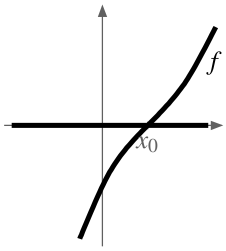
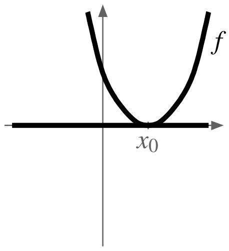
2. Intersection Multiplicities

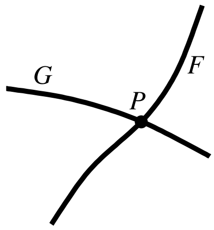
Let us start our study of curves by introducing the concept of intersection multiplicity, which will be central throughout these notes. It generalizes the well-known notion of multiplicity of a zero of a univariate polynomial: If $f \in K[x]$ is a polynomial and $x_0 \in K$ such that $f = a\,(x-x_0)^m$ for a polynomial $a \in K[x]$ with $a(x_0) \neq 0$, then f is said to have multiplicity m at x_0. As in the following two pictures on the left, a zero of multiplicity 1 means that the graph of f intersects the x-axis transversely, whereas in the case of multiplicity (at least) 2 it is tangent to it. Roughly speaking, higher multiplicities would correspond to graphs for which the x-axis is an even better approximation around x_0.



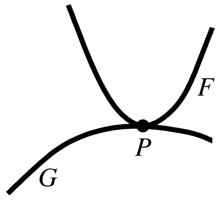
multiplicity 1
$f = a\,(x-x_0)$



multiplicity 2
$f = a\,(x-x_0)^2$



multiplicity 1



multiplicity 2

In this geometric interpretation, we have already considered how the graph of f intersects the horizontal axis locally at the given point, i. e. how the two curves $F = y - f$ and $G = y$ intersect. As in the picture above on the right, this concept should thus also make sense for arbitrary curves F and G at an intersection point P: If they intersect transversely, i. e. with different tangent directions, we want to say that they have an intersection multiplicity of 1 at P, whereas equal tangents correspond to higher multiplicities. But of course, the curves F and G might also have "singularities" as e. g. the origin in Example 0.1 (b) and (c), in which case it is not clear a priori how their intersection multiplicity can be interpreted or even defined.

So our first task must be to actually construct the intersection multiplicity for arbitrary curves. For this we need the following algebraic object that allows us to capture the local geometry of the plane around a point.

Definition 2.1 (Local rings of $\mathbb{A}^2$)**.** Let $P \in \mathbb{A}^2$ be a point.

(a) The **local ring** of $\mathbb{A}^2$ at P is defined as

$$\mathscr{O}_P := \mathscr{O}_{\mathbb{A}^2,P} := \left\{ \frac{f}{g} : f,g \in K[x,y] \text{ with } g(P) \neq 0 \right\} \quad \subset K(x,y).$$

(b) It admits a well-defined ring homomorphism

$$\mathscr{O}_P \to K,\ \frac{f}{g} \mapsto \frac{f(P)}{g(P)}$$

which we will call the **evaluation map**. Its kernel will be denoted by

$$I_P := I_{\mathbb{A}^2,P} := \left\{ \frac{f}{g} : f,g \in K[x,y] \text{ with } f(P) = 0 \text{ and } g(P) \neq 0 \right\} \quad \subset \mathscr{O}_P.$$

Remark 2.2 (Geometric and algebraic interpretation of local rings)**.** Intuitively, $\mathscr{O}_P$ describes "nice" (i. e. rational) functions that have a well-defined value at P (determined by the evaluation map), and thus also in a neighborhood of P. Note however that $\mathscr{O}_P$ does not admit similar evaluation maps at other points $Q \neq P$ since the denominator of the fractions might vanish there. This explains the name "local ring" from a geometric point of view. The ideal I_P in $\mathscr{O}_P$ describes exactly those local functions that have the value 0 at P.

Algebraically, $\mathscr{O}_P$ is a subring of $K(x,y)$ that contains $K[x,y]$. As a subring of a field it is an integral domain, and its units are precisely the fractions $\frac{f}{g}$ for which both f and g are non-zero at P. Moreover, just like $K[x,y]$ it is a factorial ring, with the irreducible elements being the irreducible polynomials that vanish at P (since the others have become units).

For those who know some commutative algebra we should mention that $\mathscr{O}_P$ is also a local ring in the algebraic sense, i. e. that it contains exactly one maximal ideal, namely I_P [G6, Definition 6.9]: If I is any ideal in $\mathscr{O}_P$ that is not a subset of I_P then it must contain an element $\frac{f}{g}$ with $f(P) \neq 0$ and $g(P) \neq 0$. But this is then a unit since $\frac{g}{f} \in \mathscr{O}_P$ as well, and hence we have $I = \mathscr{O}_P$.

In fact, in the algebraic sense $\mathscr{O}_P$ is just the localization of the polynomial ring $K[x,y]$ at the maximal ideal $\langle x-x_0, y-y_0 \rangle$ associated to the point $P=(x_0,y_0)$ — which also shows that it is a local ring [G6, Corollary 6.10].

Definition 2.3 (Intersection multiplicities)**.** For a point $P \in \mathbb{A}^2$ and two curves (or polynomials) F and G we define the **intersection multiplicity** of F and G at P to be

$$\mu_P(F,G) := \dim \mathscr{O}_P/\langle F,G \rangle \quad \in \mathbb{N} \cup \{\infty\},$$

where dim denotes the dimension as a vector space over K.

As this definition is rather abstract, we should of course figure out how to compute this number, what its properties are, and why it captures the geometric idea given above. In fact, it is not even obvious whether $\mu_P(F,G)$ is finite. But let us start with a few simple statements and examples.

Remark 2.4.

(a) It is clear from the definitions that an invertible *affine coordinate transformation* from (x,y) to

$$(x',y') = (ax+by+c, dx+ey+f) \quad \text{for } a,b,c,d,e,f \in K \text{ with } ae-bd \neq 0$$

gives us an isomorphism between the local rings $\mathscr{O}_P$ and $\mathscr{O}_{P'}$, where P' is the image point of P; and between $\mathscr{O}_P/\langle F,G \rangle$ and $\mathscr{O}_{P'}/\langle F',G' \rangle$, where F' and G' are F and G expressed in the new coordinates x' and y'. We will often use this invariance to simplify our calculations by picking suitable coordinates, e. g. such that $P=0$ is the origin.

(b) The intersection multiplicity is symmetric: We have $\mu_P(F,G) = \mu_P(G,F)$ for all F and G.

(c) For all F,G,H we have $\langle F, G+FH \rangle = \langle F,G \rangle$, and thus $\mu_P(F,G+FH) = \mu_P(F,G)$.

In Definition 2.3, we have not required a priori that P actually lies on both curves F and G. However, the intersection multiplicity is at least 1 if and only if it does:

Lemma 2.5. *Let $P \in \mathbb{A}^2$, and let F and G be two curves (or polynomials). We have:*

(a) $\mu_P(F,G) \geq 1$ *if and only if* $P \in F \cap G$;

(b) $\mu_P(F,G) = 1$ *if and only if* $\langle F,G \rangle = I_P$ *in* $\mathscr{O}_P$.

Proof. Assume first that $F(P) \neq 0$. Then F is a unit in $\mathscr{O}_P$, and thus $\langle F,G \rangle = \mathscr{O}_P$, i. e. $\mu_P(F,G) = 0$. Moreover, we then have $P \notin F$ and $F \notin I_P$, proving both (a) and (b) in this case. Of course, the case $G(P) \neq 0$ is analogous.

So we may now assume that $F(P) = G(P) = 0$, i. e. $P \in F \cap G$. Then the evaluation map at P induces a well-defined and surjective map $\mathscr{O}_P/\langle F,G \rangle \to K$. It follows that $\mu_P(F,G) \geq 1$, proving (a) in this case. Moreover, we have $\mu_P(F,G) = 1$ if and only if this map is an isomorphism, i. e. if and only if $\langle F,G \rangle$ is exactly the kernel I_P of the evaluation map. □

Example 2.6 (Intersection multiplicity of coordinate axes)**.** The kernel I_0 of the evaluation map at 0 consists exactly of the fractions $\frac{f}{g}$ such that f does not have a constant term, which is just the ideal $\langle x,y \rangle$ in $\mathscr{O}_0$. By Lemma 2.5 (b) this means that $\mu_0(x,y) = 1$, i. e. (as expected) that the two coordinate lines have intersection multiplicity 1 at the origin.

Another basic case (which we did not exclude in the definition) is when the two curves actually agree, or more generally if they have a common irreducible component through P. Although this is clearly not the main case we are interested in, it is reassuring to know that in this case the intersection multiplicity is infinite since the curves "touch at P to infinite order":

Exercise 2.7. Let F and G be two curves through a point $P \in \mathbb{A}^2$. Show:

(a) If F and G have no common component then the family $(F^n)_{n \in \mathbb{N}}$ is linearly independent in $\mathscr{O}_P / \langle G \rangle$.

(b) If F and G have a common component through P then $\mu_P(F,G) = \infty$.

For the last important basic property of intersection multiplicities we first need another easy algebraic tool.

Construction 2.8 ((Short) exact sequences)**.** We say that a sequence

$$0 \longrightarrow U \stackrel{\varphi}{\longrightarrow} V \stackrel{\psi}{\longrightarrow} W \longrightarrow 0$$

of linear maps between vector spaces (where 0 denotes the zero vector space) is **exact** if the image of each map equals the kernel of the next, i. e. if

(a) $\ker \varphi = 0$ (i. e. φ is injective);

(b) $\operatorname{im} \varphi = \ker \psi$; and

(c) $\operatorname{im} \psi = W$ (i. e. ψ is surjective).

In this case, we get a dimension formula

$$\begin{aligned} \dim U + \dim W &\stackrel{(a),(c)}{=} \dim \operatorname{im} \varphi + \dim \operatorname{im} \psi = \dim \operatorname{im} \varphi + \dim V / \ker \psi \stackrel{(b)}{=} \dim \operatorname{im} \varphi + \dim V / \operatorname{im} \varphi \\ &= \dim V. \end{aligned}$$

Proposition 2.9 (Additivity of intersection multiplicities)**.** *Let $P \in \mathbb{A}^2$, and let F, G, H be any three curves (or polynomials).*

(a) *If F and G have no common component through P there is an exact sequence*

$$0 \longrightarrow \mathscr{O}_P / \langle F,H \rangle \stackrel{\cdot G}{\longrightarrow} \mathscr{O}_P / \langle F,GH \rangle \stackrel{\pi}{\longrightarrow} \mathscr{O}_P / \langle F,G \rangle \longrightarrow 0,$$

where π is the natural quotient map.

(b) *We have $\mu_P(F,GH) = \mu_P(F,G) + \mu_P(F,H)$.*

Proof.

(a) We may assume that F and G have no common component at all, since components that do not pass through P are units in $\mathscr{O}_P$ and can therefore be dropped in the ideals.

It is checked immediately that both non-trivial maps in this sequence are well-defined, and that conditions (b) and (c) of Construction 2.8 hold. Hence we just have to show that the first multiplication map is injective: Assume that $\frac{f}{g}$ is in the kernel of this map, i. e. that

$$\frac{f}{g} \cdot G = \frac{f'}{g'} \cdot F + \frac{f''}{g''} \cdot GH$$

for certain $f', f'', g', g'' \in K[x,y]$ with $g'(P)$ and $g''(P)$ non-zero. We may assume without loss of generality that all three fractions have the same denominator, and multiply by it to obtain the equation $fG = f'F + f''GH$ in $K[x,y]$. Now G clearly divides fG and $f''GH$, hence also $f'F$, and consequently f' as F and G have no common component. So we have

$f' = aG$ for some $a \in K[x,y]$, and we see that $fG = aFG + f''GH$. Dividing by G, it follows that $f = aF + f''H$, so that f and hence also $\frac{f}{g}$ are zero in $\mathscr{O}_P/\langle F,H\rangle$. This shows the injectivity of the first map.

(b) If F and G have no common component through P the statement follows immediately from (a) by taking dimensions as in Construction 2.8. Otherwise the equation is true as $\infty = \infty$ by Exercise 2.7 (b). □

Example 2.10 (Intersection multiplicity with a coordinate axis)**.** Let F be an affine curve. We want to compute its intersection multiplicity $\mu_0(y,F)$ with this axis at the origin.

First note that if F has the x-axis y as a component, then $\mu_0(y,F) = \infty$ by Exercise 2.7 (b). So we can assume from now on that this is not the case.

By Remark 2.4 (c) we may remove all multiples of y from F, i. e. replace F by the polynomial $F(x,0) \in K[x]$, which is not the zero polynomial since y is not a component of F. We can write $F(x,0) = x^m g$ where $g \in K[x]$ is non-zero at the origin, so that m is the multiplicity of 0 in $F(x,0)$. Hence we obtain

$$\begin{aligned}
\mu_0(y,F) &= \mu_0(y,F(x,0)) && \text{(Remark 2.4 (c))}\\
&= \mu_0(y,x^m g)\\
&= m\,\mu_0(y,x) + \mu_0(y,g) && \text{(Proposition 2.9 (b))}\\
&= m && \text{(Example 2.6 and Lemma 2.5 (a)).}
\end{aligned}$$

Note that this coincides with the expectation from the beginning of this chapter: If $f \in K[x]$ is a univariate polynomial with a zero x_0 of multiplicity m (which is just $x_0 = 0$ in our current case) then the intersection multiplicity of its graph $y - f$ with the x-axis at the point $(x_0,0)$ is m.

02

We are now ready to compute the intersection multiplicity of two arbitrary curves F and G. By Remark 2.4 (a) it suffices to do this at the origin. For future reference, we formulate the following recursive algorithm for any two curves F and G through 0. Afterwards, we will prove that the algorithm actually terminates and gives a finite answer for $\mu_0(F,G)$ if F and G have no common component through 0.

Algorithm 2.11 (Computation of the intersection multiplicity $\mu_0(F,G)$)**.** Let F and G be two curves (or polynomials) with $F(0) = G(0) = 0$. We then repeat the following procedure recursively:

(a) If F and G both contain a monomial independent of y, we write

$$\begin{aligned}
F &= ax^m + (\text{terms involving } y \text{ or with a lower power of } x),\\
G &= bx^n + (\text{terms involving } y \text{ or with a lower power of } x)
\end{aligned}$$

for some $a,b \in K^*$ and $m,n \in \mathbb{N}$, where we may assume (by possibly swapping F and G) that $m \geq n$. We then set

$$F' := F - \frac{a}{b}x^{m-n}G,$$

hence canceling the x^m-term in F. By Remark 2.4 (c) we then have $\mu_0(F,G) = \mu_0(F',G)$. As $F'(0) = G(0) = 0$, we can repeat the algorithm recursively with F' and G to compute $\mu_0(F',G)$.

(b) If one of the polynomials F and G, say F, does not contain a monomial independent of y, we can factor $F = yF'$ and obtain by Proposition 2.9 (b)

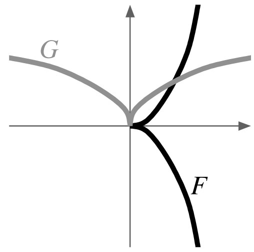
$$\mu_0(F,G) = \mu_0(y,G) + \mu_0(F',G).$$

The multiplicity $\mu_0(y,G)$ can be computed directly: By Example 2.10, it is the lowest power of x in a term of G independent of y (or ∞ if G contains y as a factor).

As for $\mu_0(F',G)$, if F' does not vanish at 0 then $\mu_0(F',G) = 0$ by Lemma 2.5 (a). So we have then computed $\mu_0(F,G)$ and stop the algorithm. Otherwise, we have $F'(0) = G(0) = 0$, and we can repeat the algorithm recursively with F' and G to compute $\mu_0(F',G)$.

Example 2.12. Let us compute the intersection multiplicity $\mu_0(F,G)$ at the origin of the two curves $F = y^2 - x^3$ and $G = x^2 - y^3$ as in the picture below on the right. We follow Algorithm 2.11 and indicate by (a) and (b) which step we performed each time:

$$\begin{aligned}
\mu_0(y^2-x^3,x^2-y^3) &\overset{(a)}{=} \mu_0(y^2-x^3+x(x^2-y^3),x^2-y^3)\\
&= \mu_0(y^2-xy^3,x^2-y^3)\\
&\overset{(b)}{=} \underbrace{\mu_0(y,x^2-y^3)}_{=2 \text{ by } 2.10} + \mu_0(y-xy^2,x^2-y^3)\\
&\overset{(b)}{=} 2 + \underbrace{\mu_0(y,x^2-y^3)}_{=2 \text{ by } 2.10} + \underbrace{\mu_0(1-xy,x^2-y^3)}_{=0 \text{ by } 2.5\text{ (a)}}\\
&= 4.
\end{aligned}$$



Fact 2.13 (Noetherian rings)**.** A ring R is called *Noetherian* if there is no infinite strictly ascending chain of ideals $I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots$ in R. It can be shown that the polynomial ring $K[x,y]$ is Noetherian [G6, Proposition 7.13], and that this property is inherited by the local rings $\mathscr{O}_P$ [G6, Exercise 7.23].

Proposition 2.14 (Finiteness of the intersection multiplicity)**.** *Let F and G be two curves (or polynomials) that have no common component through the origin. Then Algorithm 2.11 terminates with a finite answer for the intersection multiplicity $\mu_0(F,G)$.*

Proof. First note that the property of not having a common component through 0 is preserved in the algorithm: In (a) the common components of F' and G are the same as those of F and G, and in (b) F' and G clearly cannot have a common component through 0 if $F = yF'$ and G do not. Hence the case $\mu_0(y,G) = \infty$ in case (b) of the algorithm cannot occur, and thus the algorithm will give a finite answer if it terminates.

To prove termination, we now consider the ideal $\langle F,G \rangle$ in $\mathscr{O}_0$ during the algorithm.

In case (a) of the algorithm we have $\langle F',G \rangle = \langle F,G \rangle$, and the degree of the highest monomial of F independent of y strictly decreases. Hence, just as in the usual Euclidean algorithm [G1, Proposition 10.26], this case (a) can only happen finitely many times in a row before we must be in case (b).

In case (b) we then have $F = yF'$, and hence $\langle F,G \rangle \subset \langle F',G \rangle$. In fact, this inclusion is strict: Otherwise, in the exact sequence

$$0 \longrightarrow \mathscr{O}_0/\langle y,G \rangle \xrightarrow{\cdot F'} \mathscr{O}_0/\langle F,G \rangle \xrightarrow{\pi} \mathscr{O}_0/\langle F',G \rangle \longrightarrow 0$$

of Proposition 2.9 (a) the map π would be an isomorphism, and hence the first term $\mathscr{O}_0/\langle y,G \rangle$ would have to be zero, i. e. $\mu_0(y,G) = 0$ — in contradiction to Lemma 2.5 (a).

So if the algorithm did not stop, we would get an infinite strictly ascending chain of ideals in $\mathscr{O}_0$, which does not exist by Fact 2.13. □

Remark 2.15. If F and G have a common component through 0, Algorithm 2.11 might not terminate. For example, for the curves $F = x^2$ and $G = xy - x$ it yields

$$\begin{aligned}
\mu_0(x^2,xy-x) &\overset{(a)}{=} \mu_0(x^2+x(xy-x),xy-x)\\
&= \mu_0(x^2y,xy-x)\\
&\overset{(b)}{=} \underbrace{\mu_0(y,xy-x)}_{=1 \text{ by } 2.10} + \mu_0(x^2,xy-x),
\end{aligned}$$

leading to an infinite loop. However the algorithm is correct in any case, so if it does terminate (with a finite answer), then by Exercise 2.7 (b) we have proven simultaneously with this computation that F and G have no common component through P.

Exercise 2.16. Draw the real curves $F = x^2 + y^2 + 2y$ and $G = y^3x^6 - y^6x^2$, determine their irreducible decompositions, their intersection points, and their intersection multiplicities at these points.

Exercise 2.17.

(a) For the curves $F = y - x^3$ and $G = y^3 - x^4$, find a polynomial representative of $\frac{1}{x+1}$ in $\mathscr{O}_0/\langle F,G\rangle$, i. e. compute a polynomial $f \in K[x,y]$ whose class equals that of $\frac{1}{x+1}$ in $\mathscr{O}_0/\langle F,G\rangle$.

(b) Prove for arbitrary coprime F and G and any $P \in \mathbb{A}^2$ that every element of $\mathscr{O}_P/\langle F,G\rangle$ has a polynomial representative.

Is this statement still true if F and G are not coprime?

Following our algorithm, we can also give an easy and important criterion for when the intersection multiplicity is 1.

Notation 2.18 (Homogeneous parts of polynomials)**.** For a polynomial $F \in K[x,y]$ of degree d and $i = 0,\dots,d$, we define the *degree-i part* of F to be the sum of all terms of F of degree i. Hence all F_i are homogeneous, and we have $F = F_0 + \cdots + F_d$. We call F_0 the *constant part*, F_1 the *linear part*, and F_d the *leading part* of F.

Proposition 2.19 (Intersection multiplicity 1)**.** *Let F and G be two curves (or polynomials) through the origin. Then $\mu_0(F,G) = 1$ if and only if the linear parts F_1 and G_1 are linearly independent.*

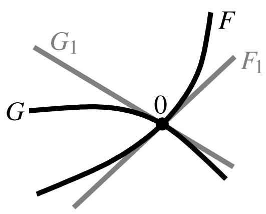
Proof. We prove the statement following Algorithm 2.11, using the notation from there.

In case (a), note that F_1' and G_1 are linearly independent if and only if F_1 and G_1 are, as either $F_1' = F_1$ (if $m > n$) or $F_1' = F_1 - \frac{a}{b}G_1$ (if $m = n$). Hence we can consider the first time we reach case (b). As $\mu_0(y,G) > 0$ by Lemma 2.5 (a), we have

$$\begin{aligned}
\mu_0(F,G) = 1 \;&\Leftrightarrow\; \mu_0(y,G) = 1 \text{ and } \mu(F',G) = 0 \\
&\Leftrightarrow\; G \text{ contains a monomial } x^1y^0 \text{ and } F' \text{ contains a constant term} \\
&\qquad\qquad \text{(by Example 2.10 and Lemma 2.5 (a))} \\
&\Leftrightarrow\; G_1 = ax + by \text{ for some } a \in K^*, b \in K, \text{ and } F_1 = cy \text{ for some } c \in K^* \\
&\Leftrightarrow\; F_1 \text{ and } G_1 \text{ are linearly independent,}
\end{aligned}$$

where the last implication "$\Leftarrow$" follows since $F = yF'$ does not contain a monomial x^1y^0. □

In fact, Proposition 2.19 has an easy geometric interpretation in the spirit of the beginning of this chapter: F_1 and G_1 can be thought of as the linear approximations of F and G around the origin. If these approximations are non-zero, hence lines, they can be thought of as the tangents to the curves as in the picture on the right, and the proposition states that the intersection multiplicity is 1 if and only if these tangent directions are not the same.



In general, it is the lowest non-zero terms of a curve F that can be considered as the best local approximation of F around 0. We can use this idea to define tangents to arbitrary curves (i. e. even if F_1 vanishes) as follows.

Definition 2.20 (Tangents and multiplicities of points)**.** Let F be a curve.

(a) The smallest $m \in \mathbb{N}$ for which the homogeneous part F_m is non-zero is called the **multiplicity** $m_0(F)$ of F at the origin. Any linear factor of F_m is called a **tangent** to F at the origin.

(b) For a general point $P = (x_0, y_0) \in \mathbb{A}^2$, tangents at P and the multiplicity $m_P(F)$ are defined by first shifting coordinates to $x' = x - x_0$ and $y' = y - y_0$, and then applying (a) to the origin $(x', y') = (0,0)$.

Exercise 2.21. Given a linear coordinate transformation that maps the origin to itself and a curve F to F', show that $m_0(F) = m_0(F')$, and that the transformation maps any tangent of F to a tangent of F'.

In particular, despite its appearance, Definition 2.20 is independent of the choice of coordinates on $\mathbb{A}^2$.

By definition, we clearly have $m_P(F) > 0$ if and only if $P \in F$. The most important case of Definition 2.20 is when $m_P(F) = 1$, i. e. if there is a non-zero local linear approximation for F around P. There is a special terminology for this case.

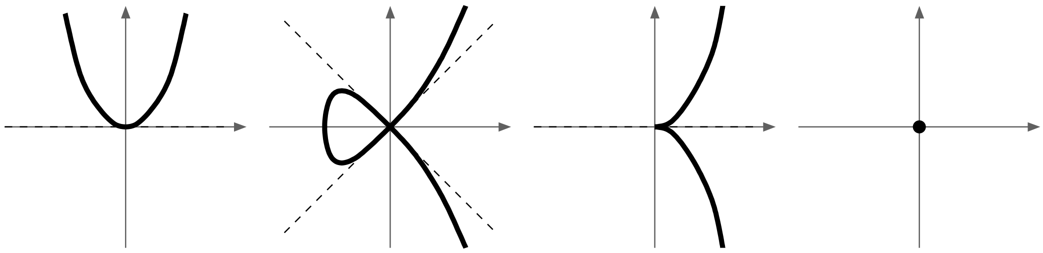
Definition 2.22 (Smooth and singular points)**.** Let F be a curve.

(a) A point P on F is called **smooth** or **regular** if $m_P(F) = 1$. Note that F has then a unique tangent at P, which we will denote by $T_P F$. For $P = 0$, it is simply given by the linear part F_1 of F.

If P is not a smooth point, i. e. if $m_P(F) > 1$, we say that P is a **singular** point or a **singularity** of F. As a special case, a singularity with $m_P(F) = 2$ such that F has two different tangents there is called a **node**.

(b) The curve F is said to be **smooth** or **regular** if all its points are smooth. Otherwise, F is called **singular**.

Example 2.23. Let us consider the origin in the real curves in the following picture.



$y - x^2$ (a) $\quad$ $y^2 - x^2 - x^3$ (b) $\quad$ $y^2 - x^3$ (c) $\quad$ $x^2 + y^2$ (d)

For the case (a), the curve $F = y - x^2$ in (a) has (no constant but) a linear term y. Hence, we have $m_0(F) = 1$, the origin is a smooth point of the curve, and its tangent there is $T_0 F = y$.

For the other three curves, the origin is a singular point of multiplicity 2. In (b), this singularity is a node, since the quadratic term is $y^2 - x^2 = (y - x)(y + x)$, and thus we have the two tangents $y - x$ and $y + x$, shown as dashed lines in the picture. The curve in (c) has only one tangent y which is of multiplicity 2. Finally, in (d) there is no tangent at all since $x^2 + y^2$ does not contain a linear factor over $\mathbb{R}$. Note that, in any case, knowing the tangents of F at the origin (which are easy to compute) tells us to some extent what the curve looks like locally around 0.

With these notations we can now reformulate Proposition 2.19.

Corollary 2.24 (Transverse intersections)**.** *Let P be a point in the intersection of two curves F and G. Then $\mu_P(F,G) = 1$ if and only if P is a smooth point of both F and G, and $T_P F \neq T_P G$.*

We say in this case that F and G intersect ***transversely*** *at P.*

Remark 2.25 (Additivity of point multiplicities)**.** Note that $m_P(FG) = m_P(F) + m_P(G)$. Hence, any point that lies on at least two (not necessarily distinct) irreducible components has multiplicity at least 2, and is thus a singular point. In particular, all points on a component of multiplicity at least 2 (in the sense of Definition 1.5 (c)) are always singular.

To check if a given curve F is smooth, i. e. whether every point $P \in F$ is a smooth point of F, there is a simple criterion that does not require to shift P to the origin first. It uses the (partial) derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ of F, which can be defined purely formally over an arbitrary ground field and then satisfy the usual rules of differentiation [G1, Exercise 9.10].

Proposition 2.26 (Affine Jacobi Criterion)**.** *Let $P = (x_0, y_0)$ be a point on an affine curve F.*

(a) *P is a singular point of F if and only if* $\frac{\partial F}{\partial x}(P) = \frac{\partial F}{\partial y}(P) = 0$.

(b) *If P is a smooth point of F the tangent to F at P is given by*

$$T_P F = \frac{\partial F}{\partial x}(P) \cdot (x - x_0) + \frac{\partial F}{\partial y}(P) \cdot (y - y_0).$$

Proof. Substituting $x = x' + x_0$ and $y = y' + y_0$, we can consider F as a polynomial in x' and y'. If we expand

$$F = ax' + by' + (\text{higher order terms in } x' \text{ and } y'),$$

then by definition F is singular at $(x', y') = (0,0)$, i. e. at P, if and only if $a = b = 0$. But by the chain rule of differentiation we have

$$\frac{\partial F}{\partial x}(P) = \frac{\partial F}{\partial x'}(0) = a \quad \text{and} \quad \frac{\partial F}{\partial y}(P) = \frac{\partial F}{\partial y'}(0) = b,$$

so that (a) follows. Moreover, if F is smooth at P then its tangent is just the term of F linear in x' and y', i. e.

$$ax' + by' = \frac{\partial F}{\partial x}(P) \cdot (x - x_0) + \frac{\partial F}{\partial y}(P) \cdot (y - y_0),$$

as claimed in (b). □

Example 2.27. Consider again the real curve $F = y^2 - x^2 - x^3$ from Example 2.23 (b). To determine its singular points, we compute the partial derivatives

$$\frac{\partial F}{\partial x} = -2x - 3x^2 \quad \text{and} \quad \frac{\partial F}{\partial y} = 2y.$$

Its common zeros are $(0,0)$ and $(-\frac{2}{3}, 0)$. But the latter does not lie on the curve, and so we conclude that the origin is the only singular point of F.

03

Smoothness of a curve F at a point P has another important algebraic consequence: It means that the containment of ideals containing F in $\mathscr{O}_P$ can be checked by a simple comparison of intersection multiplicities.

Proposition 2.28 (Comparing ideals using intersection multiplicities)**.** *Let P be a smooth point on a curve F. Then for any two curves G and H that do not have a common component with F through P we have*

$$\langle F, G \rangle \subset \langle F, H \rangle \text{ in } \mathscr{O}_P \quad \Leftrightarrow \quad \mu_P(F,G) \geq \mu_P(F,H).$$

So in particular, we have $\langle F, G \rangle = \langle F, H \rangle$ in $\mathscr{O}_P$ if and only if $\mu_P(F,G) = \mu_P(F,H)$.

Proof.

"⇒": Clearly, if $\langle F, G \rangle \subset \langle F, H \rangle$ then $\mu_P(F,G) = \dim \mathscr{O}_P / \langle F, G \rangle \geq \dim \mathscr{O}_P / \langle F, H \rangle = \mu_P(F,H)$.

"⇐": Let L be a line through P which is not the tangent $T_P F$. Then $\mu_P(F,L) = 1$ by Corollary 2.24, and hence $\mu_P(F, L^n) = n$ for all $n \in \mathbb{N}$ by Proposition 2.9. Let n be the maximum number such that $\langle F, G \rangle \subset \langle F, L^n \rangle$ in $\mathscr{O}_P$ (this exists since $\langle F, G \rangle \subset \mathscr{O}_P = \langle F, L^0 \rangle$, and $\langle F, G \rangle \subset \langle F, L^n \rangle$ requires $n \leq \mu_P(F,G)$ by the direction "⇒" that we have already shown).

We claim that then $\langle F, G \rangle = \langle F, L^n \rangle$ in $\mathscr{O}_P$, i. e. that $L^n \in \langle F, G \rangle$. To see this, note that $\langle F, G \rangle \subset \langle F, L^n \rangle$ implies $G = aF + bL^n$ for some $a, b \in \mathscr{O}_P$. If we had $b(P) = 0$ it would follow that $b \in I_P = \langle F, L \rangle$ by Lemma 2.5 (b), i. e. $b = cF + dL$ for some $c, d \in \mathscr{O}_P$, which means that $G = aF + (cF + dL)L^n \in \langle F, L^{n+1} \rangle$ and thus contradicts the maximality of n. Hence $b(P) \neq 0$, i. e. b is a unit in $\mathscr{O}_P$, and we obtain $L^n = \frac{1}{b}(G - aF) \in \langle F, G \rangle$ as desired.

Of course, now $\langle F, G \rangle = \langle F, L^n \rangle$ implies that $\mu_P(F,G) = \mu_P(F, L^n) = n$, so that we obtain $\langle F, G \rangle = \langle F, L^{\mu_P(F,G)} \rangle$. But the same holds for H instead of G, and so the inequality $\mu_P(F,G) \geq \mu_P(F,H)$ yields

$$\langle F, G \rangle = \langle F, L^{\mu_P(F,G)} \rangle \subset \langle F, L^{\mu_P(F,H)} \rangle = \langle F, H \rangle.$$ □

Example 2.29. Proposition 2.28 is false without the smoothness assumption on F: For the real curve $F = x^2 - y^2 = (x-y)(x+y)$ (i. e. the union of the two diagonals in $\mathbb{A}^2$, with singular point 0), $G = x$, and $H = y$, we have $\langle F,G \rangle = \langle x,y^2 \rangle$ and $\langle F,H \rangle = \langle y,x^2 \rangle$. Hence $\mu_0(F,G) = \mu_0(F,H) = 2$, but $\langle F,G \rangle \neq \langle F,H \rangle$ (since $y \notin \langle x,y^2 \rangle$, as otherwise we would have $\langle x,y^2 \rangle = \langle x,y \rangle$, in contradiction to $\mu_0(x,y^2) = 2 \neq 1 = \mu_0(x,y)$).

Remark 2.30 (Smooth curves over $\mathbb{R}$)**.** For the ground field $K = \mathbb{R}$, our results on smooth curves have an intuitive interpretation:

(a) The Jacobi Criterion of Proposition 2.26 (a) states that P is a smooth point of the curve F if and only if the Implicit Function Theorem [G2, Proposition 27.9] can be applied to the equation $F = 0$ around P, so that $V(F)$ is a 1-dimensional submanifold of $\mathbb{R}^2$ [G2, Definition 27.17]. Hence, in this case $V(F)$ is locally the graph of a differentiable function (expressing y as a function of x or vice versa), and thus we arrive at the intuitive interpretation of smoothness as “having no sharp corners”.

(b) To interpret Proposition 2.28, let us continue the picture of (a) and consider a local (analytic) coordinate z around P on the 1-dimensional manifold $V(F)$. In accordance with the idea of intersection multiplicity at the beginning of this chapter, a curve G should have intersection multiplicity n with F at P if on F it is locally a function of the form az^n in this coordinate, with a non-zero at P (corresponding to a unit in $\mathscr{O}_P$). Now if $n = \mu_P(F,G) \geq \mu_P(F,H) = m$ then in the same way H is of the form bz^m, so that $bz^m = H$ divides $az^n = G$. This means that $\langle G \rangle \subset \langle H \rangle$ in $\mathscr{O}_P/\langle F \rangle$ (i. e. as functions on F, a point of view that we will discuss in detail starting in Chapter 6) and thus that $\langle F,G \rangle \subset \langle F,H \rangle$ in $\mathscr{O}_P$.

Exercise 2.31 (Cusps)**.** Let P be a point on an affine curve F. We say that P is a **cusp** if $m_P(F) = 2$, there is exactly one tangent L to F at P, and $\mu_P(F,L) = 3$.

(a) Give an example of a real curve with a cusp, and draw a picture of it.

(b) If F has a cusp at P, prove that F has only one irreducible component passing through P.

(c) If F and G have a cusp at P, what is the minimum possible value for the intersection multiplicity $\mu_P(F,G)$?

Exercise 2.32.

(a) Find all singular points of the curve $F = (x^2+y^2-1)^3 + 10x^2y^2 \in \mathbb{R}[x,y]$, and determine the multiplicities and tangents to F at these points.

(b) Show that an irreducible curve F over a field of characteristic 0 has only finitely many singular points.

Can you find weaker assumptions on F that also imply that F has only finitely many singular points?

(c) Show that an irreducible cubic can have at most one singular point, and that over an algebraically closed field this singularity must be a node or a cusp as in Exercise 2.31.