Andreas Gathmann and Cedric Brendel Summer Semester 2023

Plane Algebraic Curves – Problem Set 4

due Tuesday, June 13

(1) Show by example that Max Noether's Theorem as in Corollary 4.12 is in general false …

(a) if the ground field is not algebraically closed; or

(b) if the curve F is not assumed to be smooth.

(2) Prove the following converse of Pascal's Theorem:
Let $P_1,\dots,P_6 \in \mathbb{P}^2$ be distinct points so that the six lines $\overline{P_1P_2},\overline{P_2P_3},\dots,\overline{P_5P_6},\overline{P_6P_1}$ (which can be thought of as the sides of the hexagon with vertices $P_1,\dots,P_6$) are also distinct. Let $P = \overline{P_1P_2}\cap\overline{P_4P_5}$, $Q = \overline{P_2P_3}\cap\overline{P_5P_6}$, $R = \overline{P_3P_4}\cap\overline{P_6P_1}$ be the intersection points of opposite sides of the hexagon. If P,Q,R lie on a line, then $P_1,\dots,P_6$ lie on a conic.

(3) (a) Show that a (not necessarily irreducible) reduced curve of degree d in $\mathbb{P}^2$ has at most $\binom{d}{2}$ singular points.

(b) Find an example for each d in which this maximal number of singular points is actually reached.

(4) Use Max Noether's Theorem to show the following statements about smooth projective cubics F and G over an algebraically closed field:

(a) (*Cayley-Bacharach Theorem*) Assume that F and G intersect in exactly 9 points $P_1,\dots,P_9$. Moreover, let E be another cubic that also contains the first eight points $P_1,\dots,P_8$. Prove that E then also contains P_9.

(b) Any line through two inflection points of F as in Exercise 2 of Problem Set 3 passes through a third inflection point of F.