Andreas Gathmann and Cedric Brendel Summer Semester 2023

Plane Algebraic Curves – Problem Set 3

due Tuesday, May 30

(1) For the following complex affine curves F and G, determine the points at infinity of their projective closures, and use Bézout's Theorem to read off the intersection multiplicities at all points of $F \cap G$.

(a) $F = x + y^2$ and $G = x + y^2 - x^3$;

(b) $F = y^2 - x^2 + 1$ and $G = (y + x + 1)(y - x + 1)$.

(2) For a projective curve F in the homogeneous coordinates x_0, x_1, x_2 we define the associated *Hessian* to be $H_F := \det\left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right)_{i,j=0,1,2}$.

(a) Show that the Hessian is compatible with coordinate transformations, i. e. if a projective coordinate transformation as in Exercise 3 of Problem Set 2 takes F to F' then up to multiplication with a unit it takes H_F to $H_{F'}$.

(b) Let $P \in F$ be a smooth point, and assume that the characteristic of K is 0. Show that $H_F(P) = 0$ if and only if $\mu_P(F, T_P F) \geq 3$. Such a point is called an *inflection point* of F.

(Hint: By (a) you may assume after a coordinate transformation that $P = (0:0:1)$ and $T_P F = x_1$.)

(3) Deduce the following real version of Bézout's Theorem from the complex case: If F and G are two real projective curves without common components then

$$\sum_{P \in F \cap G} \mu_P(F, G) = \deg F \cdot \deg G \mod 2.$$

In particular, two real projective curves of odd degree always intersect in at least one point.

(4) Let F be a complex irreducible projective curve of degree d, and let $P \in \mathbb{P}^2$ be a point. We set $m := m_P(F) \in \mathbb{N}$.

Show that for all but finitely many lines L in $\mathbb{P}^2$ through P, the intersection $F \cap L$ consists of exactly $d - m$ points not equal to P.