3. Projective Curves

In the last chapter we have studied the local intersection behavior of curves. Our next major goal will be to consider the global situation and ask how many intersection points two curves can have in total, i. e. how many common zeros we find for two polynomials $F,G \in K[x,y]$ (where we will count each such zero with its intersection multiplicity).

For polynomials in one variable, the corresponding question would simply be how many zeros a single polynomial $f \in K[x]$ has. At least if K is algebraically closed, so that f is a product of linear factors, the answer is then of course that we always get $\deg f$ zeros (counted with multiplicities). Hence, in our current case of two polynomials $F,G \in K[x,y]$ we would also hope for a result that depends only on $\deg F$ and $\deg G$, and not on the chosen polynomials.

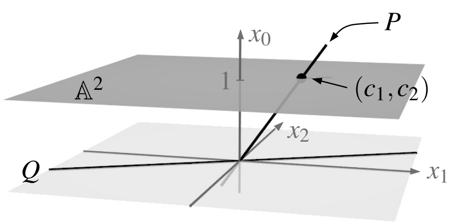
However, even in the simplest case when F and G are two distinct lines this will not work, since F and G might intersect in one point or be parallel (and hence have no intersection point). To fix this situation, the geometric idea is to add points at infinity to the affine plane $\mathbb{A}^2$, so that two lines that are parallel in $\mathbb{A}^2$ will meet there. On the other hand, two non-parallel lines (that intersect already in $\mathbb{A}^2$) should not meet at infinity any more as this would then lead to two intersection points. Hence, we have to add one point at infinity for each direction in the affine plane, so that parallel lines with the same direction meet there, whereas others do not.

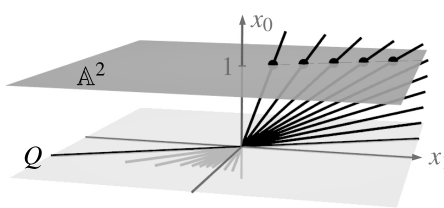
This new space with the added points at infinity will be called the *projective plane*. In the case $K = \mathbb{R}$ we can also think of it as a compactification of the affine plane $\mathbb{A}^2$. It is the goal of this chapter to study this process in detail, leading to plane curves that are "compactified" by points at infinity. For two such compactified curves we will then compute the number of intersection points in the next chapter, and the answer will then indeed depend only on the degrees of the curves.

Remark 3.1 (Geometric idea of projective spaces)**.** Algebraically, the idea for adding points at infinity is to embed the affine space $\mathbb{A}^n$ in the vector space K^{n+1} by prepending a new coordinate (typically called x_0) equal to 1, i. e. by the map

$$\mathbb{A}^n \to K^{n+1},\ (x_1,\dots,x_n) \mapsto (1,x_1,\dots,x_n),$$

and considering the 1-dimensional linear subspace in K^{n+1} spanned by this vector. For example, in this way a point $(c_1,c_2) \in \mathbb{A}^2$ corresponds to the line through the origin and $(1,c_1,c_2) \in K^3$, denoted by P in the picture below on the left.





We will define the projective plane as the set of all such 1-dimensional linear subspaces of K^3. It then consists of all lines through the origin coming from points of $\mathbb{A}^2$ as above — together with lines contained in the plane where $x_0 = 0$ that do not arise in this way, such as Q in the picture above. As shown on the right, these lines can be thought of as limits of lines coming from an unbounded sequence of points in $\mathbb{A}^2$. They can therefore be interpreted as the "points at infinity" that we were looking for.

Let us now turn this idea into a precise definition.

Definition 3.2 (Projective spaces)**.** For $n \in \mathbb{N}$, we define the **projective n-space** over K as the set of all 1-dimensional linear subspaces of K^{n+1}. It is denoted by $\mathbb{P}^n_K$ or simply $\mathbb{P}^n$.

Notation 3.3 (Homogeneous coordinates)**.** Obviously, a 1-dimensional linear subspace of K^{n+1} is uniquely determined by a spanning non-zero vector in V, with two such vectors giving the same linear subspace if and only if they are scalar multiples of each other. In other words, we have

$$\mathbb{P}^n = (K^{n+1}\backslash\{0\})\,/\sim$$

with the equivalence relation

$$(x_0,\dots,x_n) \sim (y_0,\dots,y_n) \quad :\Leftrightarrow \quad x_i = \lambda y_i \text{ for some } \lambda \in K^* \text{ and all } i.$$

The equivalence class of $(x_0,\dots,x_n)$ is usually denoted by $(x_0:\cdots:x_n) \in \mathbb{P}^n$. We call $x_0,\dots,x_n$ the **homogeneous** or **projective coordinates** of the point $(x_0:\cdots:x_n)$. Hence, in this notation for a point in $\mathbb{P}^n$ the numbers $x_0,\dots,x_n$ are not all zero, and they are defined only up to a common scalar multiple.

Remark 3.4 (Geometric interpretation of $\mathbb{P}^n$)**.** There are two ways to interpret the projective space $\mathbb{P}^n$ geometrically:

(a) As in Remark 3.1, we can embed the affine space $\mathbb{A}^n$ in $\mathbb{P}^n$ by the map

$$\mathbb{A}^n \to \mathbb{P}^n,\ (x_1,\dots,x_n) \mapsto (1:x_1:\cdots:x_n)$$

whose image is the subset $U_0 := \{(x_0:\cdots:x_n) : x_0 \neq 0\}$ of $\mathbb{P}^n$. We will often consider $\mathbb{A}^n$ as a subset of $\mathbb{P}^n$ in this way, i. e. by setting $x_0 = 1$. The other coordinates $x_1,\dots,x_n$ are then called the **inhomogeneous** or **affine coordinates** on U_0.

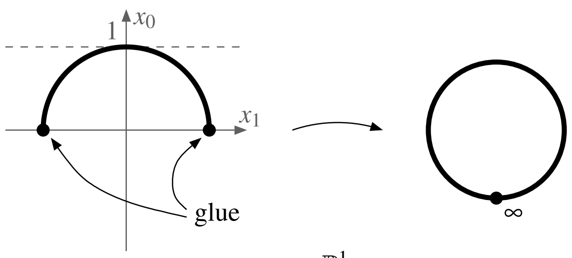
The remaining points of $\mathbb{P}^n$ are of the form $(0:x_1:\cdots:x_n)$. By forgetting their coordinate x_0 (which is zero anyway) they form a set that is naturally bijective to $\mathbb{P}^{n-1}$, corresponding to the 1-dimensional linear subspaces of K^n. As in Remark 3.1 we can regard them as *points at infinity*; there is hence one such point for each direction in K^n. In short-hand notation, one often writes this decomposition as $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$ and calls $\mathbb{A}^n$ and $\mathbb{P}^{n-1}$ the *affine* and *infinite part* of $\mathbb{P}^n$, respectively.

(b) By the symmetry of the homogeneous coordinates, the subsets $U_i := \{(x_0:\cdots:x_n) : x_i \neq 0\}$ of $\mathbb{P}^n$ are naturally bijective to $\mathbb{A}^n$ for all $i = 0,\dots,n$, in the same way as for $i = 0$ in (a). As every point of $\mathbb{P}^n$ has at least one non-zero coordinate, it lies in one of the U_i, and hence in a subset of $\mathbb{P}^n$ that just looks like the ordinary affine space $\mathbb{A}^n$. In this sense we can say that projective space "looks everywhere the same"; the fact that we interpreted the points with $x_0 = 0$ as points as infinity above was just due to our special choice of $i = 0$ in (a).

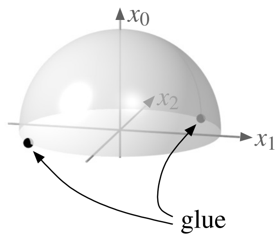
Example 3.5. By Remark 3.4 (a), we have $\mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{P}^0$. The affine part consists of the points $(1:x_1)$ for $x_1 \in K$, and the infinite part contains the single point $(0:1)$. Denoting this point at infinity by ∞, we can therefore write $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$.

Remark 3.6 (Topology of projective spaces over $\mathbb{R}$ and $\mathbb{C}$)**.** Over the real or complex numbers, every point in $\mathbb{P}^n$ has a representative on the unit sphere $\{(x_0,\dots,x_n) : |x_0|^2 + \cdots + |x_n|^2 = 1\}$ by normalizing. In other words, $\mathbb{P}^n$ can be written as the image of this compact unit sphere under the quotient map $(x_0,\dots,x_n) \mapsto (x_0:\cdots:x_n)$. In accordance with our motivation at the beginning of this chapter, this means that $\mathbb{P}^n$ is itself compact (with the quotient topology [G5, Definition 5.3 and Corollary 5.8 (c)]).

(a) For $K = \mathbb{R}$, every 1-dimensional linear subspace of K^{n+1} meets the unit sphere in exactly two points, which are negatives of each other. Hence, all points $(x_0:\cdots:x_n) \in \mathbb{P}^n$ have a representative on the upper half of the unit sphere, i. e. where $x_0 \geq 0$, and this representative is unique except for points on its boundary where $x_0 = 0$ (i. e. for points at infinity). As in the following picture, we can therefore visualize $\mathbb{P}^n_{\mathbb{R}}$ as the space obtained from the upper half unit sphere by identifying opposite points on the boundary. For $n = 1$ we have only one pair of gluing points, corresponding to one point at infinity as in Example 3.5, and obtain topologically a circle. For $n=2$, each point on the boundary of the upper half unit sphere has to be identified with its negative, which leads to a space that cannot be embedded in $\mathbb{R}^3$.

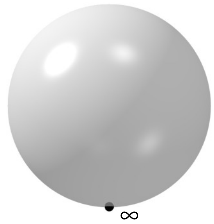


$\mathbb{P}^1_{\mathbb{R}}$



$\mathbb{P}^2_{\mathbb{R}}$

(b) For $K=\mathbb{C}$, only $\mathbb{P}^1_{\mathbb{C}}$ can be visualized in $\mathbb{R}^3$. By Example 3.5 it is just the complex plane together with a point ∞. It is therefore topologically a sphere as in the picture on the right.



$\mathbb{P}^1_{\mathbb{C}}$

Having studied projective spaces, we now want to consider subsets of $\mathbb{P}^n$ given by polynomial equations. However, polynomials in homogeneous coordinates are not well-defined functions on $\mathbb{P}^n$: For example, for the polynomial $f=x_0^2+x_1$ we have $f(1,-1)=0$ and $f(-1,1)=2$ although $(1:-1)=(-1:1)\in\mathbb{P}^1$. We can solve this problem by using homogeneous polynomials as follows.

Remark 3.7. Let

$$f=\sum_{i_0+\cdots+i_n=d} a_{i_0,\dots,i_n}\, x_0^{i_0}\cdot\;\cdots\;\cdot x_n^{i_n} \qquad \in K[x_0,\dots,x_n]$$

be a homogeneous polynomial of degree d. Then

$$f(\lambda x_0,\dots,\lambda x_n)=\sum_{i_0+\cdots+i_n=d} a_{i_0,\dots,i_n}\,\lambda^{i_0+\cdots+i_n} x_0^{i_0}\cdot\;\cdots\;\cdot x_n^{i_n}=\lambda^d f(x_0,\dots,x_n)$$

for all $\lambda\in K$. In particular, we see:

(a) Although f is not a well-defined function on $\mathbb{P}^n$, its zero locus is well-defined on $\mathbb{P}^n$, i. e. we have

$$f(\lambda x_0,\dots,\lambda x_n)=0 \quad\Leftrightarrow\quad f(x_0,\dots,x_n)=0$$

for all $\lambda\in K^*$. In the following, we will therefore write this condition simply as $f(P)=0$ for $P=(x_0:\cdots:x_n)$.

(b) If g is another homogeneous polynomial of degree d then

$$\frac{f(\lambda x_0,\dots,\lambda x_n)}{g(\lambda x_0,\dots,\lambda x_n)}=\frac{\lambda^d f(x_0,\dots,x_n)}{\lambda^d g(x_0,\dots,x_n)}=\frac{f(x_0,\dots,x_n)}{g(x_0,\dots,x_n)},$$

and so the quotient $\frac{f}{g}$ is a well-defined function on the subset of $\mathbb{P}^n$ where g does not vanish.

Definition 3.8 (Projective varieties)**.** For a subset $S\subset K[x_0,\dots,x_n]$ of homogeneous polynomials we call

$$V(S):=\{P\in\mathbb{P}^n : f(P)=0 \text{ for all } f\in S\} \quad\subset\mathbb{P}^n$$

the (projective) **zero locus** of S. Subsets of $\mathbb{P}^n$ that are of this form are called **(projective) varieties**. If $S=\{f_1,\dots,f_k\}$ is a finite set, we will write $V(S)=V(\{f_1,\dots,f_k\})$ also as $V(f_1,\dots,f_k)$. To distinguish the projective from the affine zero locus of Definition 1.3 (b), we will sometimes denote it by $V_{\mathrm{p}}(S)\subset\mathbb{P}^n$ as opposed to $V_{\mathrm{a}}(S)\subset\mathbb{A}^{n+1}$.

In this class we will mostly restrict ourselves to the case of the projective plane $\mathbb{P}^2$. We will then usually denote the homogeneous coordinates by x, y, and z, with z corresponding to the variable x_0 defining the points at infinity as in Remark 3.4 (a).

Remark 3.9. The properties of Remark 1.4 hold analogously for the projective zero locus: For any two homogeneous polynomials $f,g \in K[x,y,z]$ we have

(a) $V(f) \cup V(g) = V(fg)$;

(b) $V(f) \cap V(g) = V(f,g)$.

Exercise 3.10. By a *projective coordinate transformation* we mean a map $f\colon \mathbb{P}^n \to \mathbb{P}^n$ of the form

$$(x_0 : \cdots : x_n) \mapsto (f_0(x_0,\dots,x_n) : \cdots : f_n(x_0,\dots,x_n))$$

for linearly independent homogeneous linear polynomials $f_0,\dots,f_n \in K[x_0,\dots,x_n]$.

(a) Let $P_1,\dots,P_{n+2} \in \mathbb{P}^n$ be points such that any $n+1$ of them are linearly independent in K^{n+1}, and in the same way let $Q_1,\dots,Q_{n+2} \in \mathbb{P}^n$ be points such that any $n+1$ of them are linearly independent. Show that there is a projective coordinate transformation f with $f(P_i) = Q_i$ for all $i = 1,\dots,n+2$.

(b) Let F and G be two real smooth projective conics with non-empty set of points. Show that there is a projective coordinate transformation of $\mathbb{P}^2$ that takes F to G.

Exercise 3.11. Show:

(a) If $F,G \in K[x_0,\dots,x_n]$ are polynomials such that $F \mid G$ and G is homogeneous, then F is homogeneous.

(b) Every homogeneous polynomial in two variables over an algebraically closed field is a product of linear polynomials.

04

The definition of projective plane curves is now completely analogous to the affine case in Definition 1.5.

Definition 3.12 (Projective curves)**.**

(a) A **(projective plane algebraic) curve** (over K) is a non-constant homogeneous polynomial $F \in K[x,y,z]$ modulo units. We call $V(F) = \{P \in \mathbb{P}^2 : F(P) = 0\}$ its **set of points**.

(b) The **degree** of a projective curve is its degree as a polynomial. As in the affine case, curves of degree $1,2,3,\dots$ are called **lines**, **quadrics/conics**, **cubics**, and so on. The line z is referred to as the **line at infinity**.

(c) The notions of irreducible/reducible/reduced curves, as well as of irreducible components and their multiplicities, are defined in the same way as for affine curves in Definition 1.5 (c) (note that irreducible factors of homogeneous polynomials are always homogeneous by Exercise 3.11 (a)).

To study projective curves, we will often want to relate them to affine curves. For this we need the following construction.

Construction 3.13 (Homogenization and dehomogenization)**.**

(a) For a non-zero polynomial

$$f = \sum_{i+j \le d} a_{i,j}\, x^i y^j \quad \in K[x,y]$$

of degree d we define the **homogenization** of f as

$$f^{\mathrm{h}} := \sum_{i+j \le d} a_{i,j}\, x^i y^j z^{d-i-j} \quad \in K[x,y,z].$$

Note that f^{h} is homogeneous of degree $\deg f^{\mathrm{h}} = \deg f = d$, and that $z \nmid f^{\mathrm{h}}$ since f contains a term with $i+j=d$.

(b) For a non-zero homogeneous polynomial
$$f = \sum_{i+j+k=d} a_{i,j,k}\, x^i y^j z^k \quad \in K[x,y,z]$$
of degree d we define the **dehomogenization** of f to be
$$f^{\mathrm{i}} := f(z=1) = \sum_{i+j+k=d} a_{i,j,k}\, x^i y^j \quad \in K[x,y].$$
In general, f^{i} will be an inhomogeneous polynomial. If $z \nmid f$, i. e. if f contains a monomial without z, then this monomial will also be present in f^{i}, and thus $\deg f^{\mathrm{i}} = \deg f = d$.

In particular, there is a bijective correspondence
$$\left\{ \begin{array}{c} \text{polynomials of degree } d \\ \text{in } K[x,y] \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{homogeneous polynomials of degree } d \\ \text{in } K[x,y,z] \text{ not divisible by } z \end{array} \right\}$$
$$\begin{array}{ccc} f & \longmapsto & f^{\mathrm{h}} \\ f^{\mathrm{i}} & \longleftarrow & f. \end{array}$$

Example 3.14. For $f = y - x^2 \in K[x,y]$ we have $f^{\mathrm{h}} = yz - x^2 \in K[x,y,z]$, and then back again $(f^{\mathrm{h}})^{\mathrm{i}} = y - x^2 = f$.

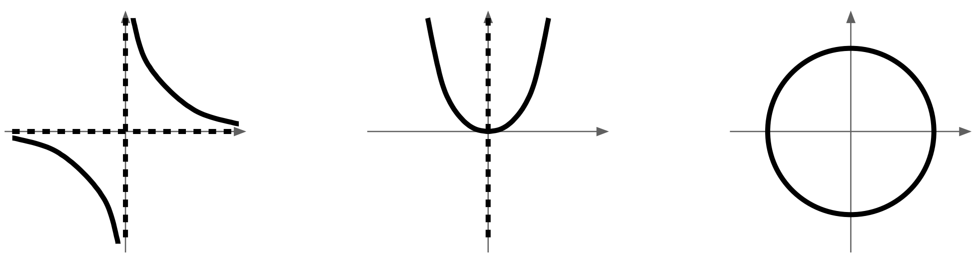
Construction 3.15 (Affine parts and projective closures)**.**

(a) For a projective curve F its affine set of points is $V_{\mathrm{p}}(F) \cap \mathbb{A}^2 = V_{\mathrm{a}}(F(z=1)) = V_{\mathrm{a}}(F^{\mathrm{i}})$. We will therefore call F^{i} the **affine part** of F. The points at infinity of F are given by $V_{\mathrm{p}}(F(z=0)) \subset \mathbb{P}^1$.

(b) For an affine curve F we call F^{h} its **projective closure**. By Construction 3.13 it is a projective curve whose affine part is again F, and that does not contain the line at infinity as a component.

However, F^{h} may contain points at infinity: If $F = F_0 + \cdots + F_d$ is the decomposition into homogeneous parts as in Notation 2.16, we have $F^{\mathrm{h}} = z^d F_0 + z^{d-1} F_1 + \cdots + F_d$ and hence $F^{\mathrm{h}}(z=0) = F_d$. So the points at infinity of F are given by the projective zero locus of the leading part of F.

Example 3.16 (Visualization of projective curves)**.** To visualize a (real) projective curve F (that does not have the line at infinity as a component), we will often just draw its affine set of points $V_{\mathrm{a}}(F^{\mathrm{i}})$, and if desired in addition its points at infinity as directions in $\mathbb{A}^2$. The following picture shows in this way the projective closures of the three types of real conics — a hyperbola, a parabola, and an ellipse (resp. a circle) — where the dashed lines correspond to the points at infinity. We see that the hyperbola has two points at infinity (namely $(0:1:0)$ and $(1:0:0)$ in the case below), the parabola has one ($(0:1:0)$ below), and the circle no such point. Note that, including these additional points, all three cases become topologically a loop, as the unbounded ends of the affine curves meet up at the corresponding points at infinity. In fact, up to a change of coordinates, we will see in Exercise 3.28 that there is essentially only one type of real projective conic.



(a) hyperbola
$F = xy - 1$
$F^{\mathrm{h}} = xy - z^2$
points at infinity: $xy = 0$

(b) parabola
$F = y - x^2$
$F^{\mathrm{h}} = yz - x^2$
points at infinity: $x^2 = 0$

(c) ellipse
$F = x^2 + y^2 - 1$
$F^{\mathrm{h}} = x^2 + y^2 - z^2$
points at infinity: $x^2 + y^2 = 0$

Remark 3.17 (Spaces of curves as projective spaces)**.** For $d \in \mathbb{N}_{>0}$, the vector space of homogeneous polynomials of degree d in $K[x,y,z]$ has dimension $\binom{d+2}{2}$, hence it is isomorphic to K^{n+1} with $n = \binom{d+2}{2} - 1$. By definition, a projective curve of degree d is then a non-zero point of this vector space modulo scalars. Hence, the space of all such curves is just the projective space $\mathbb{P}^n$, and thus itself a projective variety.

It is in fact very special to algebraic geometry — and very powerful — that the spaces of (certain) varieties are again varieties, and thus can be studied with exactly the same methods as the initial objects themselves. In other categories this is usually far from being true: The space of all groups is not a group, the space of all vector spaces is not a vector space, the space of all topological spaces is not a topological space, and so on.

For the rest of this chapter, let us transfer our results on affine curves from Chapters 1 and 2 to the projective case.

Remark 3.18 (Finiteness of zero loci)**.** Let F and G be two projective curves. The finiteness results of Lemma 1.11 and Proposition 1.12 (b) hold for the affine parts of F and G (for any choice of coordinate determining the line at infinity), and thus for F and G themselves: $V(F)$ is infinite if K is algebraically closed, $\mathbb{P}^2 \backslash V(F)$ is infinite if K is infinite, and $V(F,G)$ is finite if F and G have no common component.

Remark 3.19 (Recovering F from $V(F)$)**.** Let F be a projective curve over an algebraically closed field. We can write it as $F = z^m G$ for some $m \in \mathbb{N}$ and a curve G with $z \nmid G$. Then G can be recovered from G^{i} since $G = (G^{\mathrm{i}})^{\mathrm{h}}$ by Construction 3.13, and G^{i} can be recovered from $V_{\mathrm{a}}(G^{\mathrm{i}}) = V_{\mathrm{p}}(G) \cap \mathbb{A}^2$ and a multiplicity on each component by Remark 1.14.

As the components of F are just the components of G plus possibly the line at infinity z (with multiplicity m), this means that F can be reconstructed from $V(F)$ and a multiplicity on each component, just as in the affine case.

Construction 3.20 (Local rings of $\mathbb{P}^2$)**.** For $P \in \mathbb{P}^2$ we define the **local ring** of $\mathbb{P}^2$ at P according to Remark 3.7 (b) as

$$\mathscr{O}_P := \mathscr{O}_{\mathbb{P}^2,P} := \left\{ \frac{f}{g} : f,g \in K[x,y,z] \text{ homogeneous of the same degree with } g(P) \neq 0 \right\} \cup \{0\} \subset K(x,y,z).$$

As in Definition 2.1, these rings admit a well-defined **evaluation map**

$$\mathscr{O}_P \to K, \ \frac{f}{g} \mapsto \frac{f(P)}{g(P)}$$

with kernel

$$I_P := I_{\mathbb{P}^2,P} := \left\{ \frac{f}{g} \in \mathscr{O}_P : f(P) = 0 \right\} \ \subset \mathscr{O}_P.$$

For a point $P = (x_0 : y_0 : 1)$ in the affine part of $\mathbb{P}^2$ it is easily checked that there is an isomorphism

$$\mathscr{O}_{\mathbb{P}^2,(x_0:y_0:1)} \to \mathscr{O}_{\mathbb{A}^2,(x_0,y_0)}, \ \frac{f}{g} \mapsto \frac{f^{\mathrm{i}}}{g^{\mathrm{i}}}$$

compatible with the evaluation maps, and thus taking $I_{\mathbb{P}^2,(x_0:y_0:1)}$ to $I_{\mathbb{A}^2,(x_0,y_0)}$. Hence the local rings are still the same as in the affine case — which is of course expected, as objects that are local around a point in $\mathbb{A}^2$ should not be affected by adding points at infinity.

Construction 3.21 (Intersection multiplicities)**.** Note that homogeneous polynomials are not elements of the local ring $\mathscr{O}_{\mathbb{P}^2,P}$. But for $F_1,\dots,F_k$ homogeneous we can still define a generated ideal

$$\langle F_1,\dots,F_k \rangle = \left\{ \frac{f_1}{g_1}F_1 + \cdots + \frac{f_k}{g_k}F_k : f_i = 0 \text{ or } f_i, g_i \in K[x,y,z] \text{ homogeneous with } g_i(P) \neq 0 \text{ and } \deg(f_i F_i) = \deg g_i \text{ for all } i \right\}$$

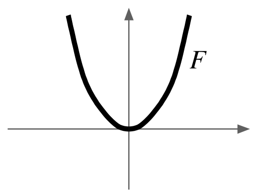
in $\mathscr{O}_P$. As in the affine case we can therefore define the **intersection multiplicity** of two curves F,G at a point $P \in \mathbb{P}^2$ as

$$\mu_P(F,G) := \dim \mathscr{O}_P/\langle F,G\rangle \quad \in \mathbb{N}\cup\{\infty\}. \tag{$*$}$$

For a point $P = (x_0:y_0:1)$ in the affine part of $\mathbb{P}^2$ one can verify directly that the isomorphism $\mathscr{O}_{\mathbb{P}^2,(x_0:y_0:1)} \cong \mathscr{O}_{\mathbb{A}^2,(x_0,y_0)}$ of Construction 3.20 takes $\langle F,G\rangle$ to $\langle F^{\mathrm{i}},G^{\mathrm{i}}\rangle$. Hence we have $\mu_{(x_0:y_0:1)}(F,G) = \mu_{(x_0,y_0)}(F^{\mathrm{i}},G^{\mathrm{i}})$, i.e. intersection multiplicities in the affine part can be computed exactly as in Chapter 2. At other points, the multiplicity can be computed similarly by choosing another (non-zero) coordinate to define the line at infinity as in Remark 3.4 (b). We will therefore probably never use the global definition $(*)$ of the multiplicity above for actual computations; its only purpose is to ensure that the result does not depend on the choice of coordinate defining the line at infinity.

Moreover, in the same way as in Remark 2.4 (a) intersection multiplicities are invariant under projective coordinate transformations as in Exercise 3.10, and they satisfy all the other properties of the multiplicities in Remark 2.4, Lemma 2.5, and Proposition 2.10.

Example 3.22. Let us compute the intersection multiplicity of the curve $F = yz - x^2$ (whose affine part is shown on the right) with the line $G = z$ at infinity at the common point $P = (0:1:0)$. For this we choose the affine part given by $y = 1$ and affine coordinates x and z. We then obtain



$$\mu_P(F,G) = \mu_{(0,0)}(z-x^2,z) = 2$$

by Example 2.11.

Construction 3.23 (Tangents and multiplicities of points, smooth and singular points)**.** The remaining concepts of Chapter 2 are also transferred easiest to a projective curve F using affine parts. So for a point $P = (x_0:y_0:1) \in \mathbb{P}^2$ in the affine part $\mathbb{A}^2$, we define the **multiplicity** $m_P(F)$ of F at P to be $m_{(x_0,y_0)}(F^{\mathrm{i}})$ in the sense of Definition 2.18. A **tangent** to F at P is the projective closure of a tangent to F^{i} at (x_0,y_0). If P is not in the affine part, we choose a different coordinate for the line at infinity as in Example 3.22 (it can be checked that this does not depend on the choice of coordinate).

We say that $P \in F$ is a **smooth** or **regular** point if $m_P(F) = 1$; its unique tangent is denoted T_PF. Otherwise, P is called a **singular** point of F. The curve F is said to be **smooth** or **regular** if all its points are smooth; otherwise F is called **singular**.

As in the affine case, there is a simple criterion to determine all singular points of a given projective curve. To prove it, we need a simple lemma first.

Lemma 3.24. *For any homogeneous polynomial $F \in K[x,y,z]$ of degree d we have*

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} = dF.$$

Proof. For $F = \sum_{i+j+k=d} a_{i,j,k}x^iy^jz^k$ we have $x\frac{\partial F}{\partial x} = \sum_{i+j+k=d} i\,a_{i,j,k}x^iy^jz^k$. An analogous formula holds for the other partial derivatives, and hence we conclude

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} = \sum_{i+j+k=d}(i+j+k)\,a_{i,j,k}x^iy^jz^k = dF. \qquad \square$$

Proposition 3.25 (Projective Jacobi Criterion)**.** *Let P be a point on a projective curve F.*

(a) *P is a singular point of F if and only if $\frac{\partial F}{\partial x}(P) = \frac{\partial F}{\partial y}(P) = \frac{\partial F}{\partial z}(P) = 0$.*

(b) *If P is a smooth point of F the tangent to F at P is given by*

$$T_PF = \frac{\partial F}{\partial x}(P)\cdot x + \frac{\partial F}{\partial y}(P)\cdot y + \frac{\partial F}{\partial z}(P)\cdot z.$$

Proof. Without loss of generality we may assume that $P = (x_0:y_0:1)$ is in the affine part of F.

(a) By the affine Jacobi criterion of Proposition 2.24 (a) we know that P is a singular point of F if and only if $\frac{\partial F^{\mathrm{i}}}{\partial x}(x_0,y_0) = \frac{\partial F^{\mathrm{i}}}{\partial y}(x_0,y_0) = 0$. As dehomogenizing F (which is just setting $z=1$) commutes with taking partial derivatives with respect to x and y, this is equivalent to $\frac{\partial F}{\partial x}(P) = \frac{\partial F}{\partial y}(P) = 0$. This is in turn equivalent to $\frac{\partial F}{\partial x}(P) = \frac{\partial F}{\partial y}(P) = \frac{\partial F}{\partial z}(P) = 0$ by Lemma 3.24 since $F(P) = 0$ by assumption.

(b) By Proposition 2.24 (b) the affine tangent to F at P is given by

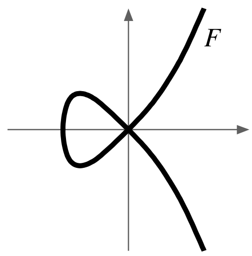
$$\begin{aligned}
&\frac{\partial F^{\mathrm{i}}}{\partial x}(x_0,y_0)\cdot(x-x_0) + \frac{\partial F^{\mathrm{i}}}{\partial y}(x_0,y_0)\cdot(y-y_0) \\
&\qquad = \frac{\partial F}{\partial x}(P)\cdot x + \frac{\partial F}{\partial y}(P)\cdot y - \Big(\frac{\partial F}{\partial x}(P)\cdot x_0 + \frac{\partial F}{\partial y}(P)\cdot y_0\Big) \\
&\qquad \overset{3.24}{=} \frac{\partial F}{\partial x}(P)\cdot x + \frac{\partial F}{\partial y}(P)\cdot y + \frac{\partial F}{\partial z}(P).
\end{aligned}$$

By definition, T_PF is now obtained by taking the projective closure, i. e. the homogenization of this polynomial. □

Remark 3.26. If the ground field K has characteristic 0, Lemma 3.24 tells us for any point $P \in \mathbb{P}^2$ that the conditions $\frac{\partial F}{\partial x}(P) = \frac{\partial F}{\partial y}(P) = \frac{\partial F}{\partial z}(P) = 0$ already imply $F(P) = 0$. In contrast to the affine case in Example 2.25, we therefore do not have to check explicitly that the point lies on the curve
05 when computing singular points with the Jacobi criterion.

Example 3.27. Let $F = y^2z - x^2z - x^3$ be the projective closure of the real affine curve $y^2 - x^2 - x^3$ of Example 2.21 (b). We have

$$\frac{\partial F}{\partial x} = -2xz - 3x^2, \quad \frac{\partial F}{\partial y} = 2yz, \quad \frac{\partial F}{\partial z} = y^2 - x^2.$$



It is checked immediately that the only common zero of these three polynomials is the point $(0\!:\!0\!:\!1)$, i. e. the origin of the affine part of F. So by Proposition 3.25 this is the only singular point of F (note that we have already seen in Example 2.25 using the affine Jacobi criterion that the origin is the only singular point of the affine part of F).

In particular, the point $(0\!:\!1\!:\!0) \in F$ at infinity is a smooth point of F, and the tangent to F there is by Proposition 3.25

$$\frac{\partial F}{\partial x}(0\!:\!1\!:\!0)\cdot x + \frac{\partial F}{\partial y}(0\!:\!1\!:\!0)\cdot y + \frac{\partial F}{\partial z}(0\!:\!1\!:\!0)\cdot z = z,$$

i. e. the line at infinity.

Exercise 3.28. Let F and G be two real smooth projective conics with non-empty set of points. Show that there is a projective coordinate transformation of $\mathbb{P}^2$ as in Exercise 3.10 that takes F to G.

Exercise 3.29. For a projective curve F in the homogeneous coordinates x_0,x_1,x_2 we define the associated *Hessian* to be $H_F := \det\big(\frac{\partial^2 F}{\partial x_i \partial x_j}\big)_{i,j=0,1,2}$.

(a) Show that the Hessian is compatible with coordinate transformations, i. e. if a projective coordinate transformation as in Exercise 3.10 takes F to F' then up to multiplication with a unit it takes H_F to $H_{F'}$.

(b) Let $P \in F$ be a smooth point, and assume that the characteristic of the ground field K is 0. Show that $H_F(P) = 0$ if and only if $\mu_P(F, T_PF) \geq 3$. Such a point is called an *inflection point* of F.

Hint: By part (a) and Exercise 3.10 you may assume after a coordinate transformation that $P = (0\!:\!0\!:\!1)$ and $T_PF = x_1$.