5. Applications of Bézout's Theorem

Bézout's Theorem as in Corollary 4.6 is our first powerful result of algebraic geometry in these notes. Let us now take some time to study several of its applications, which are in fact of very different flavors.

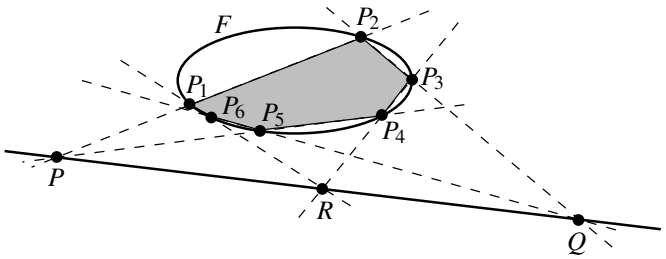
The first application is not much more than an immediate remark; it states that every smooth projective curve over an algebraically closed field is irreducible. As smoothness is easy to check using the Jacobi Criterion of Proposition 3.25 (a), this gives us a very useful sufficient criterion to determine whether a given curve is irreducible (which is usually hard to figure out).

Proposition 5.1 (Irreducibility criterion)**.** *Every smooth projective curve over an algebraically closed field is irreducible.*

Proof. Let $F = G \cdot H$ be a reducible projective curve. By Remark 4.8 (b) there is a point $P \in G \cap H$. Then $m_P(F) = m_P(G) + m_P(H) \geq 1 + 1 = 2$ by Remark 2.23, and so P is a singular point of F. □

Our next statement lies in the field of classical geometry. Over the real numbers it could in principle be proven using elementary methods (and was in fact shown in this way in the first place), but Bézout's Theorem makes the proof much simpler.

Proposition 5.2 (**Pascal's Theorem**)**.** *Let F be an irreducible projective conic with infinitely many points (e. g. over an algebraically closed field, or an ellipse over $\mathbb{R}$). Pick six distinct points $P_1, \dots, P_6$ on F (that can be thought of as the vertices of a hexagon inscribed in F). Then the intersection points of the opposite edges of the hexagon (i. e. $P = \overline{P_1P_2} \cap \overline{P_4P_5}$, $Q = \overline{P_2P_3} \cap \overline{P_5P_6}$, and $R = \overline{P_3P_4} \cap \overline{P_6P_1}$, where $\overline{P_iP_j}$ denotes the line through P_i and P_j) lie on a line.*



Proof. Consider the two (reducible) cubics $G_1 = \overline{P_1P_2} \cup \overline{P_3P_4} \cup \overline{P_5P_6}$ and $G_2 = \overline{P_2P_3} \cup \overline{P_4P_5} \cup \overline{P_6P_1}$. In accordance with Bézout's Theorem, they intersect in the 9 points $P_1, \dots, P_6, P, Q, R$.

Now pick any point $S \in F$ not equal to the previously chosen ones. Of course there are $\lambda_1, \lambda_2 \in K$, not both zero, such that the cubic $G := \lambda_1 G_1 + \lambda_2 G_2$ vanishes at S (since $G(S) = 0$ is one homogeneous linear equation in two variables λ_1, λ_2). Then F meets G in the 7 points $P_1, \dots, P_6, S$, and so by Bézout's Theorem these two curves must have a common component. As $\deg F = 2$, $\deg G = 3$, and F is irreducible, the only possibility for this is that G contains the factor F, so that $G = F \cdot L$ for a line L.

But P, Q, R lie on G (as they lie on G_1 and G_2) and not on F, so they must be on the line L. □

Exercise 5.3. Prove the following converse of Pascal's Theorem:

Let $P_1, \dots, P_6 \in \mathbb{P}^2$ be distinct points so that the six lines $\overline{P_1P_2}, \overline{P_2P_3}, \dots, \overline{P_5P_6}, \overline{P_6P_1}$ (which can be thought of as the sides of the hexagon with vertices $P_1, \dots, P_6$) are also distinct. Let $P = \overline{P_1P_2} \cap \overline{P_4P_5}$, $Q = \overline{P_2P_3} \cap \overline{P_5P_6}$, $R = \overline{P_3P_4} \cap \overline{P_6P_1}$ be the intersection points of opposite sides of the hexagon. If P, Q, R lie on a line, then $P_1, \dots, P_6$ lie on a conic.

Let us next address the question how many singular points we can have on a given projective curve. Exercise 2.30 (b) implies that, for an irreducible curve (over a field of characteristic 0), the number of singular points is always finite. Using Bézout's Theorem, we can now also give an upper bound for this number.

Example 5.4 (Singular points in low degrees)**.** Let F be an irreducible projective curve with infinitely many points (e. g. over an algebraically closed field).

(a) If $\deg F = 1$ then F is a line, which never has any singular points.

(b) If $\deg F = 2$ we claim that F has again no singular points. To show this, assume to the contrary that $P \in F$ is a singular point, and choose any other point $Q \in F$. Let G be the line through P and Q.

As P is a singular point of F, we know by Corollary 2.22 that $\mu_P(F,G) \geq 2$. Hence the total intersection multiplicity of F and G is at least

$$\mu_P(F,G) + \mu_Q(F,G) \geq 2+1 = 3,$$

which is bigger than $\deg F \cdot \deg G = 2$. So by Bézout's Theorem F and G must have a common component — which is impossible since F and G are irreducible.

(c) As for degree 3 we have already seen in Example 3.27 that the cubic $F = y^2z - x^2z - x^3$ has exactly one singular point; since F does not contain a line it is also irreducible. In fact, we will see below that a smooth irreducible cubic can have at most one singular point.

The idea to prove such bounds on the number of singular points is very similar to (b) above: Find a suitable curve G through the assumed singular points and some other points of F, and compute the total intersection multiplicity of F and G, where each singular point of F can be counted with multiplicity at least 2. If this total number exceeds $\deg F \cdot \deg G$ we arrive at a contradiction, i. e. the assumed number of singular points was too high.

In order to make this idea into an exact proof, we need an auxiliary lemma first that tells us how we can find curves (such as G above) through a given number of points.

Lemma 5.5 (Curves through given points)**.** *Let $d \in \mathbb{N}_{>0}$. For any $n := \binom{d+2}{2} - 1$ given points in $\mathbb{P}^2$ there is a projective curve of degree d passing through them.*

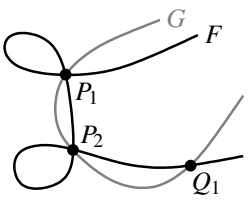
Proof. As in Remark 3.17, the vector space of all homogeneous polynomials of degree d in $K[x,y,z]$ has dimension $n+1$. Its elements are polynomials of the form $F = \sum_{i+j+k=d} a_{i,j,k} x^i y^j z^k$, where $a_{i,j,k}$ are the $n+1$ coordinates of F.

Now note that, for a given point $P = (x_0 : y_0 : z_0)$, the condition $F(P) = \sum_{i+j+k=d} a_{i,j,k} x_0^i y_0^j z_0^k \stackrel{!}{=} 0$ is just a linear equation in these coordinates. Hence, the condition that F vanishes at n given points is a system of n linear equations in $n+1$ variables. By linear algebra, such a system always has a non-trivial solution, which then is a curve of degree d passing through all the given points. □

Proposition 5.6. *Let F be an irreducible projective curve of degree d with infinitely many points (e. g. over an algebraically closed field). Then F has at most $\binom{d-1}{2}$ singular points.*

Proof. By Example 5.4 it suffices to prove the proposition for curves of degree $d \geq 3$. Assume for a contradiction that there are distinct singular points $P_1, \ldots, P_{\binom{d-1}{2}+1}$ of F. Moreover, pick $d-3$ arbitrary further distinct points $Q_1, \ldots, Q_{d-3}$ on F, so that the total number of points is

$$\binom{d-1}{2} + 1 + d - 3 = \binom{d}{2} - 1.$$



By Lemma 5.5, there is therefore a curve G of degree $d-2$ through all these points. As F is irreducible and of bigger degree than G, the curves F and G cannot have a common component. Hence Corollary 4.6 shows that F and G can intersect in at most $\deg F \cdot \deg G = d(d-2)$ points, counted with multiplicities. But the intersection multiplicity at all P_i is at least 2 by Corollary 2.22

since F is singular there. Hence the number of intersection points that we know already, counted with their respective multiplicities, is at least

$$2 \cdot \left(\binom{d-1}{2} + 1 \right) + (d-3) = d(d-2) + 1 > d(d-2),$$

which is a contradiction. □

Exercise 5.7.

(a) Show that a (not necessarily irreducible) reduced curve of degree d in $\mathbb{P}^2$ has at most $\binom{d}{2}$ singular points.

(b) Find an example for each d in which this maximal number of singular points is actually reached.

Let us now study smooth curves in more detail. An interesting topic that we have neglected entirely so far is the topology of such curves when we consider them over the real or complex numbers, e. g. their number of connected components in the usual topology. We will now see that Bézout's Theorem is able to answer such questions.

Of course, for these results we will need some techniques and statements from topology that have not been discussed in these notes. The following proofs in this chapter should therefore rather be considered as sketch proofs, which can be made into exact arguments with the necessary topological background. However, all topological results that we will need should be intuitively clear — although their exact proofs are often quite technical. Let us start with the real case, as real curves are topologically simpler than complex ones.

Remark 5.8 (Loops of real projective curves)**.** Let F be a smooth projective curve over $\mathbb{R}$. In the usual topology, its set of points $V(F)$ is then a compact 1-dimensional manifold (see Remark 2.28 (a)). This just means that $V(F)$ is a disjoint union of finitely many connected components, each of which is homeomorphic to a circle. We will refer to these components as *loops* of F. In the following pictures, we will often just draw the affine part of F; a point at infinity in such a loop will then show up as two unbounded ends of the curve. Note that the curve can consist of several loops even if it is irreducible (see Example 1.7 (c)).

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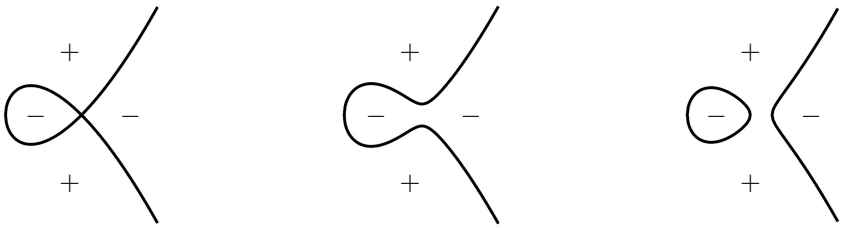
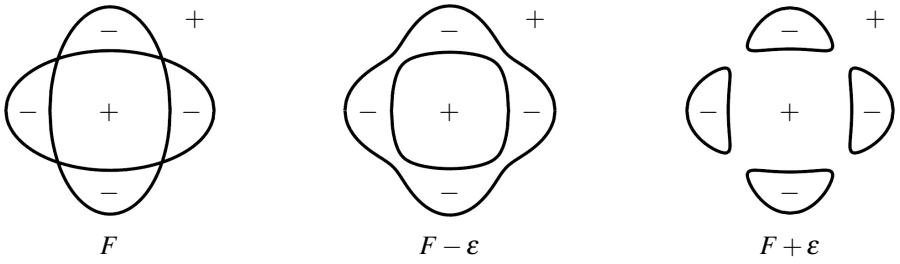
A convenient way to construct such curves is by deformations of singular curves. For example, consider (the projective closure of) the affine cubic $F = y^2 - x^2 - x^3$ with a node at the origin as in Example 2.21 (b) and the picture below on the left. In this picture, we have indicated in addition in which regions of $\mathbb{A}^2 \setminus V(F)$ the polynomial F is negative resp. positive. Together with its one point at infinity, the projective closure of F is homeomorphic to two circles glued together at a point.



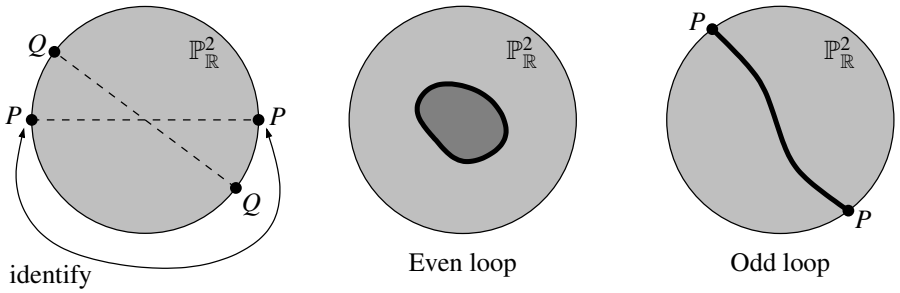
F $\quad$ $F - \varepsilon$ $\quad$ $F + \varepsilon$

Let us now perturb F and consider (the projective closures of) the curves $F \pm \varepsilon$ for a small number $\varepsilon \in \mathbb{R}_{>0}$ instead. Of course, this will only change the regions in which this polynomial is negative resp. positive by a little bit — but the origin, which was on the curve before, now lies in the negative (for $F - \varepsilon$) resp. positive (for $F + \varepsilon$) region. This leads to smooth cubics with one or two loops as in the picture above, depending on the sign of the perturbation.

The same technique applied to a singular quartic curve, e. g. the union of two ellipses given by $F = (x^2 + 2y^2 - 1)(y^2 + 2x^2 - 1)$, yields two or four loops as in the following picture.



Remark 5.9 (Even and odd loops)**.** Although all loops of real smooth curves are homeomorphic to a circle, there are two different kinds of them when we consider their embeddings in projective space. To understand this, recall from Remark 3.6 (a) that $\mathbb{P}^2_{\mathbb{R}}$ is obtained from the upper half sphere (which we will draw topologically as an disc below) by identifying opposite points on the boundary, as in the following picture on the left.



The consequence of this is that we have two different types of loops. An *even loop* is a loop such that its complement has two connected components, which we might call its "interior" (shown in dark in the picture above, homeomorphic to a disc) and "exterior" (homeomorphic to a Möbius strip), respectively. In contrast, an *odd loop* does not divide $\mathbb{P}^2_{\mathbb{R}}$ into two regions; its complement is a single component homeomorphic to a disc. Note that the distinction between even and odd is *not* whether the affine part of the curve is bounded: Whereas an odd loop always has to be unbounded, an even loop may well be unbounded, too. Instead, if you know some topology you will probably recognize that the statement being made here is just that the fundamental group $\pi_1(\mathbb{P}^2_{\mathbb{R}})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$; the two types of loops simply correspond to the two elements of this group.

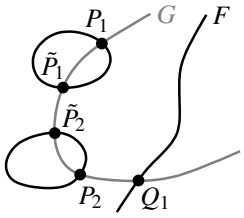
In principle, a real curve can have even as well as odd loops. There is one restriction however: As the complement of an odd loop is simply a disc, all other loops in this complement will have an interior and exterior, so that they are even. In other words, a real smooth curve can have at most one odd loop.

We are now ready to find a bound on the number of loops in an irreducible smooth curve in $\mathbb{P}^2_{\mathbb{R}}$ of a given degree. Interestingly, the idea in its proof is almost identical to that of Proposition 5.6, although the resulting statement is quite different.

Proposition 5.10 (Harnack's Theorem). *An irreducible smooth curve of degree d in $\mathbb{P}^2_{\mathbb{R}}$ has at most $\binom{d-1}{2}+1$ loops.*

Example 5.11. A line ($d=1$) has always exactly one loop. An irreducible smooth conic ($d=2$) is a hyperbola, parabola, or ellipse as in Example 3.16, so in every case the number of loops is again 1 (after adding the points at infinity). For $d=3$ Harnack's Theorem gives a maximum number of 2 loops, and for $d=4$ we get at most 4 loops. We have just seen examples of these numbers of loops in Remark 5.8. In fact, one can show that the bound given in Harnack's theorem is sharp, i. e. that for every d one can find real smooth curves of degree d with exactly $\binom{d-1}{2}+1$ loops.

Proof sketch of Proposition 5.10. Let F be a real irreducible smooth projective curve of degree d; by Example 5.11 it suffices to consider the case $d \geq 3$. Assume that the statement of the proposition is false, i. e. that there are at least $\binom{d-1}{2}+2$ loops. We have seen in Remark 5.9 that at least $\binom{d-1}{2}+1$ of these loops must be even. Hence we can pick points $P_1, \dots, P_{\binom{d-1}{2}+1}$ on distinct even loops of F, and $d-3$ more points $Q_1, \dots, Q_{d-3}$ on another loop (which might be even or odd). So, as in the proof of Proposition 5.6, we have a total of $\binom{d}{2}-1$ points.



Again as in the proof of Proposition 5.6, it now follows that there is a real curve G of degree at most $d-2$ passing through all these points. As F is irreducible and has bigger degree than G, these two curves cannot have a common component, so Bézout's Theorem as in Corollary 4.6 implies that they intersect in at most $d(d-2)$ points, counted with multiplicities. But recall from Remark 5.9 that the even loops of F containing the points P_i divide the real projective plane into two regions, hence if G enters the interior of such a loop it has to exit it again at another point $\tilde{P}_i$ of the same loop as in the picture above (it may also happen that G is singular or tangent to F at P_i, in which case $\mu_{P_i}(F,G) \geq 2$ by Corollary 2.22). So in any case the total number of intersection points, counted with their respective multiplicities, is at least

$$2 \cdot \left(\binom{d-1}{2} + 1 \right) + (d-3) = d(d-2) + 1 > d(d-2),$$

which is a contradiction. □

Let us now turn to the case of complex curves. Of course, their topology is entirely different, as they are 2-dimensional spaces and thus surfaces in the usual topology. In fact, we have seen such a case already in Example 0.2 of the introduction.

Remark 5.12 (Topology of complex curves)**.** Let F be a smooth projective curve over $\mathbb{C}$. Similarly to the real case, its set of points $V(F)$ is then a compact 1-dimensional complex manifold, and hence a compact 2-dimensional real manifold. Moreover, one can show:

(a) $V(F)$ is always an *oriented manifold*, i. e. a "two-sided surface", as opposed to e. g. a Möbius strip. To see this, note that all tangents $T_P F$ for $P \in F$ are 1-dimensional complex vector spaces after shifting P to the origin, and hence admit a well-defined multiplication with the imaginary unit i. Geometrically, this means that all tangent planes to the surface have a well-defined notion of a *positive* rotation by 90 degrees, varying continuously with P — which defines an orientation of the surface.

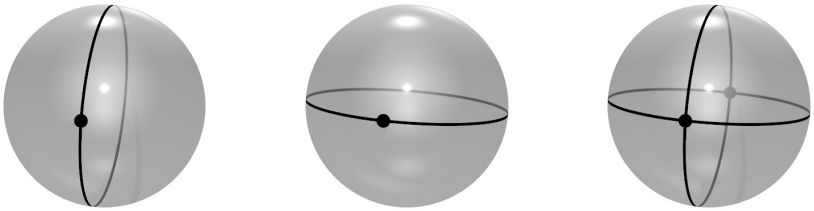
(b) In contrast to the real case that we have just studied, $V(F)$ is always *connected*. In short, the reason for this is that the notion of degree as well as Bézout's Theorem can be extended to compact oriented 2-dimensional submanifolds of $\mathbb{P}^2_{\mathbb{C}}$. Hence, if $V(F)$ had (at least) two connected components X_1 and X_2, each of them would be a compact oriented 2-dimensional manifold itself, and there would thus be well-defined degrees $\deg X_1, \deg X_2 \in \mathbb{N}_{>0}$. But then X_1 and X_2 would have to intersect in $\deg X_1 \cdot \deg X_2$ points (counted with multiplicities), which is obviously a contradiction.

Of course, the methods needed to prove Bézout's Theorem in the topological setting are entirely different from ours in Chapter 4. If you know some algebraic topology, the statement here is that the 2-dimensional homology group $H_2(\mathbb{P}^2_{\mathbb{C}}, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$. With this isomorphism, the class of a compact oriented 2-dimensional submanifold in $H_2(\mathbb{P}^2_{\mathbb{C}}, \mathbb{Z})$ is a positive number, and the intersection product $H_2(\mathbb{P}^2_{\mathbb{C}}, \mathbb{Z}) \times H_2(\mathbb{P}^2_{\mathbb{C}}, \mathbb{Z}) \to H_0(\mathbb{P}^2_{\mathbb{C}}, \mathbb{Z}) \cong \mathbb{Z}$ (using Poincaré duality) is just the product of these numbers.

It is now a (non-trivial but intuitive) topological result that a connected compact orientable 2-dimensional manifold X is always homeomorphic to a sphere with some finite number of "handles". This number of handles is called the **(topological) genus** of X. Hence every curve in $\mathbb{P}^2_{\mathbb{C}}$ can be assigned a genus that describes its topological type. The picture on the right shows a complex curve of genus 2.

We will see in Definition 8.10 that there is also an algebraic way to assign a genus to a smooth projective curve. It is then applicable to any (algebraically closed) ground field, coincides with the topological genus over $\mathbb{C}$ and plays an important role in the study of functions on the curve. Our goal for the rest of this chapter however will just be to compute the topological genus of a smooth complex projective curve in terms of its degree. To do this, we will need the following technique from topology.

Construction 5.13 (Cell decompositions)**.** Let X be a compact 2-dimensional manifold. A *cell decomposition* of X is given by writing X topologically as a finite disjoint union of points, (open) lines, and (open) discs. This decomposition should be "nice" in a certain sense, e. g. the boundary points of every line in the decomposition must be points of the decomposition. We do not want to give a precise definition here (which would necessarily be technical), but only remark that every "reasonable" decomposition that one could think of will be allowed. For example, the following picture shows three valid decompositions of the complex curve $\mathbb{P}^1_{\mathbb{C}}$, which is topologically a sphere by Remark 3.6 (b).



In the left two pictures, we have 1 point, 1 line, and 2 discs (the two halves of the sphere), whereas in the picture on the right we have 2 points, 4 lines, and 4 discs.

Of course, there are many possibilities for cell decompositions of X. But there is an important number that does not depend on the chosen decomposition:

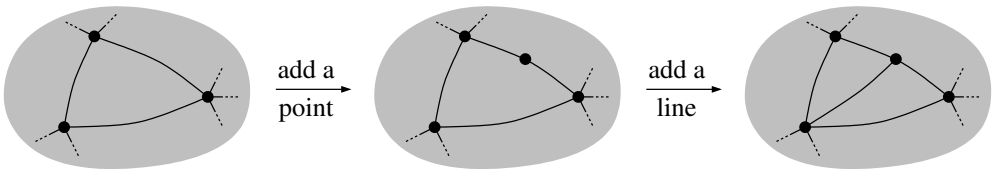
Lemma and Definition 5.14 (Euler characteristic)**.** *Let X be a compact 2-dimensional manifold. Consider a cell decomposition of X, consisting of σ_0 points, σ_1 lines, and σ_2 discs. Then the number*

$$\chi := \sigma_0 - \sigma_1 + \sigma_2$$

*depends only on X, and not on the chosen decomposition. We call it the (topological) **Euler characteristic** of X.*

Proof sketch. Let us first consider the case when we move from one decomposition to a finer one, i. e. if we add points or lines to the decomposition. Such a process is always obtained by performing the following steps a finite number of times:

- Adding another point on a line: In this case we raise σ_0 and σ_1 by 1 as in the picture below, hence the alternating sum $\chi = \sigma_0 - \sigma_1 + \sigma_2$ does not change.
- Adding another line in a disc: In this case we raise σ_1 and σ_2 by 1, so again χ remains invariant.



We conclude that the alternating sum $\sigma_0 - \sigma_1 + \sigma_2$ does not change under refinements. But any two decompositions have a common refinement — which is essentially given by taking all the points and lines in both decompositions, and maybe adding more points where two such lines intersect. For example, in Construction 5.13 the decomposition in the picture on the right is a common refinement of the other two. Hence the Euler characteristic is independent of the chosen decomposition. □

Example 5.15 (Euler characteristic $\leftrightarrow$ genus)**.** Let X be a connected compact orientable 2-dimensional manifold of genus g, and consider the cell decomposition of X as shown on the right. It has $\sigma_0 = 2g+2$ points, $\sigma_1 = 4g+4$ lines, and 4 discs, and hence we conclude that the Euler characteristic of X is

$$\chi = \sigma_0 - \sigma_1 + \sigma_2 = 2 - 2g.$$

In other words, the genus is given in terms of the Euler characteristic as $g = 1 - \frac{\chi}{2}$.

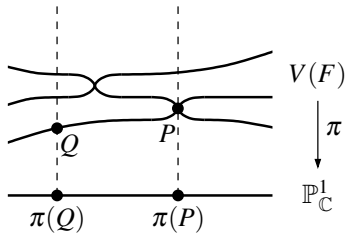
We are now ready to compute the genus of a smooth curve in $\mathbb{P}^2_{\mathbb{C}}$.

Proposition 5.16 (**Topological degree-genus formula**)**.** *A smooth curve of degree d in $\mathbb{P}^2_{\mathbb{C}}$ has topological genus $\binom{d-1}{2}$.*

Proof sketch. Let F be a smooth curve of degree d in $\mathbb{P}^2_{\mathbb{C}}$. By a projective coordinate transformation we can assume that $(0:1:0) \notin F$. Then

$$\pi\colon V(F) \to \mathbb{P}^1_{\mathbb{C}},\ (x:y:z) \mapsto (x:z)$$

is a well-defined map that can be interpreted as a projection, since in the affine part where $z = 1$ it is given by $(x,y) \mapsto x$ as in the picture on the right. Let us study its inverse images of a fixed point $(x:z) \in \mathbb{P}^1_{\mathbb{C}}$. Of course, they are given by the values of y such that $F(x,y,z) = 0$, so that there are exactly d such points — unless the polynomial $F(x,\cdot,z)$ has a multiple zero in y at a point in the inverse image, which happens if and only if F and $\frac{\partial F}{\partial y}$ are simultaneously zero there.



If we choose our original coordinate transformation general enough, exactly two of the zeros of $F(x,\cdot,z)$ will coincide at these points in the common zero locus of F and $\frac{\partial F}{\partial y}$, so that $\frac{\partial^2 F}{\partial y^2} \neq 0$ there and $\pi^{-1}(x:z)$ consists of $d-1$ instead of d points. These points, as e. g. P in the picture above, are usually called the *ramification points* of π. Note that the picture might be a bit misleading since it suggests that $V(F)$ is singular at P, which is not the case. The correct topological picture of the map is impossible to draw however since it would require the real 4-dimensional space $\mathbb{A}^2_{\mathbb{C}}$.

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At such a ramification point P we have $\mu_P(F, \frac{\partial F}{\partial y}) = 1$ by Corollary 2.22, since in affine coordinates with $P = (x_0, y_0)$ the tangents to the two curves are by Proposition 2.24 (b)

$$T_P F = \frac{\partial F}{\partial x}(P)\cdot(x-x_0) + \underbrace{\frac{\partial F}{\partial y}(P)}_{=0}\cdot(y-y_0) \quad \text{and} \quad T_P \frac{\partial F}{\partial y} = \frac{\partial^2 F}{\partial x \partial y}(P)\cdot(x-x_0) + \underbrace{\frac{\partial^2 F}{\partial y^2}(P)}_{\neq 0}\cdot(y-y_0),$$

which are clearly distinct. Hence by Bézout's Theorem there are exactly $\deg F \cdot \deg \frac{\partial F}{\partial y} = d(d-1)$ ramification points.

Let us now pick a sufficiently fine cell decomposition of $\mathbb{P}^1_{\mathbb{C}}$, containing all images of the ramification points as points of the decomposition. If $\sigma_0, \sigma_1, \sigma_2$ denote the number of points, lines, and discs in this decomposition, respectively, we have $\sigma_0 - \sigma_1 + \sigma_2 = 2$ by Example 5.15 since $\mathbb{P}^1_{\mathbb{C}}$ is topologically a sphere, i. e. of genus 0. Now lift this cell decomposition to a decomposition of $V(F)$ by taking all inverse images of the cells of $\mathbb{P}^1_{\mathbb{C}}$. By our above argument, all cells will have exactly d inverse images — except for the images of the $d(d-1)$ ramification points, which have one inverse image less. So the resulting decomposition of $V(F)$ has $d\sigma_0 - d(d-1)$ points, $d\sigma_1$ lines, and $d\sigma_2$ discs. Hence by Lemma 5.14 the Euler characteristic of $V(F)$ is

$$\chi = d\sigma_0 - d(d-1) - d\sigma_1 + d\sigma_2 = 2d - d(d-1) = 3d - d^2,$$

which means by Example 5.15 that its genus is

$$g = 1 - \frac{\chi}{2} = \frac{1}{2}(d^2 - 3d + 2) = \binom{d-1}{2}. \qquad \square$$

Example 5.17.

(a) A smooth curve of degree 1 or 2 in $\mathbb{P}^2_{\mathbb{C}}$ has topological genus 0, i. e. it is homeomorphic to a sphere. A smooth cubic has genus 1, so it is topologically a torus. We will study such cubic curves in detail in Chapter 7.

(b) Not every natural number can occur as the topological genus of a smooth complex plane curve: For example, there is no smooth complex plane curve of genus 2 since there is no $d \in \mathbb{N}$ with $\binom{d-1}{2} = 2$.