Andreas Gathmann and Cedric Brendel Summer Semester 2023

Plane Algebraic Curves – Problem Set 1

due Tuesday, May 2

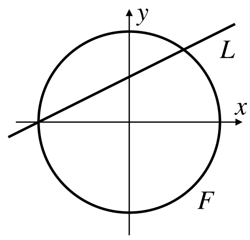
(1) Draw the real curves $F = x^2 + y^2 + 2y$ and $G = y^3x^6 - y^6x^2$, determine their irreducible decompositions, their intersection points, and their intersection multiplicity at the origin.

(Hint: To determine the intersection multiplicity, it is useful to use the additivity of the multiplicities, together with the property $\mu_0(F,G) = \mu_0(F, G+HF)$ for all polynomials F,G,H.)

(2) Let $F = x^2 + y^2 - 1 \in K[x,y]$ be the "unit circle" over K. Assume that the characteristic of K is not 2, i. e. that $1+1 \neq 0$ in K.

(a) Considering the intersection points of an arbitrary line L (with slope t) through $(-1,0)$ with F, show that the set of points of F is

$$V(F) = \{(-1,0)\} \cup \left\{ \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right) : t \in K \text{ with } 1+t^2 \neq 0 \right\}.$$



(b) Prove that the integer solutions (a,b,c) of the equation $a^2 + b^2 = c^2$ (the so-called Pythagorean triples) are, up to permuting a and b, exactly the triples of the form $\lambda(u^2 - v^2, 2uv, u^2 + v^2)$ with $\lambda, u, v \in \mathbb{Z}$.

(3) Let $F,G \in K[x,y]$ be two curves without a common component that passes through the origin. Show:

(a) There is a number $n \in \mathbb{N}$ such that $x^n = y^n = 0$ in $\mathscr{O}_0/\langle F,G\rangle$.

(b) Every element of $\mathscr{O}_0/\langle F,G\rangle$ has a polynomial representative.

(c) $\mu_0(F,G) < \infty$.

(4) Let $F,G \in K[x,y]$ be two curves that pass through the origin. Show:

(a) If F and G have no common component then the family $(F^n)_{n\in\mathbb{N}}$ is linearly independent in $\mathscr{O}_0/\langle G\rangle$.

(b) If F and G have a common component that passes through the origin then $\mu_0(F,G) = \infty$.

Please put your solutions (in groups of up to 3 people) in Cedric's mailbox next to room 48-210, or submit them in the OLAT course as a PDF file.