8. The Riemann-Roch Theorem

In the previous two chapters we have introduced and studied rational and regular functions on projective curves. As our last goal in these notes we now want to address the question *how many* such functions there are on a given projective curve (which, as before, will always be assumed to be smooth over an algebraically closed field).

But before we can try to solve this problem we first have to figure out what the precise question should be, i. e. which functions we want to consider and what exactly we mean by "how many". Note that we know already by Corollary 6.29 that global regular functions on a projective curve are always constant, and thus not very interesting. On the other hand, to obtain arbitrary rational functions we can take any quotient of two homogeneous polynomials of the same degree, so that we clearly get an infinite-dimensional vector space of such functions. Hence the most interesting question is to study something between regular and rational functions: rational functions which are everywhere regular, except for some specific points at which we allow poles of a given maximal order (or require zeros of a certain order). We will see that such functions form finite-dimensional vector spaces, so that we can then ask for their dimensions.

The conditions of allowing poles or requiring zeros at specified points is described best using the language of divisors. This leads to the following spaces that we will consider in this chapter.

Construction 8.1 ($L(D)$ and $l(D)$)**.** Let D be a divisor on a projective curve F. We set

$$L(D) := \{\varphi \in K(F)^* : \operatorname{div}\varphi + D \geq 0\} \cup \{0\}.$$

If $D = \sum_{P \in F} a_P \cdot P$, i. e. a_P denotes the coefficient of P in D, the condition $\operatorname{div}\varphi + D \geq 0$ obviously means $\mu_P(\varphi) + a_P \geq 0$, i. e. $\mu_P(\varphi) \geq -a_P$, for all points $P \in F$. Hence, except for the zero function, $L(D)$ consists by construction of all rational functions $\varphi \in K(F)^*$ that are just regular at all points of F, except that

(a) φ may have a pole of order at most a_P at P for all P with $a_P > 0$, and

(b) φ must have a zero of order at least $-a_P$ at P for all P with $a_P < 0$.

Note that $L(D)$ is a vector space over K: For all $\lambda \in K$ and $\varphi, \psi \in L(D)$, i. e. such that $\mu_P(\varphi) \geq -a_P$ and $\mu_P(\psi) \geq -a_P$ for all $P \in F$, we have

$$\mu_P(\varphi + \psi) \geq -a_P \text{ by Corollary 6.12} \quad \text{and} \quad \mu_P(\lambda\varphi) = \mu_P(\varphi) \geq -a_P \text{ by Construction 6.6 (b)}$$

for all P, and thus $\varphi + \psi \in L(D)$ and $\lambda\varphi \in L(D)$. Hence we can define

$$l(D) := \dim L(D) \quad \in \mathbb{N} \cup \{\infty\}.$$

As motivated above, it is the goal of this chapter to compute these dimensions $l(D)$. Unfortunately, we will not be able to do this for all D, since in general $l(D)$ depends on the precise position of the points occurring in D in a complicated way. However, the Riemann-Roch Theorem in Corollary 8.17 will allow to compute $l(D)$ in many cases just from the degree of D, which is of course easy to read off. The formula will also involve the *genus* of the curve — a concept that we have already seen over $\mathbb{C}$ from a topological point of view in Remark 5.12. As a byproduct of our work, we will therefore also give an algebraic definition of the genus of a curve, which is then applicable to any algebraically closed ground field.

But let us start with a few simple examples in which $l(D)$ is easy to determine.

Example 8.2.

(a) For the divisor $D = 0$ the space $L(D) = L(0)$ is by definition just the set of all rational functions that are regular at every point of the curve. Hence by Corollary 6.29 we have $L(0) = K$, and thus $l(0) = 1$.

(b) For any divisor D with $\deg D < 0$ we have $L(D) = \{0\}$ and thus $l(D) = 0$: If there was a non-zero element $\varphi \in L(D)$ we would have $\operatorname{div} \varphi + D \geq 0$, and hence $\deg \operatorname{div} \varphi + \deg D \geq 0$ by taking degrees. But this is a contradiction to $\deg D < 0$ since $\deg \operatorname{div} \varphi = 0$ by Remark 6.27 (b).

Remark 8.3. Let D be a divisor on a projective curve F.

(a) If D' is another divisor on F with $D \leq D'$ then $L(D) \subset L(D')$ and hence $l(D) \leq l(D')$, since $\operatorname{div} \varphi + D \geq 0$ clearly implies $\operatorname{div} \varphi + D' \geq \operatorname{div} \varphi + D \geq 0$.

(b) If $D' \sim D$ is linearly equivalent, i. e. $D - D' = \operatorname{div} \psi$ for a rational function $\psi \in K(F)^*$, then $L(D) \to L(D')$, $\varphi \mapsto \psi\varphi$ is an isomorphism of vector spaces (with inverse $\varphi \mapsto \frac{\varphi}{\psi}$) since the condition $\operatorname{div} \varphi + D \geq 0$ is equivalent to $\operatorname{div}(\psi\varphi) + D' \geq 0$. Hence we have $l(D) = l(D')$ in this case.

In particular, the notion $l(\,\cdot\,)$ is also well-defined for elements of the Picard group $\operatorname{Pic} F$. In the following, we will also use it in this extended way.

Many of our strategies to compute the numbers $l(D)$ will be inductive, i. e. relate $l(D)$ to $l(D \pm P)$ for a point P on the curve. Of particular importance will therefore be the following result, which tells us that $l(D)$ changes at most by 1 when adding or subtracting a point from D.

Lemma 8.4. *Let D be a divisor on a projective curve F.*

(a) *For any point $P \in F$ we have $l(D+P) = l(D)$ or $l(D+P) = l(D)+1$.*

(b) *For any divisor $D' \geq D$ we have $l(D) \leq l(D') \leq l(D) + \deg(D' - D)$.*

Proof.

(a) As $D \leq D+P$ we have $L(D) \subset L(D+P)$, and hence $l(D) \leq l(D+P)$, by Remark 8.3 (a).

Now let a_P be the coefficient of P in D, so that $a_P + 1$ is the coefficient of P in $D+P$. Consider the linear map

$$\Phi\colon L(D+P) \to K,\ \varphi \mapsto (t^{a_P+1}\varphi)(P),$$

where t is a local coordinate around P as in Proposition 6.10. Note that this evaluation of $t^{a_P+1}\varphi$ at P is well-defined, since for $\varphi \in L(D+P)\backslash\{0\}$ we have

$$\mu_P(t^{a_P+1}\varphi) = \mu_P(\varphi) + a_P + 1 \geq 0 \qquad (*)$$

(where the last inequality follows from Construction 8.1), so that $t^{a_P+1}\varphi$ is regular at P by Proposition 6.10 (b).

The kernel of Φ consists exactly of the rational functions for which $t^{a_P+1}\varphi$ has a zero at P, i. e. for which we have strict inequality in $(*)$. As this is equivalent to $\mu_P(\varphi) + a_P \geq 0$ and thus to $\operatorname{div} \varphi + D \geq 0$, we conclude that $\ker \Phi = L(D)$. The homomorphism theorem thus yields

$$L(D+P)/L(D) \cong \operatorname{im} \Phi \quad \subset K,$$

which means that $l(D+P) = l(D)$ (in case $\operatorname{im}\Phi = \{0\}$) or $l(D+P) = l(D)+1$ (in case $\operatorname{im}\Phi = K$).

(b) This follows immediately from (a) by induction on $\deg(D'-D)$, since D' is obtained from D by adding $\deg(D'-D)$ points. □

Remark 8.5. The proof of Lemma 8.4 (a) also has a simple analytic interpretation in case of the ground field $K = \mathbb{C}$. As the multiplicity of a rational function $\varphi \in L(D+P)$ at P is at least $-a_P - 1$, its Laurent expansion as in Remark 7.13 (a) can be taken to start with the power t^{-a_P-1} of an (analytic) local coordinate t. Inside $L(D+P)$, the subspace $L(D)$ now consists of exactly those functions for which the t^{-a_P-1}-coefficient of this expansion vanishes. As this coefficient is one complex number, its vanishing imposes one condition on $L(D+P)$ — which can be trivially satisfied by all elements of $L(D+P)$ already (in which case $l(D) = l(D+P)$) or not (in which case $l(D) = l(D+P) - 1$).

Corollary 8.6. *For any divisor D with $\deg D \geq 0$ on a projective curve F we have $l(D) \leq \deg D + 1$. In particular, the number $l(D)$ is always finite.*

Proof. Let $n = \deg D + 1$, and choose a point $P \in F$. Then $\deg(D - nP) = \deg D - n = -1 < 0$, so that $l(D - nP) = 0$ by Example 8.2 (b). It follows by Lemma 8.4 (b) that

$$l(D) \leq l(D - nP) + \deg(nP) = 0 + n = \deg D + 1. \qquad \square$$

Example 8.7.

(a) Let D be a divisor with $\deg D \geq 0$ on a projective curve F of degree 1 or 2. We claim that then $l(D) = \deg D + 1$, i. e. that we have equality in Corollary 8.6. In particular, together with Example 8.2 (b) this finishes the computation of all $l(D)$ on curves of degree 1 or 2.

To prove this, recall that $\operatorname{Pic}^0 F = \{0\}$ by Example 6.33, and hence $\operatorname{Pic} F \cong \mathbb{Z}$ by Remark 6.32, with an isomorphism given by the degree of divisors. If we pick any two distinct points $P, Q \in F$ this means first of all that $D \sim nP$ with $n := \deg D$. Moreover, as $P \sim Q$ there is a rational function $\varphi \in K(F)^*$ with $\operatorname{div} \varphi = Q - P$. We then have $\operatorname{div} \varphi^k = kQ - kP$ and hence $\varphi^k \in L(kP) \backslash L((k-1)P)$ for all $k \in \mathbb{N}_{>0}$, so that the inclusions

$$K \overset{8.2\,(a)}{=} L(0) \subset L(P) \subset L(2P) \subset \cdots \subset L(nP)$$

of Remark 8.3 (a) are all strict. Taking dimensions, we conclude that $l(nP) \geq n + 1$, hence in fact $l(nP) = n + 1$ by Corollary 8.6, and thus $l(D) = \deg D + 1$ by Remark 8.3 (b).

(b) Let P be a point on a projective curve F of degree at least 3. We will show that then $l(P) = 1$, i. e. that in this case we have a strict inequality in Corollary 8.6.

Consider any non-zero element $\varphi \in L(P)$. By definition, this rational function may then have a pole of order 1 at P but must be regular at all other points of F, so that $\operatorname{div} \varphi = Q - P$ for some point Q by Remark 6.27 (b). But by Proposition 6.34 this is impossible unless $Q = P$, which means that φ is a constant. Conversely, the constant functions are clearly contained in $L(P)$, and thus we see that $L(P) = K$, i. e. that $l(P) = 1$.

(c) Now consider the divisor $P - Q$ for two distinct points P and Q on a projective curve of degree at least 3. By (b) and Remark 8.3 (a) we have $L(P - Q) \subset L(P) = K$, so the elements of $L(P - Q)$ must be constant functions. But a constant does not have a zero at Q unless it is 0. Hence we see that $L(P - Q) = \{0\}$, and thus $l(P - Q) = 0$.

Exercise 8.8. Let F be a projective curve of degree d; without loss of generality we may assume that $F \neq z$. As usual, we will denote the vector space of homogeneous polynomials in x, y, z of degree n by $K[x,y,z]_n$.

For all $n \geq d$, show for the divisor $D := n \operatorname{div} z$:

(a) There is an exact sequence

$$0 \longrightarrow K[x,y,z]_{n-d} \xrightarrow{\cdot F} K[x,y,z]_n \xrightarrow{:z^n} L(D) \longrightarrow 0.$$

(b) $l(D) = \deg D + 1 - \binom{d-1}{2}$.

Remark 8.9 ($l(D)$ does not only depend on $\deg D$)**.** Note that on a projective curve F of degree at least 3 we have by Examples 8.2 (a) and 8.7 (c)

$$l(0) = 1 \quad \text{and} \quad l(P - Q) = 0$$

for any two distinct points $P, Q \in F$. In particular, as both divisors 0 and $P - Q$ have degree 0 we see that in general *the value $l(D)$ does not depend on the degree $\deg D$ alone*, but also on the exact positions of the points in D.

However, complementing the upper bound for $l(D)$ of Corollary 8.6 we can now also give a lower bound that depends only on $\deg D$. In fact, the difference between these two bounds turns out to be exactly the genus of the curve that we have already seen over $\mathbb{C}$ in Remark 5.12. We will use this observation as the definition of the genus in the algebraic setting.

Proposition and Definition 8.10. *Let F be a projective curve of degree d.*

(a) *(**Riemann's Theorem**) There is a unique smallest integer g, depending only on F, such that*

$$l(D) \geq \deg D + 1 - g \qquad (*)$$

*for any divisor D. We call g the **(algebraic) genus** of F.*

(b) *(**Algebraic degree-genus formula**) The algebraic genus of F is given by $g = \binom{d-1}{2}$.*

In particular, for $K = \mathbb{C}$ it coincides with the topological genus of Remark 5.12 and Proposition 5.16.

Proof. If we set $g = \binom{d-1}{2}$, Exercise 8.8 shows that there are divisors on F for which $(*)$ holds with equality. Hence, to prove both parts of the proposition, it suffices to prove that the inequality $(*)$ is true for every divisor D on F. To show this, note first:

(1) If $(*)$ holds for any divisor D, it also holds for any linearly equivalent divisor $D' \sim D$, since by Remarks 6.27 (b) and 8.3 (b) both sides of the inequality do not change when passing from D to D'.

(2) If $(*)$ holds for any divisor D, it also holds for any divisor $D' \leq D$: From $l(D) \geq \deg D + 1 - g$ it follows that

$$l(D') \overset{8.4}{\geq} l(D) - \deg(D - D') \geq \deg D + 1 - g - \deg(D - D') = \deg D' + 1 - g.$$

Now let D be any divisor on F, which we can write as $D = P_1 + \cdots + P_n - E$ for some points $P_1, \ldots, P_n \in F$ and an effective divisor E. As the points $P_1, \ldots, P_n$ are allowed to appear in E we may assume in this representation that $n \geq d$. For every $i = 1, \ldots, n$ choose a line l_i through P_i (which is not equal to F). Then the divisor

$$D' := D + \operatorname{div} \frac{z^n}{l_1 \cdot \cdots \cdot l_n}$$

is linearly equivalent to D, and satisfies

$$D' = P_1 + \cdots + P_n - E + n \operatorname{div} z - \sum_{i=1}^{n} \operatorname{div} l_i \leq P_1 + \cdots + P_n - E + n \operatorname{div} z - P_1 - \cdots - P_n \leq n \operatorname{div} z$$

since $\operatorname{div} l_i \geq P_i$ for all i. But now $(*)$ holds for $\operatorname{div} z^n$ by Exercise 8.8, hence also for D' by (2), and thus for D by (1). □

Summarizing, we now know by Corollary 8.6 and Proposition 8.10 (a) that

$$\deg D + 1 - g \leq l(D) \leq \deg D + 1$$

for every divisor D with $\deg D \geq 0$ on a projective curve F of genus g. We have also seen in Remark 8.9 already that we cannot expect an exact formula for $l(D)$ in terms of $\deg D$ alone. Nevertheless, one can make the above inequalities into an equality: It turns out that for every divisor D the difference between $l(D)$ and $\deg D + 1 - g$ can be identified as $l(D')$ for another divisor D' that is easily computable from D. To show this, we need the following special divisor on F.

Definition 8.11 (Canonical divisor)**.** Let F be a projective curve of degree d. For any line l (not equal to F) we call

$$K_F := (d-3) \operatorname{div} l \quad \in \operatorname{Pic} F$$

the **canonical divisor (class)** of F. (Note that for the element of $\operatorname{Pic} F$ it does not matter which line we take: For any other line l' we have $\operatorname{div} l \sim \operatorname{div} l'$ as $\operatorname{div} \frac{l}{l'} \in \operatorname{Prin} F$.)

Remark 8.12 (Canonical divisors are canonical)**.** It is hard to deny that our definition of the canonical divisor K_F of a projective curve F looks very artificial: It is not clear why divisor classes of lines and the choice of factor $d-3$ should lead to an object that plays a special role for F.

In fact, the usual definition of canonical divisors of curves in the literature is entirely different and much more natural (i. e. "canonical"): One can introduce *differential forms* on F in a way similar to the complex analytic setting in Remark 7.16, i. e. formal expressions of the form $\alpha = f\,dg$ for rational functions f and g that satisfy the usual rules of differentiation. They are natural objects on F that do not require any choices to define them, and in the same way as for rational functions one can associate multiplicities $\mu_P(\alpha)$ to a differential form α at a point $P \in F$. Combining these multiplicities for all points $P \in F$ one obtains a divisor $\operatorname{div}\alpha \in \operatorname{Div} F$, again in the same way as for rational functions.

It turns out that the divisors of any two differential forms are linearly equivalent, so that we obtain a well-defined and natural element K_F of $\operatorname{Pic} F$ as the divisor class of any differential form. This is the usual definition of the canonical divisor class K_F. It is then a computation to show that in the case of a projective plane curve this canonical divisor is equal to the one of Definition 8.11. We just took this formula as a definition of K_F in order to avoid a detailed discussion of differential forms. 13

Lemma 8.13 (Degree of the canonical divisor)**.** *For any projective curve F of genus g we have $\deg K_F = 2g-2$.*

Proof. By Remark 6.27 (a) we have for a curve F of degree d

$$\deg K_F = (d-3)\deg\operatorname{div} l = (d-3)\,d = 2\binom{d-1}{2} - 2,$$

so the result follows from the degree-genus formula $g = \binom{d-1}{2}$ of Proposition 8.10 (b). □

The key property of the canonical divisor that will allow us to make Riemann's Theorem of Proposition 8.10 into an equality is the following.

Lemma 8.14. *For any point P on a projective curve F we have $l(K_F + P) = l(K_F)$.*

Proof. If $d := \deg F \leq 2$ then $g = 0$ by Proposition 8.10 (b), and hence $\deg K_F = -2$ by Lemma 8.13. So in this case the degrees of both K_F and $K_F + P$ are negative, which means by Example 8.2 (b) that $l(K_F + P) = l(K_F) = 0$. We can therefore assume from now on that $d \geq 3$.

Choose any line l through P that is not the tangent $T_P F$. The divisor $\operatorname{div} l - P$ is then effective and does not contain P. Moreover, in this proof we will use this line l in Definition 8.11 to regard K_F as a divisor (and not just a divisor class). It then clearly suffices to prove that $L(K_F + P) = L(K_F)$. By Remark 8.3 (a) the inclusion "$\supset$" is automatic, so we will show "$\subset$".

To do this, let $\varphi = \frac{f}{g}$ be a non-zero element of $L(K_F + P)$, so that $\operatorname{div}\varphi + K_F + P \geq 0$. By Definition 8.11 this can be rewritten as

$$\operatorname{div}(f l^{d-2}) \geq \operatorname{div} g + \operatorname{div} l - P \geq \operatorname{div} g.$$

Max Noether's Theorem as in Proposition 6.28 then implies that there is a homogeneous polynomial h of degree $d-2$ with

$$\operatorname{div} h = \operatorname{div}(f l^{d-2}) - \operatorname{div} g \geq \operatorname{div} l - P. \qquad (*)$$

For all $Q \neq P$ this means that $\mu_Q(F,h) \geq \mu_Q(F,l)$, and hence $\langle F,h\rangle \subset \langle F,l\rangle$ in $\mathscr{O}_{\mathbb{P}^2,Q}$ by Proposition 2.26. But then also $\langle l,h\rangle \subset \langle F,l\rangle$ in $\mathscr{O}_{\mathbb{P}^2,Q}$, which in turn yields $\mu_Q(l,h) \geq \mu_Q(F,l)$. Taking the sum of these numbers for all $Q \neq P$ we get

$$\sum_{Q\neq P} \mu_Q(l,h) \geq \sum_{Q\neq P} \mu_Q(F,l) = \deg(\operatorname{div} l - P) = d-1.$$

But as h has degree $d-2$, Bézout's Theorem as in Corollary 4.6 implies that h must contain l as a factor. Hence we have $\operatorname{div} h \geq \operatorname{div} l$, and so by $(*)$

$$\operatorname{div}(f l^{d-2}) - \operatorname{div} g \geq \operatorname{div} l,$$

which means that $\operatorname{div}\varphi + K_F \geq 0$, and thus $\varphi \in L(K_F)$. □

Remark 8.15. Over the complex numbers, Lemma 8.14 is just a simple consequence of the Residue Theorem: In Remark 7.16 we have already seen that the sum of the residues of a differential form α on a projective curve F is 0. In particular, it follows that α cannot have exactly one non-zero residue, and thus that it is impossible for α to have exactly one pole at a point P which is in addition of order

1 (since by definition the residue would then be non-zero there). Applying this to the differential form $\varphi\alpha$ for any rational function φ this means in the language of divisors that

$$\operatorname{div}(\varphi\alpha)+P \geq 0 \quad \text{implies} \quad \operatorname{div}(\varphi\alpha) \geq 0,$$

i. e. that

$$\operatorname{div}\varphi + \operatorname{div}\alpha + P \geq 0 \quad \text{implies} \quad \operatorname{div}\varphi + \operatorname{div}\alpha \geq 0.$$

But by Construction 8.1 this is just the same as saying that $L(\operatorname{div}\alpha+P) = L(\operatorname{div}\alpha)$. Hence we have $l(\operatorname{div}\alpha+P) = l(\operatorname{div}\alpha)$ — which is exactly the statement of Lemma 8.14 since $\operatorname{div}\alpha = K_F$ by Remark 8.12.

Using Lemma 8.14 we can now finally add an additional "correction term" to the inequality in Riemann's Theorem of Proposition 8.10 to make it into an equality. Surprisingly, it turns out that it essentially suffices to prove that the inequality still holds after adding the correction term, with equality then following from this very easily.

Lemma 8.16. *Let F be a projective curve of genus g. Then*

$$l(D) - l(K_F - D) \geq \deg D + 1 - g$$

for all divisors D on F.

Proof. We will prove the statement by descending induction on $\deg D$. For the start of the induction, note that for all divisors with $\deg D > 2g-2$ we have $\deg(K_F - D) < 0$ by Lemma 8.13, hence $l(K_F - D) = 0$ by Example 8.2 (b), and so the statement is just Riemann's Theorem of Proposition 8.10.

For the induction step assume that the statement holds for a divisor D; we will show that it holds for $D-P$ for any point $P \in F$. As we already know that

$$\begin{aligned} l(D-P) - l(K_F - D + P) &\geq l(D) - 1 - (l(K_F - D) + 1) && \text{(Lemma 8.4)} \\ &\geq \deg D + 1 - g - 2 && \text{(induction assumption)} \\ &= \deg(D-P) - g \end{aligned}$$

it suffices to prove that the first inequality in this computation is strict. So assume for a contradiction that it is not, i. e. that $l(D-P) = l(D) - 1$ and $l(K_F - D + P) = l(K_F - D) + 1$. By Remark 8.3 (a) this means that $L(D-P) \subsetneq L(D)$ and $L(K_F - D) \subsetneq L(K_F - D + P)$, i. e. that there are rational functions

$$\begin{aligned} &\varphi \in L(D)\backslash L(D-P), && \text{i. e. } \operatorname{div}\varphi + D \geq 0 \text{ with equality at } P, \\ \text{and} \quad &\psi \in L(K_F - D + P)\backslash L(K_F - D), && \text{i. e. } \operatorname{div}\psi + K_F - D + P \geq 0 \text{ with equality at } P, \end{aligned}$$

where "equality at P" means that the point P appears with coefficient 0 on the left hand side of the inequalities. But then multiplying these two functions we obtain

$$\operatorname{div}(\varphi\psi) + K_F + P \geq 0 \text{ with equality at } P, \quad \text{i. e. } \varphi\psi \in L(K_F + P)\backslash L(K_F)$$

in contradiction to Lemma 8.14. □

Corollary 8.17 (Riemann-Roch). *Let D be a divisor on a projective curve F of genus g. Then*

$$l(D) - l(K_F - D) = \deg D + 1 - g.$$

Proof. Applying Lemma 8.16 to the divisor $K_F - D$ we obtain

$$l(K_F - D) - l(D) \geq \deg(K_F - D) + 1 - g \overset{8.13}{=} 2g - 2 - \deg D + 1 - g,$$

or in other words

$$l(D) - l(K_F - D) \leq \deg D + 1 - g.$$

Combining this with the statement of Lemma 8.16 for the divisor D yields immediately the desired equation. □

Remark 8.18.

(a) For the divisor $D = 0$ we have $l(0) = 1$ by Example 8.2 (a). We thus get from Corollary 8.17
$$1 - l(K_F) = \deg 0 + 1 - g,$$
and hence $g = l(K_F)$. Sometimes in the literature this is taken as the definition of the genus of a projective curve.

(b) If D is a divisor on a projective curve F of genus g with $\deg D > 2g-2$, then the divisor $K_F - D$ has negative degree by Lemma 8.13, so that $l(K_F - D) = 0$ by Example 8.2 (b), and thus we get by the Riemann-Roch Theorem
$$l(D) = \deg D + 1 - g.$$
Hence in this case of a divisor of large enough degree we can actually compute the dimension $l(D)$ just in terms of the degree of D. In fact, most applications of the Riemann-Roch theorem will just use this weaker statement.

Note that for curves of genus 0 this statement just reproduces our result for projective curves of degree at most 2 from Example 8.7 (a).

Exercise 8.19 ($l(D)$ for elliptic curves)**.** Let D be a divisor on an elliptic curve F, and denote by $\oplus$ the group structure of Chapter 7. Show that $l(D)$ is given by the following rules:

(a) If $\deg D < 0$ then $l(D) = 0$.

(b) If $\deg D > 0$ then $l(D) = \deg D$.

(c) If $\deg D = 0$ we can write $D = P_1 + \cdots + P_n - Q_1 - \cdots - Q_n$ for some $n \in \mathbb{N}$ and $P_1, \dots, P_n, Q_1, \dots, Q_n \in F$, and we have
$$l(D) = \begin{cases} 1 & \text{if } P_1 \oplus \cdots \oplus P_n = Q_1 \oplus \cdots \oplus Q_n, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 8.20. Let P be a point on a projective curve F. Prove that there is a rational function on F that has a pole (of any order) at P, and is regular at all other points of F. 14