0. INTRODUCTION

Enumerative geometry is the branch of algebraic geometry concerned with counting curves in varieties that satisfy some given conditions. We give some (classical and modern) examples of enumerative problems and sketch how they can often be reduced to the computation of intersection products on suitable moduli spaces of curves.

0.1. **What is enumerative geometry?** Let X be a complex variety, usually assumed to be smooth and projective. The goal of enumerative geometry is simply to count curves in X that satisfy some given conditions. These conditions can be of various types: we can require that the curves have specified genus, specified degree, intersect given subvarieties of X, are tangent to a given subvariety of X, have certain singularities, and so on. The only requirement is that the conditions are chosen so that we expect a *finite* number of curves satisfying them. We are then asking for this finite number.

Let us illustrate these ideas by some examples.

Example 0.1.1. Probably the easiest enumerative question that one can ask is: how many lines are there in the projective plane $\mathbb{P}^2$ through two given (distinct) points? The answer here is obviously 1.

Example 0.1.2. Let us extend example 0.1.1 to conics, i. e. plane curves of degree 2. Note that a conic is uniquely given by its equation

$$a_0x_0^2 + a_1x_0x_1 + a_2x_0x_2 + a_3x_1^2 + a_4x_1x_2 + a_5x_2^2 = 0,$$

and conversely the a_i are determined up to a common scalar by the conic. So we can think of the projective space $\mathbb{P}^5$ with homogeneous coordinates a_i as the space of all conics. We say that $\mathbb{P}^5$ is a *moduli space* for plane conics.

Now let $P \in \mathbb{P}^2$ be a given point. Then a conic (determined by the a_i) passes through P if and only if the above equation is satisfied if we set x_0, x_1, x_2 to be the coordinates of P. This is obviously one linear condition in the coordinates a_i of the moduli space $\mathbb{P}^5$. As the moduli space is 5-dimensional we see that we get a finite number of conics if we require them to pass through 5 given points in the plane. In fact, we get exactly one such conic as the intersection of the 5 linear conditions in $\mathbb{P}^5$ is a single point.

There are two potential problems here that we should mention though:

(i) We should check that the linear equations in $\mathbb{P}^5$ given by the 5 incidence conditions are in fact independent, so that their intersection is really just a point (and not a higher-dimensional subspace).
(ii) Not all points in the moduli space $\mathbb{P}^5$ describe smooth conics. Some of them correspond to unions of two lines or even double lines. In other words, the “true” moduli space of smooth conics is not $\mathbb{P}^5$ itself but rather an open subset $U \subset \mathbb{P}^5$. The complement $\mathbb{P}^5 \backslash U$ is usually called the *boundary* of the moduli space. We cannot know a priori whether the point in the moduli space that is the intersection of the 5 linear conditions lies in U or not. In other words, it may be that there is no *smooth* conic through the 5 given points.

Both problems can actually arise for some special choices of the 5 points. For example, if we choose all 5 points to lie on a line then there is no smooth conic through these points at all, but on the other hand there is an infinite family of reducible conics through them (namely the line through the points together with any other line).

We will show now however that this cannot happen if we pick the 5 points *in general position*. This means: there is a dense open subset $V \subset (\mathbb{P}^2)^5$ such that for any $(P_1, \dots, P_5) \in V$

there is precisely one smooth conic through $P_1, \dots, P_5$. In fact, in this case we can say explicitly what this open subset V looks like: all we have to require is that no three of the marked points lie on a line. It is then obvious that there is no reducible conic through the five points. Moreover, if we had two distinct smooth conics through the points then these two conics would meet in five points, which is a contradiction to Bézout's theorem.

One would probably expect in general that the above problems (intersection products of too big dimension and components of the result in the boundary of the moduli space) do not occur if we pick the conditions on the curves in a general way. This is not true however; we will see a counterexample in example 0.1.6.

Example 0.1.3. Example 0.1.2 obviously extends to curves of higher degree: plane curves of degree d are parametrized by the projectivization of the vector space of homogeneous degree-d polynomials in 3 variables, which has dimension $\binom{d+2}{2} - 1$. Arguing as above we see that there is exactly 1 curve in $\mathbb{P}^2$ of degree d that passes through $\binom{d+2}{2} - 1$ general given points (see exercise 0.2.1).

The above examples were very easy because the moduli spaces and conditions were all linear. We usually express this by saying that the curves form a *linear system*. It is obvious then that the answer to our enumerative problem must be 1 since a zero-dimensional linear space is necessarily a single point. But in general of course neither the moduli space nor the conditions need be linear, and consequently the answer to an enumerative problem need not always be 1. Let us give some examples of this.

Example 0.1.4. In this example we want to answer the following question: how many *singular* plane cubic curves are there through 8 given points?

We have seen in example 0.1.3 that plane cubics are parametrized by a projective space $\mathbb{P}^9$. The 8 point conditions are again linear conditions in this $\mathbb{P}^9$, so what we have to analyze is the new condition that the curves be singular.

To do so define the function

$$F = a_0 x_0^3 + a_1 x_0^2 x_1 + a_2 x_0 x_1^2 + \cdots + a_9 x_2^3$$

describing a general cubic curve in $\mathbb{P}^2$ with coefficients a_i. Consider the variety

$$S = Z\left(\frac{\partial F}{\partial x_0}, \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}\right) \subset \mathbb{P}^9 \times \mathbb{P}^2$$

and its projection $\pi : S \to \mathbb{P}^9$. By the projective Jacobi criterion of [G] proposition 4.4.8 (ii) the fiber of S over a point $P \in \mathbb{P}^9$ is precisely the set of singular points of the cubic curve determined by the point P in the moduli space $\mathbb{P}^9$. So the image $\pi(S) \subset \mathbb{P}^9$ is the locus of singular cubic curves. Its class is easily determined: every equation $\frac{\partial F}{\partial x_i}$ is homogeneous of degree 1 in the coordinates of $\mathbb{P}^9$ and homogeneous of degree 2 in the coordinates of $\mathbb{P}^2$. So if we denote by $H \in A_*(\mathbb{P}^9)$ and $L \in A_*(\mathbb{P}^2)$ the class of a hyperplane in $\mathbb{P}^9$ and $\mathbb{P}^2$ respectively, the class of each zero locus of $\frac{\partial F}{\partial x_i}$ is $H + 2L$ (where we use the same letters H and L to denote the pull-back classes on $\mathbb{P}^9 \times \mathbb{P}^2$. The class of S is therefore

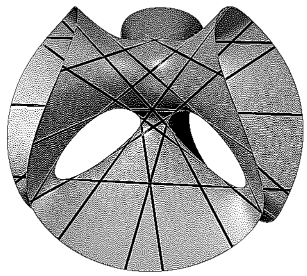
$$[S] = (H + 2L)^3 = H^3 + 6H^2L + 12HL^2 + 8L^3 \in A_8(\mathbb{P}^9 \times \mathbb{P}^2),$$

and so we conclude that

$$\pi_*[S] = 12H \in A_8(\mathbb{P}^9)$$

by the description of the push-forward of [G] construction 9.2.9. The condition of being a singular curve is therefore a hypersurface of degree 12 in the moduli space $\mathbb{P}^9$. As the 8 point conditions are again linear we see that there are 12 singular plane cubic curves through 8 given points.

Example 0.1.5. Let X be a smooth cubic surface in $\mathbb{P}^3$. We have seen in [G] section 4.5 and example 10.3.15 that there are exactly 27 lines in X. Let us briefly recall the intersection-theoretic computation that leads to this number.



Our moduli space is the 4-dimensional Grassmannian variety $G(1,3)$ of lines in $\mathbb{P}^3$. There is a tautological rank-2 subbundle F of the trivial bundle $\mathbb{C}^4$ on $G(1,3)$ whose fiber over a point $[L] \in G(1,3)$ (where $L \subset \mathbb{P}^3$ is a line) is precisely the 2-dimensional subspace of $\mathbb{C}^4$ whose projectivization is L. Dualizing, we get a surjective morphism of vector bundles $(\mathbb{C}^4)^\vee \to F^\vee$ that corresponds to restricting a linear function on $\mathbb{C}^4$ (or $\mathbb{P}^3$) to the line L. Taking the d-th symmetric power of this morphism we arrive at a surjective morphism $S^d(\mathbb{C}^4)^\vee \to S^d F^\vee$ that corresponds to restricting a homogeneous polynomial of degree d on $\mathbb{P}^3$ to L.

Now let $f = 0$ be the equation of X. By what we have just said the polynomial f determines a section of $S^3F^\vee$ whose set of zeros in $G(1,3)$ is precisely the set of lines that lie in X (i. e. the set of lines on which f vanishes). As $S^3F^\vee$ is a vector bundle of rank 4 we expect finitely many zeros of this section. Their number is given by [G] proposition 10.3.12 as the degree of the top Chern class $c_4(S^3F^\vee)$ on $G(1,3)$. This degree can be computed explicitly to be 27; see [G] example 10.3.15 for details.

It should be noted that — in the same way as in example 0.1.2 (i) — this computation does *not* show that the number of lines in X is actually finite; we have to prove this in some other way (see [G] section 4.5). Only then do we know that the Chern class computation above gives the correct answer.

Example 0.1.6. Let us now give an example of an enumerative problem where the naïve intersection-theoretic computation does not give the right answer. Consider again conics in $\mathbb{P}^2$ with associated moduli space $\mathbb{P}^5$. We have seen in example 0.1.2 that there is exactly one conic through 5 points (in general position) since incidence conditions with points are linear conditions in the moduli space $\mathbb{P}^5$.

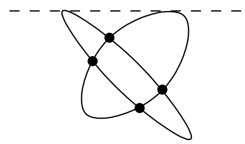
We will now replace some of the incidence conditions by tangency conditions. Let us first replace only one condition and ask: how many conics in $\mathbb{P}^2$ are tangent to a given line and moreover pass through 4 given points? Let us analyze the tangency condition. By a change of coordinates we may assume that the line is given by the equation $x_0 = 0$. Then a conic with equation

$$a_0x_0^2 + a_1x_0x_1 + a_2x_0x_2 + a_3x_1^2 + a_4x_1x_2 + a_5x_2^2 = 0$$

is tangent to this line if and only if the restriction of this equation to the line

$$a_3x_1^2 + a_4x_1x_2 + a_5x_2^2 = 0$$

has a double zero somewhere, i. e. if and only if the discriminant $a_4^2 - 2a_3a_5$ of this quadratic equation is zero. So we see that the tangency condition is a quadratic equation in the a_i. Intersecting this with 4 (linear) incidence conditions we conclude that there are exactly 2 conics that are tangent to a line and pass through 4 points.



Let us now replace all incidence conditions by tangencies and ask: how many plane conics are tangent to 5 given lines in general position? The naïve answer would be $2^5 = 32$ as the intersection of 5 quadratic conditions in $\mathbb{P}^5$. This is not true however as one can see from the theory of dual curves (see [G] exercise 4.6.11): the dual curve of a smooth conic is again a conic, and in the dual picture tangency conditions are translated into incidence conditions. So the number of conics tangent to 5 lines must be the same as the number of conics through 5 points, namely 1.

Why did the intersection-theoretic computation give the wrong answer? The problems arise from the points in $\mathbb{P}^5$ corresponding to double lines. Note that every double line intersects any other (distinct) line in one point with multiplicity two, so it counts as a tangent according to our definition above. The space of all conics tangent to the 5 given lines therefore includes the complete 2-dimensional space of double lines in $\mathbb{P}^5$. Hence the intersection-theoretic number 32 cannot be interpreted as the number of solutions to our enumerative problem.

Example 0.1.7. It happens frequently that a very simple enumerative question has a very complicated solution or is even still unsolved. For example, we can extend example 0.1.6 to higher degree and ask (similarly to example 0.1.3: how many plane curves are there that are tangent to $\binom{d+2}{2} - 1$ lines in general position? Although this question seems to be very similar to the ones that we have studied above its answer is still unknown.

Remark 0.1.8. After having studied a series of examples let us now summarize the general strategy to solve enumerative problems:

(1) Set up a moduli space that describes the curves one wants to study. The moduli space has to be compact (see below). It will therefore usually have "boundary points" that do not correspond to curves that one wants to count.
(2) Imposing the given conditions on the curve corresponds to an intersection product on the moduli space. These conditions have to be chosen so that the resulting intersection product is a cycle of dimension 0. As the moduli space is compact the degree of this 0-cycle is well-defined. It can be considered to be the "expected solution" of the enumerative problem.
(3) Finally we have to find out whether the geometric intersection of the conditions in (2) really has dimension 0 (i. e. the conditions are independent) and does not contain any points in the boundary of the moduli space (maybe for general choice of the conditions). If this is not the case then the "expected result" of (2) has to be corrected based on an explicit analysis of the geometry.

The biggest problem is usually that of finding a suitable moduli space. Note that the moduli space is certainly not uniquely defined by the problem we want to study:

- There are many ways to compactify a non-compact (moduli) space.
- We can parametrize curves in a variety X either by describing them as embedded subvarieties of X or as pairs (C, f) where C is an abstract curve and $f : C \to X$ a morphism. The resulting moduli spaces are usually different.
- If we want to count curves in a projective variety $X \subset \mathbb{P}^N$ we can either start with the moduli space of curves in $\mathbb{P}^N$ and later impose the condition that the curves actually be in X, or we can start with a moduli space of curves in X in the first place.

Of course the final answer to the enumerative problem should not depend on these choices. Different choices of moduli spaces will however lead to completely different computations: some moduli spaces may be easy to describe as a variety so that intersection products can be computed without much effort, but their boundary may be so complicated that the step from (2) to (3) above cannot be carried out. If one tries to solve this problem by picking a more sophisticated moduli space that does not give rise to complicated boundary contributions then the moduli space may become intractable as a variety, so that the intersection product (2) cannot even be computed any more.

This is in fact the point where classical enumerative geometry was stuck for a long time. For every enumerative question one had to construct and study a moduli space whose only purpose was to solve this one single problem. The situation changed only about 10 years ago with the invention of the theory of so-called stable maps that we will present in these notes.

Example 0.1.9. The transition from "classical" to "modern" enumerative geometry was in fact inspired by theoretical physicists. In 1989 the string theorists Candelas et al. claimed that they can compute the numbers n_d of genus-zero curves of degree d in a general hypersurface of degree 5 in $\mathbb{P}^4$. It is expected by a simple dimension count that these numbers are indeed finite (see exercise 0.2.4). The prediction of the physicists reads as follows. Set

$$\sum_{d\geq 0} \frac{\prod_{i=1}^{5d}(5H+i)}{\prod_{i=1}^{d}(H+i)^5}\, q^d =: F_0 + F_1 H + F_2 H^2 + \cdots \qquad (F_i \in \mathbb{Q}[[q]]).$$

Then define rational numbers N_d recursively by the equation of formal power series in q

$$F_2 = \frac{1}{2}\frac{F_1^2}{F_0} + \frac{1}{5}\sum_{d>0} d\, N_d\, q^d\, F_0 \exp\left(d\,\frac{F_1}{F_0}\right).$$

Then the enumerative numbers n_d are given by the recursion

$$N_d = \sum_{k|d} \frac{n_{d/k}}{k^3}.$$

The first few numbers are as follows.

d	n_d
1	2875
2	609250
3	317206375
4	242467530000
5	229305888887625
6	248249742118022000
7	295091050570845659250
8	375632160937476603550000
9	503840510416985243645106250
10	704288164978454686113488249750

The computation of the physicists was based on arguments from topological quantum field theory that are not understandable to mathematicians. At the time this result was published it was even a surprise to mathematicians that there is a reasonably simple generating function that computes all numbers n_d in one go. In fact, it is not even obvious that the numbers n_d defined by the recursion above are positive integers. To verify these results in classical enumerative geometry people had to find a suitable moduli space separately for every degree d. They checked the results up to degree 4, but the moduli spaces for degrees 3 and 4 were already so complicated that it was clear that no universal formula like the one above could be proven that way.

About five years ago the above formula has been proven mathematically using the theory of stable maps.

0.2. **Exercises.** Note: As we have not developed any theory yet, you are not expected to be able to solve the following problems in a mathematically precise way. Rather, they are just meant as some "food for thought" if you want to think a little further about the examples considered in this section.

Exercise 0.2.1. In example 0.1.3 check that there is in fact a unique *smooth* plane curve of degree d through $\binom{d+2}{2}-1$ given points if these points are in general position. What does "general position" mean here? Is it sufficient — as in example 0.1.2 — that no three of the points lie on a line?

Exercise 0.2.2. Generalize the statement of example 0.1.4 to plane curves of higher degree. More precisely, consider singular plane curves of degree d that pass through $n(d)$ given points. How big must $n(d)$ be so that we get a finite number of curves, and what is this number then?

Exercise 0.2.3. Solve the remaining cases of example 0.1.6: for any $a,b \geq 0$ with $a+b=5$ determine the number of plane conics that are tangent to a given lines and pass through b given points in general position. (Hint: For $a \leq 2$ show that the naïve computation gives the right answer. Then use the theory of dual curves for the cases $b \leq 2$.)

Exercise 0.2.4. Let $X \subset \mathbb{P}^N$ be a smooth hypersurface of degree e for some $e \geq 1$. Compute the expected dimension of the space of morphisms $\mathbb{P}^1 \to X$ of degree $d \geq 1$. For the case $N=4, e=5$ conclude that for every degree d one expects a finite number of curves of genus zero of degree d on a hypersurface of degree 5 in $\mathbb{P}^4$. (Hint: you may want to use corollary 1.1.5 and lemma 1.1.7.)