to our calculation above. □

Corollary 1.1.8.

(i) *If $n \le 3$ then any two smooth rational n-pointed curves are isomorphic. In particular, $M_{0,n}$ is then just a single point.*

(ii) *If $n \ge 3$, and C and C' are two smooth rational n-pointed curves that are isomorphic, then the isomorphism between them is unique.*

In particular, a smooth rational n-pointed curve has trivial automorphism group if $n \ge 3$ and infinite automorphism group if $n < 3$.

Proof. (i) follows from corollary 1.1.5 and the existence part of lemma 1.1.7. (ii) follows from the uniqueness part of lemma 1.1.7. □

1.2. **A preliminary discussion of moduli functors.** So far we have only considered the moduli spaces $M_{0,n}$ as sets. It is clear that we have to give them some further structure if we want to do any useful geometry on them. The final goal will be to give them the structure of a (smooth) variety. We will do this in this section.

The main additional structure of a variety compared to a mere set of points is that it makes sense to talk about morphisms from or to $M_{0,n}$. For example, a morphism from a scheme S to $M_{0,n}$ can be thought of as a *continuously varying* assignment of a smooth n-pointed rational curve to every point in S. We will usually call such an assignment a *family* of smooth n-pointed rational curves over S.

To study such families of smooth n-pointed rational curves we have to define first of all what exactly we mean by this — recall that so far we have only defined what a single smooth n-pointed rational curve is. Let us start by setting up a language that can be used to describe objects varying "continuously" with the points of a base scheme.

Definition 1.2.1. A **(moduli) functor** F is given by the following data:

(i) for every scheme S a set $F(S)$;

(ii) for every morphism $f : S \to S'$ of schemes a set-theoretic pull-back map $f^* : F(S') \to F(S)$.

These data must be compatible with compositions, i. e. if $f : S \to S'$ and $g : S' \to S''$ are two morphisms of schemes then $(g \circ f)^* = f^* \circ g^* : F(S'') \to F(S)$ as set-theoretic maps.

Remark 1.2.2. We should think of $F(S)$ as the set of all families of objects (e. g. smooth n-pointed rational curves, see example 1.2.4) parametrized by the base scheme S. The pull-back maps $f^* : F(S') \to F(S)$ associated to a morphism $f : S \to S'$ of schemes also have a geometric interpretation: if we are given a family in $F(S')$, i. e. an object for every point in S', then the pull-back of this family in $F(S)$ is simply the family over S that assigns to every point $P \in S$ the given object over $f(P) \in S'$.

Remark 1.2.3. There is a branch of mathematics called *category theory* that deals with the general concept of objects and morphisms between them. To define a category one simply has to say what the objects and morphisms of the category should be. Examples for categories are:

(i) the category of schemes (objects: schemes, morphisms: morphisms of schemes);
(ii) the category of sets (objects: sets, morphisms: set-theoretic maps);
(iii) the category of vector spaces (objects: vector spaces, morphisms: homomorphisms);
(iv) the category of topological spaces (objects: topological spaces, morphisms: continuous maps).

Maps from one category to another are then called functors. In this language, our definition 1.2.1 above defines functors from the category of schemes to the category of sets. We will just call them functors for short as we will not have need for functors between other categories.

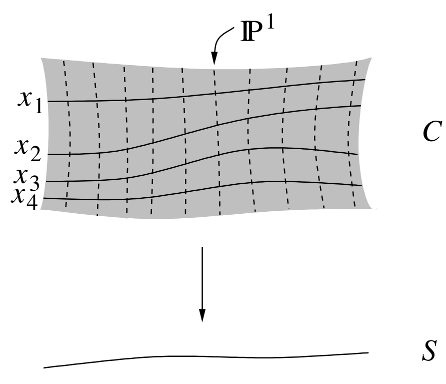
Example 1.2.4. As an example let us set up the moduli functor for $M_{0,n}$. By abuse of notation we will denote this functor by $M_{0,n}$ as well.

(i) For any base scheme S we set

$$M_{0,n}(S) = \left\{ \begin{matrix} (C, x_1, \dots, x_n) \; ; \; C \to S \text{ is a } \mathbb{P}^1\text{-bundle with} \\ \text{disjoint sections } x_1, \dots, x_n : S \to C \end{matrix} \right\} \Big/ \text{isomorphisms.}$$

(ii) For any morphism $f : S \to S'$ of schemes and any $(C', x'_1, \dots, x'_n) \in M_{0,n}(S')$ we define the pull-back $f^*(C', x'_1, \dots, x'_n)$ to be $(C, x_1, \dots, x_n) \in M_{0,n}(S)$, where $C = C' \times_{S'} S$ is simply the pull-back $\mathbb{P}^1$-bundle and $x_i = f^* x'_i$ are the pulled-back sections of C over S.

If $(C, x_1, \dots, x_n) \in M_{0,n}(S)$ note that the fiber of C over any point $P \in S$ together with the points $x_1(P), \dots, x_n(P)$ forms a smooth n-pointed rational curve. Moreover, as every smooth rational curve is isomorphic to $\mathbb{P}^1$ by corollary 1.1.5 it is clear that our definition captures precisely the idea of a "continuously varying family" of n-pointed smooth rational curves.



Example 1.2.5. For any scheme X there is an associated functor (that we will also denote by X) given by:

(i) For any base scheme S we let $X(S)$ be the set of morphisms from S to X.
(ii) For any morphism $f : S \to S'$ of schemes and any $(S' \to X) \in X(S')$ the pull-back $f^*(S' \to X) \in X(S)$ is simply defined to be the composite morphism $S \xrightarrow{f} S' \to X$.

This functor is usually called the **functor of points** of X. The name can be explained by the observation that a family over S, i. e. an element of $X(S)$, is given by a morphism $S \to X$,

i. e. by an assignment of a point of X to any point of S. So we can think of X as a "moduli space that parametrizes points in X".

Definition 1.2.6. A functor F is called **representable** by a scheme X if it agrees with the functor of points of X. In this case we say that X is a **moduli space** for F. In other words, F is representable by X if and only if there is a one-to-one correspondence between families in F over a base scheme S and morphisms $S \to X$.

Remark 1.2.7. If a functor is representable by a scheme then the scheme representing it is unique (up to canonical isomorphism). In fact, assume that X and Y are schemes whose functors of points are the same, i. e. $X(S) = Y(S)$ for any scheme S, and the pull-back maps $X(S') \to X(S)$ and $Y(S') \to Y(S)$ for any $S \to S'$ agree under this identification. We want to show that X and Y are isomorphic as schemes.

As $X(X) = Y(X)$ the identity morphism $\mathrm{id}_X : X \to X$ corresponds to a morphism $f : X \to Y$. In the same way the identity $\mathrm{id}_Y : Y \to Y$ gives rise to a morphism $g : Y \to X$ by the equality $Y(Y) = X(Y)$. Now by the equality of the pull-back g^* we know that the diagram

$$\begin{array}{ccc} X(X) & \xrightarrow{g^*} & X(Y) \\ \| & & \| \\ Y(X) & \xrightarrow{g^*} & Y(Y) \end{array}$$

is commutative. But the identity $\mathrm{id}_X \in X(X)$ is mapped to $\mathrm{id}_X \mapsto g \mapsto \mathrm{id}_Y \in Y(Y)$ by the map $X(X) \to X(Y) \to Y(Y)$, and to $\mathrm{id}_X \mapsto f \mapsto f \circ g \in Y(Y)$ by the map $X(X) \to Y(X) \to Y(Y)$. So $f \circ g = \mathrm{id}_Y$. In the same way we see that $g \circ f = \mathrm{id}_X$. Hence X and Y are canonically isomorphic.

If a functor F is representable by a scheme X we will therefore say that X is *the* moduli space for F. By abuse of notation we will then also often say that the functor "is" a scheme and denote both the functor and the scheme by the same letter.

Example 1.2.8. Let $0 \le k \le n$ be integers. The Grassmannian $G(k,n)$ of k-dimensional linear subspaces of $\mathbb{P}^n$ is defined to be the following moduli functor: to any base scheme S we associate the set of families of k-dimensional linear subspaces in a fixed $\mathbb{P}^n$ parametrized by S:

$$G(k,n)(S) = \left\{ \begin{array}{l} \pi : V \to S \ ; \ V \text{ is a } \mathbb{P}^k\text{-subbundle of} \\ \text{the trivial bundle } S \times \mathbb{P}^n \text{ over } S \end{array} \right\} \Big/ \text{isomorphisms.}$$

The pull-back maps for the functor are simply defined by pulling back the projective bundle.

It can then be shown that the functor $G(k,n)$ is (representable by) a projective variety. This variety can be constructed in several ways, e. g. by suitable gluing of affine spaces or by finding an explicit embedding in a projective space. Note however that defining the Grassmannian as a functor is much simpler than defining it as a variety.

Remark 1.2.9. Let us say a few words about the philosophy behind functors and their moduli spaces. Note that a functor is a very general concept; it does not have much structure. Consequently, it is usually very easy to set up a functor for a given moduli problem (see e. g. example 1.2.8). The downside is of course that one cannot do much useful geometry with a functor alone.

It is a remarkable fact that this situation changes drastically if we know that a functor F is representable by a scheme X. Even if we do not know how this scheme X is constructed explicitly we can deduce almost any important information about it just from the functor F:

(i) The points of X are simply $F(\mathrm{pt})$ by definition.

(ii) Thinking of a curve in X as the image of a morphism $V \to X$ from a curve V we see that we can define such a curve by an element of $F(V)$, i. e. by a family over a one-dimensional base. In the same way we can describe higher-dimensional subvarieties of X.
(iii) To find the intersection of two subvarieties of X (given by two families in F as in (ii)) one just has to figure out which objects occur in both families.
(iv) By [G] exercise 5.6.12 a tangent vector in X can be thought of as a morphism $D \to X$, where $D = \operatorname{Spec} \mathbb{C}[x]/(x^2)$ is the "double point". Hence tangent vectors in X correspond to $F(D)$, i. e. to families over the double point.
(v) In particular, knowing the tangent spaces to X allows to check whether X is smooth, or whether given subvarieties of X intersect transversally.
(vi) Recall that a scheme X is called separated if and only if "it is a Hausdorff space in the classical topology", i. e. if and only if a morphism $V \to X$ is determined by its restriction to any dense open subset $U \subset V$. This translates into the language of functors by asking whether an extension of a family over U to a family over V is unique.
(vii) Assume that X is a projective scheme. If V is a smooth curve and $P \in V$ a point on V then we have seen in lemma 1.1.2 that every morphism $f : V\backslash\{P\} \to X$ extends to a morphism $\tilde{f} : V \to X$. One can show that the following converse of this statement is true as well: if every such morphism f has an extension $\tilde{f}$ then X is proper (i. e. compact, see [G] section 9.2). So compactness of the scheme X can be tested on the functor F by checking whether any family over $V\backslash\{P\}$ (where V is a smooth curve) has an extension to V.

The conclusion is that to do computations on the moduli space X (e. g. intersection-theoretic calculations) it is often enough to know the functor F. Let us stress again however that we must know that a moduli space exists — there is e. g. nothing like intersection theory on a general functor.

Our main task is therefore to construct representable functors for the moduli problems that we want to study. In general this is not easy; in fact "most" functors that one could write down (even the ones that look reasonably well-behaved) are not representable. We will see an example of this in the next remark.

Remark 1.2.10. In example 1.2.4 we have defined the functor $M_{0,n}$ by considering $\mathbb{P}^1$-bundles with n sections *modulo isomorphisms*. What exactly do we mean by isomorphisms here? Note that there are two possible definitions that are both problematic:

(i) The most natural definition would be to say that two families $(C, x_1, \dots, x_n)$, $(C', x'_1, \dots, x'_n) \in M_{0,n}(S)$ are isomorphic if and only if there is an isomorphism between the $\mathbb{P}^1$-bundles $C \cong C'$ over S that maps the sections x_i to the sections x'_i. To see why this causes problems consider the case $n = 0$ and assume that we have two non-isomorphic $\mathbb{P}^1$-bundles C and C' (i. e. two non-isomorphic families) over S. If the functor $M_{0,0}$ is representable by a scheme X then by definition these two families must give rise to two different morphisms $S \to X$. But X is just a point by corollary 1.1.8 (i). As there is only one morphism from S to a point this is a contradiction. So the functor $M_{0,0}$ cannot be representable.
(ii) A possible way out of this problem would be to call two families $(C, x_1, \dots, x_n)$, $(C', x'_1, \dots, x'_n) \in M_{0,n}(S)$ isomorphic if and only if all their fibers over S are isomorphic as smooth n-pointed rational curves. With this definition the two non-isomorphic $\mathbb{P}^1$-bundles of (i) would be isomorphic families of the functor by definition, so we do not get a contradiction to representability. However, such a definition that uses the points of S would not make much sense for non-reduced base schemes S. For example, if $S = \operatorname{Spec} \mathbb{C}[x]/(x^2)$ is the double point then we have seen in remark 1.2.9 (iv) that the points in $M_{0,n}(S)$ correspond to tangent

vectors in the moduli space. But if any two families over S are called isomorphic when they agree on the geometric point of S this would imply that any two tangent vectors at a given point of the moduli space are the same. As this is not possible unless the moduli space is just a point we see that in general this definition will not lead to a representable functor either.

As it is quite clear that definition (ii) does not make sense (for non-reduced base schemes) we will have to stick to definition (i). Note that the problems arise here because of the automorphisms of $\mathbb{P}^1$: the existence of such automorphisms allows us to construct families that are not isomorphic although all their fibers are. (Recall that a $\mathbb{P}^1$-bundle is always locally trivial. On the overlaps these trivial bundles are glued by some automorphisms in the fibers. If there are no such automorphisms then the bundle must be globally trivial.) So to get representability we will have to require that the objects in question do not admit non-trivial automorphisms, i. e. by corollary 1.1.8 that we have at least three marked points. So we will assume this from now on.

Actually we will have to consider moduli spaces for objects with non-trivial automorphisms later on in this course. We will be able to deal with that case too; it just turns out that the language of functors as introduced in definition 1.2.1 has to be modified for this to work. (This is why we have called this section a *preliminary* discussion of moduli functors.)

After all these remarks let us now finally show that the functor $M_{0,n}$ is representable.

Lemma 1.2.11. *Let $n \geq 3$. Then the functor $M_{0,n}$ of smooth n-pointed rational curves is representable by the open subscheme of $\mathbb{A}^{n-3}$*

$$X = \{(x_4,\dots,x_n)\ ;\ x_i \notin \{0,1\} \text{ and } x_i \neq x_j \text{ for } i,j = 4,\dots,n \text{ with } i \neq j\}.$$

Proof. Let S be any scheme. The bijection $M_{0,n}(S) = X(S)$ can be written down explicitly: in one direction we have

$$\begin{array}{rcl} M_{0,n}(S) & \to & X(S) \\ (C,x_1,\dots,x_n) & \mapsto & (c(x_4;x_1,x_2,x_3),\dots,c(x_n;x_1,x_2,x_3)) \end{array}$$

where $c(\,\cdot\,;x_1,x_2,x_3)$ denotes the cross ratio of the proof of lemma 1.1.7. Note that strictly speaking the x_i are sections of a $\mathbb{P}^1$-bundle, but as the cross ratio function is invariant under automorphisms of $\mathbb{P}^1$ the number $c(\,\cdot\,;x_1,x_2,x_3)$ is well-defined. The collection of the functions $c(x_i;x_1,x_2,x_3) : S \to \mathbb{P}^1 \backslash \{0,1,\infty\}$ therefore gives rise to a morphism $S \to X$.

In the other direction we simply have

$$\begin{array}{rcl} X(S) & \to & M_{0,n}(S) \\ (x_4,\dots,x_n) & \mapsto & (S \times \mathbb{P}^1, 0, 1, \infty, x_4,\dots,x_n) \end{array}$$

where the x_i are functions from S to $\mathbb{P}^1$.

It is clear that these two constructions are inverse to each other and that they are compatible with the pull-back maps along morphisms $S \to S'$. □

1.3. **Construction of the moduli functor $\bar{M}_{0,n}$ of rational stable curves.** We have just constructed the moduli spaces $M_{0,n}$ of smooth n-pointed rational curves (for $n \geq 3$) as open subsets of $\mathbb{A}^{n-3}$. For intersection theory we will need compact moduli spaces however. So we will have to find a suitable compactification of $M_{0,n}$.

The first naïve idea might be to simply take $\mathbb{P}^{n-3}$ as a compactification as it is the easiest compact space that contains $M_{0,n} \subset \mathbb{A}^{n-3}$ as a dense open subset. Recall from remark 1.2.9 however that we finally want to do our computations with the moduli *functors* and not directly with the moduli *spaces*. So it will not help us if we have a compactification of the moduli space that does not correspond to some functor of curves. Instead we will have to "compactify the functor", i. e. to extend the functor $M_{0,n}$ to some functor $\bar{M}_{0,n}$ that contains $M_{0,n}$ (i. e. every family in $M_{0,n}$ is a family in $\bar{M}_{0,n}$) and that is representable by a compact moduli space.

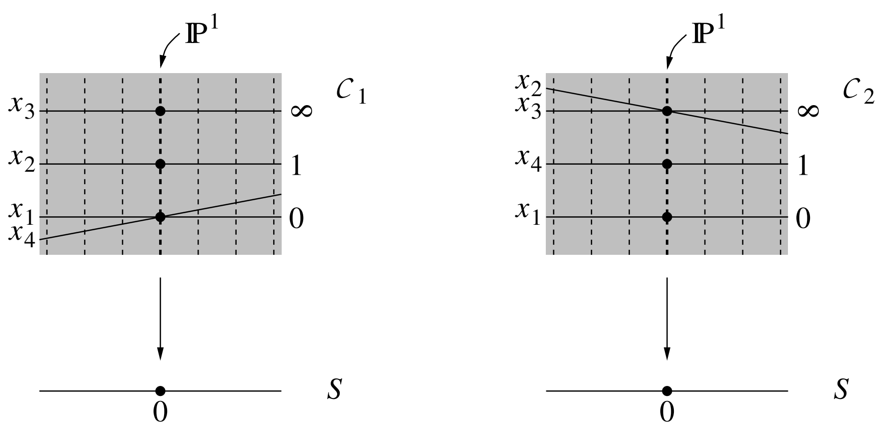
If we look at the moduli functor $M_{0,n}$ it is easy to spot why its moduli space is not compact: we have required that the marked points be distinct, which is obviously an "open condition". In other words, if we write down a family in which one of the marked points approaches another then this family has no limit at the point where the two points would coincide. Hence the moduli space is not compact.

We may therefore try to solve our problem by simply allowing the marked points to coincide. It is clear that this modified moduli problem still defines a functor. We will see in the following example however that this functor would not be representable by a nice space.

Example 1.3.1. Consider the following two families in $M_{0,4}$ over the base $S = \mathbb{A}^1 \backslash \{0,1\}$:

$$\mathcal{C}_1 = (S \times \mathbb{P}^1, 0, 1, \infty, t) \qquad \text{and} \qquad \mathcal{C}_2 = (S \times \mathbb{P}^1, 0, \frac{1}{t}, \infty, 1)$$

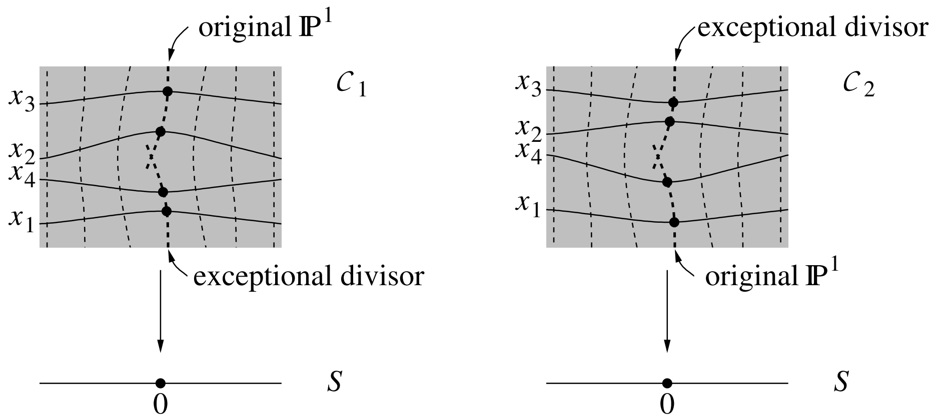
where $t \in \mathbb{A}^1 \backslash \{0,1\}$ is the coordinate on S.



Note that these two families are isomorphic in $M_{0,n}$ since they have the same cross ratio $c(t;0,1,\infty) = c(1;0,\frac{1}{t},\infty) = t$. But if we now want to extend these families to families over $\mathbb{A}^1$ by allowing the marked points to coincide then the limits for $t = 0$ would be different: in the family $\mathcal{C}_1$ the points x_1 and x_4 coincide, whereas in the family $\mathcal{C}_2$ the points x_2 and x_3 coincide. These two 4-pointed limit curves are certainly not isomorphic. By remark 1.2.9 (vi) this would mean that the moduli space could not be separated, which is certainly not desirable. So the idea of just allowing the marked points to coincide does not lead to a nice moduli functor.

Actually there would be more problems if we just allowed the marked points to coincide: as soon as fewer than 3 marked points are distinct the resulting curves would have non-trivial automorphisms again, and we would run into the same trouble as for $M_{0,n}$ in the case $n < 3$ (see remark 1.2.10).

In the above picture it is easy to see how these problems can be avoided: if we blow up the point in $S \times \mathbb{P}^1$ where the two sections x_1 and x_4 (resp. x_2 and x_3) meet then the fiber over 0 becomes reducible with two components, both of which contain two of the marked points. As the fiber and the two sections run through the blown-up point with different tangent directions their strict transforms will meet the exceptional divisor in three different points. So the new picture looks as follows:



Note that the limit curves in the two families are actually isomorphic now by the $n = 3$ case of corollary 1.1.8 (i). So we have avoided the trouble of a non-separated moduli space.

In summary the effect of the blow-up can be described as follows: when two marked points try to come together the curve sprouts off another smooth rational component (the exceptional divisor of the blow-up) that contains these two marked points. In fact this is the general idea how to compactify the moduli space $M_{0,n}$. So we will still require the marked points to be distinct, but the curves may have several irreducible components intersecting transversally. Let us now make the corresponding definition.

Definition 1.3.2. Let C be a curve. A point $P \in C$ is called a **node** of C if the tangent cone $C_{X,P}$ (see [G] remark 4.3.8) is a union of two reduced lines. Alternatively, *in the classical topology* C is locally reducible around P with two smooth components meeting transversally. The curve C is called a **nodal curve** if all points of C are either smooth points or nodes.

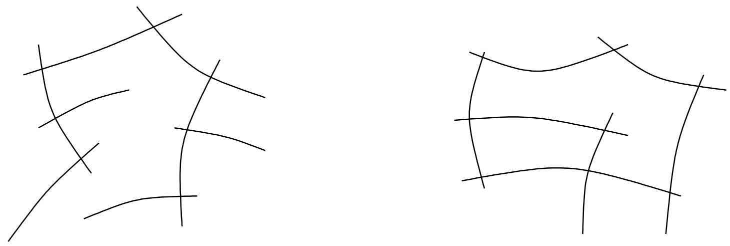
Let $n \geq 0$ be an integer. A **pre-stable n-pointed curve** is a tuple $\mathcal{C} = (C, x_1, \dots, x_n)$, where C is a nodal curve and the x_i are distinct smooth points on C. As usual for singular curves, the **genus** of $\mathcal{C}$ is defined to be $h^1(C, \mathcal{O}_C)$. As in the smooth case a pre-stable n-pointed curve is said to be rational (resp. elliptic) if its genus is 0 (resp. 1). A morphism $(C, x_1, \dots, x_n) \to (C', x'_1, \dots, x'_n)$ of pre-stable n-pointed curves is a morphism $f : C \to C'$ such that $f(x_i) = x'_i$ for all i.

A pre-stable n-pointed curve is called **stable** if its group of automorphisms is finite. For any $g \geq 0$ the set of all stable n-pointed curves of genus g modulo isomorphism is denoted $\bar{M}_{g,n}$. It will be called the moduli space of stable n-pointed curves of genus g.

Remark 1.3.3. By [G] example 8.3.6 a nodal curve obtained by gluing k smooth components of genera $g_1, \dots, g_k$ in p nodes has genus $g_1 + \cdots + g_k + p + 1 - k$. Note that we must always have $p \geq k - 1$ since the curve is connected and thus every new component must be glued to the rest of the curve in some node. So the genus of a nodal curve C can be 0 only if all g_i are zero and $p = k - 1$. This means that C is a *tree of smooth rational curves*, i. e.

(i) all components of C are isomorphic to $\mathbb{P}^1$;
(ii) there are no "loops" in the graph of C, i. e. by separating the two branches of C at any node the curve becomes disconnected.

As an example, the curve below on the left is a tree and thus has genus 0, whereas the curve on the right has genus 2 (where all irreducible components are assumed to be isomorphic to $\mathbb{P}^1$).



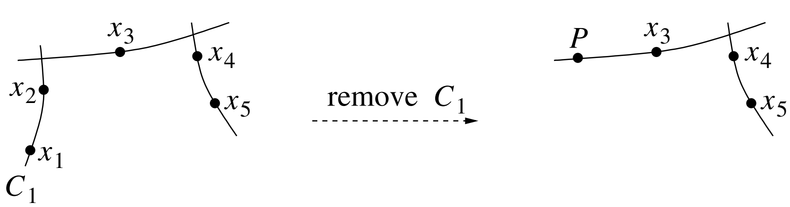
Let us now investigate the “stability condition” on the automorphism groups of the curves.

Lemma 1.3.4. *Let $\mathcal{C} = (C, x_1, \dots, x_n)$ be a rational pre-stable n-pointed curve. The following are equivalent:*

(i) *$\mathcal{C}$ is stable, i. e. it has finite automorphism group.*
(ii) *$\mathcal{C}$ has trivial automorphism group.*
(iii) *Every component of C has at least 3 special points. Here a point of C is called a* ***special point*** *if it is either a node or a marked point.*

Proof. (ii) $\Rightarrow$ (i) is trivial. The implication (i) $\Rightarrow$ (iii) is obvious as well: assume that there is a component C_1 of C with only $k < 3$ special points. Then we can regard C_1 as a smooth k-pointed rational curve. This curve has infinitely many automorphisms by corollary 1.1.8. We can now extend these automorphisms of C_1 to automorphism of C by the identity on $C \backslash C_1$.

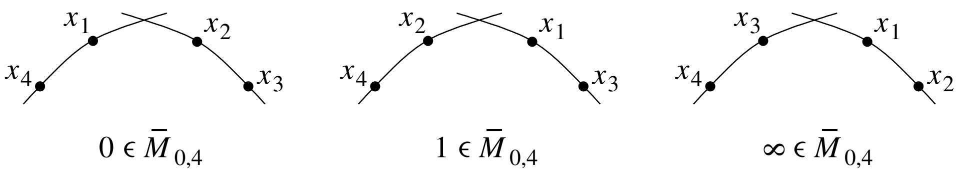
It remains to be shown that (iii) implies (ii). We will do this by induction on the number of components of C. If C is irreducible the statement is just that of corollary 1.1.8. Otherwise let C_1 be a component of C that has only one node P (there must be such a component since C is a tree). This component must then have at least two marked points, say x_1 and x_2. By definition an automorphism of $\mathcal{C}$ must keep x_1 and x_2 fixed. In particular, any automorphism of $\mathcal{C}$ must map C_1 to C_1 and keep the three special points P, x_1, and x_2 on C_1 fixed. By corollary 1.1.8 the automorphism must then be the identity on C_1. We can now remove C_1 from C and consider the remaining parts of the curve as a new pre-stable curve, where we add P to the set of marked points. The lemma now follows by the induction hypothesis applied to the remaining parts of the curve.



□

Remark 1.3.5. For curves of genus $g > 0$ all equivalences of lemma 1.3.4 will be false. In particular there are curves of higher genus with finite but non-trivial automorphism group. We have defined a stable curve to be one with finite (and not trivial) automorphism group as this turns out to be the correct generalizaton for higher genus. Note however that we have seen in remark 1.2.10 that *any* non-trivial automorphism will lead to a non-representable functor in the way we have set it up so far. This is the main reason why we will restrict ourselves to the case of rational curves for a while.

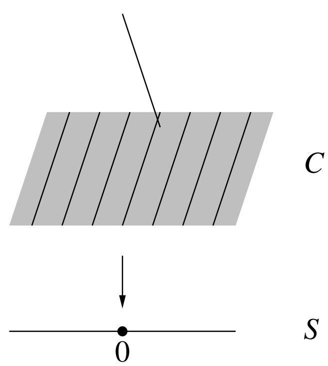
Example 1.3.6. There are precisely three rational stable 4-pointed curves that are not smooth:



In the spirit of example 1.3.1 these three curves correspond to the cases when x_4 approaches x_1, x_2, and x_3, respectively, and therefore to the cross ratios 0, 1, and ∞. So we can think of $\bar{M}_{0,4}$ as $\bar{M}_{0,4} \cup \{0,1,\infty\} = \mathbb{P}^1$. In fact, $\bar{M}_{0,4}$ will just be $\mathbb{P}^1$ as a scheme (see proposition 1.3.15). To make this statement precise however we first have to define $\bar{M}_{0,4}$ as a functor, and then prove that this functor is representable by $\mathbb{P}^1$.

So let us generalize the definition of the moduli functor $M_{0,n}$ to stable curves. Recall that a family of *smooth* rational n-pointed curves over a base scheme S was defined to be a tuple $(C, x_1, \dots, x_n)$, where $C \to S$ is a $\mathbb{P}^1$-bundle and the x_i are disjoint sections. We can think of the $\mathbb{P}^1$-bundle C as a continuously varying family of curves that are isomorphic to $\mathbb{P}^1$. Now we need to generalize this setup and allow the curves in the fibers of the morphism $C \to S$ to be nodal. We will see in the following example however that this requirement on the fibers is not enough.

Example 1.3.7. Let C be the union of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^1$ glued at the points $(0,0) \in \mathbb{P}^1 \times \mathbb{P}^1$ and $0 \in \mathbb{P}^1$. Consider the morphism $C \to S = \mathbb{P}^1$ that projects the component $\mathbb{P}^1 \times \mathbb{P}^1$ onto the first factor and contracts the component $\mathbb{P}^1$ to $0 \in \mathbb{P}^1$.



Although all fibers of this morphisms are nodal curves we can certainly not say that the fiber over $0 \in S$ is the "limit" of the nearby fibers, i. e. that we have a continuously varying family of nodal curves: the correct limit would of course be obtained by leaving out the additional component $\mathbb{P}^1$ in C.

So we need to define a property of morphisms that ensures that the curves over any point are in fact the "limits" of the nearby fibers, or in other words that there are no components of C that lie only in special fibers. This property of morphisms is called *flatness*. There is a geometric and an algebraic way to define it. We will give both definitions and explain briefly why the two notions are the same.

Definition 1.3.8. Let $f : X \to S$ be a morphism of schemes, and assume that S is reduced.

(i) If S is a smooth curve then f is called **(geometrically) flat** if no component of X is mapped to a single point in S. Here by *component* we mean an irreducible or embedded component, i. e. (in the affine picture) the subvarieties of X occurring in the primary decomposition of the ring that defines X.

(ii) For general S we say that f is (geometrically) flat if it satisfies the condition of (i) after pull-back to any smooth curve, i. e. if for every morphism $C \to S$ from a smooth curve C to S the induced morphism $X \times_S C \to C$ is (geometrically) flat as in (i).

Example 1.3.9.

(i) The family of example 1.3.1 obtained by blowing up a point in the trivial family $S \times \mathbb{P}^1 \to S$ is geometrically flat by part (i) of the definition since the blow-up of $S \times \mathbb{P}^1$ is irreducible and maps surjectively onto the base S.
(ii) The morphism of example 1.3.7 is obviously not flat.
(iii) The blow-up morphism $\pi : \tilde{\mathbb{P}}^2 \to \mathbb{P}^2$ of a point P in the plane is not flat: if $L \to \mathbb{P}^2$ is the inclusion morphism of a line in $\mathbb{P}^2$ through P then the pull-back of π to L has the exceptional divisor as an irreducible component that is mapped to P.

Note that the geometric definition of flatness above is only applicable to morphisms to a reduced scheme. The algebraic definition that we give now does not have this disadvantage. It is however not very intuitive and in general difficult to check explicitly in concrete examples.

Definition 1.3.10. Let R be a ring. An R-module M is called **flat** if for every injective R-module homomorphism $M_1 \to M_2$ the induced homomorphism $M_1 \otimes_R M \to M_2 \otimes_R M$ is also injective. In the same way we call a sheaf of $\mathcal{O}_S$-modules $\mathcal{F}$ on a scheme S flat if for every injective morphism $\mathcal{F}_1 \to \mathcal{F}_2$ of sheaves of $\mathcal{O}_S$-modules the induced morphism $\mathcal{F}_1 \otimes \mathcal{F} \to \mathcal{F}_2 \otimes \mathcal{F}$ is also injective.

A morphism $f : X \to S$ of schemes is called **(algebraically) flat** if $f_* \mathcal{O}_X$ is a flat sheaf of $\mathcal{O}_S$-modules.

Remark 1.3.11. By [G] lemma 7.2.7 (ii) algebraic flatness can be checked on affine open subsets. More precisely, if $X = \operatorname{Spec} M$ and $S = \operatorname{Spec} R$ are affine (and thus M has the structure of an R-module by the morphism f) then f is algebraically flat if and only if M is a flat R-module. In the general case it is sufficient to cover X and S by such affine open subsets and check flatness on them.

Remark 1.3.12. One can show that to check that an R-module M is flat it suffices to consider injective R-module homomorphisms of the form $\mathfrak{p} \to R$ where $\mathfrak{p}$ is a prime ideal of R, i. e. to prove that $\mathfrak{p} \otimes_R M \to M$ is injective for every prime ideal $\mathfrak{p} \subset R$. Let us briefly sketch the proof of this statement even if it uses some facts from commutative algebra that we have not developed here. Consider the exact sequence

$$0 \to \mathfrak{p} \to R \to R/\mathfrak{p} \to 0$$

for some prime ideal $\mathfrak{p}$ of R. If we tensor this exact sequence with M the resulting sequence

$$\mathfrak{p} \otimes_R M \to R \otimes_R M \to R/\mathfrak{p} \otimes_R M \to 0$$

is in general only "right exact", i. e. the first homomorphism need not be injective. Similarly to the theory of cohomology of sheaves there is a natural way to extend this sequence to the left to a long exact sequence

$$\cdots \to \operatorname{Tor}^2(R/\mathfrak{p}, M) \to \operatorname{Tor}^1(\mathfrak{p}, M) \to \operatorname{Tor}^1(R, M) \to \operatorname{Tor}^1(R/\mathfrak{p}, M) \to \mathfrak{p} \otimes_R M \to R \otimes_R M \to R/\mathfrak{p} \otimes_R M \to 0,$$

where the R-modules $\operatorname{Tor}^i(\,\cdot\,, M)$ are the so-called **torsion modules**. Torsion modules always vanish if one of their entries is the base ring, so we get an exact sequence

$$0 \to \operatorname{Tor}^1(R/\mathfrak{p}, M) \to \mathfrak{p} \otimes_R M \to R \otimes_R M \to R/\mathfrak{p} \otimes_R M \to 0.$$

So if we know that $\mathfrak{p} \otimes_R M \to M$ is injective then this means that $\operatorname{Tor}^1(R/\mathfrak{p}, M) = 0$.

Now if $M_1 \to M_2$ is any injective R-module homomorphism then one can show that there is always a so-called *composition series* (see [G] remark 9.1.1)

$$M_1 = N_0 \subset N_1 \subset \cdots \subset N_k = M_2$$

where each quotient is of the form $N_i/N_{i-1} = R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$. Tensoring the exact sequence

$$0 \to N_{i-1} \to N_i \to R/\mathfrak{p}_i \to 0$$

with M we get an exact sequence

$$\cdots \to \operatorname{Tor}^1(R/\mathfrak{p}_i, M) \to N_{i-1} \otimes_R M \to N_i \otimes_R M \to R/\mathfrak{p}_i \otimes_R M \to 0.$$

But we know that $\operatorname{Tor}^1(R/\mathfrak{p}_i, M) = 0$, hence it follows that $N_{i-1} \otimes_R M \to N_i \otimes_R M$ is injective for all i. So $M_1 \otimes_R M \to M_2 \otimes_R M$ is injective as well, i. e. M is a flat R-module.

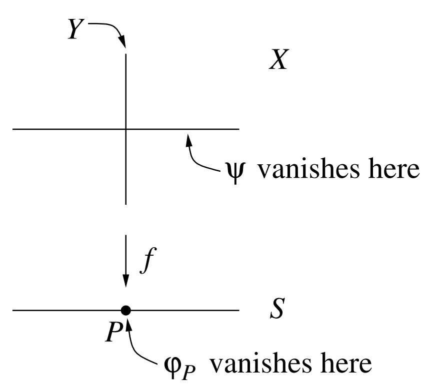
Proposition 1.3.13. *Let $f : X \to S$ be a morphism of schemes, and assume that S is reduced. Then f is geometrically flat if and only if it is algebraically flat.*

Sketch proof. We will assume for simplicity that S is a smooth curve and leave the general case as an exercise. We will also assume for simplicity that X is reduced and thus all components of X in the sense of definition 1.3.8 are irreducible components.

As flatness is a local property we can assume that both $X = \operatorname{Spec} M$ and $S = \operatorname{Spec} R$ are affine. The morphism f then gives M the structure of an R-module.

First let us assume that f is algebraically but not geometrically flat; we want to arrive at a contradiction. As f is not geometrically flat there is a component Y of X that maps to a point $P \in S$. Consider the injective R-module homomorphism $I(P) \to R$ where $I(P)$ denotes the ideal of P; we will show that the induced homomorphism $I(P) \otimes_R M \to R \otimes_R M = M$ is not injective, in contradiction to algebraic flatness.

To do so let $\varphi_P \in I(P)$ be a function on S that vanishes at P with multiplicity 1 (see [G] lemma 7.5.6), and let $\psi \in M$ be a non-zero function on X that vanishes on every component of X except Y. Then the function $\varphi_P \cdot \psi \in M$ is obviously zero, but the tensor product $\varphi_P \otimes \psi \in I(P) \otimes_R M$ is not. This shows that f cannot be algebraically flat.



Conversely, let us assume now that f is geometrically but not algebraically flat. By remark 1.3.12 there must then be a prime ideal $I \subset R$ such that $I \otimes_R M \to M$ is not injective. As S is a curve the ideal I must be of the form $I(P)$ for some point $P \in S$.

As above let $\varphi_P \in I(P)$ be a function that vanishes at P with multiplicity 1 and is non-zero at all other points of S (we may have to shrink S to achieve this). Then $I(P)$ is generated by φ_P. In other words, we can write every element of $I(P) \otimes_R M$ in the form $\varphi_P \otimes \psi$ for some $\psi \in M$.

By assumption we have $\varphi_P \cdot \psi = 0 \in M$ and $\varphi_P \otimes \psi \neq 0 \in I(P) \otimes_R M$. In particular we have $\psi \neq 0 \in M$, so ψ must be non-zero on at least one component Y of X. It follows that then φ_P is zero on Y. But this means that Y maps entirely to P, in contradiction to geometric flatness. □

Construction 1.3.14. We are now ready to set up the moduli functor $\bar{M}_{0,n}$ for stable n-pointed rational curves.

(i) For any base scheme S we set

$$\bar{M}_{0,n}(S) = \left\{ \begin{array}{l} (C, x_1, \dots, x_n) \ ; \ C \to S \text{ is a flat morphism,} \\ \text{and the } x_1, \dots, x_n : S \to C \text{ are disjoint sec-} \\ \text{tions such that all geometric fibers are sta-} \\ \text{ble } n\text{-pointed rational curves} \end{array} \right\} \Big/ \text{isomorphisms.}$$

(ii) For any morphism $f : S \to S'$ of schemes and any $(C', x'_1, \dots, x'_n) \in \bar{M}_{0,n}(S')$ we define the pull-back $f^*(C', x'_1, \dots, x'_n)$ to be $(C, x_1, \dots, x_n) \in \bar{M}_{0,n}(S)$, where $C = C' \times_{S'} S$ is simply the pull-back $\mathbb{P}^1$-bundle and $x_i = f^* x'_i$ are the pulled-back sections of C over S.

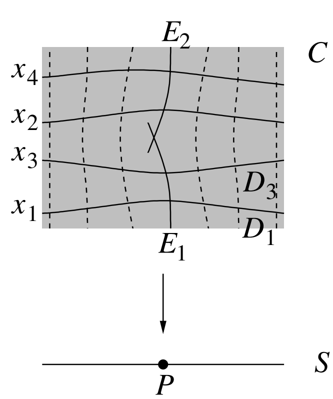
Let us start by considering the first non-trivial cases. It is clear that $\bar{M}_{0,3} = M_{0,3}$ is just a point. For 4 marked points we have the expected result from example 1.3.6:

Proposition 1.3.15. *$\bar{M}_{0,4} \cong \mathbb{P}^1$, with an isomorphism given by the cross ratio as in the proof of lemma 1.2.11, together with the three special points $0, 1, \infty$ as in example 1.3.6.*

Proof. For any base scheme S we will set up a one-to-one correspondence between morphisms $S \to \mathbb{P}^1$ and families in $\bar{M}_{0,4}(S)$.

First of all let us construct a family over $\mathbb{P}^1$ corresponding to the identity morphism $\mathrm{id}_{\mathbb{P}^1}$, i. e. a flat family whose fiber over the point $t \in \mathbb{P}^1$ is the stable 4-pointed rational curve with cross ratio t (and one of the special curves of example 1.3.6 if t is 0, 1, or ∞). In fact, this is easily done using the idea of example 1.3.1: we take the trivial bundle $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ with the 4 sections 0, 1, ∞, and t, and blow up the three points $(0,0)$, $(1,1)$, and (∞,∞) in $\mathbb{P}^1 \times \mathbb{P}^1$. If now $S \to \mathbb{P}^1$ is any morphism we can simply obtain the desired family over S by pulling back the above family over $\mathbb{P}^1$.

Conversely, let $(C, x_1, \dots, x_4) \in \bar{M}_{0,4}(S)$ be a family over S. We have to construct the corresponding morphism $f : S \to \mathbb{P}^1$. For simplicity we will assume that S is a smooth curve. The morphism f can be constructed locally around every point $P \in S$. By lemma 1.2.11 we know that the desired morphism exists if the fiber of $(C, x_1, \dots, x_4)$ over P is smooth. Moreover, the morphism clearly exists if $(C, x_1, \dots, x_4)$ is a constant family. So we may assume without loss of generality that the fiber of $(C, x_1, \dots, x_4)$ over P is reducible, say with x_1, x_3 on one component and x_2, x_4 on the other, whereas all other fibers are smooth.



If we consider C as a surface we denote the divisors $x_1(S) \subset C$ and $x_3(S) \subset C$ by D_1 and D_3, respectively. Note that C as a surface may or may not be singular at the point where the two components E_1 and E_2 of the fiber over P intersect (see exercise 1.6.2 (iii) for an example of the singular case). However C is smooth away from this point in any case. In particular, the Weil divisors D_1 and D_3 are Cartier divisors and thus correspond to line bundles $\mathcal{L}_1$ and $\mathcal{L}_3$ on C (see [G] section 9.3).

We claim that the line bundles $\mathcal{L}_1$ and $\mathcal{L}_3$ are in fact isomorphic, i. e. that D_1 and D_3 are linearly equivalent Cartier divisors. To see this let us first restrict the given family to $S\backslash\{P\}$. Over this open subset we have a family of smooth 4-pointed curves, so by lemma 1.2.11 it is isomorphic to a family of the form $S \times \mathbb{P}^1$ where x_1 and x_3 are the constant sections 0 and ∞. In particular, D_1 and D_3 are linearly equivalent on this restricted family. Filling back in the point P again we therefore see that we must have an equality

$$D_1 - D_3 = aE_1 + bE_2 \qquad (*)$$

of Cartier divisor classes on C for some $a, b \in \mathbb{Z}$. Next, let φ_P be a local function on S around P that has a simple zero at P and no other zeros (see [G] lemma 7.5.6). Pulling back this function to C we obtain a regular function with divisor $E_1 + E_2$. In other words, the divisor $E_1 + E_2$ is linearly equivalent to 0. We may therefore subtract $b(E_1 + E_2)$ from the right hand side of $(*)$ and obtain

$$D_1 - D_3 = (a-b)E_1$$

on C. Now we intersect this Cartier divisor with the Weil divisor E_2 to obtain $0 - 0 = (a-b)E_1 \cdot E_2$. As $E_1 \cdot E_2 > 0$ we conclude that $a - b = 0$ and thus that D_1 and D_3 are linearly equivalent on C. In other words, the equations σ_1 and σ_3 of D_1 and D_3 are sections of the same line bundle $\mathcal{L}$ on C.

By [G] lemma 7.5.14 we therefore obtain a morphism

$$\sigma = (\sigma_1 : \sigma_3) : C \to \mathbb{P}^1$$

(note that σ_1 and σ_3 are nowhere simultaneously zero since D_1 and D_3 do not intersect). By construction it has the property that D_1 and D_3 map to $(0:1) = 0$ and $(1:0) = \infty$, respectively. Moreover, σ must be constant on $E_2 \cong \mathbb{P}^1$ since it is nowhere zero there (i. e. E_2 does not intersect D_1). So if we define a morphism $f : S \to \mathbb{P}^1$ by

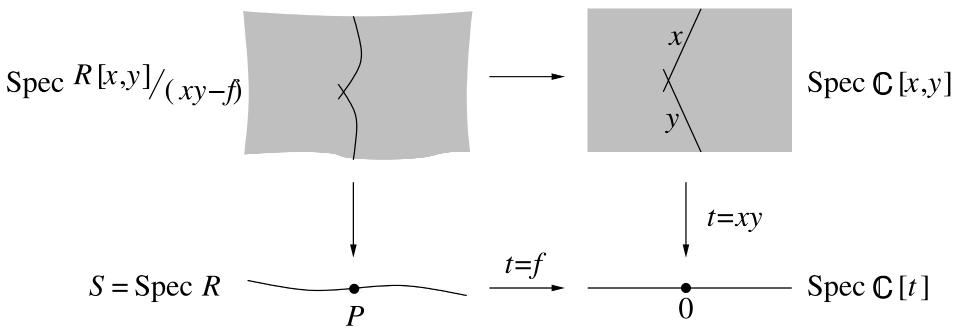
$$f(Q) = \frac{\sigma(x_4(Q))}{\sigma(x_2(Q))}$$

then $f(P) = 1$ (since σ is constant on E_2), and $f(Q)$ for $Q \neq P$ is simply the cross ratio of the four points $x_1(Q), \ldots, x_4(Q)$ (since the cross ratio of the four points $0, x_2, \infty, x_4$ on $\mathbb{P}^1$ is $\frac{x_4}{x_2}$). So we have constructed the desired morphism. □

We will extend this lemma to the case of more marked points in section 1.4.

Remark 1.3.16. In the proof of the first part of proposition 1.3.15 we have constructed a family over the variety $\bar{M}_{0,4} \cong \mathbb{P}^1$ whose fiber over a point P is precisely the stable curve parametrized by the moduli point P. Such a family is called a **universal family**. We have seen in the above proof that every other family can be obtained from this universal family by a suitable pull-back (hence the name).

Remark 1.3.17. Proposition 1.3.15 allows us to make a statement about the local structure of a family of nodal curves around a node in a fiber. More precisely, let $(C, x_1, \ldots, x_4) \in \bar{M}_{0,n}(S)$ be a family of stable 4-pointed rational curves over an affine base scheme $S = \operatorname{Spec} R$, corresponding to a morphism $S \to \bar{M}_{0,4} = \mathbb{P}^1$. Assume that $P \in S$ is a point such that the fiber of C over P is reducible with two components, i. e. such that $f(P) \in \{0, 1, \infty\}$. Let us assume for simplicity that $f(P) = 0$. We know that the family $(C, x_1, \ldots, x_4)$ must be obtained from the universal family over $\mathbb{P}^1$ by pull-back along the morphism $S \to \mathbb{P}^1$. As the universal family is simply the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ in three points it has local coordinates x and y around a node in the fiber over 0, with the morphism to the base given by $t = xy$ (where t is a local coordinate on the base $\mathbb{P}^1$). The morphism $S \to \mathbb{P}^1$ is given by the assignment $t = f$ for some $f \in R$. So we obtain the following local picture around the node in the fiber over P:



In other words, the family $(C, x_1, \dots, x_4)$ is locally around a node in a fiber of the form $C = \operatorname{Spec} R[x,y]/(xy - f)$ for some $f \in R$ with $f(P) = 0$. One can show that the same statement is true for stable rational curves with more than 4 marked points (and in fact for any family of rational nodal curves).

1.4. **Representability of the moduli functor $\bar{M}_{0,n}$.** In this section we want to extend proposition 1.3.15 to the case of more marked points, i. e. prove that the functor $\bar{M}_{0,n}$ is representable by a smooth projective variety of dimension $n-3$ for every $n \geq 3$. We will actually give an explicit construction of these varieties, although (in the spirit of remark 1.2.9) the details of this construction are not needed to do computations on the varieties $\bar{M}_{0,n}$.

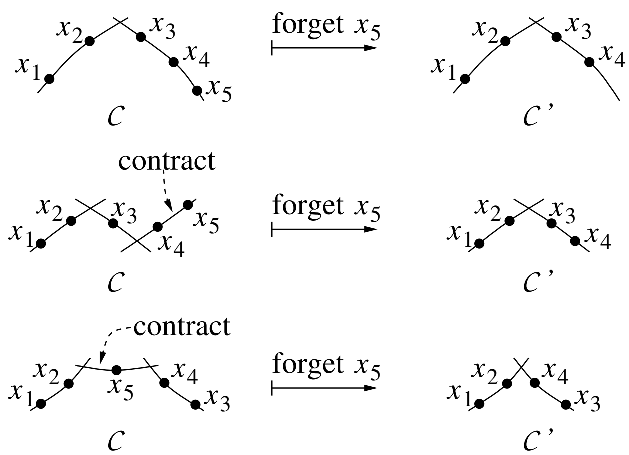
The key ingredient in the proof of representability is the construction of the so-called *forgetful maps*. Roughly speaking their idea is that we can obtain a stable n-pointed curve from an $(n+1)$-pointed curve by simply forgetting one of the marked points. One just has to be careful if forgetting the marked point leaves us with a component of the curve with only two special points, i. e. with an unstable curve.

Construction 1.4.1. Let $n \geq 3$, and let $\mathcal{C} = (C, x_1, \dots, x_{n+1})$ be a stable $(n+1)$-pointed rational curve. Denote by C_0 the irreducible component of C on which the last marked point x_{n+1} lies. We construct an associated stable n-pointed rational curve $\mathcal{C}'$ as follows:

(i) If the component C_0 has at least 4 special points then we simply set $\mathcal{C}'$ to be the curve $(C, x_1, \dots, x_n)$ obtained by forgetting the last marked point x_{n+1}.

(ii) If the component C_0 has only 3 special points (one of which must be x_{n+1}) then we let C' be the curve obtained from C by contracting C_0 to a point, and set $\mathcal{C}' = (C', x_1, \dots, x_n)$. Note that the condition $n \geq 3$ ensures that at least one of the two remaining special points of C_0 is a node, so that the contraction of C_0 cannot give rise to two coinciding marked points.

The curve $\mathcal{C}'$ is called the curve obtained from $\mathcal{C}$ by **forgetting** the last marked point. The corresponding (set-theoretic) map $\bar{M}_{0,n+1} \to \bar{M}_{0,n}$ is called the **forgetful map** that forgets the last marked point.



Obviously, there is a morphism $\mathcal{C} \to \mathcal{C}'$ in each case (i. e. a morphism $f : C \to C'$ such that $f(x_i) = x_i'$ for all i) that simply "contracts the unstable components".

Exercise 1.6.6 shows that this construction can be done in the same way in families: if $(C, x_1, \dots, x_{n+1}) \in \bar{M}_{0,n+1}(S)$ is a family of stable $(n+1)$-pointed rational curves over a base scheme S then the above construction gives rise to family $(C', x_1', \dots, x_n') \in \bar{M}_{0,n}(S)$ of stable n-pointed rational curves together with a morphism $f : C \to C'$. (Note that this is a non-trivial statement since in general contractions of components will be necessary in some but not all curves in the family. It is not clear a priori that this gives rise to a well-defined morphism of the families.)

Remark 1.4.2. Instead of the last marked point we can of course also forget any other given marked point. We can also compose several forgetful maps, i. e. forget any given subset of the marked points, as long as we keep at least 3 marked points in the end. It is easy to see that the result in this case does not depend on the order in which the selected marked points are forgotten.

Example 1.4.3.

(i) Note that $\bar{M}_{0,3}$ is just a point, i. e. every family $(C, x_1, \dots, x_4)$ of 4-pointed stable rational curves gives rise to the constant family $(\mathbb{P}^1, 0, 1, \infty)$ of 3-pointed stable rational curves after forgetting the last marked point.

(ii) Consider the family of 5-pointed stable curves as in exercise 1.6.2 (ii), possibly after extending to $t = 0$. Then the family of 4-pointed stable curves obtained by forgetting the fourth marked point is precisely the family of exercise 1.6.2 (iii).

Remark 1.4.4. Let $\mathcal{C} = (C, x_1, \dots, x_n)$ be an n-pointed rational stable curve. Note that there is a natural bijection

$$\left\{ \begin{array}{l} (n+1)\text{-pointed rational stable curves mapping to } \mathcal{C} \\ \text{when forgetting the last marked point} \end{array} \right\} \stackrel{1:1}{\longleftrightarrow} C$$

Then there is a natural bijection between C and $(n+1)$-pointed stable curves mapping to $\mathcal{C}$ when forgetting the last marked point.

This motivates the following proposition.

Proposition 1.4.5. *Assume that the functor $\bar{M}_{0,n}$ is representable, so that in particular there is a universal family $\bar{C}_{0,n} \to \bar{M}_{0,n}$. Then the functor $\bar{M}_{0,n+1}$ is representable by the scheme $\bar{C}_{0,n}$.*

Proof. We have to show that there is a one-to-one correspondence

$$\left\{ \begin{array}{l} \text{families } (C, x_1, \dots, x_{n+1}) \text{ of stable } (n+1)\text{-} \\ \text{pointed rational curves over } S \end{array} \right\} \longleftrightarrow \left\{ \text{ morphisms } S \to \bar{C}_{0,n} \right\}$$

for every base scheme S.

"$\longrightarrow$":

$$\begin{array}{ccccc} C & \longrightarrow & C' & \longrightarrow & \bar{C}_{0,n} \\ & \searrow & \downarrow & & \downarrow \\ & & S & \longrightarrow & \bar{M}_{0,n} \end{array}$$

"$\longleftarrow$": as usual it suffices to construct universal family; rest then by pull-back.

$$\begin{array}{ccc} F & \longrightarrow & \bar{C}_{0,n} \\ \downarrow & & \downarrow \\ \bar{C}_{0,n} & \longrightarrow & \bar{M}_{0,n} \end{array}$$

This is OK except where the $(n+1)$-st marked point is a node or coincides with another one. □

Corollary 1.4.6. *$\bar{M}_{0,n}$ is a smooth projective variety of dimension $n-3$ for all $n \geq 3$.*

Proof. □

1.5. **Intersection theory on $\bar{M}_{0,n}$.** Having just defined the moduli functor of stable rational n-pointed curves our next task would of course be to prove its representability. However we will rather continue in the spirit of remark 1.2.9: assume that representability holds and study the properties of the moduli space that we can read off from the functor alone.

More precisely, we will study the consequences of the following theorem whose proof we will sketch later in section 1.4:

Theorem 1.5.1. *$\bar{M}_{0,n}$ is a smooth projective variety of dimension $n-3$.*

Example 1.5.2. Pull-back of boundary divisor: sum of two parts.

Example 1.5.3. Forget more than 1 point. Both desciption of forgetful map and pull-back of divisors. Example: small WDVV, maybe for 6 marked points. Thus gives relations between boundary divisors in $\bar{M}_{0,n}$. In fact, get all relations this way.

1.6. **Exercises.**

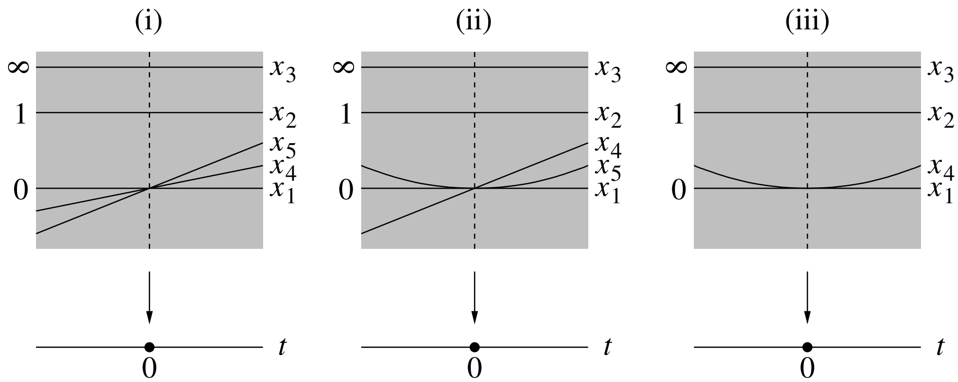
Exercise 1.6.1. What is the maximum number of irreducible components that a rational n-pointed stable curve can have?

Show that the number of rational n-pointed stable curves with this maximum number of components is finite.

Exercise 1.6.2. Consider the following families of smooth rational n-pointed curves:

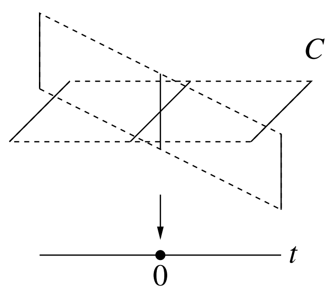
(i) $n = 5$, $(S \times \mathbb{P}^1, 0, 1, \infty, t, 2t)$;
(ii) $n = 5$, $(S \times \mathbb{P}^1, 0, 1, \infty, t, t^2)$;
(iii) $n = 4$, $(S \times \mathbb{P}^1, 0, 1, \infty, t^2)$;

where S is a suitable open subset of $\mathbb{A}^1$ with coordinate t.



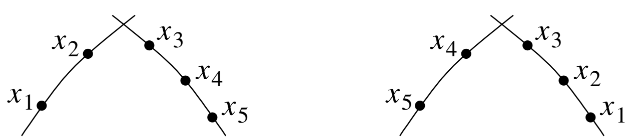
What are their "limit stable curves" as $t \to 0$? In other words, find their extensions to families of stable rational n-pointed curves over $S \cup \{0\}$.

Exercise 1.6.3. Let $C \subset \mathbb{P}^3 \times \mathbb{A}^1$ be the union of the surfaces $\{x_0 = x_1 = 0\}$ and $\{x_0 - tx_3 = x_2 = 0\}$, where x_0, x_1, x_2, x_3 and t are the coordinates of $\mathbb{P}^3$ and $\mathbb{A}^1$, respectively. Show that the morphism $C \to \mathbb{A}^1$ is a family of disconnected curves with two components approaching each other as $t \to 0$, but that the limit for $t = 0$ is not just a reducible curve with two components.



Conclude that "separating the two branches at a node of a stable curve" is *not* a flat deformation, i. e. it cannot occur in the moduli functor of stable curves.

Exercise 1.6.4. Consider the following stable 5-pointed trees:

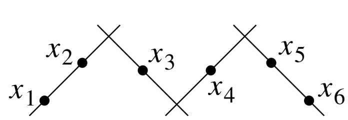


Γ_1 Γ_2

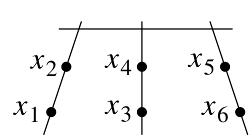
(i) What is the dimension of $[\Gamma_1]$ and $[\Gamma_2]$ in $\bar{M}_{0,5}$?
(ii) List all stable curves with more than one node that are contained in $[\Gamma_1]$ resp. $[\Gamma_2]$. Conclude that $[\Gamma_1]$ and $[\Gamma_2]$ intersect in exactly one point in $\bar{M}_{0,5}$.
(iii) Find an open neighborhood of this point in $\bar{M}_{0,5}$. Prove that $[\Gamma_1]$ and $[\Gamma_2]$ intersect transversally in this point (i. e. with multiplicity 1).
(iv) Are $[\Gamma_1]$ and $[\Gamma_2]$ the same cycle in the Chow group $A_*(\bar{M}_{0,5})$?

Exercise 1.6.5.

(i) Write the cycles



and



in $A_0(\bar{M}_{0,6})$ as an intersection of three boundary divisors in $\bar{M}_{0,6}$.

(ii) Show in general that every cycle of the form $[\Gamma] \in A_*(\bar{M}_{0,n})$ for a stable n-pointed tree Γ with k nodes can be written uniquely as a product of k boundary divisors.

Exercise 1.6.6. In this exercise we will construct the dualizing sheaf of a nodal curve and use it to prove that forgetting a marked point as in construction 1.4.1 is well-defined in families. The key idea is contained in part (v) below.

(i) Let $\varphi \in \Omega_C \otimes K(C)$ be a rational differential form on a smooth curve C. Pick a point $P \in C$ such that $\operatorname{ord}_P \varphi \geq -1$, i. e. φ is either regular or has a pole of order 1 at P. Choose a regular function φ_P in a neighborhood U of P with a simple zero at P and no other zeros on U (see [G] lemma 7.5.6). Show that we can then write $\varphi = f \cdot \varphi_P^{-1} d\varphi_P$ on U for a regular function $f \in \mathcal{O}_C(U)$, and that the value $f(P)$ does not depend on the choice of φ_P. This value is then called the **residue** of φ at P and denoted $\operatorname{res}_P \varphi$.

(ii) Let C be a rational nodal curve, and denote by $\operatorname{Pic} C$ the group of line bundles on C. Let $C_1, \dots, C_r$ be the irreducible components and $P_1, \dots, P_k$ the nodes of C. Show that there is an isomorphism

$$\operatorname{Pic} C \cong \operatorname{Pic} C_1 \oplus \cdots \oplus \operatorname{Pic} C_r.$$

In other words, giving a line bundle on C is the same thing as giving line bundles on all components C_i.

Can you find a corresponding statement if the nodal curve C is not necessarily rational?

(iii) Using (i) and (ii) show that every nodal curve C has a unique line bundle $\omega_C \in \operatorname{Pic} C$ whose sections over an open subset $U \subset C$ are given by collections of rational differential forms φ_i on $C_i \cap U$ such that for every $P \in U$ we have:
- if $P \in C_i$ is a smooth point of C then φ_i is regular at P;
- if $P \in C_i \cap C_j$ is a node of C then $\operatorname{ord}_P \varphi_i \geq -1$, $\operatorname{ord} \varphi_j \geq -1$, and $\operatorname{res}_P \varphi_i + \operatorname{res}_P \varphi_j = 0$.

The line bundle ω_C is called the **dualizing sheaf** of C.

(iv) (The result of this part is not needed for the rest of the exercise.) The dualizing sheaf ω_C can be thought of as a generalization of the canonical bundle of a smooth curve in the following sense:
- $h^0(\omega_C) = h^1(\mathcal{O}_C)$ is the genus g of the curve;
- for every line bundle $\mathcal{L}$ on C we have the Riemann-Roch theorem

$$h^0(C, \mathcal{L}) - h^0(C, \omega_C \otimes \mathcal{L}^\vee) = \deg \mathcal{L} + 1 - g.$$

(v) Let $(C, x_1, \dots, x_{n+1})$ be a stable $(n+1)$-pointed rational curve. Show that the stable n-pointed rational curve obtained by forgetting the last marked point x_{n+1} is

$$C' = \operatorname{Proj}\left(\bigoplus_{k \geq 0} H^0(C, \omega_C(x_1 + \cdots + x_n)^{\otimes k}) \right),$$

and that there is a morphism $f : C \to C'$ that contracts the unstable component (if there is any).

(vi) Let X be a variety, and let $\mathcal{L}_1, \mathcal{L}_2$ be two line bundles on X. If $Y \subset X$ is a subvariety of codimension at least 2 such that $\mathcal{L}_1|_{X \setminus Y} \cong \mathcal{L}_2|_{X \setminus Y}$ then $\mathcal{L}_1 \cong \mathcal{L}_2$.

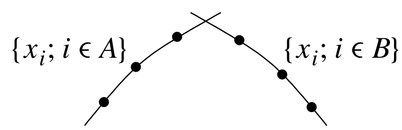
(vii) Now let $(C, x_1, \dots, x_{n+1}) \in \bar{M}_{0,n+1}(S)$ be a family of stable $(n+1)$-pointed rational curves over a base variety S. Show that there is a unique line bundle $\omega_{C/S}$ (called the **relative dualizing sheaf**) such that
- the restrictions of $\omega_{C/S}$ to the fibers of the morphism $C \to S$ agree with the dualizing sheaves constructed above;

- $\omega_{C/S}$ is isomorphic to the sheaf $\Omega_{C/S}$ of relative differential forms away from the nodes of the fibers of the morphism $C \to S$.

(Hint: If $S = \operatorname{Spec} R$ is affine then by remark 1.3.17 the morphism $C \to S$ is locally around a node of a fiber of the form $\operatorname{Spec} R[x,y]/(xy - f) \to \operatorname{Spec} R$ for some $f \in R$. On such an open neighborhood the subsheaf of $\Omega_{C/S} \otimes K(C)$ generated by $\frac{dx}{x}$ and $\frac{dy}{y}$ gives a line bundle with the desired properties.)

(viii) Using the above results conclude that forgetting a marked point is well-defined in families, i. e. if $(C, x_1, \dots, x_{n+1}) \in \bar{M}_{0,n+1}(S)$ is a family of stable $(n+1)$-pointed rational curves over a base variety S then forgetting the last marked point gives rise to a family $(C', x'_1, \dots, x'_n) \in \bar{M}_{0,n}(S)$ of stable n-pointed rational curves together with a morphism $f : C \to C'$ that contracts the unstable components in the fibers.

Exercise 1.6.7. For any decomposition $A \cup B = \{1, \dots, n\}$ with $|A|, |B| \geq 2$ denote by $D(A;B) \in A_{n-4}(\bar{M}_{0,n})$ the class of the boundary divisor with marked points $\{x_i \,;\, i \in A\}$ on one component and $\{x_i \,;\, i \in B\}$ on the other.



(i) For any A, B as above compute the push-forward $\pi_* D(A;B) \in A_{n-4}(\bar{M}_{0,n-1})$, where $\pi : \bar{M}_{0,n} \to \bar{M}_{0,n-1}$ is the morphism that forgets the last marked point.

(ii) For any A, B as above compute the pull-back $\pi^* D(A;B) \in A_{n-3}(\bar{M}_{0,n+1})$, where $\pi : \bar{M}_{0,n+1} \to \bar{M}_{0,n}$ is the morphism that forgets the last marked point.

(iii) Show that
$$\sum_{1,2 \in A;\, 3,4 \in B} D(A;B) = \sum_{1,3 \in A;\, 2,4 \in B} D(A;B)$$
in $A_{n-4}(\bar{M}_{0,n})$.

Exercise 1.6.8. As usual let $\bar{C}_{0,n} \to \bar{M}_{0,n}$ be the universal curve. For any $i = 1, \dots, n$ the sheaf $x_i^* \Omega_{\bar{C}_{0,n}/\bar{M}_{0,n}}$ is a line bundle on $\bar{M}_{0,n}$ whose fiber at a point $(C, x_1, \dots, x_n) \in \bar{M}_{0,n}$ is canonically isomorphic to the cotangent space $T^\vee_{C,x_i}$. We denote the divisor corresponding to this line bundle by $\psi_{i,n}$ (or ψ_i if the number n of marked points is clear from the context). It is usually called the i-th **cotangent line class**.

(i) Compute the degree of the divisor ψ_1 on $\bar{M}_{0,4} \cong \mathbb{P}^1$.

(ii) Let $\pi : \bar{M}_{0,n} \to \bar{M}_{0,n-1}$ be the morphism that forgets the last marked point. Show that
$$\psi_{1,n} = \pi^* \psi_{1,n-1} + D(\{1,n\}; \{2, \dots, n-1\})$$
in $A_{n-4}(\bar{M}_{0,n})$, and conclude that
$$\psi_{1,n} = \sum_{1 \in A;\, 2,3 \in B} D(A;B).$$

(iii) Let $D := D(\{1,2\};\, \{3, \dots, n\})$. Show that
$$D \cdot D = -\psi \cdot D$$
in $A_{n-5}(\bar{M}_{0,n})$, where ψ denotes the cotangent line class at the gluing point in $D \cong \bar{M}_{0,n-1}$.

Exercise 1.6.9. Let $k_1, \dots, k_n$ be non-negative integers with $k_1 + \dots + k_n = n - 3$. Show by induction on n that
$$\psi_1^{k_1} \cdot \cdots \cdot \psi_n^{k_n} = \frac{(n-3)!}{k_1! \cdots k_n!}$$

on $\bar{M}_{0,n}$. (Hint: At least one of the numbers k_i must obviously be 0 or 1, say k_n. Using the morphism $\bar{M}_{0,n} \to \bar{M}_{0,n-1}$ that forgets the last marked point one can then reduce the given intersection product to a similar product on $\bar{M}_{0,n-1}$.)

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[KV] J. Kock, I. Vainsencher, *Kontsevich's formula for rational plane curves*, electronic preprint (1999), `http://www.dmat.ufpe.br/~israel/kontsevich.html`.