5. Tensor Products

In the last two chapters we have developed powerful methods to work with modules and linear maps between them. However, in practice bilinear (or more generally multilinear) maps are often needed as well, so let us have a look at them now. Luckily, it turns out that to study bilinear maps we do not have to start from scratch, but rather can reduce their theory to the linear case. More precisely, for given R-modules M and N we will construct another module named $M \otimes N$ — the so-called *tensor product* of M and N — such that *bilinear* maps from $M \times N$ to any other R-module P are in natural one-to-one correspondence with *linear* maps from $M \otimes N$ to P. So instead of bilinear maps from $M \times N$ we can then always consider linear maps from the tensor product, and thus use all the machinery that we have developed so far for homomorphisms.

As the construction of this tensor product is a bit lengthy, let us first give an easy example that should show the idea behind it.

Example 5.1 (Idea of tensor products)**.** Let M and N be finitely generated free modules over a ring R, and choose bases $B = (b_1, \dots, b_m)$ and $C = (c_1, \dots, c_n)$ of M and N, respectively. Then every bilinear map $\alpha : M \times N \to P$ to a third R-module P satisfies

$$\alpha(\lambda_1 b_1 + \cdots + \lambda_m b_m, \mu_1 c_1 + \cdots + \mu_n c_n) = \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_i \mu_j \, \alpha(b_i, c_j) \qquad (*)$$

for all $\lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_n \in R$. Hence α is uniquely determined by specifying the values $\alpha(b_i, c_j) \in P$. Conversely, any choice of these values gives rise to a well-defined bilinear map $\alpha : M \times N \to P$ by the above formula.

Now let F be a free R-module of rank $m \cdot n$. We denote a basis of this space by $b_i \otimes c_j$ for $i = 1, \dots, m$ and $j = 1, \dots, n$ — so at this point this is just a name for a basis of this module, rather than an actual operation between elements of M and N. By the same argument as above, a linear map $\varphi : F \to P$ can then be specified uniquely by giving arbitrary images $\varphi(b_i \otimes c_j) \in P$ of the basis elements [G2, Corollary 16.27]. Putting both results together, we see that there is a one-to-one correspondence between bilinear maps $\alpha : M \times N \to P$ and linear maps $\varphi : F \to P$, given on the bases by $\alpha(b_i, c_j) = \varphi(b_i \otimes c_j)$: both maps can be specified by giving $m \cdot n$ arbitrary elements of P. So in the above sense F is a tensor product of M and N.

If M and N are no longer free, but still finitely generated, we can at least pick generators $(b_1, \dots, b_m)$ and $(c_1, \dots, c_n)$ of M and N, respectively. Then $(*)$ shows that any bilinear map $\alpha : M \times N \to P$ is still determined by the values $\alpha(b_i, c_j)$. But these values can no longer be chosen independently; they have to be compatible with the relations among the generators. For example, if we have the relation $2b_1 + 3b_2 = 0$ in M, we must have $2\alpha(b_1, c_j) + 3\alpha(b_2, c_j) = 0$ for all j in order to get a well-defined bilinear map α. For the tensor product, this means the following: if G is the submodule of F generated by all relations — so in our example we would take $2\, b_1 \otimes c_j + 3\, b_2 \otimes c_j$ for all j — then bilinear maps $\alpha : M \times N \to P$ now correspond exactly to those linear maps $\varphi : F \to P$ that are zero on G. As these are the same as linear maps from F/G to P, we can now take F/G to be our tensor product of M and N.

In fact, this idea of the construction of the tensor product should be clearly visible in the proof of Proposition 5.5 below. The main difference will be that, in order to avoid unnatural choices of generators, we will just take *all* elements of M and N as a generating set. This will lead to a huge module F, but also to a huge submodule G of relations among these generators, and so the quotient F/G will again be what we want.

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But let us now see how to obtain the tensor product $M \otimes N$ rigorously. There are two options for this: we can either construct it directly and then prove its properties, or define it to be a module having the desired property — namely that linear maps from $M \otimes N$ are the same as bilinear maps from $M \times N$ — and show that such an object exists and is uniquely determined by this property. As this property is actually much more important than the technical construction of $M \otimes N$, we will take the latter approach.

Definition 5.2 (Tensor products)**.** Let M, N, and P be R-modules.

(a) A map $\alpha : M \times N \to P$ is called R-**bilinear** if $\alpha(\,\cdot\,, n) : M \to P$ and $\alpha(m, \cdot\,) : N \to P$ are R-linear for all $m \in M$ and $n \in N$.

(b) A **tensor product** of M and N over R is an R-module T together with a bilinear map $\tau : M \times N \to T$ such that the following *universal property* holds: for every bilinear map $\alpha : M \times N \to P$ to a third module P there is a unique linear map $\varphi : T \to P$ such that $\alpha = \varphi \circ \tau$, i. e. such that the diagram on the right commutes. The elements of a tensor product are called **tensors**.

$$\begin{tikzcd} M \times N \arrow[r, "\alpha"] \arrow[d, "\tau"'] & P \\ T \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

Remark 5.3. In the above notation, Definition 5.2 (b) just means that there is a one-to-one correspondence

$$\begin{array}{ccc} \{\text{bilinear maps } M \times N \to P\} & \overset{1:1}{\longleftrightarrow} & \{\text{homomorphisms } T \to P\} \\ \alpha & \longmapsto & \varphi \\ \varphi \circ \tau & \longleftarrow\!\shortmid & \varphi \end{array}$$

as explained in the motivation above.

Proposition 5.4 (Uniqueness of tensor products)**.** *A tensor product is unique up to unique isomorphism in the following sense: if T_1 and T_2 together with bilinear maps $\tau_1 : M \times N \to T_1$ and $\tau_2 : M \times N \to T_2$ are two tensor products for M and N over R, there is a unique R-module isomorphism $\varphi : T_1 \to T_2$ such that $\tau_2 = \varphi \circ \tau_1$.*

$$\begin{tikzcd} M \times N \arrow[r, "\tau_2"] \arrow[d, "\tau_1"'] & T_2 \\ T_1 \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

Proof. Consider the universal property of Definition 5.2 (b) for the first tensor product: as $\tau_2 : M \times N \to T_2$ is bilinear, there is a unique morphism $\varphi : T_1 \to T_2$ with $\tau_2 = \varphi \circ \tau_1$. In the same way, reversing the roles of the tensor products we get a unique morphism $\psi : T_2 \to T_1$ with $\tau_1 = \psi \circ \tau_2$.

Now apply the universal property for the first tensor product again, this time for the bilinear map $\tau_1 : M \times N \to T_1$ as shown on the right. Note that we have $\psi \circ \varphi \circ \tau_1 = \psi \circ \tau_2 = \tau_1$ as well as $\mathrm{id}_{T_1} \circ \tau_1 = \tau_1$, so that both $\psi \circ \varphi$ and id_{T_1} make the diagram commute. Hence, by the uniqueness part of the universal property we conclude that $\psi \circ \varphi = \mathrm{id}_{T_1}$. In the same way we see that $\varphi \circ \psi = \mathrm{id}_{T_2}$, and thus φ is an isomorphism. □

$$\begin{tikzcd} M \times N \arrow[r, "\tau_1"] \arrow[d, "\tau_1"'] & T_1 \\ T_1 \arrow[ur, dashed, bend left, "\mathrm{id}"] \arrow[ur, dashed, bend right, "\psi \circ \varphi"'] & \end{tikzcd}$$

Proposition 5.5 (Existence of tensor products)**.** *Any two R-modules have a tensor product.*

Proof. Let M and N be R-modules. We denote by F the R-module of all finite formal linear combinations of elements of $M \times N$, i. e. formal sums of the form

$$a_1\,(m_1, n_1) + \cdots + a_k\,(m_k, n_k)$$

for $k \in \mathbb{N}$, $a_1, \ldots, a_k \in R$, and distinct $(m_i, n_i) \in M \times N$ for $i = 1, \ldots, k$. More precisely, F can be modeled as the set of maps from $M \times N$ to R that have non-zero values at most at finitely many elements, where the values at these elements $(m_1, n_1), \ldots, (m_k, n_k) \in M \times N$ are $a_1, \ldots, a_k$ in the above notation. In this picture, the R-module structure of F is then given by pointwise addition and scalar multiplication. It is more intuitive however to think of the elements of F as linear combinations of elements of $M \times N$ as above (rather than as functions from $M \times N$ to R), and so we will use this notation in the rest of the proof.

Note that the various elements $(m, n) \in M \times N$ are by definition all independent in F — e. g. for given $a \in R$, $m \in M$, and $n \in N$ the linear combinations $a\,(m, n)$, $1\,(am, n)$, and $1\,(m, an)$ are in general all different elements of F. In order to construct the tensor product we now want to enforce just enough relations so that the formal linear combinations become bilinear: let G be the submodule of F generated by all expressions

$$(m_1+m_2,n)-(m_1,n)-(m_2,n), \qquad (am,n)-a\,(m,n),$$
$$(m,n_1+n_2)-(m,n_1)-(m,n_2), \qquad \text{and } (m,an)-a\,(m,n)$$

for all $a \in R$, $m,m_1,m_2 \in M$, and $n,n_1,n_2 \in N$, and set $T := F/G$. Then the map

$$\tau : M \times N \to T, \ \ (m,n) \mapsto \overline{(m,n)}$$

is R-bilinear by the very definition of these relations.

We claim that T together with τ is a tensor product for M and N over R. So to check the universal property let $\alpha : M \times N \to P$ be an R-bilinear map. Then we can define a homomorphism $\varphi : T \to P$ by setting $\varphi\big(\overline{(m,n)}\big) := \alpha(m,n)$ and extending this by linearity, i. e.

$$\varphi\big(a_1\,\overline{(m_1,n_1)}+\cdots+a_k\,\overline{(m_k,n_k)}\big) = a_1\,\alpha(m_1,n_1)+\cdots+a_k\,\alpha(m_k,n_k).$$

Note that φ is well-defined since α is bilinear, and we certainly have $\alpha = \varphi \circ \tau$. Moreover, it is also obvious that setting $\varphi\big(\overline{(m,n)}\big) = \alpha(m,n)$ is the only possible choice such that $\alpha = \varphi \circ \tau$. Hence the universal property is satisfied, and T together with τ is indeed a tensor product. □

Notation 5.6 (Tensor products)**.** Let M and N be R-modules. By Propositions 5.4 and 5.5 there is a unique tensor product of M and N over R up to isomorphism, i. e. an R-module T together with a bilinear map $\tau : M \times N \to T$ satisfying the universal property of Definition 5.2 (b). We write T as $M \otimes_R N$ (or simply $M \otimes N$ if the base ring is understood), and $\tau(m,n)$ as $m \otimes n$. The element $m \otimes n \in M \otimes N$ is often called the tensor product of m and n.

Remark 5.7.

(a) Tensors in $M \otimes N$ that are of the form $m \otimes n$ for $m \in M$ and $n \in N$ are called **pure** or **monomial**. As we can see from the proof of Proposition 5.5, not every tensor in $M \otimes N$ is pure — instead, the pure tensors generate $M \otimes N$ as an R-module, i. e. a general element of $M \otimes N$ can be written as a finite linear combination $\sum_{i=1}^{k} a_i\,(m_i \otimes n_i)$ for $k \in \mathbb{N}$, $a_1,\dots,a_k \in R$, $m_1,\dots,m_k \in M$, and $n_1,\dots,n_k \in N$.

Note that these generators are not independent, so that there are in general many different ways to write a tensor as a linear combination of pure tensors. This makes it often a non-trivial task to decide whether two such linear combinations are the same tensor or not.

(b) The tensor product of two elements of M and N is bilinear by Definition 5.2 (b), i. e. we have

$$(m_1+m_2)\otimes n = m_1 \otimes n + m_2 \otimes n \quad \text{and} \quad a\,(m \otimes n) = (am)\otimes n$$

in $M \otimes N$ for all $a \in R$, $m,m_1,m_2 \in M$, and $n \in N$, and similarly for the second factor. In fact, the tensor product has been defined in such a way that the relations among tensors are *exactly* those bilinear ones.

(c) Of course, using multilinear instead of bilinear maps, we can also define tensor products $M_1 \otimes \cdots \otimes M_k$ of more than two modules in the same way as above. We will see in Exercise 5.9 however that the result is nothing but a repeated application of the tensor product for bilinear maps.

Before we give some examples of tensor product spaces, let us first prove a few simple properties that will also make the study of the examples easier.

Lemma 5.8. *For any R-modules M, N, and P there are natural isomorphisms*

(a) $M \otimes N \cong N \otimes M$;

(b) $M \otimes R \cong M$;

(c) $(M \oplus N) \otimes P \cong (M \otimes P) \oplus (N \otimes P)$.

Proof. The strategy for all three proofs is the same: using the universal property of Definition 5.2 (b) we construct maps between the tensor products from bilinear maps, and then show that they are inverse to each other.

(a) The map $M \times N \to N \otimes M$, $(m,n) \mapsto n \otimes m$ is bilinear by Remark 5.7 (b), and thus induces a (unique) linear map

$$\varphi : M \otimes N \to N \otimes M \quad \text{with} \quad \varphi(m \otimes n) = n \otimes m \quad \text{for all } m \in M \text{ and } n \in N$$

by the universal property of $M \otimes N$. In the same way we get a morphism $\psi : N \otimes M \to M \otimes N$ with $\psi(n \otimes m) = m \otimes n$. Then $(\psi \circ \varphi)(m \otimes n) = m \otimes n$ for all $m \in M$ and $n \in N$, so $\psi \circ \varphi$ is the identity on pure tensors. But the pure tensors generate $M \otimes N$, and so we must have $\psi \circ \varphi = \mathrm{id}_{M \otimes N}$. In the same way we see that $\varphi \circ \psi = \mathrm{id}_{N \otimes M}$. Hence φ is an isomorphism.

(b) From the bilinear map $M \times R \to M$, $(m,a) \mapsto am$ we obtain a linear map

$$\varphi : M \otimes R \to M \quad \text{with} \quad \varphi(m \otimes a) = am \quad \text{for all } m \in M \text{ and } a \in R$$

by the universal property. Furthermore, there is a linear map $\psi : M \to M \otimes R$, $m \mapsto m \otimes 1$. As

$$(\psi \circ \varphi)(m \otimes a) = am \otimes 1 = m \otimes a \quad \text{and} \quad (\varphi \circ \psi)(m) = m$$

for all $m \in M$ and $a \in R$, we conclude as in (a) that φ is an isomorphism.

(c) The bilinear map $(M \oplus N) \times P \to (M \otimes P) \oplus (N \otimes P)$, $((m,n),p) \mapsto (m \otimes p, n \otimes p)$ induces a linear map

$$\varphi : (M \oplus N) \otimes P \to (M \otimes P) \oplus (N \otimes P) \quad \text{with} \quad \varphi((m,n) \otimes p) = (m \otimes p, n \otimes p)$$

as above. Similarly, we get morphisms $M \otimes P \to (M \oplus N) \otimes P$ with $m \otimes p \mapsto (m,0) \otimes p$ and $N \otimes P \to (M \oplus N) \otimes P$ with $n \otimes p \mapsto (0,n) \otimes p$, and thus by addition a linear map

$$\psi : (M \otimes P) \oplus (N \otimes P) \to (M \oplus N) \otimes P \quad \text{with} \quad \psi(m \otimes p, n \otimes q) = (m,0) \otimes p + (0,n) \otimes q.$$

Is is verified immediately that φ and ψ are inverse to each other on pure tensors, and thus also on the whole tensor product space. □

Exercise 5.9 (Associativity of tensor products)**.** Let M, N, and P be three R-modules. Prove that there are natural isomorphisms

$$M \otimes N \otimes P \cong (M \otimes N) \otimes P \cong M \otimes (N \otimes P)$$

where $M \otimes N \otimes P$ is the tensor product constructed from trilinear maps as in Remark 5.7 (c).

Example 5.10.

(a) Let M and N be free R-modules of ranks m and n, respectively. Then $M \cong R^m$ and $N \cong R^n$, and so by Lemma 5.8 we have

$$M \otimes N \cong R^m \otimes (\underbrace{R \oplus \cdots \oplus R}_{n \text{ times}}) \cong (R^m \otimes R) \oplus \cdots \oplus (R^m \otimes R) \cong R^m \oplus \cdots \oplus R^m \cong R^{mn},$$

as expected from Example 5.1. So after a choice of bases the elements of $M \otimes N$ can be described by $m \times n$-matrices over R.

(b) Let I and J be coprime ideals in a ring R. Then there are elements $a \in I$ and $b \in J$ with $a + b = 1$, and so we obtain in the tensor product $R/I \otimes R/J$ for all monomial tensors $\overline{r} \otimes \overline{s}$ with $r, s \in R$

$$\overline{r} \otimes \overline{s} = (a+b)\,(\overline{r} \otimes \overline{s}) = \overline{ar} \otimes \overline{s} + \overline{r} \otimes \overline{bs} = 0$$

since $\overline{ar} = 0 \in R/I$ and $\overline{bs} = 0 \in R/J$. But these monomial tensors generate the tensor product, and so we conclude that $R/I \otimes R/J = 0$. As a concrete example, we get $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}_q = 0$ for any two distinct primes p and q.

This shows that (in contrast to (a)) a tensor product space need not be "bigger" than its factors, and might be 0 even if none of its factors are.

(c) In the tensor product $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2$ we have

$$2 \otimes \overline{1} = 1 \otimes \overline{2} = 0$$

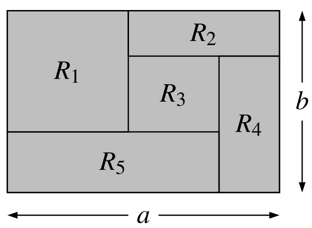
by bilinearity. However, if we now consider $2 \otimes \overline{1}$ as an element of $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2$, the above computation is invalid since $1 \notin 2\mathbb{Z}$. So is it still true that $2 \otimes \overline{1} = 0$ in $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2$?

We can answer this question with Lemma 5.8 (b): we know that $2\mathbb{Z}$ is isomorphic to $\mathbb{Z}$ as a $\mathbb{Z}$-module by sending 2 to 1, and so $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2$ is isomorphic to $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2 \cong \mathbb{Z}_2$ by the map $\varphi : 2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2 \to \mathbb{Z}_2$, $2a \otimes \overline{b} \mapsto \overline{ab}$. But now $\varphi(2 \otimes \overline{1}) = \overline{1} \neq 0$, and so indeed we have $2 \otimes \overline{1} \neq 0$ in $2\mathbb{Z} \otimes \mathbb{Z}_2$.

The conclusion is that, when writing down tensor products $m \otimes n$, we have to be very careful to specify which tensor product space we consider: if $n \in N$ and m lies in a submodule M' of a module M, it might happen that $m \otimes n$ is non-zero in $M' \otimes N$, but zero in $M \otimes N$. In other words, for a submodule M' of M it is not true in general that $M' \otimes N$ is a submodule of $M \otimes N$ in a natural way! We will discuss this issue in more detail in Proposition 5.22 (b) and Remark 5.23.

Exercise 5.11. Compute the tensor products $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$, $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, and $\mathbb{Q}[x] \otimes_{\mathbb{Q}} \mathbb{C}$.

Exercise 5.12. Assume that we have n rectangles $R_1, \dots, R_n$ in the plane, of size $a_i \times b_i$ for $i = 1, \dots, n$, that fit together to form a rectangle R of size $a \times b$ as in the picture on the right. Prove:



(a) $a \otimes b = \sum_{i=1}^n a_i \otimes b_i$ in $\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}$.

(b) If each of the rectangles $R_1, \dots, R_n$ has at least one side with a rational length, then R must also have at least one side with a rational length.

Exercise 5.13 (Dual vector spaces)**.** Let V and W be vector spaces over a field K. We call $V^* := \operatorname{Hom}_K(V, K)$ the *dual vector space* to V. Moreover, denote by $\operatorname{BLF}(V)$ the vector space of bilinear forms $V \times V \to K$.

(a) Show that there are (natural, i. e. basis-independent) linear maps

$$\begin{aligned} \Phi &: V^* \otimes W \to \operatorname{Hom}(V, W) && \text{such that } \Phi(\varphi \otimes w)(v) = \varphi(v) \cdot w, \\ \Psi &: V^* \otimes V^* \to \operatorname{BLF}(V) && \text{such that } \Psi(\varphi \otimes \psi)(v, v') = \varphi(v) \cdot \psi(v'), \\ T &: V^* \otimes V \to K && \text{such that } T(\varphi \otimes v) = \varphi(v). \end{aligned}$$

(b) Prove that Φ and Ψ are injective.

(c) Prove that Φ and Ψ are in fact isomorphisms if V and W are finite-dimensional, but not in general for arbitrary vector spaces.

(d) Assume that $\dim_K V = n < \infty$, so that $V^* \otimes V$ is naturally isomorphic to $\operatorname{Hom}_K(V, V)$ by (c), which in turn is isomorphic to $\operatorname{Mat}(n \times n, K)$ in the standard way after choosing a basis of V. Using these isomorphisms, T becomes a linear map $\operatorname{Mat}(n \times n, K) \to K$ that is invariant under a change of basis. Which one?

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To define homomorphisms between tensor products the following construction will be useful.

Construction 5.14 (Tensor product of homomorphisms)**.** Let $\varphi : M \to N$ and $\varphi' : M' \to N'$ be two homomorphisms of R-modules. Then the map $M \times M' \to N \otimes N'$, $(m, m') \mapsto \varphi(m) \otimes \varphi'(m')$ is bilinear, and thus by the universal property of the tensor product gives rise to a homomorphism

$$\varphi \otimes \varphi' : M \otimes M' \to N \otimes N' \quad \text{such that} \quad (\varphi \otimes \varphi')(m \otimes m') = \varphi(m) \otimes \varphi'(m')$$

for all $m \in M$ and $m' \in M'$. We call $\varphi \otimes \varphi'$ the **tensor product** of φ and φ'.

Remark 5.15. Note that for $\varphi \in \operatorname{Hom}_R(M,N)$ and $\varphi' \in \operatorname{Hom}_R(M',N')$ we have already defined a tensor product $\varphi \otimes \varphi'$ as an element of $\operatorname{Hom}_R(M,N) \otimes_R \operatorname{Hom}_R(M',N')$ in Notation 5.6 — whereas Construction 5.14 gives an element of $\operatorname{Hom}_R(M \otimes_R M', N \otimes_R N')$. In fact, it is easy to see by the universal property of the tensor product that there is a natural homomorphism

$$\operatorname{Hom}_R(M,N) \otimes_R \operatorname{Hom}_R(M',N') \to \operatorname{Hom}_R(M \otimes_R M', N \otimes_R N')$$

that sends $\varphi \otimes \varphi'$ in the sense of Notation 5.6 to the morphism of Construction 5.14. It should therefore not lead to confusion if we denote both constructions by $\varphi \otimes \varphi'$.

One application of tensor products is to extend the ring of scalars for a given module. For vector spaces, this is a process that you know very well: suppose that we have e. g. a real vector space V with $\dim_{\mathbb{R}} V = n < \infty$ and want to study the eigenvalues and eigenvectors of a linear map $\varphi : V \to V$. We then usually set up the matrix $A \in \operatorname{Mat}(n \times n, \mathbb{R})$ corresponding to φ in some chosen basis, and compute its characteristic polynomial. Often it happens that this polynomial does not split into linear factors over $\mathbb{R}$, and that we therefore want to pass from the real to the complex numbers.

But while it is perfectly possible to consider A as a complex matrix in $\operatorname{Mat}(n \times n, \mathbb{C})$ instead and talk about complex eigenvalues and eigenvectors of A, it is not clear what this means in the language of the linear map φ: in the condition $\varphi(x) = \lambda x$ for an eigenvector of φ it certainly does not make sense to take x to be a “complex linear combination” of the basis vectors, since such an element does not exist in V, and so φ is not defined on it. We rather have to extend V first to a complex vector space, and φ to a $\mathbb{C}$-linear map on this extension. It turns out that the tensor product of V with $\mathbb{C}$ over $\mathbb{R}$ is exactly the right construction to achieve this in a basis-independent language.

Construction 5.16 (Extension of scalars)**.** Let M be an R-module, and R' an R-algebra (so that R' is a ring as well as an R-module). Moreover, for any $a \in R'$ we denote by $\mu_a : R' \to R'$, $s \mapsto as$ the multiplication map, which is obviously a homomorphism of R-modules. If we then set $M_{R'} := M \otimes_R R'$, we obtain a scalar multiplication with R' on $M_{R'}$

$$\begin{aligned} R' \times M_{R'} &\to M_{R'} \\ (a, m \otimes s) &\mapsto a \cdot (m \otimes s) := (1 \otimes \mu_a)(m \otimes s) = m \otimes (as) \end{aligned}$$

which turns $M_{R'}$ into an R'-module. We say that $M_{R'}$ is obtained from M by an **extension of scalars** from R to R'. Note that any R-module homomorphism $\varphi : M \to N$ then gives rise to an “extended” R'-module homomorphism $\varphi_{R'} := \varphi \otimes \mathrm{id} : M_{R'} \to N_{R'}$.

Example 5.17 (Complexification of a real vector space)**.** Let V be a real vector space. Extending scalars from $\mathbb{R}$ to $\mathbb{C}$, we call $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ the *complexification* of V. By Construction 5.16, this is now a complex vector space.

Let us assume for simplicity that V is finitely generated, with basis $(b_1, \dots, b_n)$. Then $V \cong \mathbb{R}^n$ by an isomorphism that maps b_i to the i-th standard basis vector e_i for $i = 1, \dots, n$, and consequently

$$V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C} \cong (\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C})^n \cong \mathbb{C}^n$$

as $\mathbb{R}$-modules by Lemma 5.8. But by definition of the scalar multiplication with $\mathbb{C}$ from Construction 5.16 this is also a $\mathbb{C}$-module homomorphism, and thus an isomorphism of complex vector spaces. Moreover, $b_i \otimes 1$ maps to e_i under this chain of isomorphisms for all i, and so as expected the vectors $b_1 \otimes 1, \dots, b_n \otimes 1$ form a basis of the complexified vector space $V_{\mathbb{C}}$.

Finally, let us consider a linear map $\varphi : V \to V$ described by the matrix $A \in \operatorname{Mat}(n \times n, \mathbb{R})$ with respect to the basis $(b_1, \dots, b_n)$, i. e. we have $\varphi(b_i) = \sum_{j=1}^n a_{j,i} b_j$ for all i. Then $\varphi_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$ from Construction 5.16 is an endomorphism of the complexified vector space, and since

$$\varphi_{\mathbb{C}}(b_i \otimes 1) = \varphi(b_i) \otimes 1 = \sum_{j=1}^n a_{j,i}\,(b_j \otimes 1)$$

we see that the matrix of $\varphi_{\mathbb{C}}$ with respect to the basis $(b_1 \otimes 1, \dots, b_n \otimes 1)$ is precisely A again, just now considered as a complex matrix.

Of course, the same constructions can not only be used to pass from the real to the complex numbers, but also for any field extension.

Exercise 5.18. Let M and N be R-modules, and R' an R-algebra. Show that the extension of scalars commutes with tensor products in the sense that there is an isomorphism of R'-modules

$$(M \otimes_R N)_{R'} \cong M_{R'} \otimes_{R'} N_{R'}.$$

In Construction 5.16 we have considered the case when one of the two factors in a tensor product over a ring R is not only an R-module, but also an R-algebra. If both factors are R-algebras, we can say even more: in this case, the resulting tensor product will also be an R-algebra in a natural way:

Construction 5.19 (Tensor product of algebras)**.** Let R be a ring, and let R_1 and R_2 be R-algebras. Then the map $R_1 \times R_2 \times R_1 \times R_2 \to R_1 \otimes R_2$, $(s,t,s',t') \mapsto (ss') \otimes (tt')$ is multilinear, and so by the universal property of the tensor product (and its associativity as in Exercise 5.9) it induces a homomorphism

$$(R_1 \otimes R_2) \otimes (R_1 \otimes R_2) \to R_1 \otimes R_2 \quad \text{with} \quad (s \otimes t) \otimes (s' \otimes t') \mapsto (ss') \otimes (tt')$$

for all $s,s' \in R_1$ and $t,t' \in R_2$. Again by the universal property of the tensor product this now corresponds to a bilinear map

$$(R_1 \otimes R_2) \times (R_1 \otimes R_2) \to R_1 \otimes R_2 \quad \text{with} \quad (s \otimes t, s' \otimes t') \mapsto (s \otimes t) \cdot (s' \otimes t') := (ss') \otimes (tt').$$

It is obvious that this multiplication makes $R_1 \otimes R_2$ into a ring, and thus into an R-algebra. So the tensor product of two R-algebras has again a natural structure of an R-algebra.

Example 5.20 (Multivariate polynomial rings as tensor products)**.** Let R be a ring. We claim that $R[x,y] \cong R[x] \otimes_R R[y]$ as R-algebras, i. e. that polynomial rings in several variables can be thought of as tensor products of polynomial rings in one variable.

In fact, there are R-module homomorphisms

$$\varphi : R[x] \otimes_R R[y] \to R[x,y], \ \ f \otimes g \mapsto fg$$

(by the universal property of the tensor product) and

$$\psi : R[x,y] \mapsto R[x] \otimes R[y], \ \ \sum_{i,j} a_{i,j} x^i y^j \mapsto \sum_{i,j} a_{i,j} x^i \otimes y^j.$$

As

$$\begin{aligned} (\psi \circ \varphi)(x^i \otimes y^j) &= \psi(x^i y^j) = x^i \otimes y^j \\ \text{and} \quad (\varphi \circ \psi)(x^i y^j) &= \varphi(x^i \otimes y^j) = x^i y^j \end{aligned}$$

for all $i,j \in \mathbb{N}$ and these elements $x^i \otimes y^j$ and $x^i y^j$ generate $R[x] \otimes_R R[y]$ and $R[x,y]$ as an R-module, respectively, we see that φ and ψ are inverse to each other. Moreover, φ is also a ring homomorphism with the multiplication in $R[x] \otimes_R R[y]$ of Construction 5.19, since

$$\varphi((f \otimes g) \cdot (f' \otimes g')) = \varphi((ff') \otimes (gg')) = ff'gg' = \varphi(f \otimes g) \cdot \varphi(f' \otimes g').$$

Hence $R[x,y] \cong R[x] \otimes_R R[y]$ as R-algebras.

Exercise 5.21.

(a) Let I and J be ideals in a ring R. Prove that $R/I \otimes_R R/J \cong R/(I+J)$ as R-algebras.

(b) Let $X \subset \mathbb{A}^n_K$ and $Y \subset \mathbb{A}^m_K$ be varieties over a field K, so that $X \times Y \subset K^{n+m}$. Show that $X \times Y$ is again a variety, and $A(X \times Y) \cong A(X) \otimes_K A(Y)$ as K-algebras.

Finally, to conclude this chapter we want to study how tensor products behave in exact sequences. The easiest way to see this is to trace it back to Exercise 4.9, in which we applied $\mathrm{Hom}_R(\cdot N)$ to an exact sequence.

Proposition 5.22 (Tensor products are right exact)**.**

(a) *For any R-modules M, N, and P we have* $\mathrm{Hom}(M, \mathrm{Hom}(N,P)) \cong \mathrm{Hom}(M \otimes N, P)$.

(b) *Let*

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \longrightarrow 0$$

be an exact sequence of R-modules. Then for any R-module N the sequence

$$M_1 \otimes N \xrightarrow{\varphi_1 \otimes \mathrm{id}} M_2 \otimes N \xrightarrow{\varphi_2 \otimes \mathrm{id}} M_3 \otimes N \longrightarrow 0$$

is exact as well.

Proof.

(a) A natural isomorphism can be constructed by identifying $\alpha \in \mathrm{Hom}(M, \mathrm{Hom}(N,P))$ with $\beta \in \mathrm{Hom}(M \otimes N, P)$, where

$$\alpha(m)(n) = \beta(m \otimes n) \quad \in P$$

for all $m \in M$ and $n \in N$. Note that this equation can be used to define α in terms of β to get a map $\mathrm{Hom}(M \otimes N, P) \to \mathrm{Hom}(M, \mathrm{Hom}(N,P))$, and also (by the universal property of the tensor product) to define β in terms of α in order to get a map in the opposite direction. Obviously, these two maps are then R-linear and inverse to each other.

(b) Starting from the given sequence, Exercise 4.9 (a) gives us an exact sequence

$$0 \longrightarrow \mathrm{Hom}(M_3, \mathrm{Hom}(N,P)) \longrightarrow \mathrm{Hom}(M_2, \mathrm{Hom}(N,P)) \longrightarrow \mathrm{Hom}(M_1, \mathrm{Hom}(N,P))$$

for all R-modules N and P, where the two non-trivial maps send $\alpha_i \in \mathrm{Hom}(M_i, \mathrm{Hom}(N,P))$ to $\alpha_{i-1} \in \mathrm{Hom}(M_{i-1}, \mathrm{Hom}(N,P))$ with $\alpha_{i-1}(m_{i-1})(p) = \alpha_i(\varphi_{i-1}(m_{i-1}))(p)$ for $i \in \{2,3\}$ and all $m_1 \in M_1$, $m_2 \in M_2$, $p \in P$. Using the isomorphism of (a), this is the same as an exact sequence

$$0 \longrightarrow \mathrm{Hom}(M_3 \otimes N, P) \longrightarrow \mathrm{Hom}(M_2 \otimes N, P) \longrightarrow \mathrm{Hom}(M_1 \otimes N, P),$$

where the maps are now $\beta_i \mapsto \beta_{i-1}$ with

$$\beta_{i-1}(m_{i-1} \otimes p) = \beta_i(\varphi_{i-1}(m_{i-1}) \otimes p) = \beta_i((\varphi_{i-1} \otimes \mathrm{id})(m_{i-1} \otimes p))$$

for $i \in \{2,3\}$. But using Exercise 4.9 (a) again, this means that the sequence

$$M_1 \otimes N \xrightarrow{\varphi_1 \otimes \mathrm{id}} M_2 \otimes N \xrightarrow{\varphi_2 \otimes \mathrm{id}} M_3 \otimes N \longrightarrow 0$$

is also exact. □

Remark 5.23. Similarly to the case of $\mathrm{Hom}(\cdot, N)$ in Exercise 4.9, the statement of Proposition 5.22 (b) is in general not true with an additional zero module at the left, i. e. if φ_1 is injective it does not necessarily follow that $\varphi_1 \otimes \mathrm{id}$ is injective. In fact, we know this already from Example 5.10 (c), where we have seen that for a submodule M_1 of M_2 we cannot conclude that $M_1 \otimes N$ is a submodule of $M_2 \otimes N$ in a natural way.

In analogy to Exercise 4.9 we say that taking tensor products is *right exact*, but not exact. However, we have the following result:

Exercise 5.24. Let $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ be a short exact sequence of R-modules. Show that the sequence $0 \longrightarrow M_1 \otimes N \longrightarrow M_2 \otimes N \longrightarrow M_3 \otimes N \longrightarrow 0$ is also exact if one of the following assumptions hold:

(a) the sequence $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ is split exact;

(b) N is a finitely generated, free R-module.

As an example of how Proposition 5.22 can be used, let us prove the following statement.

Corollary 5.25. *Let I be an ideal in a ring R, and let M be an R-module. Then* $M/IM \cong M \otimes_R R/I$.

Proof. By Example 4.3 (b) we know that the sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

is exact. So by Proposition 5.22 (b) it follows that

$$M \otimes_R I \xrightarrow{\varphi} M \otimes_R R \xrightarrow{\psi} M \otimes_R R/I \longrightarrow 0$$

is also exact, where ψ is the projection in the second factor, $M \otimes_R R \cong M$ by Lemma 5.8 (b), and $\varphi(m \otimes a) = am$ using this isomorphism. In particular, the image of φ is just IM by Definition 3.12 (a), and so we conclude by the exactness of the sequence and the homomorphism theorem

$$M \otimes_R R/I = \operatorname{im} \psi \cong M/\ker \psi = M/\operatorname{im} \varphi = M/IM. \qquad \square$$

Example 5.26 (Derivatives in terms of tensor products)**.** Let V and W be normed real vector spaces [G2, Definition 23.1], and let $f : U \to W$ be a function on an open subset $U \subset V$. Then for any point $a \in U$ the derivative $f'(a)$ of f in a (if it exists) is an element of $\operatorname{Hom}_{\mathbb{R}}(V,W)$: it is just the homomorphism such that $x \mapsto f(a) + f'(a)\,(x-a)$ is the affine-linear approximation of f at a [G2, Definition 25.3 and Remark 25.6].

If we now want to define the second derivative f'' of f, the most natural way to do this is to take the derivative of the map $f' : U \to \operatorname{Hom}_{\mathbb{R}}(V,W)$, $a \mapsto f'(a)$, with a suitable norm on $\operatorname{Hom}_{\mathbb{R}}(V,W)$. By the same reasoning as above, this will now lead to an element $f''(a)$ of $\operatorname{Hom}_{\mathbb{R}}(V, \operatorname{Hom}_{\mathbb{R}}(V,W))$ [G2, Remark 26.5 (a)]. Similarly, the third derivative $f'''(a)$ is an element of $\operatorname{Hom}_{\mathbb{R}}(V, \operatorname{Hom}_{\mathbb{R}}(V, \operatorname{Hom}_{\mathbb{R}}(V,W)))$, and so on.

With Proposition 5.22 (a) we can now rephrase this in a simpler way in terms of tensor products: the k-th derivative $f^{(k)}(a)$ of f in a point $a \in U$ is an element of $\operatorname{Hom}_{\mathbb{R}}(V \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} V, W)$ for $k \in \mathbb{N}_{>0}$, where we take the k-fold tensor product of V with itself. So the higher derivatives of f can again be thought of as linear maps, just with a tensor product source space. Of course, if V and W are finite-dimensional with $n = \dim_{\mathbb{R}} V$ and $m = \dim_{\mathbb{R}} W$, then $\operatorname{Hom}_{\mathbb{R}}(V \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} V, W)$ is of dimension $n^k m$, and the coordinates of $f^{(k)}(a)$ with respect to bases of V and W are simply the n^k partial derivatives of order k of the m coordinate functions of f.

Exercise 5.27. Show that $l(M \otimes N) \leq l(M) \cdot l(N)$ for any two R-modules M and N (where an expression $0 \cdot \infty$ on the right hand side is to be interpreted as 0). Does equality hold in general?

09 (Hint: It is useful to consider suitable exact sequences.)