Andreas Gathmann and Cedric Brendel Summer Semester 2023

Plane Algebraic Curves – Problem Set 5

due Tuesday, June 27

(1) Consider the rational function $\varphi = \frac{x^2}{y^2+yz}$ on the projective curve $F = y^2z + x^3 - xz^2$. Moreover, let $P = (0:0:1) \in F$.

(a) Compute the order $n = \mu_P(\varphi)$.

(b) Determine a local parameter $t \in \mathscr{O}_{F,P}$.

(c) Give an explicit description of φ in the form $\varphi = ct^n$ for some $c \in \mathscr{O}^*_{F,P}$, where c should be written as $\frac{f}{g}$ for some homogeneous $f, g \in S(F)$ of the same degree with $f(P) \neq 0$ and $g(P) \neq 0$.

(2) (a) Let P be a point on an affine curve F. Show that there is a rational function $\varphi \in K(F)$ which has exactly one pole which is simple and at P, i. e. such that $\mu_P(\varphi) = -1$ and $\mu_Q(\varphi) \geq 0$ for all $Q \neq P$.

(b) Let P_1 and P_2 be distinct points on a projective conic F. Show that there is a rational function $\varphi \in K(F)$ with $\mu_{P_1}(\varphi) = 1$, $\mu_{P_2}(\varphi) = -1$, and $\mu_P(F) = 0$ at all other points P of F.

(3) Let P be a point on an affine curve F. Show that there are ring isomorphisms

(a) $\mathscr{O}_{F,P} \cong \mathscr{O}_{\mathbb{A}^2,P}/\langle F \rangle$;

(b) $K(F) \cong K(F^{\mathrm{h}})$.

(4) (a) Let F be a projective curve, and let f be a homogeneous polynomial with $\operatorname{div} f = D + E$ for two divisors D and E on F. Show: If D' is linearly equivalent to D and $D' + E$ is effective then there is a homogeneous polynomial g with $\operatorname{div} g = D' + E$.

(b) Let P, Q, R, S be four distinct points on a cubic curve F. Show that $P + Q \sim R + S$ if and only if the intersection point of the lines $\overline{PQ}$ and $\overline{RS}$ lies on F.