Andreas Gathmann and Cedric Brendel Summer Semester 2023

Plane Algebraic Curves – Problem Set 2

due Tuesday, May 16

(1) (a) Find all singular points of the curve $F = (x^2+y^2-1)^3 + 10x^2y^2 \in \mathbb{R}[x,y]$, and determine the multiplicities and tangents to F at these points.

(b) Show that an irreducible curve F over a field of characteristic 0 has only finitely many singular points.

Can you find weaker assumptions on F that also imply that F has only finitely many singular points?

(2) Show:

(a) If $F, G \in K[x_0, \dots, x_n]$ are polynomials such that $F \mid G$ and G is homogeneous, then F is homogeneous.

(b) Every homogeneous polynomial in two variables over an algebraically closed field is a product of linear polynomials.

(3) By a *projective coordinate transformation* we mean a map $f\colon \mathbb{P}^n \to \mathbb{P}^n$ of the form

$$(x_0 : \cdots : x_n) \mapsto (f_0(x_0,\dots,x_n) : \cdots : f_n(x_0,\dots,x_n))$$

for linearly independent homogeneous linear polynomials $f_0, \dots, f_n \in K[x_0, \dots, x_n]$.

(a) Let $P_1, \dots, P_{n+2} \in \mathbb{P}^n$ be points such that any $n+1$ of them are linearly independent in K^{n+1}, and in the same way let $Q_1, \dots, Q_{n+2} \in \mathbb{P}^n$ be points such that any $n+1$ of them are linearly independent. Show that there is a projective coordinate transformation f with $f(P_i) = Q_i$ for all $i = 1, \dots, n+2$.

(b) Let F and G be two real smooth projective conics with non-empty set of points. Show that there is a projective coordinate transformation of $\mathbb{P}^2$ that takes F to G.

(4) Let P be a point on an affine curve F. We say that P is a *cusp* if $m_P(F) = 2$, there is exactly one tangent L to F at P, and $\mu_P(F,L) = 3$.

(a) Give an example of a real curve with a cusp, and draw a picture of it.

(b) If F has a cusp at P, prove that F has only one irreducible component passing through P.

(c) If F and G have a cusp at P, what is the minimum possible value for the intersection multiplicity $\mu_P(F,G)$?