

4. Bézout's Theorem

Let F and G be two projective curves without common component. We have seen already in Remark 3.19 that the intersection $F \cap G$ is finite in this case. Bézout's Theorem, which is the main goal of this chapter, will determine the number of these intersection points, where each such point P will be counted with its intersection multiplicity $\mu_P(F, G)$.

In the same way as for the number of zeros of a univariate polynomial, the result will only be nice (i. e. depend only on the degree of the polynomials) if we assume that the underlying ground field is algebraically closed. To use this assumption we will need the following result from commutative algebra that extends the defining property of an algebraically closed field to polynomials in several variables.

Fact 4.1 (Hilbert's Nullstellensatz). Recall that a field K is called algebraically closed if every univariate polynomial $f \in K[x]$ without a zero in K is constant.

An obvious generalization of this statement to the multivariate case (which can be proven easily by induction of the number of variables) would be that every polynomial $f \in K[x_1, \dots, x_n]$ without a zero in \mathbb{A}^n is constant. However, there is a much stronger statement that also applies to several polynomials at once, or more precisely to the ideal generated by them: *Any ideal I in $K[x_1, \dots, x_n]$ with $V(I) = \emptyset$ over an algebraically closed field K is the unit ideal $I = \langle 1 \rangle$.* This statement is called by its German name **Hilbert's Nullstellensatz** ("theorem of the zeros") [G6, Remark 10.12]. Obviously, in the case $n = 1$ of polynomials in one variable, the ideal I must be generated by a single polynomial f as $K[x_1]$ is a principal ideal domain, and thus Hilbert's Nullstellensatz just reduces to the original statement that f must be constant if it does not have a zero.

Although Bézout's Theorem requires projective curves (as we have already motivated at the beginning of Chapter 3), it is actually more convenient to perform almost all steps required in its proof for the affine case. Our first step will be to compute the sum $\sum_{P \in F \cap G} \mu_P(F, G)$ of the local intersection multiplicities of two affine curves F and G and express it in terms of one global object. In fact, in the same way as $\mu_P(F, G)$ is by definition the dimension of the quotient of the *local ring* \mathcal{O}_P by the ideal $\langle F, G \rangle$, the sum of these multiplicities is just the dimension of the quotient of the *global* polynomial ring $K[x, y]$ by $\langle F, G \rangle$:

Lemma 4.2. *Let F and G be two affine curves over K with no common component. We consider the natural ring homomorphism*

$$\varphi: K[x, y]/\langle F, G \rangle \rightarrow \prod_{P \in F \cap G} \mathcal{O}_P/\langle F, G \rangle$$

that sends the class of a polynomial $f \in K[x, y]$ to the class of $f \in \mathcal{O}_P$ in each factor $\mathcal{O}_P/\langle F, G \rangle$.

- (a) *The morphism φ is surjective.*
- (b) *If K is algebraically closed then φ is an isomorphism.*

In particular, we have $\sum_P \mu_P(F, G) \leq \dim K[x, y]/\langle F, G \rangle$, with equality if K is algebraically closed.

Proof.

- (a) Let $F \cap G = \{P_0, \dots, P_m\}$. We show the surjectivity of φ in three steps, each producing polynomials with increasingly general images under φ .

Step 1: If $P_i = (x_i, y_i)$ for $i = 0, \dots, m$, and we set

$$f := \prod_{i: x_i \neq x_0} (x - x_i) \cdot \prod_{i: y_i \neq y_0} (y - y_i) \in K[x, y],$$

then by construction $f(P_0) \neq 0$, and $f(P_i) = 0$ for all $i > 0$.

Step 2: Now choose $n > \mu_{P_i}(F, G)$ for all i ; we claim that the polynomial f of Step 1 satisfies $f^n = 0$ in $\mathcal{O}_{P_i}/\langle F, G \rangle$ for all $i > 0$. In fact, the chain of ideals

$$\langle F, G, f^0 \rangle \supset \langle F, G, f^1 \rangle \supset \cdots \supset \langle F, G, f^n \rangle$$

in \mathcal{O}_{P_i} leads to a chain of inequalities

$$\dim \mathcal{O}_{P_i}/\langle F, G, f^0 \rangle \leq \dim \mathcal{O}_{P_i}/\langle F, G, f^1 \rangle \leq \cdots \leq \dim \mathcal{O}_{P_i}/\langle F, G, f^n \rangle.$$

But all of these numbers are at most $\dim \mathcal{O}_{P_i}/\langle F, G \rangle$ and thus less than n . Hence, we must have equality at some stage, i. e. $\langle F, G, f^k \rangle = \langle F, G, f^{k+1} \rangle$ in \mathcal{O}_{P_i} for some $k \in \{0, \dots, n-1\}$ (depending on i). This means that $f^k = aF + bG + cf^{k+1}$ for some $a, b, c \in \mathcal{O}_{P_i}$, and thus $f^k(1 - cf) \in \langle F, G \rangle$. But $1 - cf$ has value 1 at P_i and hence is a unit in \mathcal{O}_{P_i} . We conclude that $f^k \in \langle F, G \rangle$ in \mathcal{O}_{P_i} , which implies $f^n = 0 \in \mathcal{O}_{P_i}/\langle F, G \rangle$ for all $i > 0$ as desired.

Step 3: Finally, let $a \in \mathcal{O}_{P_0}$ be arbitrary. By Exercise 2.17 (b) we know that $\frac{a}{f^n}$ has a polynomial representative g in \mathcal{O}_{P_0} . Then the morphism φ maps gf^n to $\frac{a}{f^n} \cdot f^n = a$ in the first factor $\mathcal{O}_{P_0}/\langle F, G \rangle$, and to 0 in all other factors since $f^n = 0$ there. Hence all elements of $\prod_{P \in F \cap G} \mathcal{O}_P/\langle F, G \rangle$ with arbitrary entry in the P_0 -component and zero entries in all other components lie in the image of φ . By symmetry, the same holds for all other points P_i , and as the image of φ must be a subring we conclude that φ is surjective.

- (b) In view of (a) it remains to be shown that φ is injective. So let $f \in \ker \varphi$, and consider the set $I := \{g \in K[x, y] : gf \in \langle F, G \rangle\}$. This is clearly an ideal containing $\langle F, G \rangle$ (usually called the *ideal quotient* $\langle F, G \rangle : \langle f \rangle$). By the Nullstellensatz of Fact 4.1 it suffices to prove that $V(I) = \emptyset$, since then $I = K[x, y]$, hence $1 \in I$, i. e. $f \in \langle F, G \rangle$, and thus $f = 0 \in K[x, y]/\langle F, G \rangle$.

So assume that there is a point $P \in V(I)$. As $F, G \in I$ we know that $P \in F \cap G$. Hence P is one of the points in the product in the target space of φ , and so $f = 0 \in \mathcal{O}_P/\langle F, G \rangle$ as $f \in \ker \varphi$. This means that $f = \frac{a}{g}F + \frac{b}{g}G$ for some polynomials $a, b, g \in K[x, y]$ with $g(P) \neq 0$. But then $gf = aF + bG$, hence $g \in I$, and as $P \in V(I)$ we arrive at the contradiction $g(P) = 0$. \square

Remark 4.3. There are two ways to interpret the statement of Lemma 4.2:

- (a) A case that often occurs in Lemma 4.2 is that F and G intersect transversely, i. e. that the intersection multiplicities $\mu_P(F, G)$ at all $P \in F \cap G$ are equal to 1. In this case every factor $\mathcal{O}_P/\langle F, G \rangle$ is isomorphic to K by Definition 2.3, and the morphism φ is just the combined evaluation map at all points of $F \cap G$. The assertion of Lemma 4.2 (a) is then simply the interpolation statement that we can always find a polynomial having prescribed values at these points — which is probably not surprising, and is in fact already achieved by a suitable linear combination of polynomials as in Step 1 in the proof. If the intersection is not transverse and $\mu_P(F, G) > 1$ at some point P , then the map φ remembers more information at P on the polynomial than just its value, such as the values of some of its partial derivatives at P .
- (b) If you have some commutative algebra background then you probably know the statement of Lemma 4.2 already: As $V(F, G)$ is 0-dimensional, the ring $K[x, y]/\langle F, G \rangle$ is Artinian, and thus by the Structure Theorem on Artinian rings it is isomorphic to the product of its localizations at its various maximal ideals [G6, Proposition 7.20]. If K is algebraically closed then these maximal ideals all correspond to points in \mathbb{A}^2 [G6, Corollary 10.10], and so the map φ of the lemma is an isomorphism. If K is not necessarily algebraically closed then there are maximal ideals of $K[x, y]/\langle F, G \rangle$ that are not of this form and thus “missing” in the target space of φ , so that φ is only surjective.

Of course, our goal must now be to compute the dimension of the quotient $K[x, y]/\langle F, G \rangle$. In order to do this, we need a lemma first that tells us how polynomials in the ideal $\langle F, G \rangle$ of $K[x, y]$ can be represented.

Lemma 4.4. *Let F and G be two affine curves of degrees $m := \deg F$ and $n := \deg G$, respectively, such that their leading parts F_m and G_n (as in Notation 2.18) have no common component.*

Then every $f \in \langle F, G \rangle \subset K[x, y]$ of degree $d := \deg f$ can be written as $f = aF + bG$ for two polynomials a and b with $\deg a \leq d - m$ and $\deg b \leq d - n$.

Proof. As $f \in \langle F, G \rangle$ we can write $f = aF + bG$ for some $a, b \in K[x, y]$; choose such a representation with $\deg a$ minimal.

Assume for a contradiction that $\deg a > d - m$ or $\deg b > d - n$. Then aF or bG contains a term of degree bigger than d . As $f = aF + bG$ has degree d this means that the leading terms of aF and bG must cancel in f . Hence, if a_* and b_* denote the leading terms of a and b , respectively, we have $a_*F_m = -b_*G_n$. But F_m and G_n have no common component by assumption, and so we must have $a_* = cG_n$ and $b_* = -cF_m$ for some homogeneous polynomial c . This gives us a new representation

$$f = (a - cG)F + (b + cF)G$$

in which the leading term a_* of a cancels the leading term cG_n of cG in the first bracket. Hence $\deg(a - cG) < \deg a$, contradicting the minimality of $\deg a$. \square

Lemma 4.5. *Let F and G be affine curves with no common component, of degrees $m := \deg F$ and $n := \deg G$.*

- (a) $\dim K[x, y]/\langle F, G \rangle \leq mn$.
- (b) *If the leading parts F_m and G_n have no common component either then equality holds in (a).*

Proof.

- (a) For all $d \geq m + n$ consider the sequence of vector space homomorphisms

$$\begin{aligned} K[x, y]_{\leq d-m} \times K[x, y]_{\leq d-n} &\xrightarrow{\alpha} K[x, y]_{\leq d} \xrightarrow{\pi} K[x, y]/\langle F, G \rangle \\ (a, b) &\longmapsto aF + bG \end{aligned}$$

where $K[x, y]_{\leq d}$ denotes the vector subspace of $K[x, y]$ of all polynomials of degree at most d , which has dimension $\binom{d+2}{2}$, and π is the quotient map.

The kernel of α consists of all pairs (a, b) of polynomials of degrees at most $d - m$ and $d - n$, respectively, with $aF = -bG$. As F and G have no common component, this is equivalent to $a = cG$ and $b = -cF$ for some $c \in K[x, y]_{\leq d-m-n}$, so that

$$\ker \alpha = K[x, y]_{\leq d-m-n} \cdot (G, -F). \quad (1)$$

Moreover, it is obvious that

$$\operatorname{im} \alpha \subset \ker \pi. \quad (2)$$

So we conclude with the homomorphism theorem

$$\begin{aligned} \dim \operatorname{im} \pi &= \binom{d+2}{2} - \dim \ker \pi \\ &\stackrel{(2)}{\leq} \binom{d+2}{2} - \dim \operatorname{im} \alpha \\ &= \binom{d+2}{2} - \binom{d-m+2}{2} - \binom{d-n+2}{2} + \dim \ker \alpha \\ &\stackrel{(1)}{=} \binom{d+2}{2} - \binom{d-m+2}{2} - \binom{d-n+2}{2} + \binom{d-m-n+2}{2} \\ &= mn. \end{aligned}$$

Note that this bound is independent of d (as long as $d \geq m + n$), and thus also holds for the projection map $\pi: K[x, y] \rightarrow K[x, y]/\langle F, G \rangle$ from the full polynomial ring, which is surjective. It follows that $\dim K[x, y]/\langle F, G \rangle \leq mn$.

- (b) It suffices to establish equality in (2) in the proof of (a) above, i. e. that $\ker \pi \subset \operatorname{im} \alpha$. But this is precisely the statement of Lemma 4.4. \square

We can now switch back to the projective case and prove the main result of this chapter.

Corollary 4.6 (Bézout's Theorem). *Let F and G be projective curves without common component over an infinite field K . Then*

$$\sum_{P \in F \cap G} \mu_P(F, G) \leq \deg F \cdot \deg G.$$

Moreover, equality holds if K is algebraically closed.

Proof. By Lemma 1.11 (b) there is a point Q in the affine part of \mathbb{P}^2 which does not lie on $F^i \cup G^i$, i. e. neither on F nor on G . Moreover, as K is infinite but $F \cap G$ finite by Proposition 1.12, we can pick a line L through Q which does not intersect $F \cap G$. Now we make a projective coordinate transformation so that L becomes the line at infinity. Then neither F nor G contains the line at infinity as a component (so that $\deg F^i = \deg F$ and $\deg G^i = \deg G$), and all intersection points of F and G lie in the affine part (i. e. they are also intersection points of the affine curves F^i and G^i).

Applying Lemma 4.2 (a) and 4.5 (a) to F^i and G^i then yields

$$\sum_{P \in F \cap G} \mu_P(F, G) = \sum_{P \in F^i \cap G^i} \mu_P(F^i, G^i) \stackrel{4.2}{\leq} \dim K[x, y] / \langle F^i, G^i \rangle \stackrel{4.5}{\leq} \deg F^i \cdot \deg G^i = \deg F \cdot \deg G. \quad (*)$$

Now let K be algebraically closed. Then the first inequality is actually an equality by Lemma 4.2 (b). Moreover, the leading parts of F^i and G^i are homogeneous polynomials in two variables, and hence a product of linear factors by Exercise 3.11 (b). But these factors correspond exactly to the points at infinity of the two curves by Remark 3.15 (b). As there are no such common points by our choice of L , we conclude that the leading parts of F^i and G^i have no common component, and thus by Lemma 4.5 (b) that the second inequality in (*) is actually an equality as well. \square

Remark 4.7 (Bézout's Theorem over arbitrary ground fields). It can be shown that (the inequality part of) Bézout's Theorem holds in fact over arbitrary fields. The assumption of an infinite ground field was only necessary for the strategy of our proof to choose coordinates so that all intersection points of the curves lie in the affine part — which would not be possible over finite fields, since the two curves might then intersect in every point of \mathbb{P}^2 (without having a common component).

Remark 4.8. Let F and G be two projective curves without common component over an infinite ground field K .

- (a) As the intersection multiplicity at each point of $F \cap G$ is at least 1, it follows from Bézout's Theorem that F and G intersect in at most $\deg F \cdot \deg G$ points (disregarding the multiplicities).
- (b) If K is algebraically closed, Bézout's Theorem implies in particular that F and G intersect in at least one point. Note that already this statement is non-trivial — and clearly false for general ground fields, as then already $V(F)$ might be empty.

Exercise 4.9. For the following complex affine curves F and G , determine the points at infinity of their projective closures, and use Bézout's Theorem to read off the intersection multiplicities at all points of $F \cap G$.

- (a) $F = x + y^2$ and $G = x + y^2 - x^3$;
- (b) $F = y^2 - x^2 + 1$ and $G = (y + x + 1)(y - x + 1)$.

Exercise 4.10. Deduce the following real version of Bézout's Theorem from the complex case: If F and G are two real projective curves without common components then

$$\sum_{P \in F \cap G} \mu_P(F, G) = \deg F \cdot \deg G \pmod{2}.$$

In particular, two real projective curves of odd degree always intersect in at least one point.

Exercise 4.11. Let F be a complex irreducible projective curve of degree d , and let $P \in \mathbb{P}^2$ be a point. We set $m := m_P(F) \in \mathbb{N}$.

Show that for all but finitely many lines L in \mathbb{P}^2 through P , the intersection $F \cap L$ consists of exactly $d - m$ points not equal to P .

We will discuss many examples and applications of Bézout’s Theorem in the next chapter. Instead, at the end of this chapter let us prove another theorem that can be obtained by very similar methods and that will be useful later on. It considers a *smooth* projective curve F over an algebraically closed field and states roughly that, for any two other curves G and H such that F intersects H everywhere with at least the same multiplicity as G , the “remaining multiplicities” $\mu_P(F, H) - \mu_P(F, G)$ can be obtained by intersecting F with another curve.

Corollary 4.12 (Noether’s Theorem). *Let F be a smooth projective curve over an algebraically closed field. Moreover, let G and H be two projective curves that do not have a common component with F .*

If $\mu_P(F, G) \leq \mu_P(F, H)$ for all points $P \in \mathbb{P}^2$ then there are homogeneous polynomials A and B (of degrees $\deg H - \deg F$ resp. $\deg H - \deg G$ if non-zero), such that

- (a) $H = AF + BG$;
- (b) $\mu_P(F, H) = \mu_P(F, G) + \mu_P(F, B)$ for all $P \in \mathbb{P}^2$.

Proof. As in the proof of Corollary 4.6 we may assume by a projective coordinate transformation that none of the curves contain the line at infinity as a component, and that all points of $F \cap G$ lie in the affine part of \mathbb{P}^2 . We then have again $\deg F^i = \deg F$, $\deg G^i = \deg G$, $\deg H^i = \deg H$, and the leading parts of F^i and G^i have no common component.

Now F is assumed to be smooth, and hence — working with the affine curves for a moment — the assumption $\mu_P(F^i, G^i) \leq \mu_P(F^i, H^i)$ implies $\langle F^i, H^i \rangle \subset \langle F^i, G^i \rangle$ in \mathcal{O}_P for all $P \in F \cap G$ by Proposition 2.28, and thus in particular $H^i \in \langle F^i, G^i \rangle$ in \mathcal{O}_P . By Lemma 4.2 (b) we then have $H^i \in \langle F^i, G^i \rangle$ in $K[x, y]$ as well. But as the leading parts of F^i and G^i have no common component, Lemma 4.4 gives us an equation

$$H^i = aF^i + bG^i$$

for some polynomials a and b of degrees at most $\deg H - \deg F$ and $\deg H - \deg G$, respectively. Homogenizing this yields H , so a homogeneous polynomial of degree $\deg H$, and thus

$$H = \underbrace{z^{\deg H - \deg F - \deg a} a^h F}_{=:A} + \underbrace{z^{\deg H - \deg G - \deg b} b^h G}_{=:B},$$

which proves (a). But this also implies part (b), since by (the projective version of) the properties of intersection multiplicities we have

$$\mu_P(F, H) = \mu_P(F, AF + BG) \stackrel{2.4(c)}{=} \mu_P(F, BG) \stackrel{2.9(b)}{=} \mu_P(F, B) + \mu_P(F, G). \quad \square$$

Exercise 4.13 (Cayley-Bacharach). Let F and G be smooth projective cubics over an algebraically closed field that intersect in exactly 9 points P_1, \dots, P_9 . Moreover, let E be another cubic that also contains the first eight points P_1, \dots, P_8 . Prove that E then also contains P_9 .

(Hint: Apply Noether’s Theorem to a suitable curve H .)

Exercise 4.14. Show by example that Noether’s Theorem is in general false ...

- (a) if the ground field is not algebraically closed; or
- (b) if the curve F is not assumed to be smooth.