Algebraic Topology of Smooth Manifolds

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0. Introduction

We all know that a donut has a hole. But what exactly does that mean, how can we prove it, and what are the consequences?

In an introductory class on topology one usually learns that the so-called fundamental group of a topological space can be used to capture this idea: Its elements are homotopy classes of maps from loops to the given space, so it measures whether every loop in the space can be contracted to a point. A donut is then shown to have a non-trivial fundamental group, and this is a precise way to phrase the existence of its hole. Moreover, you might have seen that the fundamental group can be used to prove many nice results. For example, the arguably most intuitive way to prove the fundamental theorem of algebra (i. e. that every non-constant complex polynomial has a zero) is a topological one that uses the non-contractibility of loops in $\mathbb{C}\backslash\{0\} = \mathbb{R}^2\backslash\{0\}$, and thus the fundamental group.

But not all holes are created equal: You would probably agree that the space $\mathbb{R}^3\backslash\{0\}$ has a hole in the origin as well. However, every loop in it can be contracted to a point, i. e. its fundamental group is trivial and does not see the existence of the hole. So our initial question remains, which can actually be thought of as the fundamental question of algebraic topology: What exactly does it mean that a topological space has a hole, how can we prove it, and what are the consequences?

From the example $\mathbb{R}^3\backslash\{0\}$ one can easily read off one possible idea to solve this problem: We should probably pass from loops to higher-dimensional test objects. More precisely, it seems intuitive that the unit 2-sphere in $\mathbb{R}^3\backslash\{0\}$ cannot be contracted to a point in the same way as the unit 1-sphere in $\mathbb{R}^2\backslash\{0\}$ cannot be contracted to a point. This idea would lead to the so-called higher *homotopy groups* generalizing the fundamental group to higher dimensions. Unfortunately, it turns out however that these homotopy groups are very complicated. Already the homotopy groups of spheres are surprisingly wild and hard to compute [H, Chapter 4.1].

The good news is that there are many different other and easier ways to detect higher-dimensional holes. And by this I mean *completely different* – using combinatorics, algebra, topology, or even analysis with its (multivariate) differential and integral calculus. This is what these notes are about. We will construct the "hole-detecting" *homology* and *cohomology groups* (which will actually be vector spaces over a given ground field in our case) using several entirely different methods, but arriving at the same result in the end. Of course, this is always when things get really interesting since it shows deep relations between different fields of mathematics that seemed to be unrelated before.

Let us quickly explain the intuition behind the algebraic resp. topological and the analytic idea behind homology and cohomology, considering the example $\mathbb{R}^2\backslash\{0\}$ of the punctured plane:

- In *algebra* (resp. *topology*) we consider again the unit circle, just as for the fundamental group. This time however we do not care about its contractibility, but note that it is – roughly speaking – a "closed one-dimensional cycle" (i. e. a loop, as we said above) that is not the boundary of a two-dimensional compact region. In $\mathbb{R}^2$ (without the hole) it would be the boundary of the two-dimensional unit disk, but in $\mathbb{R}^2\backslash\{0\}$ this does not work, showing the existence of the hole. The idea to construct algebraic homology groups is thus to consider "closed cycles modulo boundaries".
- In *analysis*, let us consider the question whether a given smooth (i. e. infinitely differentiable) vector field $f\colon \mathbb{R}^2\backslash\{0\} \to \mathbb{R}^2$ with coordinate functions f_1 and f_2 is the (total) differential of a scalar function $F\colon \mathbb{R}^2\backslash\{0\} \to \mathbb{R}$, i. e. whether there is such a function F with $\partial_1 F = f_1$ and $\partial_2 F = f_2$. Of course, the commutativity of partial derivatives gives the necessary condition $\partial_2 f_1 = \partial_2\partial_1 F = \partial_1\partial_2 F = \partial_1 f_2$ (called the *integrability condition*), but is this also sufficient?

It turns out that on $\mathbb{R}^2$ this would be true, but on $\mathbb{R}^2\backslash\{0\}$ it is not due to the hole in the domain of definition: The "derivative of the angle in polar coordinates", i. e.

$$f(x) = \left(\arctan\frac{x_2}{x_1}\right)' = \left(\frac{-x_2}{x_1^2+x_2^2} \quad \frac{x_1}{x_1^2+x_2^2}\right),$$

would require F to be the angle itself up to an additive constant (which is impossible globally as running around the origin once would pick up another additive constant 2π), although it satisfies the integrability condition since it is locally a derivative. Hence, the analytic idea to detect holes is to consider functions satisfying an integrability condition (i. e. functions that locally look as if they are derivatives) modulo functions that actually are derivatives globally.

To study these approaches and their applications, these notes are divided into four parts:

The first part up to Chapter 5 is purely combinatorial resp. algebraic in nature. In this part we will only consider topological spaces that can be given by combinatorial data in a certain way – similarly to graphs, which can be thought of as one-dimensional topological spaces obtained by gluing finitely many edges at their ends in a specified way. The resulting homology theory is called **simplicial homology**; we will give its construction and study its properties in detail. It is probably the easiest homology theory and thus gives a good introduction to the general ideas and techniques in algebraic topology. Moreover, it has the advantage that everything eventually reduces to linear algebra for finite-dimensional vector spaces, and is thus very accessible for explicit computations.

In the second part from Chapter 6 to 9 we will study **singular homology**, the most general homology theory that is applicable to arbitrary topological spaces. Its construction is very similar to that of simplicial homology and thus mostly algebraic, and in fact with our preparation from the first part many results will just carry over without much effort. We will see that singular homology agrees with simplicial homology when both are defined, and discuss several applications.

We then turn to the analytic setting in the third part. Of course, general topological spaces will not suffice here since they do not have enough structure to define derivatives. From Chapter 10 to 12 we will therefore digress from homology theory and give a short introduction to **smooth manifolds**, which roughly speaking are topological spaces that locally look like $\mathbb{R}^n$. They will allow us, for example, to define tangent spaces at points, leading to the general concept of vector bundles that describe vector spaces varying with a point on a manifold.

In the last part starting in Chapter 13 we will then follow the analytic idea presented above to construct **de Rham cohomology**, the cohomology theory for smooth manifolds. While it shares many of the formal properties of singular cohomology (the dual of singular homology), the proofs of these statements are now analytic, of course, and thus entirely different. Nevertheless we will eventually see that de Rham cohomology agrees with singular cohomology in the cases when both are defined. At the end, we will then use our combined results to prove Poincaré duality – a non-trivial relation among the cohomology groups of oriented manifolds – and to show that cohomology has a product structure that can be described as an "intersection product" given geometrically by the intersection of submanifolds.

As you can see, these notes cover a substantial amount of material – but as usual in topology most statements should be very intuitive. In fact, although singular homology works for arbitrary topological spaces, all topological spaces that we will ever consider in these notes will be very nice in the sense that they can be glued from subsets of $\mathbb{R}^n$ with the standard topology. In other words, our topological spaces will be interesting because they have a non-trivial global structure coming from cutting and gluing processes, and not because they carry a wild topology locally. As a consequence, all our pictures of topological spaces will actually visualize correctly what is going on, and many results that we will prove are probably already plausible from the pictures.

We have mentioned already that algebraic topology has many relations to a wide variety of mathematical areas – which is after all what makes it so interesting! In order to keep these notes accessible we will therefore usually refrain from discussing every topic in full breadth and generality, include many examples, and keep the prerequisites to a minimum. Let us quickly list them.

Remark 0.1 (Prerequisites)**.** In these notes we will roughly assume the knowledge of a first-year mathematics class on algebra and analysis, together with an introductory course in topology. More precisely, we will assume familiarity with the following topics:

(a) *Algebra:* permutation groups, the finite fields $\mathbb{Z}/p\mathbb{Z}$, vector spaces and linear maps, determinants, dual vector spaces.

(b) *Analysis:* multivariate differentiation, implicit functions, multivariate integration in $\mathbb{R}^n$. As for integration, Riemann and Lebesgue integrals will be equally fine since all our functions will be nice enough so that the difference does not matter.

(c) *Topology:* topological spaces and their basic properties, quotient spaces, homotopies and homotopy equivalences. Fundamental groups might help since they share the general idea with homology groups of assigning algebraic objects to topological spaces, but they will not actually be used.

Many connections to other fields of mathematics will be mentioned throughout the notes, but prior knowledge of them will not be required: Simplicial complexes are basically just a straightforward generalization of *graph theory*. Relating mathematical objects of different types such as assigning vector spaces to topological spaces is one of the main ideas of *category theory*. The chain complexes and exact sequences needed to construct homology are a topic from *commutative* resp. *homological algebra* – but clearly motivated by topology, so if you have seen them already then our topological point of view will probably make them more intuitive. A byproduct of our analytic study of cohomology will be the multivariate integral theorems from *vector analysis* such as the divergence theorem of Gauss and the curl theorem of Stokes. The study of topological properties of manifolds is an important part of *algebraic geometry*. Finally, on the applied mathematics side, *topological data analysis* uses algebraic topology in the form of persistent homology as its backbone.

Notation 0.2. Throughout these notes we will adopt the following common notations and conventions:

- The set $\mathbb{N}$ of natural numbers contains 0.
- For two sets S and T the inclusion relation $T \subset S$ allows the case $T = S$; we write $T \subsetneq S$ if equality is not allowed.
- K will always be a fixed ground field. Our usual choice will be $K = \mathbb{R}$, and not much will be lost if you just always assume that. Sometimes we will also consider the finite field $\mathbb{Z}/p\mathbb{Z}$ for a prime number p, which we denote by $\mathbb{Z}_p$.
- For a finite index set I with cardinality $n \in \mathbb{N}$, we denote by
$$K^I := \{x = (x_i)_{i\in I} : x_i \in K \text{ for all } i \in I\}$$
the n-dimensional K-vector space with coordinates labeled by I. Its standard basis is $(e_i)_{i\in I}$, where $e_i \in K^I$ for $i \in I$ is the standard unit vector whose i-th component is 1, and all others are 0. If $J \subset I$ is another index set, we regard K^J as a vector subspace of K^I by setting all coordinates in $J \backslash I$ to 0.

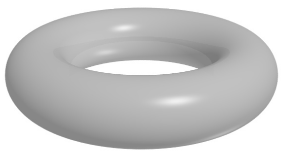
  Of course, the most common case will be $K^n := K^{\{1,\dots,n\}}$ for $n \in \mathbb{N}$. For $n = 0$, the zero vector space K^0 is often just written as 0 instead of $\{0\}$.
- More generally, for any (not necessarily finite) set B we can construct the vector space of all tuples $(x_b)_{b\in B}$ with $x_b \in K$ for all $b \in B$ such that only finitely many x_b are non-zero. Usually, for a given $b \in B$ the tuple with $x_b = 1$ and all other coordinates being 0 is then just written as b, so that every element of the vector space can be thought of as a finite linear combination $\sum_{b\in B} x_b \cdot b$ (i. e. with almost all $x_b = 0$). In particular, this means that B itself is a basis of this vector space. We will refer to it as *the vector space with basis* B.
- For a vector space V, its dual space will be written as $V^\vee$ (writing it as V^*, which is common in many areas of mathematics, would be confusing later on since the star notation will already have a different meaning as e. g. in Construction 5.6).

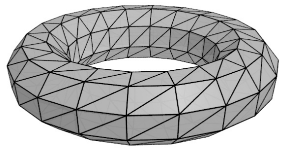
- For $n \in \mathbb{N}$, the vector spaces $\mathbb{R}^n$ and any of their subsets will always be equipped with the standard topology coming from the usual (Euclidean) norm $\|\cdot\|_2$, which we just write as $\|\cdot\|$. As special subsets, we denote by $S^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ the unit n-sphere in $\mathbb{R}^{n+1}$, and by $D^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ the unit n-disk (or n-ball) in $\mathbb{R}^n$. The open ball with radius $r \in \mathbb{R}_{>0}$ and center point $a \in \mathbb{R}^n$ will be written as $U_r(a)$.

Simplicial Homology

1. Simplicial Complexes

As we mentioned already in the introduction, the goal of this course is the global study of topological spaces, or intuitively, to define and search for "holes" in them. However, in order to keep things simple for the beginning, we will restrict our attention in the first part of these notes up to Chapter 5 to topological spaces called *simplicial complexes* that can be described entirely combinatorially. Intuitively, this means for example that a torus will for the moment not look for us like in the following picture on the left, but rather as in the picture on the right – obtained from triangles by gluing them along their edges.

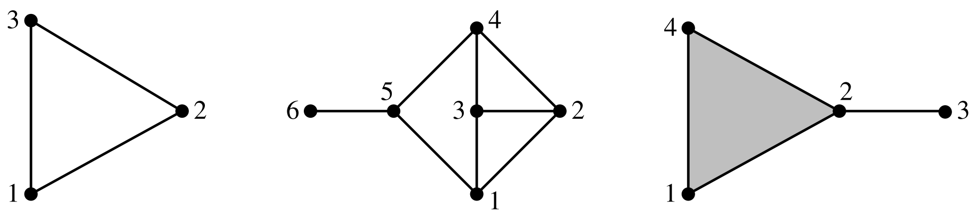




Another important goal of this first part of the notes is to introduce all the algebraic concepts from homological algebra that we will need throughout the course. In fact, in many cases simplicial complexes can really be thought of as a geometric interpretation of homological algebra, so they will also help us to understand the idea behind these algebraic concepts.

In this first chapter, we will just construct simplicial complexes rigorously and study some basic constructions with them. The question of how to find holes in them will then be addressed in Chapter 2.

Remark 1.1 (Graphs)**.** In your (mathematical) life you have probably come across *graphs*. Intuitively, graphs are finite combinatorial structures that are made up from *vertices* (sometimes called *nodes* or just *points*) and *edges* connecting them. In the following picture, S_1 and S_2 are examples of graphs, with the dots representing the vertices and the lines representing the edges.



S_1 S_2 S_3

Formally, a graph can be given by assigning labels to the vertices as in the picture, and considering each edge as a two-element set whose elements are just the labels of the connected vertices. For our purposes it will be convenient to add the vertices themselves as one-element sets to the structure, so that we could e. g. write the graph S_1 above as $S_1 = \{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\}$.

Simplicial complexes generalize this idea in the sense that we are also allowed to connect more than two vertices, e. g. to connect three vertices by a "triangle" as for S_3 in the picture above.

Definition 1.2 (Abstract simplices and simplicial complexes)**.**

(a) An **(abstract) simplex** is just a non-empty finite set α. Any simplex β with $\beta \subset \alpha$ will be called a **face** of α; it is a **proper face** if $\beta \subsetneq \alpha$.

(b) An **(abstract) simplicial complex** is a finite set S of simplices that is closed under taking faces, i. e. such that for all simplices $\beta \subset \alpha$ with $\alpha \in S$ we have $\beta \in S$.

(c) Any element of a simplex α is called a **vertex** of α. Any element of a simplex in a simplicial complex S is called a vertex of S. We denote the set of all vertices of S by $V(S)$.

(d) If a simplex α has exactly $p+1$ vertices we say that the **dimension** $\dim \alpha$ of α is p, and that α is a p**-simplex**. For a simplicial complex S, its **dimension** $\dim S$ is defined to be the maximum dimension of its simplices. A 1-simplex is called an **edge**, and a simplicial complex of dimension at most 1 is called a **graph**.

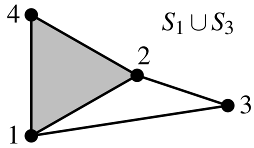
Example 1.3. Let us reconsider S_1, S_2, and S_3 as in Remark 1.1.

(a) S_1, S_2, and S_3 are simplicial complexes. While S_1 and S_2 are graphs, the simplicial complex S_3 is 2-dimensional due to its 2-simplex $\{1,2,4\} \in S_3$ with the three vertices 1, 2, and 4. The vertex set of S_3 is $V(S_3) = \{1,2,3,4\}$.

(b) Clearly, intersections and unions of simplicial complexes are again simplicial complexes. For example, we have

$$S_1 \cap S_3 = \{\{1\},\{2\},\{3\},\{1,2\},\{2,3\}\},$$

and the union $S_1 \cup S_3$ is shown in the picture on the right.



Definition 1.4 (Simplicial subcomplexes)**.** A **(simplicial) subcomplex** of a simplicial complex S is just a simplicial complex $T \subset S$. We say that it is **full** if for any $\alpha \in S$ with $\alpha \subset V(T)$ we have $\alpha \in T$.

Remark 1.5. By definition, a subcomplex $T \subset S$ being full means that a simplex in S is a simplex in T if and only if all its vertices lie in T. Of course, this makes dealing with such subcomplexes much easier since they can be specified by giving just their vertices. But we will also see e. g. in Proposition 1.24 that full subcomplexes have nicer properties than arbitrary ones.

Example 1.6.

(a) For the simplicial complexes from Remark 1.1, S_1 is a full subcomplex of S_2, but not a subcomplex of S_3 (since the edge $\{1,3\}$ is in S_1, but not in S_3). The subcomplex S_3 of $S_1 \cup S_3$ (shown in Example 1.3 (b)) is not full since the edge $\{1,3\} \in S_1 \cup S_3$ is a subset of $V(S_3)$, but does not lie in S_3.

(b) The intersection $T_1 \cap T_2$ of two full subcomplexes T_1 and T_2 of a simplicial complex S is again a full subcomplex. However, the subcomplex $T_1 \cup T_2$ need not be full: $\{\{1\}\}$ and $\{\{2\}\}$ are full subcomplexes of $\{\{1\},\{2\},\{1,2\}\}$, but their union $\{\{1\},\{2\}\}$ is not.

As a first property of simplicial complexes, let us study the intuitive notion of connectedness.

Definition 1.7 (Connectedness)**.** For a non-empty simplicial complex S, the equivalence classes of the equivalence relation

$$\alpha \sim \alpha' \quad :\Leftrightarrow \quad \begin{array}{l} \text{there are } \alpha_0,\dots,\alpha_n \in S \text{ with } n \in \mathbb{N} \text{ such that } \alpha = \alpha_0,\ \alpha' = \alpha_n, \\ \text{and } \alpha_{k-1} \cap \alpha_k \neq \emptyset \text{ for all } k = 1,\dots,n \end{array}$$

on S are called the **connected components** of S. We say that S is **connected** if it has only one connected component.

Clearly, S is then the disjoint union of all its connected components; let us now check that these components are in fact full subcomplexes. Moreover, as $\{i\} \in S$ for every vertex $i \in V(S)$, every vertex is a vertex of a unique connected component. We will see now that on vertices the equivalence relation of Definition 1.7 is completely analogous to the notion of path-connectedness of a topological space.

Lemma 1.8. *Let S be a simplicial complex.*

(a) *Every connected component of S is a full subcomplex of S.*

(b) *Two vertices $i, i' \in V(S)$ are vertices of the same connected component of S if and only if they can be connected by a chain of edges in S, i. e. if there are $n \in \mathbb{N}$ and edges $\alpha_0, \dots, \alpha_n$ with $i \in \alpha_0$, $i' \in \alpha_n$, and $\alpha_{k-1} \cap \alpha_k \neq \emptyset$ for all $k = 1, \dots, n$.*

Proof.

(a) Let $T \subset S$ be a connected component, i. e. a class of the equivalence relation of Definition 1.7. As $\beta \sim \alpha$ for every face β of α it is clear that T is a subcomplex. To show that it is full, let $\alpha = \{i_0, \dots, i_p\} \in S$ be a p-simplex with $i_0, \dots, i_p \in V(T)$. Then $\alpha \sim \{i_0\} \in T$, so it follows that $\alpha \in T$.

(b) The direction "$\Leftarrow$" is clear from the definition. For the direction "$\Rightarrow$" assume that i and i' are two vertices of the same connected component, i. e. that there are $\alpha_0, \dots, \alpha_n \in S$ with $\{i\} = \alpha_0$, $\{i'\} = \alpha_n$, and $\alpha_{k-1} \cap \alpha_k \neq \emptyset$ for all $k = 1, \dots, n$. Pick a vertex $i_k \in \alpha_{k-1} \cap \alpha_k$ for all $k = 1, \dots, n$. Then $i = i_1$, $i' = i_n$, and $\{i_{k-1}, i_k\} \subset \alpha_{k-1} \in S$ for all $k = 2, \dots, n$, so $\{i_1, i_2\}, \{i_2, i_3\}, \dots, \{i_{n-1}, i_n\}$ gives the desired chain. □

Example 1.9. Relabeling the vertices $1,2,3,4$ in the simplicial complex S_3 of Remark 1.1 to $1', 2', 3', 4'$ to obtain the simplicial complex S_3' makes $S_1 \cup S_3'$ into a disconnected simplicial complex with connected components S_1 and S_3'.

So far we have considered a simplicial complex S only as a combinatorial structure. But for our purposes we also want to realize it as a topological space, e. g. to interpret the pictures in Remark 1.1 as topological subspaces of $\mathbb{R}^2$ obtained by picking some points for the vertices of S and joining them by lines or triangles according to the simplices in S. In arbitrary dimensions, these building blocks will be called *geometric simplices*. Of course, for this idea to work we have to pick the points for the vertices general enough so that e. g. three points are not on a line if we want to join them by a triangle and two lines do not intersect if they do not have a vertex in common. Let us quickly go through the constructions necessary to achieve this.

Definition 1.10 (Affine independence)**.** We say that finitely many points $v_0, \dots, v_p \in W$ in a real vector space W are called **affinely independent** if for all $x_0, \dots, x_p \in \mathbb{R}$ with $x_0 v_0 + \cdots + x_p v_p = 0$ and $x_0 + \cdots + x_p = 0$ we have $x_0 = \cdots = x_p = 0$.

Remark 1.11.

(a) Using that $x_0 = -x_1 - \cdots - x_p$ we can rephrase the condition of Definition 1.10 by saying that $x_1(v_1 - v_0) + \cdots + x_p(v_p - v_0) = 0$ with $x_1, \dots, x_p \in \mathbb{R}$ implies $x_1 = \cdots = x_p = 0$. Hence the affine independence of $v_0, \dots, v_p$ just means that shifting one of the vectors to the origin makes the other ones linearly independent, or in other words that $v_0, \dots, v_p$ span a shifted p-dimensional linear subspace.

(b) Obviously, linear independence implies affine independence, but not vice versa (as e. g. 1 and 2 are affinely, but not linearly independent in $\mathbb{R}$).

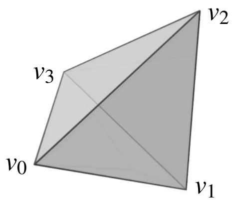
Construction 1.12 (Geometric simplices and barycentric coordinates)**.** Let $v_0, \dots, v_p \in W$ be points in a real vector space W.

(a) We denote the convex hull spanned by $v_0, \dots, v_p$ by

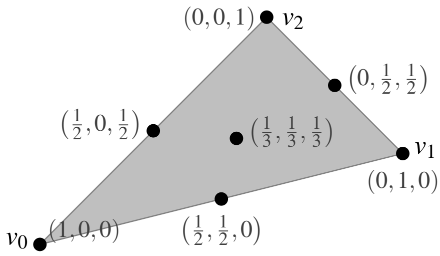
$$\langle v_0, \dots, v_p \rangle := \{x_0 v_0 + \cdots + x_p v_p : x_0, \dots, x_p \in \mathbb{R}_{\geq 0} \text{ with } x_0 + \cdots + x_p = 1\} \quad \subset W.$$

If $v_0, \dots, v_p$ are affinely independent, we call this set the **geometric p-simplex** spanned by $v_0, \dots, v_p$. A **face** of it is any geometric simplex spanned by a subset of the given points, and it is a **proper face** if it is spanned by a proper subset. We call the union of all proper faces the **boundary** and its complement in the geometric simplex the **interior** of $\langle v_0, \dots, v_p \rangle$.

For example, a geometric 3-simplex $\langle v_0, v_1, v_2, v_3 \rangle$ is a tetrahedron as in the following picture on the left. Two of its proper faces are the geometric 0-simplex $\langle v_0 \rangle = \{v_0\}$ (which is just a point) and the geometric 2-simplex $\langle v_0, v_1, v_2 \rangle$ (which is a triangle).



$\langle v_0, v_1, v_2, v_3 \rangle$



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(b) Note that in (a) the values of $x_0, \dots, x_p$ leading to a point $x = x_0 v_0 + \dots + x_p v_p$ in the geometric p-simplex spanned by $v_0, \dots, v_p$ are uniquely determined: For another representation $x = x_0' v_0 + \dots + x_p' v_p$ with $x_0' + \dots + x_p' = 1$ we obtain

$$(x_0 - x_0')v_0 + \dots + (x_p - x_p')v_p = 0 \quad \Rightarrow \quad x_i = x_i' \text{ for all } i$$

since $\sum_{i=0}^p (x_i - x_i') = \sum_{i=0}^p x_i - \sum_{i=0}^p x_i' = 1 - 1 = 0$ and $v_0, \dots, v_p$ are affinely independent. We call $(x_0, \dots, x_p)$ the **barycentric coordinates** of x (the name comes from the word "barycenter" meaning "center of mass", and in fact x is by definition the weighted average of $v_0, \dots, v_p$ with the weights $x_0, \dots, x_p$).

In summary, barycentric coordinates of a point in $\langle v_0, \dots, v_p \rangle$ are uniquely defined, non-negative, and sum up to 1. The interior of $\langle v_0, \dots, v_p \rangle$ consists exactly of the points whose barycentric coordinates are all positive. The picture above on the right shows some points in the triangle $\langle v_0, v_1, v_2 \rangle$ with their barycentric coordinates (x_0, x_1, x_2).

Let us now assume that we are given a simplicial complex S and a "realization map" $r \colon V(S) \to W$ to a real vector space that tells us at which points we want the vertices to be. We can then try to connect these points by simplices according to S to obtain a geometric realization of S.

Construction 1.13 (Geometric realizations)**.** For a simplicial complex S and a map $r \colon V(S) \to W$ to a real vector space W we set

$$r(\alpha) := \langle r(i_0), \dots, r(i_p) \rangle \quad \subset W \quad \text{for any } p\text{-simplex } \alpha = \{i_0, \dots, i_p\} \in S, \text{ with } r(\emptyset) := \emptyset$$

$$\text{and} \quad r(S) := \bigcup_{\alpha \in S} r(\alpha) \quad \subset W.$$

We call r a **geometric realization** of S if $r(\alpha) \cap r(\beta) = r(\alpha \cap \beta)$ for any $\alpha, \beta \in S$. By abuse of notation, we will then often also say that $r(S)$ is the geometric realization of S determined by r.

Remark 1.14 (Topological realizations)**.** Let $r \colon V(S) \to W$ be a geometric realization of a simplicial complex S.

(a) Clearly, $r(\beta)$ is a face of $r(\alpha)$ for any two simplices $\beta \subset \alpha$ in S.

(b) If $\alpha \in S$ is an abstract p-simplex then $r(\alpha)$ is a geometric p-simplex, i. e. the points $r(i)$ for $i \in \alpha$ are affinely independent: Assume for a contradiction that we have coordinates $x_i \in \mathbb{R}$ for $i \in \alpha$ with $\sum_{i \in \alpha} x_i \, r(i) = 0$ and $\sum_{i \in \alpha} x_i = 0$, where at least one x_i is non-zero. For $\alpha_+ := \{i \in \alpha : x_i > 0\}$ and $\alpha_- := \{i \in \alpha : x_i < 0\}$ we then have

$$x := \sum_{i \in \alpha_+} x_i \, r(i) = \sum_{i \in \alpha_-} (-x_i) \, r(i).$$

The coordinates on both sides are now positive and sum up to the same number due to the condition $\sum_{i \in \alpha} x_i = 0$, so by normalizing we may assume that the sum of the coordinates is 1 on both sides. But then it follows that $x \in r(\alpha_+) \cap r(\alpha_-) = r(\alpha_+ \cap \alpha_-) = r(\emptyset) = \emptyset$, which is a contradiction.

(c) It follows by (a) and (b) that the geometric realization $r(S)$ can also be constructed *as a topological space* by a gluing process: We can start with the *disjoint* union of all geometric simplices $r(\alpha)$ for $\alpha \in S$, and then glue any two geometric simplices $r(\alpha)$ and $r(\beta)$ along their common face $r(\alpha \cap \beta)$, giving the resulting space the quotient topology [G3, Chapter

5]. Note that this captures precisely the idea how we wanted to consider abstract simplicial complexes as topological spaces.

Moreover, as any two p-simplices are homeomorphic this means that any two geometric realizations of S are homeomorphic. If we think of $r(S)$ this way we will also refer to it as the **topological realization** of S.

It just remains to be shown that every simplicial complex admits a geometric realization at all. In fact, we will now show that it even has a canonically defined one.

Proposition 1.15 (Standard geometric realization)**.** *Let S be a simplicial complex. Then the map $r\colon V(S) \to \mathbb{R}^{V(S)}$, $i \mapsto e_i$ is a geometric realization of S. We call it the **standard geometric realization** of S, and denote its images in $R^{V(S)}$ by $|\alpha| := r(\alpha)$ and $|S| := r(S)$ for a simplex $\alpha \in S$.*

Proof. Let α and β be two simplices of S; we have to show that $r(\alpha) \cap r(\beta) = r(\alpha \cap \beta)$.

"$\subset$": Let $x \in r(\alpha) \cap r(\beta)$, so there are barycentric coordinates x_i for $i \in \alpha$ and x'_i for $i \in \beta$ with $x = \sum_{i\in\alpha} x_i e_i = \sum_{i\in\beta} x'_i e_i$. As all unit vectors are independent this means that x_i and x'_i can only be non-zero for $i \in \alpha \cap \beta$; hence $x \in r(\alpha \cap \beta)$.

"$\supset$": This follows immediately from Remark 1.14 (a) as $\alpha \cap \beta$ is a face of both α and β. □

Remark 1.16. The standard geometric realization $|S|$ of a simplicial complex S has the disadvantage of having a high-dimensional target space, so that it is usually not useful for visualization. On the other hand, it has several advantages:

(a) It does not depend on any choices.

(b) It has barycentric coordinates that are defined globally and not just on a fixed simplex: By construction, we have

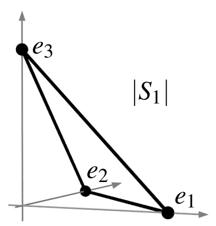
$$|S| = \left\{ x \in \mathbb{R}^{V(S)}_{\geq 0} : \sum_{i \in V(S)} x_i = 1 \text{ and } \{i \in V(S) : x_i > 0\} \in S \right\} \quad \subset \mathbb{R}^{V(S)}.$$

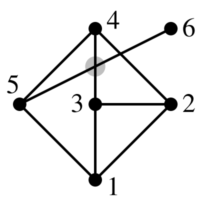
(c) Any other geometric realization $r\colon V(S) \to W$ is obtained from it by a linear map that is injective on $|S|$, namely by the linear map $\mathbb{R}^{V(S)} \to W$ determined by $e_i \mapsto r(i)$ for all $i \in V(S)$.

Hence, for topological purposes (when the realization does not matter by Remark 1.14 (c)), we will usually use the standard geometric realization.

Example 1.17. Consider again the simplicial complexes S_1, S_2, and S_3 of Remark 1.1.

(a) The picture below on the left shows the standard geometric realization $|S_1| \subset \mathbb{R}^{\{1,2,3\}}$ of S_1. All pictures in Remark 1.1 show valid geometric realizations of the given simplicial complexes when we consider them as subsets of $\mathbb{R}^2$, with the realization map r given by mapping each vertex i to the point labeled i in the picture. In contrast, the map $r\colon V(S_2) \to \mathbb{R}^2$ shown below on the right is not a geometric realization of S_2 since $r(\{3,4\}) \cap r(\{5,6\})$ contains the gray point, whereas $r(\{3,4\} \cap \{5,6\}) = r(\emptyset) = \emptyset$.





(b) Let α be a p-simplex. Then α together with all of its faces, i. e.

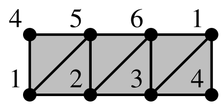
$$D^\alpha := \{\beta : \emptyset \neq \beta \subset \alpha\}$$

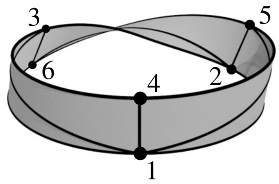
is a simplicial complex whose topological realization is a p-dimensional disk D^p. Similarly, for $p > 0$ the set of all proper faces

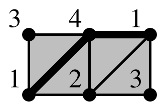
$$S^\alpha := \{\beta : \emptyset \neq \beta \subsetneq \alpha\}$$

is a simplicial complex whose topological realization is the sphere S^{p-1}. For example, the simplicial complex S_1 is just $S^{\{1,2,3\}}$.

(c) To construct a topological realization it is sometimes useful to only use the gluing construction of Remark 1.14 (c) without any embedding in a real vector space. For example, the **Möbius strip** is the topological space obtained from the unit square $[0,1] \times [0,1] \subset \mathbb{R}^2$ by gluing the left to the right side in opposite directions, giving the resulting space the quotient topology [G3, Example 5.9 (b)]. It is visibly the topological realization of the simplicial complex shown in the picture below on the left, where the identification of the left and right side in opposite directions is implied by the numbering. Note that it is not a subspace of $\mathbb{R}^2$, but can be visualized as a topological space (not as a geometric realization!) in $\mathbb{R}^3$ as in the middle picture.

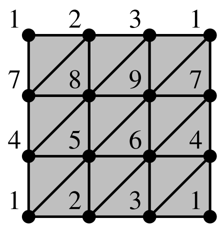




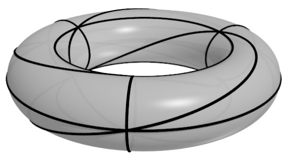


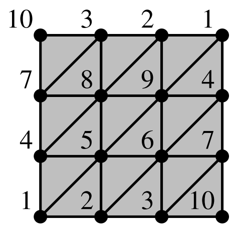
Also note that subdividing the unit square horizontally into only two instead of three parts as in the right picture would not have been sufficient: This is not a valid simplicial complex as the vertices 1 and 4 are joined by two edges (drawn in bold) – which is not allowed since a simplicial complex is just a set and cannot contain a simplex multiple times.

(d) Similarly to (b), we can obtain a **torus** by taking a unit square and identifying the left with the right and the top with the bottom side in the same direction [G3, Example 5.9 (c)]. A simplicial complex achieving this is shown in the picture below on the left; it consists of 9 zero-dimensional, 27 one-dimensional, and 18 two-dimensional simplices. As a topological space this is just the product $S^1 \times S^1$ of two circles in the middle picture.



Torus





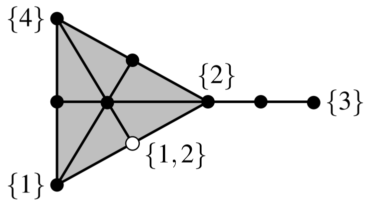
Real projective plane

In contrast, the picture above on the right shows a simplicial complex realizing the topological space obtained from gluing opposite sides of the unit square in opposite directions. Note that it needs one vertex more since now not all four corners of the square are identified, but only opposite ones. Although this simplicial complex might look very similar to that for the torus, it gives a fundamentally different topological space called the **real projective plane** [G3, Example 5.9 (c)]. We will consider it in more detail later in Construction 8.3.

Exercise 1.18. Draw an abstract simplicial complex whose topological realization is the real projective plane and that has fewer than 10 vertices.

Exercise 1.19. Let $T_1, \dots, T_n$ be the connected components of a simplicial complex S. Show that $|T_1|, \dots, |T_n|$ are the path-connected components of the topological space $|S|$.

Of course, it may happen that different simplicial complexes have homeomorphic topological realizations. The most obvious way to achieve this is by subdividing the simplices in a simplicial complex S. A symmetric approach to this construction is to add a vertex in the barycenter (i. e. midpoint) of each simplex $\alpha \in S$ and subdivide accordingly. For example, after this subdivision process the simplicial complex S_3 of Remark 1.1 would look like the picture on the right, with the hollow vertex being the barycenter of the edge $\{1,2\}$.



Note that in this way the vertex set of the subdivided complex can be identified with the set S of simplices in the original simplicial complex. In doing so, the old vertices $i \in V(S)$ will be relabeled by the 0-simplices $\{i\} \in S$ as in the picture above. Let us now define this subdivision process rigorously.

Definition 1.20 (Barycentric subdivision)**.** For a simplicial complex S, the simplicial complex

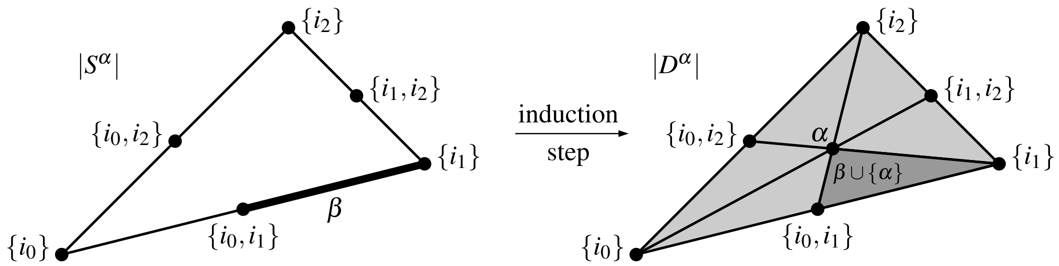
$$\mathrm{Sd}(S) := \big\{\{\alpha_0,\dots,\alpha_p\} : p \in \mathbb{N} \text{ and } \emptyset \neq \alpha_0 \subsetneq \cdots \subsetneq \alpha_p \in S\big\}$$

with vertex set $V(\mathrm{Sd}(S)) = S$ is called the **barycentric subdivision** of S.

Proposition 1.21 (Barycentric realization)**.** *Let S be an abstract simplicial complex. Then the map $r\colon V(\mathrm{Sd}(S)) \to \mathbb{R}^{V(S)}$ that sends each p-simplex $\alpha = \{i_0,\dots,i_p\} \in S$ to its barycenter $r(\alpha) := \frac{1}{p+1}(e_{i_0} + \cdots + e_{i_p})$ is a geometric realization of $\mathrm{Sd}(S)$ with $r(\mathrm{Sd}(S)) = |S|$. We call it the **barycentric realization** of $\mathrm{Sd}(S)$.*

Sketch proof. We show the statement by induction on the cardinality of S, with the case $\dim S = 0$ being obvious. So assume that $\dim S > 0$, choose a simplex $\alpha = \{i_0,\dots,i_p\}$ of maximum dimension p in S, and set $S' := S \backslash \{\alpha\}$. By induction, we know that r is a geometric realization of $\mathrm{Sd}(S')$ with $r(\mathrm{Sd}(S')) = |S'|$.

Note that all proper faces of α must already be in S'. So S' contains the sphere S^α, and hence $\mathrm{Sd}(S')$ contains the subdivided sphere $\mathrm{Sd}(S^\alpha)$, which again by induction is realized by r with image $|S^\alpha|$. A picture of this situation in the case $p = 2$ is shown below on the left (where the simplices in $\mathrm{Sd}(S') \backslash \mathrm{Sd}(S^\alpha)$ are not shown for simplicity).



By construction, the transition from S' to S now just adds the simplex α, hence by Definition 1.20 the transition from $\mathrm{Sd}(S')$ to $\mathrm{Sd}(S)$ adds the simplices $\{\alpha_0,\dots,\alpha_k\}$ for $k \in \mathbb{N}$ and $\emptyset \neq \alpha_0 \subsetneq \cdots \subsetneq \alpha_k = \alpha$. This is precisely the 0-simplex $\{\alpha\}$ together with all simplices of the form $\beta \cup \{\alpha\}$ for $\beta \in \mathrm{Sd}(S^\alpha)$. As in the picture above on the right, placing $\alpha \in V(\mathrm{Sd}(S))$ at its barycenter the transition from $\mathrm{Sd}(S')$ to $\mathrm{Sd}(S)$ thus fills in the sphere $|S^\alpha|$ to a disk $|D^\alpha|$. Hence r realizes $\mathrm{Sd}(S)$ with image $|S'| \cup |D^\alpha| = |S'| \cup |\alpha| = |S|$. $\square$

Remark 1.22.

(a) As one can see from the sketch proof above, it actually does not matter that we place the vertices of the subdivision in the barycenters – any point in the interior of the corresponding geometric simplex would work equally well. Also, in the same way we could start with any (non-standard) geometric realization r of a simplicial complex S and then construct a barycentric realization of $\mathrm{Sd}(S)$ with the same image as $r(S)$.

(b) Obviously, barycentric subdivisions behave nicely with respect to subcomplexes, intersections, and unions: If T is a subcomplex of a simplicial complex S then $\mathrm{Sd}(T)$ is a subcomplex of $\mathrm{Sd}(S)$; and if S and T are arbitrary simplicial complexes then $\mathrm{Sd}(S\cap T)=\mathrm{Sd}(S)\cap\mathrm{Sd}(T)$ and $\mathrm{Sd}(S\cup T)=\mathrm{Sd}(S)\cup\mathrm{Sd}(T)$.

Barycentric subdivisions will come up at various places, but to see their usefulness now let us prove as a first result that they can be used to make subcomplexes full, and to construct open neighborhoods of subcomplexes with nice properties.

Lemma 1.23. *If T is a subcomplex of a simplicial complex S, then $\mathrm{Sd}(T)$ is a full subcomplex of $\mathrm{Sd}(S)$.*

Proof. By definition, $\mathrm{Sd}(T)$ is clearly a simplicial subcomplex of $\mathrm{Sd}(S)$. To show that it is full, let $\beta=\{\alpha_0,\dots,\alpha_p\}\in\mathrm{Sd}(S)$, i. e. $\emptyset\neq\alpha_0\subsetneq\cdots\subsetneq\alpha_p\in S$, such that $\alpha_0,\dots,\alpha_p\in V(\mathrm{Sd}(T))=T$. Then $\emptyset\neq\alpha_0\subsetneq\cdots\subsetneq\alpha_p\in T$, and hence $\beta\in\mathrm{Sd}(T)$. □

Proposition 1.24 (Standard neighborhoods)**.** *For a subcomplex T of a simplicial complex S we set*

$$N_{T/S}:=\{x\in|S|:x_i>0\text{ for some }i\in V(T)\}.$$

(a) *$N_{T/S}$ is an open subset of $|S|$ that contains $|T|$ (i. e. an open neighborhood of $|T|$ in $|S|$).*

(b) *If T is full then $|T|$ is a deformation retract of $N_{T/S}$. In particular, $N_{T/S}$ is then homotopy equivalent to $|T|$.*

We call $N_{T/S}$ the ***standard neighborhood*** *of $|T|$ in $|S|$.*

Proof.

(a) It is clear from the definition that $N_{T/S}$ is open. Moreover, every $x\in|T|$ has barycentric coordinates x_i with $i\in V(T)$ that sum up to 1. Hence, at least one of them is non-zero, and it follows that $x\in N_{T/S}$.

(b) Let us consider the map

$$\pi\colon N_{T/S}\to\mathbb{R}^{V(T)},\ x\mapsto\frac{\sum_{i\in V(T)}x_ie_i}{\sum_{i\in V(T)}x_i}$$

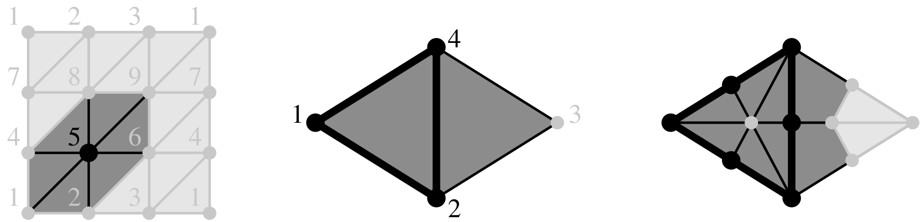
that projects to $\mathbb{R}^{V(T)}\subset\mathbb{R}^{V(S)}$ and normalizes. It is in fact a well-defined continuous map as the denominator will never be 0 on $N_{T/S}$. We claim that its image lies in $|T|$. To see this, note first that for any $x\in N_{T/S}\subset|S|$ the coordinates of $\pi(x)$ are clearly still non-negative and sum up to 1. Moreover, by definition of $|S|$ we have $x\in|\alpha|$ for some $\alpha\in S$, and thus $\pi(x)\in|\alpha\cap V(T)|$. But $\alpha\cap V(T)$ is a simplex in S by Definition 1.2 (b) and has its vertices in $V(T)$, so it is a simplex of T since T is full. It follows that $\pi(x)\in|T|$, i. e. we have constructed a continuous map $\pi\colon N_{T/S}\to|T|$. Finally, π is in fact a deformation retraction since

$$H\colon N_{T/S}\times[0,1]\to N_{T/S},\ (x,t)\mapsto\frac{\sum_{i\in V(T)}x_ie_i+t\sum_{i\notin V(T)}x_ie_i}{\sum_{i\in V(T)}x_i+t\sum_{i\notin V(T)}x_i}$$

(which moves each point of $\mathbb{R}^{V(S)}$ linearly to its projection to $\mathbb{R}^{V(T)}$ and normalizes) is a homotopy between $H(\cdot,0)=\pi$ and $H(\cdot,1)=\mathrm{id}_{N_{T/S}}$. □

Example 1.25. By Construction 1.12 (b), for a vertex i of a simplicial complex S the set of all $x\in|S|$ with $x_i>0$ consists of all interiors of simplices containing the vertex i. This explains the shape of the standard neighborhoods $N_{T/S}$ in the picture below, where the simplicial complex S is drawn in light color, the subcomplex T in bold, and $N_{T/S}$ in dark color.

(a) Let S be the torus of Example 1.17 (d), and let T be the 0-dimensional subcomplex of S with the single vertex 5. The standard neighborhood $N_{T/S}$ is shown in the picture on the left and can be thought of as a simplicial version of an open ball neighborhood in analysis.



(b) As in the middle picture, consider the simplicial complex $S = D^{\{1,2,4\}} \cup D^{\{2,3,4\}}$ with the subcomplex $T = S^{\{1,2,4\}}$. The standard neighborhood $N_{T/S}$ is then all of $|S|$ except the vertex 3. While this clearly is a neighborhood of $|T|$, it is not homotopy equivalent to $|T|$ since it is contractible whereas the circle $|T|$ is not. However, as in the picture above on the right, if we perform a barycentric subdivision first then the subcomplex becomes full by Lemma 1.23, and consequently its geometric realization is now a deformation retract of its standard neighborhood by Proposition 1.24 (with the retraction going in the direction of the thin black lines). Intuitively, the reason for this is that the subdivision makes the simplicial structure fine enough so that the standard neighborhood does not completely fill in the hole of the circle $|T|$.

Exercise 1.26. Prove that for any two subcomplexes T_1 and T_2 of a simplicial complex S we have:

(a) $N_{(T_1 \cup T_2)/S} = N_{T_1/S} \cup N_{T_2/S}$;

(b) $N_{(T_1 \cap T_2)/S} \subset N_{T_1/S} \cap N_{T_2/S}$, with equality if $T_1 \cup T_2$ is full.

Let us conclude this chapter with a definition of morphisms of simplicial complexes, which is also completely combinatorial but captures the idea of a continuous map between the topological realizations.

Definition 1.27 (Morphisms of simplicial complexes)**.** Let S and T be simplicial complexes.

(a) A **morphism** f from S to T is a map $f\colon V(S) \to V(T)$ such that for any $\alpha \in S$ we have $f(\alpha) \in T$. We will write this as $f\colon S \to T$.

(b) Note that any such morphism $f\colon S \to T$ induces a unique continuous map from $|S|$ to $|T|$ that sends e_i to $e_{f(i)}$ for all $i \in V(S)$ and is affine-linear on every simplex of S. We will denote it by $|f|\colon |S| \to |T|$.

Example 1.28.

(a) Clearly, the identity $\mathrm{id}_S\colon S \to S$ given by $\mathrm{id}_S(i) = i$ for all $i \in V(S)$ is a morphism. More generally, for any subcomplex T of S there is an inclusion morphism $f\colon T \to S$ given by $f(i) = i$ for all $i \in V(T)$.

(b) With the labeling of the vertices as in Example 1.17 (c) and (d) there is a morphism f from the Möbius strip to the real projective plane given by $f(i) = i+3$ for all $i \in \{1,\dots,6\}$.

(c) Note that a morphism of simplicial complexes need not be injective, and it does not even have to preserve the dimension of simplices. For example, we could map the torus with the labeling as in Example 1.17 (d) to the circle $S^{\{1,2,3\}}$ by sending the vertex $3i+j$ to j for all $i = \{0,1,2\}$ and $j \in \{1,2,3\}$. This would map e. g. the 2-simplex $\{1,2,5\}$ of the torus to the edge $\{1,2,2\} = \{1,2\}$ of the circle. Topologically, this morphism just corresponds to the projection $S^1 \times S^1 \to S^1$ of the torus to one of its factors.

(d) Clearly, the composition of morphisms of simplicial complexes is again a morphism.

Definition 1.29 (Isomorphisms of simplicial complexes)**.** A morphism $f\colon S \to T$ of simplicial complexes is an **isomorphism** if there is a morphism $g\colon T \to S$ with $g \circ f = \mathrm{id}_S$ and $f \circ g = \mathrm{id}_T$. Two simplicial complexes are called **isomorphic** if there is an isomorphism between them.

Example 1.30.

(a) Relabeling the vertices of a simplicial complex produces an isomorphic simplicial complex.

(b) Of course, any isomorphism $f\colon S\to T$ of simplicial complexes induces a homeomorphism $|f|\colon |S|\to|T|$ between their topological realizations. We have seen already however that non-isomorphic simplicial complexes may have homeomorphic topological realizations, e. g. any simplicial complex and its barycentric subdivision.

In fact, in this course we will often construct certain mathematical objects (such as simplicial complexes in this chapter) and structure-preserving morphisms between them, and relate them to other objects of the same type (e. g. subdivide a simplicial complex) or of different types (e. g. associate a topological space to a simplicial complex). The first questions and properties for such constructions are always the same, so it is useful to introduce a unified language to deal with them. This is
02 achieved by the notion of *categories* and *functors* that we will now briefly explain.

Notation 1.31 (Categories)**.** A **category** C is given by specifying

- a class of objects of C; and
- for any two objects of C a class of morphisms between them, such that morphisms can be composed, composition is associative, and any object has an identity morphism.

Examples that we already know and that will be relevant for us are:

(a) the category Vect whose objects are vector spaces over a fixed ground field, and morphisms are linear maps;

(b) the category Top whose objects are topological spaces, and morphisms are continuous maps;

(c) the category Simp whose objects are simplicial complexes, and morphisms are as in Definition 1.27 (a).

In a category, as in Definition 1.29 an *isomorphism* will always be a morphism that has a two-sided inverse, and two objects will be called *isomorphic* if there is an isomorphism between them.

Notation 1.32 (Functors)**.** For two categories C and D a **(covariant) functor** $F\colon \mathsf{C}\to\mathsf{D}$ is given by:

- for every object X of C an object $F(X)$ of D;
- for every morphism $f\colon X\to Y$ of objects of C a morphism $F(f)\colon F(X)\to F(Y)$ of objects of D, such that $F(g\circ f)=F(g)\circ F(f)$ and $F(\mathrm{id}_X)=\mathrm{id}_{F(X)}$ for any composition $g\circ f$ of morphisms in C and any object X of C.

Again, we have already seen some examples of functors in this chapter:

(a) the *topological realization functor* $|\cdot|\colon \mathsf{Simp}\to\mathsf{Top}$ that sends a simplicial complex S to its topological realization $|S|$;

(b) the *barycentric subdivision functor* $\mathrm{Sd}\colon \mathsf{Simp}\to\mathsf{Simp}$ that sends a simplicial complex S to its barycentric subdivision $\mathrm{Sd}(S)$.

Note that in both cases we have just specified the functor on objects, since its action on morphisms is clear: For (a) it was given in Definition 1.27 (b), and for (b) a morphism $f\colon S\to T$ between simplicial complexes is mapped to the morphism $\mathrm{Sd}(f)\colon\ \mathrm{Sd}(S)\to\mathrm{Sd}(T)$ given on a vertex $\alpha\in V(\mathrm{Sd}(S))=S$ by $\mathrm{Sd}(f)(\alpha)=f(\alpha)$. In both cases, checking the required properties is straightforward.

Sometimes, one also has assignments on objects such that the correspondence for morphisms works in the opposite direction, i. e. so that for a morphism $f\colon X\to Y$ in C we have $F(f)\colon F(Y)\to F(X)$, and consequently $F(g\circ f)=F(f)\circ F(g)$ for any composition $g\circ f$ of morphisms in C. In this case, F is called a **contravariant functor**. A well-known example for this is the dualization functor ${}^\vee\colon \mathsf{Vect}\to\mathsf{Vect}$ that sends a vector space V to its dual $V^\vee$, and a linear map $f\colon V\to W$ to its dual map $f^\vee\colon W^\vee\to V^\vee$.

There is a whole branch of mathematics called *category theory* that studies the properties of general categories. We are not going to pursue this here; instead we will just use the notations introduced above as a convenient language. For example, it is clear from the above constructions that functors send isomorphisms to isomorphisms and can be composed to give new functors, so that from now on remarks as e. g. in Example 1.30 (b) do not need special mention since they are automatic.

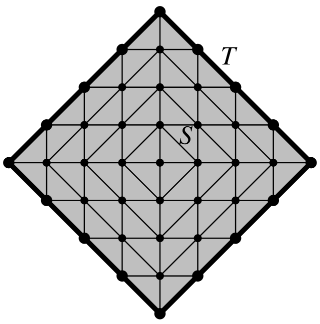
Remark 1.33 (Variants of simplicial complexes)**.** There are two variants of simplicial complexes that often appear in the literature, but that we do not want to consider here to keep notations simple:

(a) In the literature, simplicial complexes are usually allowed to have infinitely many simplices, maybe even of unbounded dimension (although each simplex always has to be finite-dimensional). By allowing this, one would also be able to obtain non-compact spaces as geometric realizations, e. g. the set of simplices $\{\{n\},\{n,n+1\} : n \in \mathbb{Z}\}$ would construct the real line as an "infinite chain of edges". However, we will not do so in these notes since infinite simplicial complexes would often make computations and (inductive) proofs more complicated or even impossible. The restriction to compact geometric realizations at this point will actually turn out not to be that bad, since our final theory will be invariant under homotopy equivalence, and almost all topological spaces occurring in practice will visibly be homotopy equivalent to a compact space.

(b) By modeling simplicial complexes as sets, the vertices of each simplex have to be distinct, and it is not possible to include the same simplex in a simplicial complex multiple times. For example, in the case of graphs, we cannot connect a vertex to itself by an edge to form a loop, and we cannot connect two vertices by more than one edge (see also Example 1.17 (c)). To waive these restrictions, one can consider so-called Δ-*complexes* [H, Chapter 2.1], or *multigraphs* in the 1-dimensional case.

Exercise 1.34. Let S be the 2-dimensional simplicial complex in the picture below, with vertices labeled (i,j) for $i,j \in \mathbb{Z}$ and $|i|+|j| \leq 4$ according to their position in the plane. We denote its "boundary" by T, i. e. the full 1-dimensional subcomplex drawn in bold containing the vertices (i,j) with $|i|+|j| = 4$. Obviously, the topological realizations of S and T are the 2-dimensional disk D^2 and its boundary circle S^1, respectively.

The following two facts are known from topology:



- There is no continuous map $f\colon D^2 \to S^1$ with $f|_{S^1} = \mathrm{id}$.
- Every continuous map $f\colon D^2 \to D^2$ has a fixed point.

Are the following simplicial versions of these statements true or false? If a statement is true, give a purely simplicial proof as well as a proof deriving it from the corresponding topological fact above. If a statement is false, provide a counterexample.

(a) There is no morphism $f\colon S \to T$ with $f|_T = \mathrm{id}$.

(b) Every morphism $f\colon S \to S$ has a fixed vertex.

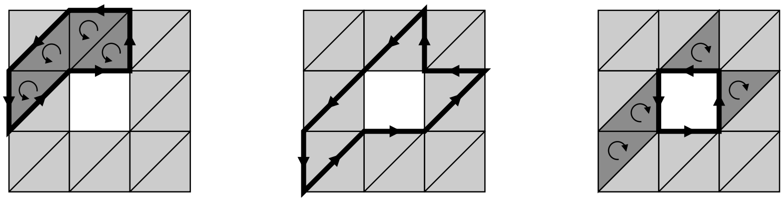
Exercise 1.35. Prove that every simplicial complex S admits a geometric realization in $\mathbb{R}^n$ with $n = 2\dim S + 1$.

(Hint: Start with the standard geometric realization and try to project it linearly to smaller-dimensional subspaces as long as possible. It might be useful that the complement of a finite union of proper affine subspaces of a real vector space is a dense open subset.)

2. The Construction of Simplicial Homology

In the last chapter we have introduced simplicial complexes as a combinatorial way to describe certain topological spaces. Let us now proceed to construct the *simplicial homology groups* which will be able to detect "holes" in such spaces. It should be said clearly that this construction (and in fact everything we do from now on until Chapter 5) is entirely combinatorial resp. algebraic and does not use topology in any way. However, the questions we ask and the methods we use will strongly be motivated by topology, so the pictures we draw will often look very topological. In fact, we will see in Chapter 8 that simplicial complexes with homeomorphic topological realizations will have isomorphic simplicial homology groups – but this is not at all obvious from the definitions and will require substantial effort to prove.

As already mentioned in the introduction, the idea to detect holes is to consider "closed cycles modulo boundaries". Consider for example the 2-dimensional simplicial complex realizing an annulus shown in the picture below. In the left picture, the thick lines show a loop (which we will call a closed 1-chain, or a 1-cycle) which is the boundary of the two-dimensional dark region (which we will call a 2-chain). Hence this loop will vanish in homology when we consider cycles modulo boundaries, which means intuitively that this loop does not detect a hole in the simplicial complex. On the other hand, the picture in the middle shows a 1-cycle which cannot be written as a boundary, so it runs around a hole and will be a non-trivial element in homology. Finally, the same holds for the 1-cycle in the picture on the right, but the boundary of the dark region is the difference of it and the 1-cycle from the middle picture, so these two 1-cycles are the same modulo boundaries. Hence they will represent the same homology class, corresponding to the intuitive statement that they detect the same hole in the space.



You have probably noticed that the above explanation implicitly used *orientations*, at the latest when we said that the boundary of the dark region in the right picture is the *difference* of two 1-cycles. This is more or less forced on us if we want the spaces describing homology to be groups, and corresponds to the fact that a loop in the fundamental group of a topological space becomes its inverse if we reverse its orientation. Formally, an orientation of a simplex will be an ordering of its vertices modulo even permutations. So for 1-simplices it just is an ordering of its vertices (we have indicated this above by the arrows on the loops), and for 2-simplices it is a cyclic ordering (indicated by the circular arrows above). In higher dimensions, it would be harder to visualize.

Also, to obtain groups we need to be able to add and subtract chains, and thus we have to consider sums of oriented simplices at least with integer coefficients. This is how homology groups are usually constructed in the literature, but for our purposes it is more convenient to allow coefficients in a chosen ground field: Firstly, when we move on to analysis later on we will have to pass to real coefficients anyway, and secondly, this makes all our spaces into vector spaces and thus accessible to linear algebra. Hence, our homology groups will actually be "homology vector spaces", but we will stick to the term "homology group" as this is the common name in the literature.

Let us now start with the formal definition of chains, following the ideas above. From now on, let K always denote a fixed ground field.

Definition 2.1 (Ordered simplicial chains)**.** Let S be a simplicial complex, and $p \in \mathbb{N}$.

(a) An **ordered p-simplex** is just an arbitrary $(p+1)$-tuple $(i_0,\dots,i_p)$. We say that it is an ordered p-simplex in S if $\{i_0,\dots,i_p\} \in S$. Note that we do not require the vertices $i_0,\dots,i_p$ to be distinct.

(b) We denote by $C_p^{\text{ord}}(S)$ the K-vector space with basis given by all ordered p-simplices in S. Its elements (which are then K-linear combinations of ordered p-simplices in S) will be called **ordered simplicial p-chains** in S.

For our purposes, it will be convenient to define the vector space $C_p^{\text{ord}}(S)$ for all $p \in \mathbb{Z}$, so we set $C_p^{\text{ord}}(S) := 0$ for all $p < 0$. If we want to indicate the ground field in the notation we will write $C_p^{\text{ord}}(S)$ also as $C_p^{\text{ord}}(S,K)$.

Definition 2.2 (Oriented simplicial chains)**.** Again let S be a simplicial complex, and let $p \in \mathbb{N}$. We denote by $C_p(S)$ the quotient vector space of $C_p^{\text{ord}}(S)$ modulo the subspace generated by:

- all ordered p-simplices $(i_0,\dots,i_p)$ for which $i_0,\dots,i_p$ are not pairwise distinct; and
- all ordered p-chains $(i_0,\dots,i_p) - \operatorname{sign}(\pi)\cdot(i_{\pi(0)},\dots,i_{\pi(p)})$ for permutations π of $\{0,\dots,p\}$.

The class of an ordered p-simplex $(i_0,\dots,i_p)$ in $C_p(S)$ is called an **oriented p-simplex** in S and will be written as $[i_0,\dots,i_p]$. Elements of $C_p(S)$ will be called **(oriented) simplicial p-chains** in S.

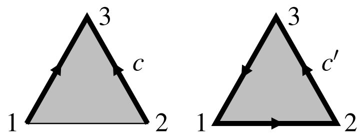
As before, we set $C_p(S) = 0$ for all $p < 0$, and write $C_p(S)$ as $C_p(S,K)$ if we want to indicate the ground field in the notation.

Remark 2.3. Let S be a simplicial complex, and let $p \in \mathbb{Z}$.

(a) By construction, $C_p(S)$ is a vector space whose dimension is the number of p-simplices in S. For any such p-simplex $\{i_0,\dots,i_p\}$ there is a corresponding basis vector $[i_0,\dots,i_p] \in C_p(S)$ that changes its sign if we apply an odd permutation to the vertices. In particular, $C_p(S)$ will be the zero vector space if $p < 0$ or $p > \dim S$. Oriented simplices with repeated vertices are allowed, but equal to 0 in $C_p(S)$ by definition.

(b) The vector spaces $C_p^{\text{ord}}(S)$ are still finite-dimensional, but much bigger since permutations of the vertices in an ordered simplex lead to new ordered simplices. In fact, as repeated vertices are allowed in ordered simplices, $C_p^{\text{ord}}(S)$ will even be non-zero for arbitrary $p \in \mathbb{N}$ unless S is empty. Hence, it would be computationally much more expensive to work with $C_p^{\text{ord}}(S)$ than with $C_p(S)$. Fortunately, working with $C_p(S)$ will be fine for us for the moment; we will come back to the ordered simplicial chains in Construction 4.23.

Example 2.4. In the simplicial disk $D^{\{1,2,3\}}$, the picture on the right shows two oriented simplicial 1-chains: $c = [1,3]+[2,3]$ and $c' = [1,2]+[2,3]+[3,1]$. Note that we could also write c' as $[1,2]-[1,3]+[2,3]$, and that $c+c' = [1,2]+2\,[2,3]$. In particular, in case of the ground field $\mathbb{Z}_2$ we would have $c+c' = [1,2]$.



Let us now construct the boundary map for oriented chains. By definition, the boundary of a p-simplex is always meant to be the chain of its $(p-1)$-dimensional faces (so no faces of smaller dimensions are involved). This makes the following definition straightforward, we just have to get the orientations right:

Definition 2.5 (Boundary map for simplicial chains)**.** Let S be a simplicial complex.

(a) For $p > 0$, the **boundary** of an oriented p-simplex $[i_0,\dots,i_p]$ in S is the $(p-1)$-chain

$$\partial_p[i_0,\dots,i_p] := \sum_{k=0}^{p} (-1)^k \cdot [i_0,\dots,\widehat{i_k},\dots,i_p] \quad \in C_{p-1}(S),$$

where the notation $\widehat{i_k}$ means that this vertex is left out.

By linearity, we extend this to a linear **boundary map** $\partial_p \colon C_p(S) \to C_{p-1}(S)$. To extend it to all integers, we define $\partial_p \colon C_p(S) \to C_{p-1}(S)$ to be the zero map for $p \le 0$. Often we will drop the index and write the boundary map ∂_p just as ∂.

(b) A p-chain $c \in C_p(S)$ is called **closed**, or a p**-cycle**, if $\partial c = 0$. It is called a **boundary** if $c = \partial c'$ for some $c' \in C_{p+1}(S)$.

Remark 2.6. Whenever we define something on oriented simplices such as in Definition 2.5 (a), we have of course to check that it is well-defined, i. e. that the definition is compatible with vertex permutations. To do this for Definition 2.5 (a), we write its right hand side for given $0 \le a < b \le p$ as

$$(-1)^a \cdot \underbrace{[\dots,\widehat{i_a},\dots,i_b,\dots]}_{\text{(A)}} + (-1)^b \cdot \underbrace{[\dots,i_a,\dots,\widehat{i_b},\dots]}_{\text{(B)}} + \sum_{k \neq a,b} (-1)^k \cdot \underbrace{[\dots,\widehat{i_k},\dots]}_{\text{(C)}}. \qquad (*)$$

Now consider the two cases of Definition 2.2:

- If $i_a = i_b$ then (C) $= 0$, and (B) is obtained from (A) by moving i_b to the left $b-a-1$ places, giving a sign of $(-1)^{b-a-1}$. Hence, $(*) = ((-1)^a + (-1)^b(-1)^{b-a-1})\,(\mathrm{A}) + (\mathrm{C}) = 0$.
- Exchanging i_a and i_b in $[i_0,\dots,i_p]$ transforms (C) to $-$(C), (A) (again by moving the b-th position $b-a-1$ places to the left) to $(-1)^{b-a-1}$(B), and similarly (B) to $(-1)^{b-a-1}$(A). So $(*)$ becomes

  $$(-1)^a(-1)^{b-a-1}(\mathrm{B}) + (-1)^b(-1)^{b-a-1}(\mathrm{A}) - (\mathrm{C}) = -(*),$$

  i. e. $(*)$ is multiplied by -1, which is exactly the sign of the transposition exchanging i_a and i_b. The general statement then follows since every permutation is a composition of transpositions.

Note that the sign $(-1)^k$ in Definition 2.5 (a) was important for this calculation to work out.

As in this example, checking well-definedness for constructions involving oriented simplices is usually just a straightforward computation. In the future, we will mostly omit such calculations.

Example 2.7. In Example 2.4, we have

$$\partial c = ([3]-[1]) + ([3]-[2]) = 2\,[3]-[1]-[2]$$
$$\text{and} \quad \partial c' = ([2]-[1]) + ([3]-[2]) + ([1]-[3]) = 0$$

in $C_0(D^{\{1,2,3\}})$. Hence, c' is a 1-cycle in $D^{\{1,2,3\}}$. In fact, it is also a boundary since

$$\partial[1,2,3] = [2,3]-[1,3]+[1,2] = c'.$$

Intuitively, this just says that c' describes a closed loop which is the boundary of the triangle.

With our constructions, we have now defined a sequence of vector spaces $C_p(S)$ for $p \in \mathbb{Z}$ for every simplicial complex S, with linear maps $\partial_p\colon C_p(S) \to C_{p-1}(S)$ between them. We will see many more sequences of similar type later on, so let us formalize this concept.

Definition 2.8 (Sequences of vector spaces)**.**

(a) A **sequence of vector spaces** (C,∂) is a collection of vector spaces $C = (C_p)_{p\in\mathbb{Z}}$ and linear maps $\partial = (\partial_p)_{p\in\mathbb{Z}}$ with $\partial_p\colon C_p \to C_{p-1}$ for all $p \in \mathbb{Z}$, i. e. we can write this as

$$\cdots \longrightarrow C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \longrightarrow \cdots.$$

We will usually write such a sequence just as C if ∂ is clear from the context.

(b) For two such sequences (C,∂) and (D,∂'), a **morphism** f between them is given by a collection of linear maps $f_p\colon C_p \to D_p$ for all $p \in \mathbb{Z}$ such that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{p+1} \arrow[r, "\partial_{p+1}"] \arrow[d, "f_{p+1}"] & C_p \arrow[r, "\partial_p"] \arrow[d, "f_p"] & C_{p-1} \arrow[r] \arrow[d, "f_{p-1}"] & \cdots \\
\cdots \arrow[r] & D_{p+1} \arrow[r, "\partial'_{p+1}"] & D_p \arrow[r, "\partial'_p"] & D_{p-1} \arrow[r] & \cdots
\end{tikzcd}$$

is **commutative**. This means that any two paths along arrows with the same source and target in this diagram are the same map, i. e. in this case that $f_{p-1} \circ \partial_p = \partial'_p \circ f_p$ for all $p \in \mathbb{Z}$.

Often, the linear maps of the sequences C and D will be boundary maps of some kind, and thus all be denoted by ∂. Moreover, as for ∂ we will usually omit the index from f, and thus often write the commutativity condition just as $f \circ \partial = \partial \circ f$ (which also explains the name "commutativity").

This makes sequences of vector spaces into a category, which we will denote by Seq.

Construction 2.9 (Pushforwards and the simplicial chain functor)**.** Rephrasing our constructions in Definition 2.2 and 2.5 in the language of Definition 2.8, we have now associated to every simplicial complex S a sequence of vector spaces

$$\cdots \longrightarrow C_{p+1}(S) \xrightarrow{\partial_{p+1}} C_p(S) \xrightarrow{\partial_p} C_{p-1}(S) \longrightarrow \cdots,$$

which we will just denote by $C(S)$. Let us make this assignment $S \mapsto C(S)$ into a functor from Simp to Seq by also assigning to a morphism $f\colon S \to T$ of simplicial complexes an induced morphism $C(S) \to C(T)$ of sequences of vector spaces. The official name of this induced morphism in the language of Notation 1.32 would be $C(f)$, but it is common to denote it by $f_*\colon C(S) \to C(T)$ and call it the **pushforward** by f (referring to the fact that it moves a simplicial chain on S to one on T, i. e. "forward" in the same direction as the morphism f). Its construction is straightforward: We just define it by

$$f_*[i_0,\dots,i_p] := [f(i_0),\dots,f(i_p)] \quad \text{for all } [i_0,\dots,i_p] \in C_p(S),$$

extended linearly to a map $C_p(S) \to C_p(T)$ for all $p \in \mathbb{Z}$. Intuitively, if we think of a p-chain as a "p-dimensional object" in the topological space $|S|$, its pushforward can be thought of as its image under the continuous map $|f|\colon |S| \to |T|$.

It is immediate to see that this is well-defined in the sense of Remark 2.6 and commutes with the boundary maps as in Definition 2.8, so we have actually constructed a pushforward morphism of sequences of vector spaces $f_*\colon C(S) \to C(T)$. Moreover, it is obvious that the assignments $S \mapsto C(S)$ and $f \mapsto f_*$ give rise to a functor $\mathsf{Simp} \to \mathsf{Seq}$, i. e. that they are compatible with compositions and the identity. We call it the *simplicial chain functor*.

Example 2.10. Consider again the morphism f of Example 1.28 (c) sending the torus S of Example 1.17 (d) with vertices labeled $1,\dots,9$ to the sphere $S^{\{1,2,3\}}$ by mapping the vertex $3i+j$ to j for all $i = \{0,1,2\}$ and $j \in \{1,2,3\}$. For this map, we have e. g. for the 2-chain $[1,2,5] \in C_2(S)$

$$\begin{aligned} f_*\partial[1,2,5] &= f_*([2,5]-[1,5]+[1,2]) = [2,2]-[1,2]+[1,2] = 0 \\ \text{and also} \quad \partial f_*\partial[1,2,5] &= \partial[1,2,2] = 0 \end{aligned}$$

in $C_1(S^{\{1,2,3\}})$, verifying in this example that f_* is a morphism of sequences. Note that this makes use of the fact that oriented simplices with repeated vertices are zero.

Remark 2.11. If we have a sequence of vector spaces (C,∂), the most natural first question is probably how long non-zero elements can "survive" in it, i. e. given a non-zero element $c \in C_p$ for some $p \in \mathbb{Z}$, and forming $\partial c \in C_{p-1}$, $\partial^2 c := \partial\partial c \in C_{p-2}$, and so on, what is the first $k \in \mathbb{N}$ such that $\partial^k c = 0$ in C_{p-k}? We will study this question in general in Proposition 9.13 when discussing persistent homology, but in our case at hand the situation is actually quite simple as every element in this sequence will die after at most two steps:

Lemma 2.12. *For any simplicial complex S and $p \in \mathbb{Z}$, the composition $\partial^2\colon C_{p+1}(S) \to C_{p-1}(S)$ of the boundary map with itself is the zero map.*

Proof. We may assume that $p > 0$, since otherwise $C_{p-1}(S) = 0$. As in Remark 2.6, the statement now depends crucially on the sign $(-1)^k$ in Definition 2.5: For any oriented $(p+1)$-simplex

$[i_0,\dots,i_{p+1}]$ we have

$$\begin{aligned}
\partial^2[i_0,\dots,i_{p+1}] &= \partial \sum_{k=0}^{p+1} (-1)^k [\dots,\widehat{i_k},\dots] \\
&= \sum_{k,l:\,l<k} (-1)^l (-1)^k [\dots,\widehat{i_l},\dots,\widehat{i_k},\dots] + \sum_{k,l:\,l>k} (-1)^{l-1} (-1)^k [\dots,\widehat{i_k},\dots,\widehat{i_l},\dots] \qquad (1) \\
&= 0, \qquad (2)
\end{aligned}$$

where the exponent $l-1$ in (1) comes from the fact that the entry i_l is actually at position $i_l - 1$ as i_k is already left out, and (2) follows since the first sum in (1) is just the negative of the second due to the symmetry $k \leftrightarrow l$. □

In fact, this situation of a sequence of vector spaces in which already the composition of any two of its linear maps is zero is so ubiquitous in mathematics that it has a special name (borrowed from algebraic topology).

Definition 2.13 (Chain complexes)**.**

(a) A sequence (C,∂) of vector spaces is called a **chain complex** if $\partial_p \circ \partial_{p+1} = 0$ for all $p \in \mathbb{Z}$, i. e. if $\operatorname{im}\partial_{p+1} \subset \ker \partial_p$.

(b) Morphisms of chain complexes are exactly morphisms as sequences as in Definition 2.8 (b). This makes chain complexes into a category, which we denote by Ch.

(c) We carry over the notations from Definition 2.2 and 2.5: The elements of C_p for $p \in \mathbb{Z}$ will be called **p-chains**, and the maps ∂_p are referred to as **boundary maps**. Chains in the kernel of ∂ are called **closed**, or **cycles**, and chains in the image of ∂ are called **boundaries**.

Example 2.14. Lemma 2.12 says precisely that the simplicial chain functor of Construction 2.9 is actually a functor from Simp to Ch. We will therefore call $C(S)$ the **simplicial chain complex** of a
03 simplicial complex S.

Now in the setting of chain complexes, the defining property $\operatorname{im}\partial_{p+1} \subset \ker\partial_p$ means that boundaries are always cycles, so it makes sense to apply our original idea to consider the quotient spaces of cycles modulo boundaries. In fact, almost all homology theories (and clearly all homology theories that we will see in these notes) are obtained by taking cycles modulo boundaries in a suitable chain complex, so let us define homology now in this general situation.

Definition 2.15 (Homology of a chain complex)**.** For a chain complex (C,∂) and $p \in \mathbb{Z}$, its p-th **homology group** is the quotient vector space

$$H_p(C) := \ker\partial_p / \operatorname{im}\partial_{p+1}.$$

Its dimension $b_p(C) := \dim H_p(C) \in \mathbb{N} \cup \{\infty\}$ is called the p-th **Betti number** of C.

Notation 2.16. By construction, the p-th homology group $H_p(C)$ of a chain complex C is a so-called *subquotient* of C_p, i. e. a quotient space of a subspace of C_p. To keep the notations simple, we will not indicate the equivalence classes of this quotient space in any way, i. e. for a p-chain $c \in C_p$ we will just write $c \in H_p(C)$ to indicate that it is closed and determined only up to boundaries.

Lemma 2.17 (Homology as a functor)**.** *For any $p \in \mathbb{Z}$, the p-th homology sending a chain complex C to $H_p(C)$ is a functor from Ch to Vect. In other words, for any morphism $f\colon C \to D$ of chain complexes there is a well-defined induced linear map $H_p(C) \to H_p(D)$, $c \mapsto f(c)$ that is compatible with compositions and the identity. (As in Notation 2.16, we will usually denote it by the same letter f.)*

Proof. As in Definition 2.8, the morphism f gives us a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{p+1} \arrow[r, "\partial"] \arrow[d, "f"] & C_p \arrow[r, "\partial"] \arrow[d, "f"] & C_{p-1} \arrow[r] \arrow[d, "f"] & \cdots \\
\cdots \arrow[r] & D_{p+1} \arrow[r, "\partial"] & D_p \arrow[r, "\partial"] & D_{p-1} \arrow[r] & \cdots .
\end{tikzcd}$$

Now let a homology class in $H_p(C)$ be given by $c \in C_p$, i. e. we have $\partial c = 0$. By commutativity of the right square above, we then have $\partial f(c) = f(\partial c) = f(0) = 0$, i. e. $f(c) \in D_p$ is a cycle as well. To check well-definedness, let now c be a boundary, i. e. $c = \partial c'$ for some $c' \in C_{p+1}$. Then, by commutativity of the left square we have $f(c) = f(\partial c') = \partial f(c')$, i. e. $f(c)$ is a boundary as well. Hence, the assignment $c \mapsto f(c)$ gives a well-defined linear map from $H_p(C)$ to $H_p(D)$. It is obvious that it is compatible with compositions and the identity. □

As you can see from the previous proof, checking functoriality is often just a straightforward computation. In these cases, we will usually omit such calculations in the future.

But let us come back to topology. Of course, the idea is now to take the homology of the simplicial chain complex of a simplicial complex:

Definition 2.18 (Simplicial homology)**.** For a simplicial complex S and $p \in \mathbb{Z}$, the p-th **simplicial homology group** is defined to be the vector space $H_p(S) := H_p(C(S))$, i. e. the p-th homology group of the simplicial chain complex of S. If we want to indicate the chosen ground field K in the notation, we will write it as $H_p(S,K) = H_p(C(S,K))$. The corresponding Betti numbers are called the **Betti numbers** of S and denoted by

$$b_p(S) := b_p(S,K) = \dim H_p(S,K) \quad \in \mathbb{N} \cup \{\infty\}.$$

Notation 2.19 (Pushforwards for simplicial homology)**.** As the composition of the simplicial chain functor with the homology functor, assigning to a simplicial complex S its p-th homology group $H_p(S)$ for a given $p \in \mathbb{Z}$ is a functor from Simp to Vect. This means that for any morphism $f\colon S \to T$ of simplicial complexes there is an induced morphism $H_p(S) \to H_p(T)$ compatible with compositions and the identity, which is given by sending the class of a p-cycle $c \in C_p(S)$ to the class of the pushforward $f_*(c) \in C_p(T)$ as in Construction 2.9. In accordance with Notation 2.16 and Lemma 2.17, we will write this morphism also just as $f_*\colon H_p(S) \to H_p(T)$ and still call it the *pushforward* by f.

Remark 2.20.

(a) Of course, for any simplicial complex S we have $b_p(S) = 0$ for $p < 0$ or $p > \dim S$: In this case, we already have $C_p(S) = 0$, which implies that $H_p(S) = 0$.

(b) As we have already said in the introduction to this chapter, we will see later that the simplicial homology groups of two simplicial complexes with homeomorphic topological realizations are naturally isomorphic, so we have in fact just constructed vector spaces that see the topological (and not the combinatorial) properties of a simplicial complex. It is almost impossible however to prove this statement directly from our definitions – in fact, we will construct a completely new homology theory called *singular homology* in Chapter 6 which from the very start only needs a topological space as its input, and for which we can then show that it agrees with simplicial homology for topological realizations of simplicial complexes (see Proposition 8.1).

(c) Note that all our constructions to arrive at the simplicial homology groups are phrased in terms of finite-dimensional vector spaces and linear maps between them, all of which are explicitly given by Definition 2.2 and 2.5. Hence, the simplicial homology of a simplicial complex is readily computable using just standard algorithms of linear algebra. In fact, this is the main advantage of simplicial homology – all other homology theories that we will see later on work with infinite-dimensional vector spaces and thus lack this immediate computability. Moreover, the matrices describing the boundary maps in Definition 2.5 are actually very simple: Most of its entries are 0, and all others are ± 1. However, for simplicial complexes occurring in practice the matrices can easily become huge: already for the torus as in Example 1.17 (d) we had 27 one-dimensional simplices, so we would have to deal with vector spaces of this dimension. Hence, we still rely on clever techniques to actually perform the computations with reasonable effort. We will develop such techniques now and in Chapters 3 and 4. Let us start with some easy results by showing that simplicial homology splits on connected components and considering simplicial homology in dimension 0.

Lemma 2.21 (Simplicial homology of connected components)**.** *For a simplicial complex S with connected components $S_1, \dots, S_n$ we have*

$$H_p(S) \cong H_p(S_1) \times \cdots \times H_p(S_n)$$

for all $p \in \mathbb{Z}$ (with an isomorphism given by the pushforwards $H_p(S_i) \to H_p(S)$ for the inclusion morphisms $S_i \to S$ for all $i = 1, \dots, n$). In particular, this means that $b_p(S) = b_p(S_1) + \cdots + b_p(S_n)$.

Proof. By construction of the simplicial chain groups it is clear that $C_p(S) = C_p(S_1) \times \cdots \times C_p(S_n)$ for all $p \in \mathbb{Z}$. As the boundary maps act on each factor separately, it follows immediately by taking cycles modulo boundaries that $H_p(S) \cong H_p(S_1) \times \cdots \times H_p(S_n)$. □

Lemma 2.22 (Degree of a 0-cycle)**.** *For any non-empty simplicial complex S, there is a well-defined surjective linear map*

$$\deg\colon H_0(S) \to K$$

*determined by sending an oriented 0-simplex $[i]$ to 1 for all $i \in V(S)$. We call $\deg c$ the **degree** of $c \in H_0(S)$ (we can think of it as just counting the points in c with their respective coefficients).*

Proof. There is clearly a unique surjective linear map $\deg\colon C_0(S) \to K$ sending each oriented 0-simplex $[i]$ to 1 for $i \in V(S)$ as these 0-simplices form a basis of $C_0(S)$. To see that it passes to a surjective linear map on $H_0(S)$ note first that all of these 0-simplices are also closed since the degree map $\partial\colon C_0(S) \to C_{-1}(S)$ is trivial. Hence it just remains to prove that deg is the zero map on boundaries. But this is obvious since for every oriented edge $[i_0, i_1]$ in S we have $\deg \partial[i_0, i_1] = \deg([i_1] - [i_0]) = 1 - 1 = 0$. □

Corollary 2.23 (b_0 for a simplicial complex)**.** *For any simplicial complex S, the Betti number $b_0(S)$ is the number of connected components of S. More precisely, a basis of $H_0(S)$ is given by the 0-simplices $[i]$ where we take one arbitrary vertex i from each connected component of S. In particular, the degree map of Lemma 2.22 is an isomorphism if S is connected.*

Proof. By Lemma 2.21 we may assume that S is connected. Note that $\operatorname{im}\partial_1$ is the vector space generated by the 0-chains $\partial[i_0, i_1] = [i_1] - [i_0]$ for all edges $\{i_0, i_1\} \in S$. This means that two oriented 0-simplices $[i_0]$ and $[i_1]$ in $C_0(S)$ are identified in $H_0(S)$ whenever they are connected by an edge in S. But the connectedness of S means that any two vertices are connected by edges by Lemma 1.8 (b), so the class of any one oriented 0-simplex in S suffices to generate $H_0(S)$. In particular, this means that $b_0(S) \leq 1$. As Lemma 2.22 implies that we must also have $b_0(S) \geq 1$, the result follows. □

Another trick to obtain some Betti numbers easily is the Euler characteristic; it is a certain combination of the Betti numbers that can be computed without considering the boundary maps at all.

Definition 2.24 (Euler characteristic)**.**

(a) Let C be a chain complex for which all Betti numbers are finite, and only finitely many of them are non-zero. Then

$$\chi(C) := \sum_{p \in \mathbb{Z}} (-1)^p b_p(C)$$

is called the **Euler characteristic** of C.

(b) The Euler characteristic of a simplicial complex S is defined to be $\chi(S) := \chi(C(S))$, i. e. the Euler characteristic of the simplicial chain complex of S.

Lemma 2.25. *Let C be a chain complex such that all vector spaces C_p for $p \in \mathbb{Z}$ are finite-dimensional, and almost all of them are zero. Then*

$$\chi(C) = \sum_{p \in \mathbb{Z}} (-1)^p \dim C_p.$$

Proof. This follows immediately from the dimension formulas of linear algebra, noting that the following sums are actually all finite by assumption:

$$\begin{aligned}
\chi(C) &= \sum_{p\in\mathbb{Z}} (-1)^p \dim(\ker\partial_p/\operatorname{im}\partial_{p+1}) \\
&= \sum_{p\in\mathbb{Z}} (-1)^p \dim\ker\partial_p + \sum_{p\in\mathbb{Z}} (-1)^{p+1} \dim\operatorname{im}\partial_{p+1} \\
&= \sum_{p\in\mathbb{Z}} (-1)^p \dim\ker\partial_p + \sum_{p\in\mathbb{Z}} (-1)^p \dim\operatorname{im}\partial_p \qquad \text{(by index shifting)} \\
&= \sum_{p\in\mathbb{Z}} (-1)^p \dim C_p. \qquad \square
\end{aligned}$$

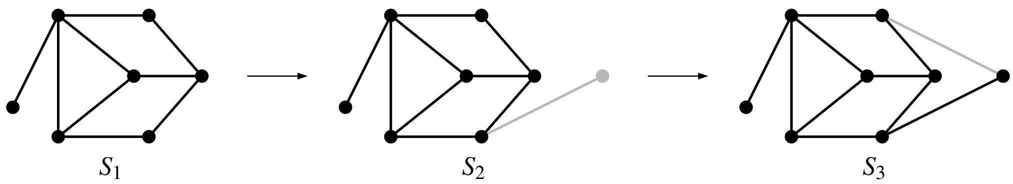
For the remaining part of this chapter, let us now study several examples in which we can already compute the simplicial homology groups with our current methods.

Example 2.26 (Simplicial homology of graphs)**.** Let S be a graph, i. e. a simplicial complex with $\dim S \leq 1$. Lemma 2.21 tells us that it is sufficient to consider the case when S is connected, hence $b_0(S) = 1$ by Corollary 2.23. By Remark 2.20 (a) the only other Betti number that is not trivially zero is then $b_1(S)$, and we can compute it using the Euler characteristic as

$$b_1(S) = b_0(S) - (b_0(S) - b_1(S)) = 1 - \chi(S) \overset{2.25}{=} 1 - \dim C_0(S) + \dim C_1(S),$$

i. e. as 1 minus the number of vertices plus the number of edges of S. This number $b_1(S)$ is usually called the *genus* of the graph S. Let us study its geometric interpretation by inductively building up the graph, adding one edge at a time. To do this, we start with a graph S having only one vertex and no edges, which has genus $b_1(S) = 1-1+0 = 0$. Now there are two types of adding an edge, shown in the picture below where we add the gray edge to a given graph:

- We can add an edge that is connected to the previous graph at only one vertex, as for passing from S_1 to S_2 in the picture. This adds one 0-simplex and one 1-simplex to the graph, hence keeps the genus constant. Topologically, we just add an end to the graph somewhere, not affecting any holes in it.
- We can add an edge that is connected to the previous graph at both vertices, as for the transition from S_2 to S_3 in the picture. This only adds one 1-simplex, and thus raises the genus by 1. Geometrically, we close a new circle in the graph in this way. If the graphs are realized by a map $r \colon V(S) \to \mathbb{R}^2$ as in the picture below, we can also interpret this process as raising the number of connected components of $\mathbb{R}^2 \backslash r(S)$ by 1.

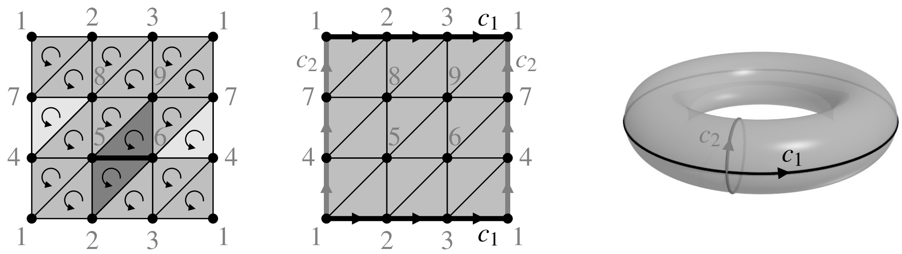


In summary, this means that we can interpret the genus of a graph S as the number of independent circles in the graph, or in the case of a graph realized in $\mathbb{R}^2$ as the number of bounded connected components of $\mathbb{R}^2 \backslash r(S)$ (which we can also view as the number of holes in the graph). In this way, we can e. g. readily see without counting vertices and edges that the genus of S_1 and S_2 above is 3, whereas it is 4 for S_3.

Example 2.27 (Simplicial homology of a torus)**.** Let S be the simplicial complex of Example 1.17 (d) realizing a torus, shown again in the picture below on the left. As S is connected, we have $b_0(S) = 1$ again by Corollary 2.23.

Next, we claim that $b_2(S) = 1$ as well. To see this, we first show that any closed 2-chain $c \in C_2(S)$ contains all 2-simplices of S with the same coefficient, taken with the orientation as indicated by the circular arrows. Consider any inner edge of S (i. e. not at the border of the square), such as the thick edge $\{5,6\}$ in the picture. In the boundary $\partial c \in C_1(S)$ it gets a contribution from exactly two 2-simplices, in the example below these are the dark 2-simplices $\{2,5,6\}$ and $\{5,6,9\}$. But $\partial c = 0$, and hence these contributions must cancel. This means that the dark simplices must occur in c with the same coefficient, with the orientation as indicated in the picture. Using this argument for all inner edges, we see that c must in fact contain all 2-simplices with the same coefficient.

Now note that this cancellation of the edges in the boundary ∂c actually works out at the border of the square as well: For example, the edge $\{4,7\}$ (which appears both on the left and right side of the square due to the identification of these sides) gets a contribution in ∂c from the two light-colored 2-simplices $\{4,7,8\}$ and $\{4,6,7\}$. As we already know that they have the same coefficient in c, and the orientations work out as well (the boundary of $[8,7,4]$ contains $[7,4]$, and the boundary of $[4,7,6]$ contains $[4,7] = -[7,4]$), these contributions cancel as well. Hence, we see that the space of closed 2-chains in S is in fact one-dimensional, generated by the cycle containing all 2-simplices of S with coefficient 1 in the orientation below. This 2-cycle is clearly not a boundary since $C_3(S) = 0$, and hence we see that $H_2(S)$ is one-dimensional, i. e. $b_2(S) = 1$.



The computation of $b_1(S)$ is now simple: As the torus has 9 zero-dimensional, 27 one-dimensional and 18 two-dimensional simplices, its Euler characteristic is $\chi(S) = 9-27+18 = 0$ by Lemma 2.25, and thus we obtain

$$b_1(S) = b_0(S) + b_2(S) - \chi(S) = 1+1-0 = 2.$$

In fact, we claim that a basis of $H_1(S)$ is given by c_1 and c_2 in the middle picture above (which correspond topologically to the two independent loops in the torus as in the right picture). As c_1 and c_2 are clearly closed and we know already that $\dim H_1(S) = b_1(S) = 2$, it suffices for a proof of this claim to see that these two cycles are linearly independent. Again, with a neat trick we can prove this without huge matrix computations: Let $\lambda_1, \lambda_2 \in K$ with

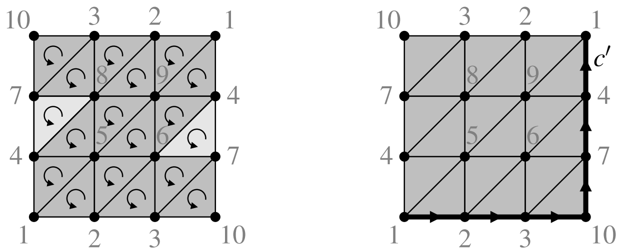
$$\lambda_1 c_1 + \lambda_2 c_2 = 0 \quad \in H_1(S). \tag{$*$}$$

By Example 1.28 (c) there is a morphism $f\colon S \to S^{\{1,2,3\}}$ to the circle $S^{\{1,2,3\}}$ corresponding to the vertical projection in the middle picture above. Its pushforward $f_*\colon H_1(S) \to H_1(S^{\{1,2,3\}})$ as in Notation 2.19 clearly sends c_2 to 0, so applying f_* to $(*)$ yields

$$\lambda_1 f_* c_1 = 0 \quad \in H_1(S^{\{1,2,3\}}).$$

But $f_* c_1 = [1,2]+[2,3]+[3,1]$ is a generator of the 1-dimensional homology group $H_1(S^{\{1,2,3\}})$ by Example 2.26, so we conclude that $\lambda_1 = 0$. Using the same argument for the horizontal projection gives $\lambda_2 = 0$ as well, thus rigorously proving without any matrix computations that c_1 and c_2 are a basis of $H_1(S)$ – as expected from topology.

Example 2.28 (Simplicial homology of the real projective plane)**.** Let us perform the same analysis as in Example 2.27 for the real projective plane S of Example 1.17 (d), shown again in the picture below on the left. It might seem at first that everything works very much in the same way as before, but there is in fact an unexpected twist.



As before, we have $b_0(S) = 1$ since S is connected. It is also still true by the same argument as in Example 2.27 that any 2-cycle in $C_2(S)$ must contain all 2-simplices of S with the same coefficient, in the orientation as in the picture. However, as opposite sides of the square are now identified in opposite directions, the cancellation for the outer edges does not work out any more: For example, the edge $\{4,7\}$ gets a contribution from the light-colored 2-simplices $\{4,7,8\}$ and $\{4,6,7\}$, but this time both contributions from $[8,7,4]$ and $[7,4,6]$ are $[7,4]$, so they get multiplied by 2 instead of canceling out. In other words, if $c \in C_2(S)$ denotes the 2-chain of all 2-simplices with multiplicity 1 then ∂c is now $2c'$ as in the picture above on the right, instead of being zero.

Actually, what this means now *depends on the characteristic of the ground field*:

- If the characteristic is not 2, i. e. if $2 \neq 0$ in K such as e. g. in the case $K = \mathbb{R}$, then c is not a cycle. There are thus no non-zero cycles in $C_2(S)$ at all, i. e. we have $b_2(S,K) = 0$. The Euler characteristic of S is $\chi(S) = 10 - 27 + 18 = 1$, so we obtain
$$b_1(S,K) = b_0(S,K) + b_2(S,K) - \chi(S) = 1 + 0 - 1 = 0.$$
- If the characteristic is 2, i. e. if $2 = 0$ in K such as e. g. in the case $K = \mathbb{Z}_2$, then $\partial c = 2c' = 0$, and thus we obtain $b_2(S,K) = 1$ as in Example 2.27. The Euler characteristic is still the same (in fact, by Lemma 2.25 it is always independent of the ground field), so now we have
$$b_1(S,K) = b_0(S,K) + b_2(S,K) - \chi(S) = 1 + 1 - 1 = 1.$$

Intuitively, the reason for this is that the real projective plane – similarly to the Möbius strip – is *not orientable*. This means that there is no way orient the 2-simplices of S coherently as we have seen above, usually leading to $b_2(S,K) = 0$. However, considering a field of characteristic 2 with $-1 = 1$ essentially means that we ignore orientations (note that Definition 2.2 then identifies all permutations of a simplex), so that then the real projective plane just looks like a closed surface with $b_2(S,K) = 1$.

Exercise 2.29 (Simplicial homology of the Klein bottle)**.** Similarly to Example 1.17 (d), the *Klein bottle* is obtained as a topological space by taking the unit square and identifying the top with the bottom side in the same direction, and the left with the right side in opposite directions.

Starting from a subdivision of the unit square into 18 triangles as in this example, construct a simplicial complex realizing the Klein bottle, and compute its Betti numbers. Do they depend on the chosen ground field?

Exercise 2.30.

(a) Let C be a complex of finite-dimensional vector spaces, with only finitely many C_p non-zero, and let $f\colon C \to C$ be a morphism. Show that
$$\sum_{p\in\mathbb{Z}} (-1)^p \operatorname{tr}(f\colon C_p \to C_p) = \sum_{p\in\mathbb{Z}} (-1)^p \operatorname{tr}(f\colon H_p(C) \to H_p(C)), \qquad (*)$$
where tr denotes the trace of a linear map. If f is the pushforward $C(S) \to C(S)$ by a morphism $g\colon S \to S$ of simplicial complexes, we call $(*)$ the *Lefschetz number* of g.

(b) Show that any morphism $g\colon S \to S$ of simplicial complexes with non-zero Lefschetz number has a fixed simplex.

(c) Now suppose that S is a connected simplicial complex with Betti numbers $b_p(S) = 0$ for all $p > 0$. Show that any simplicial morphism $g\colon S \to S$ must fix at least one simplex. Use this to show that any simplicial morphism from a tree (i. e. a connected graph of genus 0) to itself must have a fixed simplex.

Exercise 2.31 (Reduced homology)**.** Sometimes it is more natural to slightly modify our very first Definition 1.2 (a) and drop the condition that a simplex has to be non-empty. The empty set $\emptyset$ is then a valid (-1)-simplex that is a proper face of every other simplex. For a simplicial complex S we can form the *augmented simplicial complex* $\tilde{S} := S \cup \{\emptyset\}$, satisfying again our Definition 1.2 (b) of a simplicial complex with the new notion of faces. As for the constructions in this chapter, every augmented simplicial complex has an oriented (-1)-simplex $[\,]$, and the formula for the boundary map in Definition 2.5 (a) is extended to $p \geq 0$ in the natural way by setting $\partial[i_0] = [\,]$ for all 0-simplices $\{i_0\}$.

If we leave all other constructions of this chapter unchanged, the resulting simplicial homology of the augmented simplicial complex $\tilde{S}$ is called the *reduced simplicial homology* of S and denoted by
04 $\tilde{H}_p(S) := H_p(\tilde{S})$ for all $p \in \mathbb{Z}$. How it is related to $H_p(S)$?

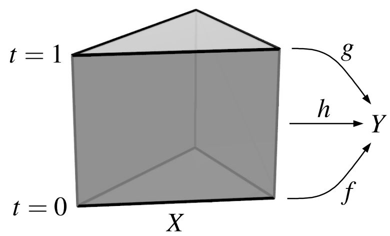
3. Chain Homotopies

In the previous chapter we have defined the simplicial homology groups of a simplicial complex. Due to their explicit construction using only finite-dimensional vector spaces, we saw that they can be computed with standard linear algebra techniques. However, the dimensions of these vector spaces are usually very large, and thus the brute-force algorithmic approach to determine the homology groups is often infeasible for real-world examples, and also unsuitable for proving theoretical results.

For example, at the moment we cannot even compute the simplicial homology of the disk D^α for a p-simplex α for all p: As α has $p+1$ vertices it has $\binom{p+1}{k+1}$ faces (i. e. subsets of cardinality $k+1$) of dimension $k \in \{0,\dots,p\}$, and thus the simplicial chain complex $C(D^\alpha)$ is already quite complicated. This might be surprising since the topological realization of D^α is just the disk D^p, which is of course a contractible space (i. e. homotopy equivalent to a point) and thus does not have any holes, i. e. it should have the (trivial) homology of a point. In fact, we will see in this chapter that this is true, but proving this statement directly by explicit computations with the simplicial chain complex would be messy.

But from this topological argument of contractibility we can already see a possible solution for our problems: We should talk about homotopy. In the same way as the fundamental group, the homology groups will be invariant under homotopy equivalences, so it will be a perfectly valid argument that the homology of a disk is trivial since it is homotopy equivalent to a point. However, homotopies are usually defined topologically in terms of continuous maps, so what does this concept mean in our current algebraic simplicial setting?

To see this, let us first look at the topological picture. As shown on the right, two continuous maps $f,g\colon X \to Y$ between topological spaces are called homotopic if there is a continuous *homotopy map* $h\colon X \times [0,1] \to Y,\ (x,t) \mapsto h(x,t)$ on the product space $X \times [0,1]$ (which is geometrically a prism with base X) such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$ for all $x \in X$. Intuitively, this means that f can be deformed continuously into g.



To translate this concept into algebra, let us first of all simplify the picture above by just looking at the prism, ignoring the maps to Y for a moment – our constructions will be functorial, so once we have described the prism it will be no problem for us to map this situation to some target space. Next, let us pass to the simplicial picture and regard X as the topological realization of a simplicial p-chain, so that the prism $X \times [0,1]$ should be the topological realization of a $(p+1)$-chain. In the above picture, we have drawn this situation for a single 2-simplex α: We can imagine $f(\alpha)$ as the bottom face of the prism, $g(\alpha)$ as its top face, and $h(\alpha)$ as the whole prism (let us postpone for a moment how we can realize this prism as a simplicial $(p+1)$-chain; we will address this problem in Construction 3.6).

Now let us phrase this geometric situation in terms of boundary maps: The boundary of the prism $\partial h(\alpha)$ consists of its bottom face $f(\alpha)$, its top face $g(\alpha)$, and its sides. The key observation is now that these sides are just another prism whose base is the boundary $\partial\alpha$, so they can be written as $h(\partial\alpha)$. In short, $\partial h(\alpha)$ is a combination of $f(\alpha)$, $g(\alpha)$, and $h(\partial\alpha)$. To get the signs right in this linear combination, note first that $f(\alpha)$ and $g(\alpha)$ should clearly have opposite orientations as they sit on opposite faces of the prism. But the remaining signs are just conventional as they can be absorbed by the precise construction of the homotopy map h (note that in $\partial h(\alpha)$ it takes prisms over p-chains and in $h(\partial\alpha)$ over $(p-1)$-chains, so these are actually two different prism maps that can also be given independent signs). The standard convention is to just take the sign to be positive for both $\partial h(\alpha)$ and $h(\partial\alpha)$, so we arrive at the equation

$$\partial h(\alpha) + h(\partial\alpha) = g(\alpha) - f(\alpha).$$

In fact, this is the purely algebraic description of homotopy that we were looking for. It makes sense to formulate it in terms of arbitrary chain complexes in this way, so let us write down the corresponding definition.

Definition 3.1 (Chain homotopies)**.** Two morphisms $f,g\colon C \to D$ of chain complexes as shown in the diagram below are called **(chain) homotopic** if there is a collection $h = (h_p)_{p\in\mathbb{Z}}$ of linear maps $h_p\colon C_p \to D_{p+1}$ with $\partial_{p+1}\circ h_p + h_{p-1}\circ\partial_p = g_p - f_p$ for all $p \in \mathbb{Z}$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{p+1} \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & C_p \arrow[r, "\partial_p"] \arrow[dl, "h_p"'] \arrow[d, shift right, "f_p"'] \arrow[d, shift left, "g_p"] & C_{p-1} \arrow[r] \arrow[dl, "h_{p-1}"] \arrow[d, shift right] \arrow[d, shift left] & \cdots \\
\cdots \arrow[r] & D_{p+1} \arrow[r, "\partial_{p+1}"'] & D_p \arrow[r] & D_{p-1} \arrow[r] & \cdots
\end{tikzcd}$$

The collection h is then called a **homotopy** or **prism map** between f and g. To simplify notations, we will usually write all maps h_p just as h, so that the homotopy equation becomes $\partial\circ h + h\circ\partial = g - f$.

Remark 3.2.

(a) It is checked immediately that homotopy is an equivalence relation on the set of all morphisms between given chain complexes.

(b) Although a homotopy h between two morphisms $f,g\colon C \to D$ of chain complexes maps the vector spaces of C to the vector spaces of D, it is clearly not a morphism from C to D in the sense of Definition 2.8 (b):

- It shifts the dimension index, i. e. does not map C_p to D_p, but to D_{p+1}.
- It does not form a commutative diagram of maps in any way, but instead satisfies the homotopy equation $\partial\circ h + h\circ\partial = g - f$.
- It does not induce maps on homology between $H_p(C)$ and $H_{p+1}(D)$.

Instead, as explained above its importance lies in the fact that its existence proves that f and g are homotopic, and thus define the same map on homology:

Proposition 3.3. *Two homotopic morphisms $f,g\colon C \to D$ of chain complexes induce the same map on homology, i. e. we have $f_p = g_p$ as maps from $H_p(C)$ to $H_p(D)$ for all $p \in \mathbb{Z}$.*

Proof. Let $c \in H_p(C)$ be a cycle, i. e. $\partial c = 0$. For a homotopy h between f and g,

$$f(c) - g(c) = \partial h(c) + h(\partial c) = \partial h(c)$$

is then a boundary, hence zero in $H_p(D)$. □

As in topology, this leads immediately to the definition of homotopy equivalence.

Definition 3.4 (Homotopy equivalence)**.** A morphism $f\colon C \to D$ of chain complexes is a **homotopy equivalence** if there is a morphism $g\colon D \to C$ such that $g\circ f$ is homotopic to id_C and $f\circ g$ is homotopic to id_D. In this case, g is called a **homotopy inverse** of f.

Corollary 3.5. *Any homotopy equivalence $f\colon C \to D$ of chain complexes induces isomorphisms $f_p\colon H_p(C) \to H_p(D)$ between their homology groups for all $p \in \mathbb{Z}$.*

Proof. Let g be a homotopy inverse of f. As homology is a functor by Lemma 2.17, we already know that there are induced maps $f_p\colon H_p(C) \to H_p(D)$ and $g_p\colon H_p(D) \to H_p(C)$ for all $p \in \mathbb{Z}$ whose compositions are $g_p\circ f_p\colon H_p(C) \to H_p(C)$ and $f_p\circ g_p\colon H_p(D) \to H_p(D)$. But these maps are just the identity by Proposition 3.3 as $g\circ f$ and $f\circ g$ are homotopic to the identity by assumption. Hence, f_p is an isomorphism with inverse g_p. □

So far, our setting for homotopy was purely algebraic. To apply this to simplicial homology, we now have to construct prisms in the category of simplicial complexes. Note that it is not immediately obvious how to do this since the product of a simplex with an interval – as in the picture in the introduction to this chapter – is not directly a simplex again. However, there are ways to subdivide this space into simplices, and this will be sufficient for our purposes. We will see one possibility for such a subdivision now; another one will be considered in Construction 7.2.

Construction 3.6 (Prism construction)**.** To see how to construct the prism over a simplicial complex, let us first consider as an example the $(p-1)$-simplex $\alpha = \{1,2,3,\dots,p\}$ with its standard geometric realization $|\alpha| \subset \mathbb{R}^{\{1,\dots,p\}}$. By Construction 1.12 and Proposition 1.15, the prism over α is then given in barycentric coordinates for α as

$$P = \{(x_1,\dots,x_p,t) : x_1,\dots,x_p \in \mathbb{R}_{\geq 0}, x_1+\cdots+x_p = 1, 0 \leq t \leq 1\} \quad \subset \mathbb{R}^{\{1,\dots,p\}} \times \mathbb{R}.$$

It has $2p$ vertices $(e_i,0)$ and $(e_i,1)$ for $i = 1,\dots,p$, which we label by i and i', respectively; the case $p = 3$ is shown in the picture below on the left. The idea to subdivide it into simplices is to consider the p inequalities

$$0 \leq x_p \leq x_{p-1}+x_p \leq x_{p-2}+x_{p-1}+x_p \leq \cdots \leq x_1+\cdots+x_p = 1$$

and subdivide P according to where the number $t \in [0,1]$ fits in. For $k = 1,\dots,p$ let us denote by $P_k \subset P$ the subset where t fits into the i-th inequality counted from the right, i. e. where

$$0 \leq x_p \leq \cdots \leq x_{k-1}+\cdots+x_p \leq t \leq x_k+\cdots+x_p \leq \cdots \leq x_1+\cdots+x_p = 1. \qquad (*)$$

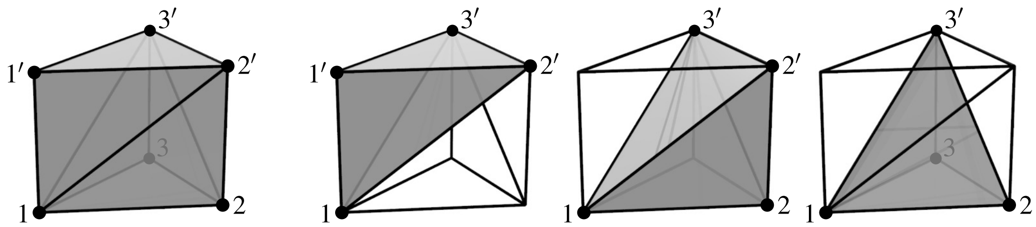
One can show that this is actually the geometric realization of a p-simplex, with vertices given by the points where only one of the $p+1$ inequalities in $(*)$ is not an equality. Now for $i = 1,\dots,p+1$ the point where only the i-th inequality (again counted from the right) is not an equality is given by

$$0 = \cdots = t = \cdots = x_{i-1}+\cdots+x_p \leq x_i+\cdots+x_p = \cdots = 1$$

for $i \leq k$, which is just $(e_i,0)$, and by

$$0 = \cdots = x_{i-2}+\cdots+x_p \leq x_{i-1}+\cdots+x_p = \cdots = t = \cdots = 1$$

for $i > k$, which is $(e_{i-1},1)$. Hence, P_k is the geometric realization (in our chosen coordinates) of the simplex $\{1,\dots,k,k',\dots,p\}$. The picture below on the right shows how our prism P in the case $p = 3$ is subdivided into the simplices $\{1,1',2',3'\}$, $\{1,2,2',3'\}$, and $\{1,2,3,3'\}$.



Luckily, we do not need a full exact proof that this procedure really works and extends to simplicial complexes. We just take the computations above as a motivation for the following simplicial definition of the prism map, for which we then do prove rigorously that it has the desired properties. Note that in order for this to work we have to choose an order on the vertices as we needed this to set up the inequalities $(*)$ above.

Definition 3.7 (Simplicial prism map)**.** Let S be a simplicial complex, for which we fix an order on its set of vertices $V(S)$.

(a) The **prism** over S is defined to be the simplicial complex

$$P(S) := \big\{\{i_0,\dots,i_{k-1},i'_k,\dots,i'_p\} : \\ k \leq p \text{ and } \{i_0,\dots,i_p\} \in S \text{ with } i_0 < \cdots < i_{k-1} \leq i_k < \cdots < i_p\big\},$$

with vertices labeled i and i' for $i \in V(S)$. It comes together with inclusion morphisms $f,g\colon S \to P(S)$ of the bottom resp. top face given by $f(i) = i$ and $g(i) = i'$ for all $i \in V(S)$.

(b) The associated **prism map** for $p \in \mathbb{Z}$ is the linear map $h_p \colon C_p(S) \to C_{p+1}(P(S))$ given on p-simplices $\{i_0,\dots,i_p\} \in S$ with $i_0 < \cdots < i_p$ by

$$h_p([i_0,\dots,i_p]) = \sum_{k=0}^{p} (-1)^k \cdot [i_0,\dots,i_k,i'_k,\dots,i'_p] \quad \in C_{p+1}(P(S)).$$

Lemma 3.8 (Simplicial homotopies)**.** *Let S be a simplicial complex with a fixed ordering of its set of vertices $V(S)$. With notations as in Definition 3.7, the prism map h is a homotopy between the morphisms of simplicial chain complexes induced by the bottom and top face inclusion of S in $P(S)$, i. e. we have*

$$\partial \circ h + h \circ \partial = g_* - f_*$$

as morphisms $C(S) \to C(P(S))$ of chain complexes. In particular, f_* and g_* agree as linear maps $H_p(S) \to H_p(P(S))$ on homology for all $p \in \mathbb{Z}$ by Proposition 3.3.

Proof. This is just a computation, and by our motivation from Construction 3.6 it should not be surprising that it works out: For any p-simplex $\{i_0,\dots,i_p\} \in S$ with $i_0 < \cdots < i_p$ we have

$$\begin{aligned} \partial h([i_0,\dots,i_p]) &= \partial \sum_{k=0}^{p} (-1)^k \cdot [i_0,\dots,i_k,i'_k,\dots,i'_p] \\ &= \sum_{k,l:\,l\le k} (-1)^{k+l} \cdot [i_0,\dots,\widehat{i_l},\dots,i_k,i'_k,\dots,i'_p] + \sum_{k,l:\,l\ge k} (-1)^{k+l+1} \cdot [i_0,\dots,i_k,i'_k,\dots,\widehat{i'_l},\dots,i'_p] \end{aligned}$$

and

$$\begin{aligned} h(\partial[i_0,\dots,i_p]) &= h\left(\sum_{l=0}^{p} (-1)^l \cdot [i_0,\dots,\widehat{i_l},\dots,i_p] \right) \\ &= \sum_{k,l:\,k<l} (-1)^{k+l} \cdot [i_0,\dots,i_k,i'_k,\dots,\widehat{i'_l},\dots,i'_p] + \sum_{k,l:\,k>l} (-1)^{k+l-1} \cdot [i_0,\dots,\widehat{i_l},\dots,i_k,i'_k,\dots,i'_p]. \end{aligned}$$

Note that in the sum of these two expressions all summands cancel out except the ones for $k=l$ (which are only present in the first expression), and thus we get as desired

$$\begin{aligned} \partial h([i_0,\dots,i_p]) + h(\partial[i_0,\dots,i_p]) &= \sum_{k=0}^{p} [i_0,\dots,i_{k-1},i'_k,\dots,i'_p] - \sum_{k=0}^{p} [i_0,\dots,i_k,i'_{k+1},\dots,i'_p] \\ &= [i'_0,\dots,i'_p] - [i_0,\dots,i_p] \\ &= g_*([i_0,\dots,i_p]) - f_*([i_0,\dots,i_p]). \end{aligned}$$

□

As a reality check, Lemma 3.8 is just the algebraic simplicial manifestation of the obvious topological statement that the bottom face of a prism can be deformed continuously to the top face by moving upwards. In particular, so far we have just considered the prism over a simplicial complex itself. Just as in the topological setting, to obtain useful results we now have to map this prism to another simplicial complex and use that the restrictions to the bottom and top face of the prism are homotopic.

In topology, the simplest application of this type is arguably the statement that a star-shaped set is contractible. Recall that a topological subspace $X \subset \mathbb{R}^n$ is called star-shaped with respect to $a \in X$ if for every point $x \in X$ the line segment connecting a and x lies in X as well. We can then construct a homotopy equivalence between X and the one-point space $\{a\}$ by moving every point $x \in X$ on a straight line to a. In fact, the same idea also works in the simplicial setting as follows.

Definition 3.9 (Star-shaped simplicial complexes)**.** A simplicial complex S is called **star-shaped** if there is a vertex $i \in V(S)$ such that $\alpha \cup \{i\} \in S$ for all $\alpha \in S$. We then say that S is star-shaped with respect to i.

Example 3.10. Let $\alpha = \{i_0,\dots,i_p\}$ be an arbitrary p-simplex.

(a) The disk D^α is clearly star-shaped (with respect to any of its vertices). In contrast, the sphere S^α is not star-shaped: For any $k \in \{0,\dots,p\}$ we have $\{i_0,\dots,\widehat{i_k},\dots,i_p\} \in S^\alpha$, but $\{i_0,\dots,\widehat{i_k},\dots,i_p\} \cup \{i_k\} \notin S^\alpha$.

(b) The barycentric subdivision $\mathrm{Sd}(D^\alpha)$ of the disk D^α as in Definition 1.20 is star-shaped with respect to its barycenter $\{i_0,\dots,i_p\}$ (but not with respect to any other of its vertices).

Proposition 3.11 (Betti numbers of star-shaped simplicial complexes)**.** *Any star-shaped simplicial complex S has the homology of a point, i. e. Betti numbers $b_0(S) = 1$ and $b_p(S) = 0$ for all $p \neq 0$. We will say that a simplicial complex with these Betti numbers has* ***trivial homology****.*

Proof. Let S be star-shaped with respect to $j \in S$. Consider the morphism $h\colon P(S) \to S$ given by $h(i) = i$ and $h(i') = j$ for all $i \in V(S)$, i. e. h is the identity on the bottom face and the constant map j on the top face of the prism $P(S)$. To see that this is in fact well-defined we have to show that the image of any simplex $\{i_0,\dots,i_{k-1},i'_k,\dots,i'_p\} \in P(S)$ lies in S. This is obvious if $k = 0$ or $k = p+1$. For $0 < k < p+1$ note that $\{i_0,\dots,i_p\} \in S$ by definition of $P(S)$, hence also $\{i_0,\dots,i_{k-1}\} \in S$, and thus

$$h(\{i_0,\dots,i_{k-1},i'_k,\dots,i'_p\}) = \{i_0,\dots,i_{k-1},j,\dots,j\} = \{i_0,\dots,i_{k-1}\} \cup \{j\} \in S$$

since S is star-shaped with respect to j. Thus, the map $h\colon P(S) \to S$ is in fact well-defined.

Now let $f,g\colon S \to P(S)$ be the inclusion of the bottom and top face as in Definition 3.7 (a). Then $h \circ f = \mathrm{id}_S$, and $h \circ g$ is the constant map j. Hence, for the pushforward maps $H_p(S) \to H_p(S)$ by these compositions we have for all $p \in \mathbb{Z}$

$$b_p(S) = \mathrm{rk}\,\mathrm{id}_{H_p(S)} = \mathrm{rk}(h_* \circ f_*)_p \overset{3.8}{=} \mathrm{rk}(h_* \circ g_*)_p.$$

But the pushforward $(h_* \circ g_*)_p\colon H_p(S) \to H_p(S)$ by the constant map j is just zero for $p \neq 0$ and $\alpha \mapsto (\deg \alpha) \cdot [j]$ of rank 1 for $p = 0$ (with the degree as in Lemma 2.22), so the result follows. □

Example 3.12 (Simplicial homology of disks)**.** By Example 3.10, Proposition 3.11 implies that for any p-simplex α the disk D^α and the subdivided disk $\mathrm{Sd}(D^\alpha)$ have trivial homology – as already mentioned in the introduction to this chapter and expected from topology (since the topological realization of both simplicial complexes is just the contractible disk D^p).

05

Example 3.13 (Simplicial homology of spheres)**.** Using Example 3.12, we can now also derive the simplicial homology of a p-dimensional sphere S^α for a $(p+1)$-simplex $\alpha = \{i_0,\dots,i_{p+1}\}$: Note that $D^\alpha = S^\alpha \cup \{\alpha\}$, so

$$\dim C_k(S^\alpha) = \begin{cases} \dim C_k(D^\alpha) & \text{if } k \neq p+1, \\ \dim C_k(D^\alpha) - 1 & \text{if } k = p+1. \end{cases} \tag{$*$}$$

A similar result holds for the simplicial homology: As the definition of $H_k(S^\alpha)$ only uses the part $C_{k+1}(S^\alpha) \to C_k(S^\alpha) \to C_{k-1}(S^\alpha)$ of the simplicial chain complex, and this agrees with that of D^α for $k \notin \{p,p+1,p+2\}$ by $(*)$, the resulting homology groups must then be the same. But S^α does not even have simplices of dimensions $p+1$ and $p+2$, so these Betti numbers are clearly zero – just as for D^α. Hence, the only Betti number of S^α that can differ from that of D^α is b_p, which means that we can obtain it easily from an Euler characteristic computation: By Lemma 2.25 we have

$$(-1)^p(b_p(S^\alpha) - b_p(D^\alpha)) = \chi(S^\alpha) - \chi(D^\alpha) = (-1)^{p+1} \underbrace{(\dim C_{p+1}(S^\alpha) - \dim C_{p+1}(D^\alpha))}_{=-1 \text{ by } (*)},$$

and thus $b_p(S^\alpha) = b_p(D^\alpha) + 1$. In summary, this means that for $p > 0$ we have

$$b_k(S^\alpha) = \begin{cases} 1 & \text{for } k = 0 \text{ or } k = p, \\ 0 & \text{otherwise.} \end{cases}$$

The case $p = 0$ is special and gives $b_0(S^\alpha) = 2$ and $b_k(S^\alpha) = 0$ otherwise by the same argument, but this should not be surprising since then $S^\alpha = \{\{i_0\},\{i_1\}\}$ is just a disjoint union of two points.

Staying in the case $p > 0$, it is also easy to write down a generator of the one-dimensional vector space $H_p(S^\alpha)$: The p-chain

$$\partial[i_0,\dots,i_{p+1}] = \sum_{k=0}^{p+1} (-1)^k \cdot [i_0,\dots,\widehat{i_k},\dots,i_{p+1}] \quad \in C_p(S^\alpha)$$

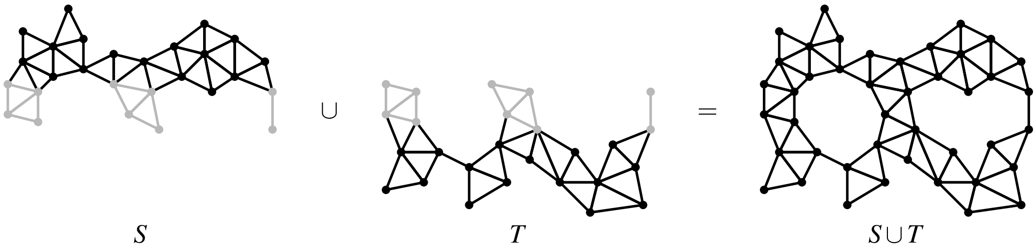
is closed by Lemma 2.12 and not a boundary as S^α has no $(p+1)$-simplices, so it must be a generator. Geometrically, it just corresponds to "the sphere S^α itself" considered as a p-cycle in S^α.

Note however that this generator is not entirely natural since a change of orientation of the vertices would replace it by its negative. This happens for example for the isomorphism $S^\alpha \to S^\alpha$ exchanging two of the vertices i_k and i_l with $k < l$, which in the standard geometric realization corresponds to a reflection at the hyperplane spanned by $e_0, \dots, \widehat{e_k}, \dots, \widehat{e_l}, \dots, e_p$, and $e_k + e_l$.

Exercise 3.14. Let $f, g \colon S \to T$ be two morphisms of simplicial complexes with $f(\alpha) \cup g(\alpha) \in T$ for any $\alpha \in S$. Show that the pushforwards f_* and g_* agree as maps in homology $H_p(S) \to H_p(T)$ for all $p \in \mathbb{Z}$.

4. The Mayer-Vietoris Sequence in Simplicial Homology

In the picture below, the number of holes in the graphs S and T (i. e. their genus in the language of Example 2.26) is $b_1(S) = 24$ and $b_1(T) = 20$, respectively, and the intersection $S \cap T$ is drawn in gray. Can you figure out from this information – without explicit counting – the number of holes in $S \cup T$, i. e. the Betti number $b_1(S \cup T)$?



This is not too hard: As a first approximation, the holes in $S \cup T$ are the ones in S or T, so we start with the sum $b_1(S) + b_1(T)$. But there are two corrections to this number:

(a) We have counted the holes in the gray part $S \cap T$ twice, so we have to subtract $b_1(S \cap T) = 4$.

(b) In addition to the small triangular holes, we have created two big holes in the "shape ∞" of $S \cup T$ since we glued S and T in three connected components – i. e. since $b_0(S \cap T)$ is 3 and not 1.

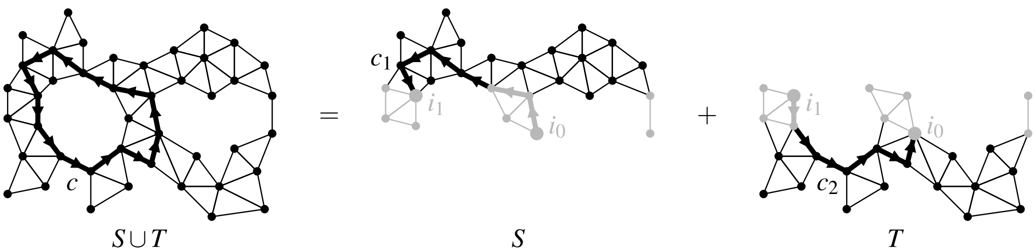
In summary, we thus obtain $b_1(S \cup T) = 24 + 20 - 4 + 2 = 42$ (an exact proof of this computation will be given in Example 4.13).

From a formal point of view, it might seem strange at first that a Betti number $b_1(S \cup T)$ in dimension 1 receives a contribution from a Betti number $b_0(S \cap T)$ in dimension 0; so let us try to understand this in more detail. Consider a hole in $S \cup T$ coming from (b) above, such as the 1-cycle c in the picture below on the left. We cannot write it as a hole in S or in T, but we can split it onto the two components in the following way (indicated in the picture on the right):

(1) Write it as a sum $c = c_1 + c_2$ for two 1-chains c_1 and c_2 in S and T, respectively.

(2) Take the boundaries of these chains; in our example case we have $\partial c_1 = [i_1] - [i_0]$ and $\partial c_2 = [i_0] - [i_1]$.

(3) Note that $\partial c_2 = \partial(c - c_1) = -\partial c_1$ as c is closed. Hence, ∂c_1 and ∂c_2 carry the same information, and they are clearly 0-cycles in $S \cap T$. Pick one of them, $\partial c_1 \in H_0(S \cap T)$ say, and assign it to c.



In fact, this assignment $c \mapsto \partial c_1$ gives us a well-defined map $h\colon H_1(S \cup T) \to H_0(S \cap T)$ in homology: We might have picked a different 1-cycle representing the same hole in $S \cup T$ (i. e. the same class in $H_1(S \cup T)$) as c, and we might have chosen different splitting points (e. g. instead of i_1 in the picture above we could have taken the vertex right below it), but the result would always be the class of one vertex in the left minus one vertex in the middle component of $S \cap T$ (i. e. the same homology class by Corollary 2.23). This map $h\colon H_1(S \cup T) \to H_0(S \cap T)$ is usually called the *connecting map* since it connects homology groups in different dimensions.

Note that as in Notation 2.19 there are also natural pushforward maps $H_1(S) \to H_1(S \cup T)$ and $H_1(T) \to H_1(S \cup T)$ by the inclusions $S \to S \cup T$ and $T \to S \cup T$. Adding them, we obtain a linear map $g\colon H_1(S) \times H_1(T) \to H_1(S \cup T)$, and thus together with the connecting map we have obtained a short "sequence" of maps

$$H_1(S) \times H_1(T) \xrightarrow{g} H_1(S \cup T) \xrightarrow{h} H_0(S \cap T) \qquad (*)$$

that we can interpret as in the geometric argument above by saying that the holes in $S \cup T$ (in the middle term) are related to the holes in S or T (in the left term) and the connected components of $S \cap T$ (in the right term).

But there is one last step missing: Of course, just that these homology groups are related by linear maps does not yet mean that the middle term $H_1(S \cup T)$ is actually is made up from contributions from the two sides – after all, the maps in the sequence $(*)$ could e. g. just be the zero maps. It turns out however that this sequence has a special property:

- Coming from the left term $H_1(S) \times H_1(T)$, i. e. from holes entirely in S or T, the construction of the connecting map above is actually trivial in the sense that we do not have to introduce splitting points at all, and thus the resulting class in $H_0(S \cap T)$ will be zero. More formally, this means that $h \circ g = 0$, or that $\operatorname{im} g \subset \ker h$. In other words, we have a short version of a chain complex as in Definition 2.13.
- Conversely, if we have a cycle $c \in \ker h$, i. e. a cycle for which the connecting map does not see any splitting onto the connected components of $S \cap T$, it is intuitive that c must come from holes already present in S and T, i. e. that $c \in \operatorname{im} g$.

So altogether we see that $\operatorname{im} g = \ker h$ in $(*)$. Sequences with this property will be called *exact*, and they allow us to think about their maps as giving contributions to their vector spaces: Considering an element $c \in H_1(S \cup T)$, it does *not* give rise to anything in $H_0(S \cap T)$ (i. e. we have $c \in \ker h$) *if and only if* it comes from something in $H_1(S) \times H_1(T)$ (i. e. we have $c \in \operatorname{im} g$). This is precisely what it means to say that $H_1(S \cup T)$ "is made up from contributions from $H_1(S) \times H_1(T)$ and $H_0(S \cap T)$".

In fact, we can also interpret this in terms of the "lifespan" of elements in the sequence $(*)$ in the sense of Remark 2.11: In contrast to chain complexes, which just say that every element in the sequence survives at most two stages in the sequence, an exact sequence means that every element survives *exactly* two stages. So every element in the middle of $(*)$ comes from the left or from the right, but not both. We will pursue this perspective further in Example 9.14.

As you might already suspect, the goal of this chapter is to generalize all this and make it precise. We will see that for any two simplicial complexes S and T we can compute the homology of $S \cup T$ from that of S, T, and $S \cap T$; and to compute the homology in dimension p of $S \cup T$ for some $p \in \mathbb{Z}$ we do not only need the homology of S, T, and $S \cap T$ in dimension p, but there will also be connecting maps giving rise to contributions from dimension $p-1$.

Just as for homology and homotopy in the two preceding chapters, exact sequences in homology are a purely algebraic concept that we will encounter in many more cases than just for simplicial complexes. So let us now start the chapter again with some algebra. Generalizing sequences such as $(*)$ to arbitrary dimension of the chains will make them into sequences whose terms are actually chain complexes themselves and not just vector spaces, so let us start by introducing this concept analogously to Definition 2.8.

Definition 4.1 (Sequences of chain complexes)**.** A **sequence of chain complexes** is a collection of chain complexes $(C^{(k)}, \partial^{(k)})_{k \in \mathbb{Z}}$ and morphisms $f^{(k)}\colon C^{(k)} \to C^{(k+1)}$ between them for all $k \in \mathbb{Z}$:

$$\cdots \longrightarrow C^{(k-1)} \xrightarrow{f^{(k-1)}} C^{(k)} \xrightarrow{f^{(k)}} C^{(k+1)} \longrightarrow \cdots.$$

By Definition 2.8 we could write this as a commutative diagram of vector spaces and morphisms as follows, with the chain complexes $C^{(k)}$ running down.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & C_p^{(k-1)} \arrow[r, "f_p^{(k-1)}"] \arrow[d, "\partial_p^{(k-1)}"] & C_p^{(k)} \arrow[r, "f_p^{(k)}"] \arrow[d, "\partial_p^{(k)}"] & C_p^{(k+1)} \arrow[r] \arrow[d, "\partial_p^{(k+1)}"] & \cdots \\
\cdots \arrow[r] & C_{p-1}^{(k-1)} \arrow[r, "f_{p-1}^{(k-1)}"] \arrow[d] & C_{p-1}^{(k)} \arrow[r, "f_p^{(k)}"] \arrow[d] & C_{p-1}^{(k+1)} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Often the chain complexes $C^{(k)}$ will be zero outside a small range for k, in which case we will only write this finite part of the sequence. Moreover, as in Definition 2.8, in the notation we will often drop the lower index that indicates the position within each chain complex. However, in order to avoid confusion we will always distinguish between the individual complexes and the maps between them in the notation, either by an upper index or by using different letters. Also, we will always write the individual chain complexes vertically in the columns, and the different chain complexes horizontally in the rows.

Remark 4.2. It might seem strange that we chose the index in a sequence to count downward in Definition 2.8, but upward in Definition 4.1. The reason for this is that counting upward is probably the more natural choice, but the fact that the boundary map in homology reduces dimension made the downward counting more suggestive in Chapter 2. Of course, at the end of the day it does not matter whether we count up or down, and in fact in the next chapter when discussing cohomology we will also encounter boundary maps that increase the dimension.

Example 4.3 (The simplicial Mayer-Vietoris sequence)**.** Our main example of a sequence of chain complexes in this chapter is the following: Let S and T be simplicial complexes. Then there is a three-term sequence of simplicial chain complexes $C(S\cap T)\to C(S)\times C(T)\to C(S\cup T)$ which in each row p is just given by

$$\begin{array}{ccccc}
C_p(S\cap T) & \longrightarrow & C_p(S)\times C_p(T) & \longrightarrow & C_p(S\cup T) \\
c & \longmapsto & (c,-c) & & \\
 & & (c,c') & \longmapsto & c+c'.
\end{array}$$

We call it the **simplicial Mayer-Vietoris sequence** for S and T. The reason for the sign in the first map is that the composition of the two chain maps is zero this way – this makes the sequence into a "chain complex of chain complexes". In fact, it is our first example of an exact sequence already mentioned in the introduction to this chapter:

Definition 4.4 (Exact sequences)**.**

(a) A sequence of vector spaces

$$\cdots \longrightarrow C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \longrightarrow \cdots$$

is called **exact** if $\ker\partial_p = \operatorname{im}\partial_{p+1}$ for all $p\in\mathbb{Z}$.

(b) A sequence of chain complexes

$$\cdots \longrightarrow C^{(k-1)} \xrightarrow{f^{(k-1)}} C^{(k)} \xrightarrow{f^{(k)}} C^{(k+1)} \longrightarrow \cdots$$

is called exact if each of its rows as in Definition 4.1

$$\cdots \longrightarrow C_p^{(k-1)} \xrightarrow{f_p^{(k-1)}} C_p^{(k)} \xrightarrow{f_p^{(k)}} C_p^{(k+1)} \longrightarrow \cdots$$

for $p \in \mathbb{Z}$ is exact in the sense of (a).

Remark 4.5.

(a) By definition, a sequence of vector spaces is exact if and only if it is a chain complex with all homology groups being zero.

(b) If $(C_p)_{p\in\mathbb{Z}}$ is an exact sequence of vector spaces, it follows directly from (a) that its Euler characteristic is $\chi(C) = 0$. Hence, if moreover all C_p are finite-dimensional and almost all of them are zero, Lemma 2.25 implies that $\sum_{p\in\mathbb{Z}}(-1)^p \dim C_p = 0$.

Example 4.6 (Short exact sequences)**.** As already mentioned in the introduction to this chapter, an exact sequence of vector spaces can be thought of as a sequence in which "every term is made up from contributions from its left and right neighbor". This is particularly evident for sequences that are short and/or contain zero vector spaces at their ends:

(a) A sequence $0 \to A \xrightarrow{f} B$ of vector spaces (with the first map of course being the zero map) is exact if and only if f is injective. Intuitively, as A cannot obtain any contribution from the left, all of A must also be contained in B. Similarly, a sequence $A \xrightarrow{f} B \to 0$ is exact if and only if f is surjective (all of B must come from A).

(b) It follows immediately from (a) that an exact sequence of vector spaces $0 \to A \to 0$ means that $A = 0$, and that an exact sequence $0 \to A \xrightarrow{f} B \to 0$ means that f is an isomorphism.

(c) The first non-trivial case is an exact sequence of vector spaces $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ with three arbitrary terms, bounded by 0; they are usually referred to as **short exact sequences**. In this case, Remark 4.5 (b) implies that $\dim B = \dim A + \dim C$ if all vector spaces are finite-dimensional, which again confirms the idea that B is made up from contributions from A and C. More naturally, f being injective means that A can be regarded as a subspace of B, and by the homomorphism theorem we have $C = \operatorname{im} g \cong B/\ker g = B/\operatorname{im} f = B/A$.

In general, not all exact sequences that we will encounter are of these types; in fact, many of them are rather long. However, most of the terms in these sequences will then be 0, and if there are at most three terms between two zero entries we can then still apply the reasoning of (b) or (c) to this part of the sequence.

Example 4.7 (The simplicial Mayer-Vietoris sequence is exact)**.** The simplicial Mayer-Vietoris sequence of Example 4.3 is in fact a short exact sequence of chain complexes

$$0 \longrightarrow C(S\cap T) \longrightarrow C(S)\times C(T) \longrightarrow C(S\cup T) \longrightarrow 0$$

for any two simplicial complexes S and T. This follows directly from the definitions: Clearly, any p-chain in $C_p(S\cup T)$ can be written as $c+c'$ for $c \in C_p(S)$ and $c' \in C_p(T)$, and such an expression $c+c'$ is zero if and only if $c' = -c$, which is only possible for $c = -c' \in C_p(S\cap T)$.

Of course, we now want to use this sequence of simplicial chain complexes to compute their homology. This is again a purely algebraic process that we will often see in these notes:

Proposition 4.8 (The long exact homology sequence)**.** *For any short exact sequence*

$$0 \longrightarrow A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \longrightarrow 0$$

of chain complexes there are linear connecting maps $h\colon H_p(C) \to H_{p-1}(A)$ for all $p \in \mathbb{Z}$ giving rise to an induced long exact sequence in homology

$$\cdots \longrightarrow H_p(A) \xrightarrow{f} H_p(B) \xrightarrow{g} H_p(C) \xrightarrow{h} H_{p-1}(A) \xrightarrow{f} H_{p-1}(B) \xrightarrow{g} H_{p-1}(C) \xrightarrow{h} H_{p-2}(A) \longrightarrow \cdots.$$

Proof. The central part of the proof is the construction of the connecting maps $h\colon H_p(C) \to H_{p-1}(A)$ for $p \in \mathbb{Z}$. For their definition, we proceed analogously to the steps (1), (2), (3) in the introduction to this chapter for the Mayer-Vietoris sequence of Example 4.3, following the dashed arrow in the diagram below. Elements in the vector spaces not on this dashed path are marked with the numbers in the gray circles whenever they appear. Start with a closed chain $c_p \in C_p$.

(1) The map $g\colon B_p \to C_p$ is surjective, so we can choose $b_p \in B_p$ with $g(b_p) = c_p$.

(2) Take the boundary map of B to obtain $b_{p-1} := \partial b_p \in B_{p-1}$.

(3) Note that $g(b_{p-1}) = g(\partial b_p) = \partial g(b_p) = \partial c_p = 0 \in C_{p-1}$ since c_p is closed. 1 As the row $p-1$ is exact, we therefore have $b_{p-1} = f(a_{p-1})$ for a unique $a_{p-1} \in A_{p-1}$.

Set $h(c_p) := a_{p-1}$.

$$\begin{tikzcd}
 & & B_{p+1}^{\text{5}} \arrow[r, "g"] \arrow[d, "\partial"] & C_{p+1}^{\text{4}} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_p^{\text{6}} \arrow[r, "f"] \arrow[d, "\partial"] & B_p \arrow[r, "g"] \arrow[d, "\partial"] & C_p \arrow[r] \arrow[d, "\partial"] \arrow[dll, dashed, bend right] & 0 \\
0 \arrow[r] & A_{p-1} \arrow[r, "f"'] \arrow[d, "\partial"] & B_{p-1} \arrow[r, "g"'] \arrow[d, "\partial"] & C_{p-1}^{\text{1}} \arrow[r] & 0 \\
0 \arrow[r] & A_{p-2}^{\text{3}} \arrow[r, "f"'] & B_{p-2}^{\text{2}} & &
\end{tikzcd}$$

Let us show that this map $h\colon H_p(C) \to H_{p-1}(A)$ is actually well-defined:

- a_{p-1} is closed: We have $f(\partial a_{p-1}) = \partial f(a_{p-1}) = \partial b_{p-1} = \partial^2 b_p = 0 \in B_{p-2}$, 2 which means that $\partial a_{p-1} = 0 \in A_{p-2}$ since f is injective. 3
- $a_{p-1} \in H_{p-1}(A)$ is independent of choices: Let $c'_p \in C_p$ be another representative of the same class as c_p, and $b'_p \in B_p$ with $g(b'_p) = c'_p$. Correspondingly, set $b'_{p-1} = \partial b'_p$ and $a'_{p-1} \in A_{p-1}$ with $f(a'_{p-1}) = b'_{p-1}$.

  Then $c_p - c'_p = \partial c$ for some $c \in C_{p+1}$. 4 As g is surjective we can choose $b \in B_{p+1}$ with $g(b) = c$. 5 We obtain
  $$g(b_p - b'_p - \partial b) = c_p - c'_p - \partial g(b) = c_p - c'_p - \partial c = 0,$$
  so $b_p - b'_p - \partial b = f(a)$ for some $a \in A_p$ since row p is exact. 6 Hence,
  $$f(a_{p-1} - a'_{p-1}) = b_{p-1} - b'_{p-1} = \partial(b_p - b'_p) = \partial(f(a) + \partial b) = \partial f(a) + \underbrace{\partial^2 b}_{=0} = f(\partial a),$$
  which by injectivity of f means that $a_{p-1} - a'_{p-1} = \partial a$ is a boundary (i. e. zero in homology).

For obvious reasons, the technique that we applied here is usually called *diagram chasing*.

It now remains to prove that the connecting maps h give rise to an exact homology sequence as stated in the proposition. In fact, it is obvious from the constructions that this homology sequence is at least a chain complex:

- $g \circ f$ is zero on homology since this already holds for the initial chain complexes.
- $h \circ g$ is zero on homology: Applying the construction of h above to $g(b_p)$ for a *cycle* b_p means that $\partial b_p = 0$, so we will get 0 in step (2).
- $f \circ h$ is zero on homology: Applying f to the cycle a_{p-1} in the construction above gives b_{p-1}, which is a boundary by (2), and hence zero in homology.

The proof that the long homology sequence is actually exact, i. e. that also the kernel of any of its maps is included in the image of the previous map, is somewhat long and tedious (it is a long sequence after all). But it is just straightforward diagram-chasing, and we will also see a visual explanation for this statement later in Remark 9.15, so we will omit the proof here and leave it as an exercise. □

Exercise 4.9. In the situation of Proposition 4.8, show that in the long homology sequence the kernel of any map is included in the image of the previous map, thus completing the proof of the proposition.

06

Construction 4.10 (The long exact homology sequence is functorial)**.** As one would probably expect, the long exact homology sequence of Proposition 4.8 is functorial in the following sense: Consider a *morphism of short exact sequences of chain complexes*, by which of course we mean a commutative diagram of chain complexes

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "k^{(1)}"] & B \arrow[r, "g"] \arrow[d, "k^{(2)}"] & C \arrow[r] \arrow[d, "k^{(3)}"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0 \end{tikzcd}$$

with exact rows (note that this means that the vertical maps $k^{(i)}$ for $i \in \{1,2,3\}$ commute both with f and g as well as with the boundary maps of the chain complexes). Then the rows of this diagram induce a long exact homology sequence as in Proposition 4.8, and the columns pass to maps in homology by Lemma 2.17. We claim that these maps fit into a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & H_p(A) \arrow[r, "f"] \arrow[d, "k^{(1)}"] & H_p(B) \arrow[r, "g"] \arrow[d, "k^{(2)}"] & H_p(C) \arrow[r, "h"] \arrow[d, "k^{(3)}"] & H_{p-1}(A) \arrow[r] \arrow[d, "k^{(1)}"] & \cdots \\ \cdots \arrow[r] & H_p(A') \arrow[r, "f'"] & H_p(B') \arrow[r, "g'"] & H_p(C') \arrow[r, "h'"] & H_{p-1}(A') \arrow[r] & \cdots \end{tikzcd}$$

with exact rows, where h and h' are the connecting maps (i. e. into a morphism of exact sequences of vector spaces). In fact, the commutativity in the squares involving f, f', g, and g' is just the usual functoriality of homology in Lemma 2.17, so it only remains to prove that $h'(k^{(3)}(c_p)) = k^{(1)}(h(c_p))$ for all cycles $c_p \in C_p$. But this can easily be seen by following the steps (1), (2), (3) in the proof of Proposition 4.8 to construct the connecting map h' for the element $k^{(3)}(c_p) \in C'_p$, using the notations as in the proof:

(1) We can choose $k^{(2)}(b_p) \in B'_p$ since $g'(k^{(2)}(b_p)) = k^{(3)}(g(b_p)) = k^{(3)}(c_p)$.

(2) Applying the boundary map gives $\partial k^{(2)}(b_p) = k^{(2)}(\partial b_p) = k^{(2)}(b_{p-1}) \in B'_{p-1}$.

(3) We have $k^{(2)}(b_{p-1}) = k^{(2)}(f(a_{p-1})) = f'(k^{(1)}(a_{p-1}))$, so the construction gives that h' maps $k^{(3)}(c_p)$ to $k^{(1)}(a_{p-1}) = k^{(1)}(h(c_p)) \in A'_{p-1}$ as required.

Corollary 4.11. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of chain complexes for which all homology groups are finite-dimensional, and almost all of them are zero. Then $\chi(B) = \chi(A) + \chi(C)$.*

Proof. This follows from Remark 4.5 (b) applied to the long exact homology sequence of Proposition 4.8. □

Corollary 4.12 (The simplicial Mayer-Vietoris homology sequence)**.** *For any two simplicial complexes S and T, the sequence*

$$\cdots \longrightarrow H_p(S \cap T) \longrightarrow H_p(S) \times H_p(T) \longrightarrow H_p(S \cup T) \longrightarrow H_{p-1}(S \cap T) \longrightarrow \cdots$$

(with morphisms as in Example 4.3) is exact. In particular, we have

$$\chi(S \cup T) + \chi(S \cap T) = \chi(S) + \chi(T).$$

Proof. The long exact sequence follows from Proposition 4.8 applied to the Mayer-Vietoris sequence of Example 4.7, the statement about the Euler numbers then from Corollary 4.11. □

Example 4.13 (Gluing graphs)**.** We are now ready to give a proof for the intuitive computation at the beginning of this chapter: Let S and T be two graphs, and assume for simplicity that S, T, and $S \cup T$ are non-empty and connected (so that $b_0(S) = b_0(T) = b_0(S \cup T) = 1$ by Corollary 2.23). Then Corollary 4.12 implies that the first Betti number of $S \cup T$ is

$$\begin{aligned} b_1(S \cup T) &= 1 - \chi(S \cup T) = 1 - \chi(S) - \chi(T) + \chi(S \cap T) \\ &= b_1(S) + b_1(T) - b_1(S \cap T) + b_0(S \cap T) - 1. \end{aligned}$$

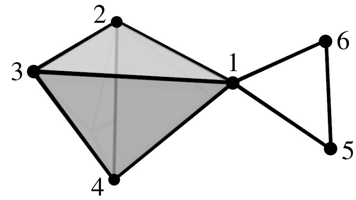
For the graphs in the introduction to this chapter we had $b_1(S \cup T) = 24 + 20 - 4 + (3 - 1) = 42$.

We could also obtain this result from the exact Mayer-Vietoris homology sequence directly: In our case of a simplicial complex of dimension 1, this sequence reads

$$0 \to H_1(S\cap T) \to H_1(S)\times H_1(T) \to H_1(S\cup T) \to H_0(S\cap T) \to H_0(S)\times H_0(T) \to H_0(S\cup T) \to 0.$$

Using Remark 4.5 (b), it leads to the same dimension computation for $b_1(S\cup T)$ as before, and the geometric interpretation of the connecting map $H_1(S\cup T) \to H_0(S\cap T)$ has already been given in the introduction.

Construction 4.14 (Wedge sums)**.** Let S and T be two simplicial complexes that share exactly one vertex, such as e. g. $S = S^{\{1,2,3,4\}}$ and $T = S^{\{1,5,6\}}$ in the picture on the right. Their union $S\cup T$ is then usually called a *wedge sum* of the two simplicial complexes. Let us assume for simplicity that S and T are connected, so that $S\cup T$ is clearly connected as well, and we are only interested in the Betti numbers $b_p(S\cup T)$ for $p > 0$.



As we have $H_p(S\cap T) = 0$ for $p > 0$, the long exact Mayer-Vietoris homology sequene of Corollary 4.12 contains exact parts

$$0 \longrightarrow H_p(S)\times H_p(T) \longrightarrow H_p(S\cup T) \longrightarrow 0 \tag{1}$$

for $p > 1$. In contrast, for $p = 1$ this exact sequence continues as

$$0 \longrightarrow H_1(S)\times H_1(T) \longrightarrow H_1(S\cup T) \longrightarrow H_0(S\cap T).$$

But the last map here is injective by Corollary 2.23, so the previous map must be the zero map since the sequence is exact. This means that we can equally well end this exact sequence with $\cdots \to H_1(S\cup T) \to 0$, so that (1) holds for $p = 1$ as well. By Example 4.6 (b) this means that $H_p(S)\times H_p(T) \cong H_p(S\cup T)$, and thus

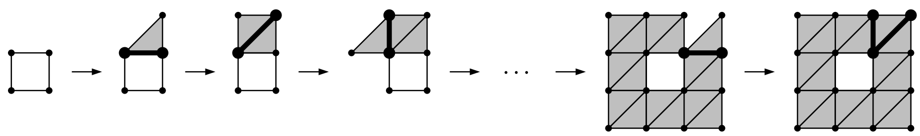
$$b_p(S\cup T) = b_p(S) + b_p(T) \quad \text{for all } p > 0. \tag{2}$$

So, in the example of the picture above, we would have $b_2(S\cup T) = b_1(S\cup T) = b_0(S\cup T) = 1$ by Example 3.13, with all other Betti numbers being zero.

Let us discuss two interesting consequences of this result:

(a) For any given collection of numbers $b_p \in \mathbb{N}$ for $p \in \mathbb{N}$, with $b_0 > 0$ and only finitely many being non-zero, there is a simplicial complex S that has these Betti numbers for any ground field: By Example 3.13, we can just take the wedge sum of b_p copies of a simplicial p-sphere for $p > 0$, and form the disjoint union with $b_0 - 1$ isolated vertices to reach b_0 as well.

(b) Obviously, the result (2) holds with the same argument also if $S\cap T$ is not just a single vertex, but has the same trivial homology as a single vertex. Specializing to the case when T has trivial homology as well, we see that passing from S to $S\cup T$ if both T and $S\cap T$ have trivial homology (e. g. for star-shaped simplicial complexes as in Proposition 3.11) does not change homology. For example, in the following picture every simplicial complex is obtained from the previous one by attaching a simplicial 2-disk along a star-shaped subcomplex (drawn in bold), which gives a rigorous proof (without even looking at the actual simplicial chain complexes) that all these spaces have isomorphic homology. In particular, the simplicial annulus on the right has the same Betti numbers as the graph on the left, namely zeroth and first Betti number 1 and all others zero. Of course, this fits well with the geometric intuition all the spaces in the picture just have one hole.



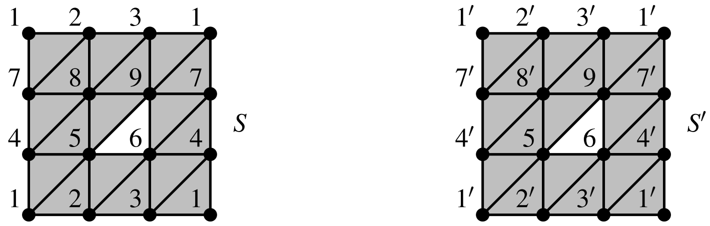
Exercise 4.15. Let S be a simplicial complex, and let $*$ and $\circ$ be new vertices not contained in S. We construct the new simplicial complex

$$\Sigma(S) := \{\{*\},\{\circ\}\} \cup \{\alpha, \alpha\cup\{*\}, \alpha\cup\{\circ\} : \alpha \in S\}.$$

(a) Compute the Betti numbers of $\Sigma(S)$ in terms of those of S.

(b) Starting with the 0-dimensional simplicial complex $S = \{\{0\},\{1\}\}$, which well-known topological space is the topological realization of the n-fold construction $\Sigma^n(S)$ of (a) for $n \in \mathbb{N}$?

Exercise 4.16. Let S and S' be the two simplicial complexes in the picture below, obtained by removing a 2-simplex from a torus as in Example 1.17 (c), and such that the boundary of this 2-simplex agrees in S and S'.



(a) Compute bases of the homology groups $H_p(S)$ for $p = 0,1,2$.

(b) Compute bases of the homology groups $H_p(S \cup S')$ for $p = 0,1,2$.

Exercise 4.17. Suppose that a simplicial complex S is the union of subcomplexes $S_1,\dots,S_n$ with $n \geq 2$ such that every intersection of any number of these subcomplexes is either empty or has the homology of a point.

(a) Show that $b_p(S) = 0$ for all $p \geq n-1$.

(b) Show for all n that the inequality of (a) cannot be improved.

There is one more important thing that we can see from Construction 4.14 (b): The Mayer-Vietoris sequence cannot only be used to simplify computations, but also opens up the path for inductive proofs of statements about the homology of simplicial complexes by writing them as unions of smaller subcomplexes. The way in which we usually use this kind of argument is the following.

Lemma 4.18 (Simplicial Mayer-Vietoris induction)**.** *Suppose that we want to prove an assertion about all simplicial complexes. Then it suffices to show:*

(a) *(Induction start) The assertion holds for the empty simplicial complex and any simplicial disk D^α.*

(b) *(Induction step) If S and T are simplicial complexes such that the assertion holds for S, T, and $S\cap T$, then it also holds for $S \cup T$.*

Proof. We show that the assertion holds for any simplicial complex S by induction on the number of its simplices. For $S = \emptyset$ we know that the assertion holds by (a), so let us assume that S is non-empty. Let α be a simplex of maximum dimension in S. Then D^α is a subcomplex of S by Definition 1.2 (a), and $S\backslash\{\alpha\}$ is a subcomplex as well. If $S = D^\alpha$ the assertion again holds by (a), so we may assume that $D^\alpha \subsetneq S$. But then $S = (S\backslash\{\alpha\}) \cup D^\alpha$ is a union of subcomplexes with fewer simplices, so by induction the assertion holds for $S\backslash\{\alpha\}$, D^α, and $(S\backslash\{\alpha\}) \cap D^\alpha$. Hence the assertion holds for the union S as well by (b). □

Of course, the reason why this is called Mayer-Vietoris induction is that the induction step (b) in Lemma 4.18 will usually be proved using the long exact Mayer-Vietoris sequence. But there is one last problem before we can actually use this strategy in concrete cases: The Mayer-Vietoris sequence does not actually compute the homology of $S \cup T$ directly from that of S, T, and $S \cap T$, but only includes it in a long exact sequence in which every third term is a homology group of $S\cup T$, and the other terms are known. How can we use this information to extract statements about the homology of $S \cup T$? In fact, most of our statements will be that the homology of a simplicial complex is isomorphic to something else, and for such statements the following lemma solves this problem when we apply it to long exact Mayer-Vietoris homology sequences.

Lemma 4.19 (5-Lemma)**.** *Consider a commutative diagram*

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & A_3 \arrow[r, "f_3"] \arrow[d, "h_3"] & A_4 \arrow[r, "f_4"] \arrow[d, "h_4"] & A_5 \arrow[d, "h_5"] \\
B_1 \arrow[r, "g_1"] & B_2 \arrow[r, "g_2"] & B_3 \arrow[r, "g_3"] & B_4 \arrow[r, "g_4"] & B_5
\end{tikzcd}$$

of vector spaces with exact rows. If h_1, h_2, h_4, and h_5 are isomorphisms, then so is h_3.

Proof. We will show that h_3 is injective and surjective. Again, this is just tedious but straightforward diagram chasing. In both parts of the proof, the gray element in the diagram is the element to be chased, and the numbers in the gray circles denote the order of the chase.

(a) h_3 is injective: Let $a_3 \in A_3$ 1 with $h_3(a_3) = 0$. 2 Then $g_3(h_3(a_3)) = 0$, 3 so $h_4(f_3(a_3)) = 0$, and since h_4 is injective we have $f_3(a_3) = 0$. 4 As the top row is exact, this means $a_3 = f_2(a_2)$ for some $a_2 \in A_2$. 5 Let $b_2 = h_2(a_2)$, 6 then $g_2(b_2) = g_2(h_2(a_2)) = h_3(f_2(a_2)) = h_3(a_3) = 0$. As the bottom row is exact, it follows that $b_2 = g_1(b_1)$ for some $b_1 \in B_1$. 7 Now h_1 is surjective, so $b_1 = h_1(a_1)$ for some $a_1 \in A_1$. 8 This element satisfies $f_1(a_1) = a_2$, since $h_2(f_1(a_1)) = g_1(h_1(a_1)) = g_1(b_1) = b_2 = h_2(a_2)$ and h_2 is injective. But $f_2 \circ f_1 = 0$, so $a_3 = f_2(f_1(a_1)) = 0$.

$$\begin{tikzcd}
{}^{a_1\,8}A_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & {}^{a_2\,5}A_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & {}^{a_3\,1}A_3 \arrow[r, "f_3"] \arrow[d, "h_3"] & {}^{0\,4}A_4 \arrow[r, "f_4"] \arrow[d, "h_4"] & A_5 \arrow[d, "h_5"] \\
{}^{b_1\,7}B_1 \arrow[r, "g_1"] & {}^{b_2\,6}B_2 \arrow[r, "g_2"] & {}^{0\,2}B_3 \arrow[r, "g_3"] & {}^{0\,3}B_4 \arrow[r, "g_4"] & B_5
\end{tikzcd}$$

(b) h_3 is surjective: Let $b_3 \in B_3$, 1 and set $b_4 = g_3(b_3)$. 2 Then $g_4(b_4) = g_4(g_3(b_3)) = 0$ since $g_4 \circ g_3 = 0$. 3 As h_4 is surjective, we have $b_4 = h_4(a_4)$ for some $a_4 \in A_4$. 4 Then $h_5(f_4(a_4)) = g_4(h_4(a_4)) = g_4(b_4) = 0$, hence $f_4(a_4) = 0$ as h_5 is injective. 5 As the top row is exact, we have $a_4 = f_3(a_3)$ for some $a_3 \in A_3$. 6 Be do not necessarily have $b_3 = h_3(a_3)$, but $g_3(b_3 - h_3(a_3)) = g_3(b_3) - g_3(h_3(a_3)) = b_4 - h_4(f_3(a_3)) = b_4 - h_4(a_4) = b_4 - b_4 = 0$, so as the bottom row is exact we can write $b_3 - h_3(a_3) = g_2(b_2)$ for some element $b_2 \in B_2$. 7 Now we have $b_2 = h_2(a_2)$ for some $a_2 \in A_2$ since h_2 is surjective. 8 It follows that $h_3(a_3 + f_2(a_2)) = h_3(a_3) + h_3(f_2(a_2)) = h_3(a_3) + g_2(h_2(a_2)) = h_3(a_3) + g_2(b_2) = b_3$, i. e. h_3 is surjective.

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & {}^{8}A_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & {}^{a_3\,6}A_3 \arrow[r, "f_3"] \arrow[d, "h_3"] & {}^{a_4\,4}A_4 \arrow[r, "f_4"] \arrow[d, "h_4"] & {}^{0\,5}A_5 \arrow[d, "h_5"] \\
B_1 \arrow[r, "g_1"] & {}^{7}B_2 \arrow[r, "g_2"] & {}^{b_3\,1}B_3 \arrow[r, "g_3"] & {}^{b_4\,2}B_4 \arrow[r, "g_4"] & {}^{0\,3}B_5
\end{tikzcd}$$

□

Remark 4.20. The 5-Lemma is clearly an important statement that will make our Mayer-Vietoris induction arguments work. But as it is easy to misread this statement it is equally important to understand what the 5-Lemma *does not* say:

(a) The 5-Lemma does not say that a diagram as in Lemma 4.19 in which the middle vertical map f_3 is initially missing (commutative, with exact rows, and the other vertical maps being isomorphisms) can be filled in uniquely to the complete commutative with an isomorphism f_3. Instead, it is important that we must have the map f_3 already beforehand; the 5-Lemma then just ensures that it is an isomorphism. For example, the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{R} & \xrightarrow{f_2} & \mathbb{R}^2 & \xrightarrow{f_3} & \mathbb{R} & \longrightarrow & 0 \\
\| & & \| & & & & \| & & \| \\
0 & \longrightarrow & \mathbb{R} & \xrightarrow{f_2} & \mathbb{R}^2 & \xrightarrow{f_3} & \mathbb{R} & \longrightarrow & 0
\end{array}$$

with $f_2\colon x_1 \mapsto \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ and $f_3 \mapsto \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_2$ can be filled in with a middle vertical isomorphism $\mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + \lambda x_2 \\ x_2 \end{pmatrix}$ for any $\lambda \in \mathbb{R}$ to form a commutative diagram.

(b) Without the vertical maps being given, it is not even true that the dimensions of the outer four vector spaces in a five-term exact sequence determine the dimension of the middle space: For example, the sequences

$$\begin{aligned}
&\mathbb{R} \xrightarrow{\mathrm{id}} \mathbb{R} \xrightarrow{0} \mathbb{R} \xrightarrow{\mathrm{id}} \mathbb{R} \xrightarrow{0} \mathbb{R} \\
\text{and}\quad &\mathbb{R} \xrightarrow{\mathrm{id}} \mathbb{R} \xrightarrow{0} 0 \xrightarrow{0} \mathbb{R} \xrightarrow{\mathrm{id}} \mathbb{R}
\end{aligned}$$

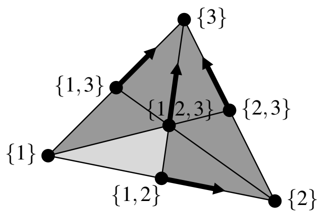
are both exact (but by the 5-Lemma there is no way to map one to the other so that the resulting diagram is commutative and the outer four maps are isomorphisms – which is also easy to see directly).

In short, it is vital for the application of the 5-Lemma that all vertical maps are known to exist and form a commutative diagram beforehand.

After this preparation, we want to finish this chapter with two interesting and clearly non-trivial statements that we can now prove by Mayer-Vietoris induction. The first is the statement that barycentric subdivision does not change the homology of a simplicial complex – which is clearly expected since by Proposition 1.21 the topological realization does not change. We have just learned in Remark 4.20 that for the Mayer-Vietoris argument to work it will be important to have a morphism between the two spaces to start with, so let us construct this first.

Construction 4.21. Let S be a simplicial complex. For any simplex $\alpha \in S$, pick an arbitrary element $f(\alpha) \in \alpha$. This defines a morphism of simplicial complexes $f\colon \mathrm{Sd}(S) \to S$: For any p-simplex $\{\alpha_0, \dots, \alpha_p\} \in \mathrm{Sd}(S)$ with $\alpha_0 \subsetneq \cdots \subsetneq \alpha_p \in S$ we have $f(\alpha_k) \in \alpha_k \subset \alpha_p$ for all $k \in \{0, \dots, p\}$, hence $f(\{\alpha_0, \dots, \alpha_p\})$ is a subset of α_p, and thus a simplex in S.

The picture on the right shows this map by arrows in the example of the 2-disk $S = D^{\{1,2,3\}}$ and the morphism $f\colon \mathrm{Sd}(S) \to S$ given on vertices of $\mathrm{Sd}(S)$ by $f(\alpha) := \max \alpha$. The light-colored 2-simplex $\{\{1\},\{1,2\},\{1,2,3\}\} \in \mathrm{Sd}(S)$ is the only one that maps to a 2-simplex $\{1,2,3\} \in S$; all other 2-simplices of $\mathrm{Sd}(S)$ are contracted to the boundary of the triangle.



Proposition 4.22 (Invariance of simplicial homology under barycentric subdivision)**.** *For every simplicial complex S, any morphism $f\colon \mathrm{Sd}(S) \to S$ as in Construction 4.21 induces an isomorphism $f_*\colon H_p(\mathrm{Sd}(S)) \to H_p(S)$ for all $p \in \mathbb{Z}$.*

Proof. We will prove the assertion using Mayer-Vietoris induction as in Lemma 4.18. For the induction start, note that the statement is obvious for $S = \emptyset$. For a simplicial disk, we know by Example 3.12 that $b_p(\mathrm{Sd}(S)) = b_p(S)$ for all $p \in \mathbb{Z}$, with this Betti number being 1 for $p = 0$ and 0 otherwise. Hence, the statement of the proposition holds in this case as well (for $p = 0$ note that f_* maps any vertex to a vertex, hence it is an isomorphism by Corollary 2.23).

For the induction step let S and T be simplicial complexes such that the proposition holds for S, T, and $S \cap T$. Noting that $\mathrm{Sd}(S \cap T) = \mathrm{Sd}(S) \cap \mathrm{Sd}(T)$ and $\mathrm{Sd}(S \cup T) = \mathrm{Sd}(S) \cup \mathrm{Sd}(T)$ by Remark

1.22 (b), we then obtain a commutative diagram of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & C(\mathrm{Sd}(S\cap T)) \arrow[r] \arrow[d] & C(\mathrm{Sd}(S)) \times C(\mathrm{Sd}(T)) \arrow[r] \arrow[d] & C(\mathrm{Sd}(S\cup T)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C(S\cap T) \arrow[r] & C(S)\times C(T) \arrow[r] & C(S\cup T) \arrow[r] & 0,
\end{tikzcd}$$

with exact rows as in Example 4.7, and the the vertical maps being pushforwards by f. Hence, Construction 4.10 gives us commutative diagrams of vector spaces

$$\begin{tikzcd}
H_p(\mathrm{Sd}(S\cap T)) \arrow[r] \arrow[d] & H_p(\mathrm{Sd}(S))\times H_p(\mathrm{Sd}(T)) \arrow[r] \arrow[d] & H_p(\mathrm{Sd}(S\cup T)) \arrow[r] \arrow[d] & H_{p-1}(\mathrm{Sd}(S\cap T)) \arrow[r] \arrow[d] & H_{p-1}(\mathrm{Sd}(S))\times H_{p-1}(\mathrm{Sd}(T)) \arrow[d] \\
H_p(S\cap T) \arrow[r] & H_p(S)\times H_p(T) \arrow[r] & H_p(S\cup T) \arrow[r] & H_{p-1}(S\cap T) \arrow[r] & H_{p-1}(S)\times H_{p-1}(T)
\end{tikzcd}$$

for all $p \in \mathbb{Z}$ with exact rows, with the vertical maps again being pushforwards by f. As the outer four vertical maps are isomorphisms by the induction assumption, we can conclude by the 5-Lemma 4.19 that the middle vertical map is an isomorphism as well. □ 07

Our second application of Mayer-Vietoris induction is probably more surprising and goes back to the very definition of the homology groups of a simplicial complex S. Why exactly did we construct it using *oriented* simplicial chains $C_p(S)$ in Definition 2.2, and not just with *ordered* ones $C_p^{\mathrm{ord}}(S)$ as in Definition 2.1? While the notion of orientation simplified computations and seemed geometrically intuitive in our pictures, it adds an algebraic level of complexity since it makes the simplicial chain groups into quotient spaces. So let us study what would happen if we set everything up using ordered simplices instead – which is in fact what we will do for the singular homology of topological spaces introduced in Chapter 6. Fortunately, our algebraic approach admitting arbitrary chain complexes allows us to figure this out without much effort.

Construction 4.23 (Ordered simplicial homology)**.** Let S be a simplicial complex. For any $p \in \mathbb{Z}$ we define the boundary maps $\partial_p \colon C_p^{\mathrm{ord}}(S) \to C_{p-1}^{\mathrm{ord}}(S)$ for ordered chains by the same formula as in Definition 2.5, just replacing oriented simplices $[i_0,\dots,i_p]$ by ordered ones $(i_0,\dots,i_p)$. Note that the proof of Lemma 2.12 that $\partial^2 = 0$ never actually permutes the vertices in a simplex or uses that a simplex with repeated vertices vanishes, so it still holds in our new ordered setting with literally the same argument. Hence, we obtain an *ordered simplicial chain complex* $C^{\mathrm{ord}}(S)$. Let us call its homology groups $H_p(C^{\mathrm{ord}}(S))$ the *ordered simplicial homology* of S, denoted by $H_p^{\mathrm{ord}}(S)$.

Example 4.24 (Ordered simplicial homology of a point)**.** Already for the 0-dimensional simplicial complex $S = \{\{i_0\}\}$ consisting of a single vertex i_0 the ordered simplicial chain groups are completely different from the oriented ones: As repeated vertices in simplices are now allowed, every $C_p^{\mathrm{ord}}(S)$ for $p \geq 0$ has dimension 1, generated by the p-simplex $(i_0,\dots,i_0)$. In particular, there are now infinitely many non-zero terms in the complex $C^{\mathrm{ord}}(S)$. Due to the alternating sign in its definition, the boundary map on $(i_0,\dots,i_0) \in C_p^{\mathrm{ord}}(S)$ becomes

$$\partial_p(i_0,\dots,i_0) = \sum_{k=0}^{p} (-1)^k \cdot (i_0,\dots,i_0) = \begin{cases} (i_0,\dots,i_0) \in C_{p-1}^{\mathrm{ord}}(S) & \text{if } p \text{ is even,} \\ 0 & \text{if } p \text{ is odd.} \end{cases}$$

Hence, using the isomorphism $C_p^{\mathrm{ord}}(S) \to K,\ 1 \mapsto (i_0,\dots,i_0)$ for $p \geq 0$ the ordered simplicial chain complex $C^{\mathrm{ord}}(S)$ is

$$\cdots \longrightarrow \underbrace{K}_{p=4} \xrightarrow{\mathrm{id}} \underbrace{K}_{p=3} \xrightarrow{0} \underbrace{K}_{p=2} \xrightarrow{\mathrm{id}} \underbrace{K}_{p=1} \xrightarrow{0} \underbrace{K}_{p=0} \longrightarrow \underbrace{0}_{p=-1} \longrightarrow \underbrace{0}_{p=-2} \longrightarrow \cdots,$$

and in fact, the homology $H_p^{\mathrm{ord}}(S)$ of this complex is 1-dimensional for $p = 0$ and zero otherwise – just as for oriented simplicial homology!

Especially given the completely different form of $C^{\mathrm{ord}}(S)$, this result is clearly surprising. It does not really reflect a geometric idea but rather an algebraic coincidence, namely that the boundary map has some sign canceling that works out in our favor. Nevertheless, our algebraic machinery is robust enough so that this coincidence almost automatically carries over to all simplicial complexes:

Proposition 4.25 (Ordered = oriented simplicial homology)**.** *For any simplicial complex S, the natural maps $C_p^{\text{ord}}(S) \to C_p(S)$, $(i_0,\dots,i_p) \mapsto [i_0,\dots,i_p]$ induce isomorphisms $H_p^{\text{ord}}(S) \to H_p(S)$ in homology for all $p \in \mathbb{Z}$.*

Proof. We will prove the statement using the Mayer-Vietoris induction of Lemma 4.18. Of course, the statement is obvious for $S = \emptyset$.

Now let S be a simplicial disk. Note that the proof of Lemma 3.8 on simplicial homotopies using the prism $P(S)$ never permutes vertices in simplices or uses that simplices with repeated vertices vanish, so it holds in the ordered setting as well. Hence, the proof of Proposition 3.11 on Betti numbers of star-shaped simplicial complexes also carries over and shows that S has the ordered homology of a point – which is trivial by Example 4.24, just as in the oriented case. This finishes the start of the induction.

Finally, for the induction step let S and T be simplicial complexes such that the proposition holds for S, T, and $S \cap T$. As in the proof of Proposition 4.22 we have a commutative diagram of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & C^{\text{ord}}(S\cap T) \arrow[r] \arrow[d] & C^{\text{ord}}(S) \times C^{\text{ord}}(T) \arrow[r] \arrow[d] & C^{\text{ord}}(S\cup T) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C(S\cap T) \arrow[r] & C(S) \times C(T) \arrow[r] & C(S\cup T) \arrow[r] & 0,
\end{tikzcd}$$

with exact rows as in Example 4.7 (and its corresponding ordered version), where the vertical maps are given by mapping each ordered simplex to the corresponding oriented one. Construction 4.10 thus gives us commutative diagrams of vector spaces

$$\begin{tikzcd}
H_p^{\text{ord}}(S\cap T) \arrow[r] \arrow[d] & H_p^{\text{ord}}(S) \times H_p^{\text{ord}}(T) \arrow[r] \arrow[d] & H_p^{\text{ord}}(S\cup T) \arrow[r] \arrow[d] & H_{p-1}^{\text{ord}}(S\cap T) \arrow[r] \arrow[d] & H_{p-1}^{\text{ord}}(S) \times H_{p-1}^{\text{ord}}(T) \arrow[d] \\
H_p(S\cap T) \arrow[r] & H_p(S) \times H_p(T) \arrow[r] & H_p(S\cup T) \arrow[r] & H_{p-1}(S\cap T) \arrow[r] & H_{p-1}(S) \times H_{p-1}(T)
\end{tikzcd}$$

for all $p \in \mathbb{Z}$ with exact rows. As the four outer vertical maps are isomorphisms by the induction assumption, it follows from the 5-Lemma 4.19 that the middle vertical map is an isomorphism as well. □

5. Simplicial Cohomology

In this last chapter on simplicial complexes we want to briefly discuss *simplicial cohomology*, which is obtained from simplicial homology by dualizing – i. e. by replacing all vector spaces and linear maps in our sequences by their duals. The goal of this is twofold: First of all, we want to set up the general algebraic framework of dualizing and study its properties since we will need this several times in the future. Secondly, and probably more importantly, from a geometric point of view this will give us a strong hint how to interpret our previous results in terms of analysis: Simplicial chains can be thought of as objects mapping *into* a simplicial complex S (in a topological realization, any simplex in S gives a subset of $|S|$ after all), so by dualizing we will study maps *from* a simplicial complex, i. e. (K-valued) functions on S. But functions on some geometric space are some of the main objects studied in analysis; and thus when we introduce algebraic topology in analytic terms in Chapter 13 it will actually be much more natural to do this in terms of cohomology instead of homology. Moreover, we will also see in this chapter that dualizing our simplicial boundary maps makes them into operations that can be interpreted as a discrete version of differentiation on S. Using this correspondence, we will then be able to establish from the properties of these boundary maps a simplicial version of the multivariate integral theorems that you might know from vector analysis, such as the divergence theorem of Gauss and the curl theorem of Stokes.

This might sound like a big project, but actually it is not since really nothing much happens when dualizing vector spaces. In case you know some more algebra, you might realize that this would be completely different for groups or modules (see e. g. [H, Chapter 3.1]) – and in fact this is one of the main reasons why we restrict to homology over *fields* in these notes. For vector spaces however we will see e. g. in Remark 5.5 already that the simplicial version of the integral theorems from vector analysis mentioned above is basically just a reformulation of the definitions.

But let us start by recalling the basic constructions for dualizing vector spaces that we will use from linear algebra.

Remark 5.1 (Dualizing vector spaces)**.** Let V be a K-vector space. From linear algebra we know [G1, Chapter 21.D]:

(a) The vector space $V^\vee := \operatorname{Hom}(V,K)$ of all linear functions from V to K is called its *dual vector space*.

(b) There is a natural bilinear pairing
$$V^\vee \times V \to K,\ (\varphi,v) \mapsto \varphi(v)$$
given by just applying the linear function $\varphi\colon V \to K$ to $v \in V$.

(c) For a linear map $f\colon V \to W$ to another vector space W, there is an associated *dual map*
$$f^\vee\colon W^\vee \to V^\vee,\ \varphi \mapsto \varphi \circ f.$$
This makes dualizing into a contravariant functor ${}^\vee\colon \mathsf{Vect} \to \mathsf{Vect}$ in the sense of Notation 1.32, i. e. we have $(g\circ f)^\vee = f^\vee \circ g^\vee$ and $\mathrm{id}_V^\vee = \mathrm{id}_{V^\vee}$ for any two composable linear maps f and g.

(d) If V is finite-dimensional with $n := \dim V$, then $\dim V^\vee = n$ as well, so $V^\vee$ is isomorphic to V (although there is no *natural* isomorphism between those spaces, i. e. you have to pick bases to construct one). However, given a basis $(v_1,\dots,v_n)$ of V there is an associated *dual basis* $(v_1^\vee,\dots,v_n^\vee)$ of $V^\vee$ defined in terms of the pairing (b) by
$$v_k^\vee(v_l) = \begin{cases} 1 & \text{if } k = l, \\ 0 & \text{if } k \neq l. \end{cases}$$

If $f\colon V \to W$ is a linear map between finite-dimensional vector spaces, its matrix with respect to fixed bases for V and W is just the transpose of the matrix of $f^\vee\colon W^\vee \to V^\vee$ with respect to the dual bases.

Let us now apply this dualization process to chain complexes. By its contravariant nature, this will reverse the directions of all linear maps, and thus the order of the indices in sequences.

Construction 5.2 (Cochain complexes)**.**

(a) For the purposes of dualizing, we will often allow a sequence of vector spaces C as in Definition 2.8 (a) to have increasing instead of decreasing indices. In this case it is customary to write its vector spaces as C^p with an upper index and its linear maps by $d_p\colon C^p \to C^{p+1}$, so that the sequence $C = ((C^p)_{p\in\mathbb{Z}}, (d_p)_{p\in\mathbb{Z}})$ reads

$$\cdots \longrightarrow C^{p-1} \xrightarrow{d_{p-1}} C^p \xrightarrow{d_p} C^{p+1} \longrightarrow \cdots.$$

As before, we will often drop the index p at the map d_p if it is clear from the context on which space it acts.

If $d^2 := d \circ d = 0$, the sequence C is called a **cochain complex** in accordance with Definition 2.13, with the prefix "co" just reminding us of the ascending indices. Correspondingly, the elements of C^p are called **p-cochains**, and the maps d_p are referred to as **coboundary maps**. Cochains in the kernel of d are called **cocycles** or **closed**, and cochains in the image of d are called **coboundaries**.

A morphism $f\colon C \to D$ of cochain complexes is still a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C^{p-1} \arrow[r] \arrow[d, "f_{p-1}"] & C^p \arrow[r] \arrow[d, "f_p"] & C^{p+1} \arrow[r] \arrow[d, "f_{p+1}"] & \cdots \\
\cdots \arrow[r] & D^{p-1} \arrow[r] & D^p \arrow[r] & D^{p+1} \arrow[r] & \cdots
\end{tikzcd}$$

as in Definition 2.8 (b). This makes cochain complexes into a category, which we denote by $\mathsf{Ch}^\vee$.

(b) For any chain complex $C = ((C_p)_{p\in\mathbb{Z}}, (\partial_p)_{p\in\mathbb{Z}})$ as in Definition 2.13, Remark 5.1 (c) tells us that we obtain a **dual cochain complex** $C^\vee := ((C^p)_{p\in\mathbb{Z}}, (d_p)_{p\in\mathbb{Z}})$ by setting $C^p = C_p^\vee$ and $d_p = \partial_{p+1}^\vee\colon C_p^\vee \to C_{p+1}^\vee$, i. e.

$$\cdots \longrightarrow C_{p-1}^\vee \xrightarrow{\partial_p^\vee} C_p^\vee \xrightarrow{\partial_{p+1}^\vee} C_{p+1}^\vee \longrightarrow \cdots.$$

Note that this is actually a cochain complex (since $d \circ d = \partial^\vee \circ \partial^\vee = (\partial \circ \partial)^\vee = 0$) and gives rise to a contravariant dualization functor ${}^\vee\colon \mathsf{Ch} \to \mathsf{Ch}^\vee$. Hence, we can also dualize a sequence of chain complexes as in Definition 4.1 to obtain a reversed sequence of dual cochain complexes.

Of course, our most important example for the moment will come from dualizing simplicial chain complexes.

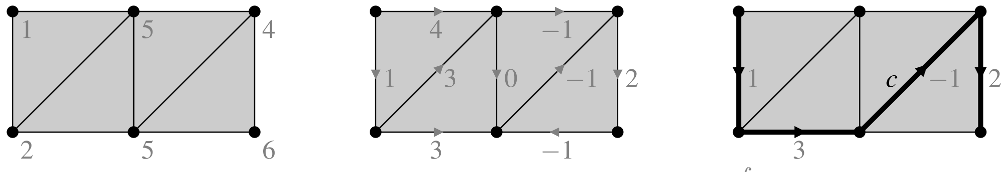
Construction 5.3 (Simplicial cochain complexes)**.** For any simplicial complex S we define the **simplicial cochain complex** as the dual cochain complex $C^\vee(S)$ of the simplicial chain complex $C(S)$ of Construction 2.9 and Example 2.14, i. e. by $C^p(S) := (C_p(S))^\vee$ and the corresponding dual boundary maps $d_p\colon C^p(S) \to C^{p+1}(S)$ according to Remark 5.1 (c) given by $d\varphi(c) = \varphi(\partial c)$ for all $\varphi \in C^p(S)$ and $c \in C_{p+1}(S)$, which by Definition 2.5 (a) means

$$d\varphi([i_0,\dots,i_{p+1}]) = \sum_{k=0}^{p+1} (-1)^k \cdot \varphi([i_0,\dots,\widehat{i_k},\dots,i_{p+1}]) \quad \in K$$

for any oriented $(p+1)$-simplex $[i_0,\dots,i_{p+1}]$ of S.

According to Construction 5.2 (a), the elements of $C^p(S)$ for $p \in \mathbb{Z}$ are called **simplicial p-cochains** of S, and the coboundary maps $d_p \colon C^p(S) \to C^{p+1}(S)$ are called the **simplicial coboundary maps** of S. Elements of $C^p(S)$ are called **simplicial p-cocycles** or **simplicial p-coboundaries** of S if they are in the kernel or image of the simplicial coboundary map, respectively.

Example 5.4 (Discrete version of differentiation and integration)**.** The picture below on the left shows a simplicial 0-cochain φ on a 2-dimensional simplicial complex S, and the middle picture shows a simplicial 1-cochain ψ on S. The gray numbers are the values of the cochains on the various simplices. Due to cochains being defined on *oriented* simplices, we had to choose an arbitrary orientation of each edge (indicated by the gray arrows) to specify ψ; reversing the orientation of an edge would replace the value of ψ on that edge by its negative.



$\varphi \in C^0(S)$ $\qquad$ $\psi = d\varphi \in C^1(S)$ $\qquad$ $\int_c \psi = 1+3-1+2 = 5$

It might seem that simplicial cochains are absolutely the same as simplicial chains, just assigning a number to every oriented simplex of the given dimension. But the geometric interpretation of this number is entirely different: For example, the picture for $\varphi \in C^0(S)$ above represents a function on the set of vertices of S (which could roughly be thought of as a discrete approximation of a function on the topological realization $|S|$), whereas the same numbers interpreted as a chain in $C_0(S)$ would correspond to a collection of points on S, having the gray number of points at the corresponding vertices. Similarly, the 1-cochain ψ is not to be thought of as an oriented path through S with some multiplicities (as it would be the case for a 1-chain), but as a function on on the oriented edges of S. For example, if we think of the simplicial complex S as a geographic map, a 0-cochain might describe the elevation at the various points, whereas a 1-cochain might describe the elevation gain when traveling along an edge.

Algebraically, the difference between chains and cochains is that their (co-)boundary maps are completely different. For example, whereas the boundary of a 0-chain is always zero by definition, by Construction 5.3 the coboundary $d\varphi \in C^1(S)$ of a cochain $\varphi \in C^0(S)$ is given by

$$d\varphi([i_0,i_1]) = \varphi([i_1]) - \varphi([i_0]) \quad \text{for all } \{i_0,i_1\} \in S,$$

i. e. it takes the difference of a 0-cocycle at adjacent vertices and can thus be thought of as a *discrete version of differentiation*. In fact, in the picture above the 1-cochain ψ is just equal to $d\varphi$, i. e. for a geographic map with elevation profile φ the 1-cocycle ψ describes the elevation gain along the edges.

On the other hand, the pairing $\varphi(c)$ of Remark 5.1 (b) by definition evaluates a p-cochain φ on every simplex in a p-chain c and takes the weighted sum of all these values, which can be interpreted geometrically as a *discrete version of integration* of a function φ on a space c, which we might write suggestively as

$$\int_c \varphi := \varphi(c) \quad \in K.$$

In fact, in the analytic analogue of our current setting this pairing will be an actual integration (see Construction 14.11). The picture above on the right shows a concrete example of a 1-chain c going from the upper left to the lower right corner of S. Note that in this case we had chosen the orientation of the gray arrows for ψ so that they match with the orientation of the edges of c, so for the integral we only have to sum up the values of ψ on the edges of c. In the geometric interpretation in terms of elevation profiles above, the integral $\psi(c)$ just computes the sum of the elevation differences when traveling along the path, so the result is if course the difference between the elevation at the end (which is 6) and at the start (which is 1). As $\psi = d\varphi$ in the example, this also follows from the defining equation $d\varphi(c) = \varphi(\partial c)$ of the coboundary map in Construction 5.3, which in integral notation becomes

$$\int_c d\varphi = \int_{\partial c} \varphi, \qquad \text{i. e.} \int_{[i_0,i_1]} d\varphi = \varphi([i_1]) - \varphi([i_0]) \quad \text{for } \varphi \in C_0(S).$$

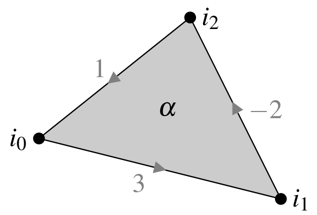
Note that this formula for 0-cochains can be understood as a discrete version of the fundamental theorem of calculus.

Remark 5.5 (Integral theorems in higher dimensions)**.** To obtain intuitive interpretations of the equation $\int_c d\varphi = \int_{\partial c} \varphi$ for $\varphi \in C^p(S)$ and $c \in C_{p+1}(S)$ with $p > 0$ on a simplicial complex S, we have to understand the geometric meaning of the coboundary operator in these cases. Let us quickly do this in two simple cases, leading to discrete versions of integral theorems that you might know from vector analysis and that we will prove later in Proposition 14.13 and Example 14.14.

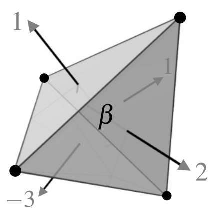
(a) For a first interpretation specifically for $p = 1$, note that by Construction 5.3 we obtain for a 1-cochain $\varphi \in C^1(S)$ and a 2-simplex $[i_0, i_i, i_2] \in S$

$$d\varphi([i_0,i_1,i_2]) = \varphi([i_0,i_1]) + \varphi([i_1,i_2]) + \varphi([i_2,i_0]).$$

Hence, e. g. in the picture below on the left we have $d\varphi(\alpha) = 3 - 2 + 1 = 2$ for $\alpha = [i_0,i_1,i_2]$. To interpret this picture geometrically, imagine that α is a triangle floating on a surface of water, and that the 1-cochain φ describes the direction and strength of the flow of water around the triangle. The number $d\varphi(\alpha)$ then determines whether the triangle starts rotating: In our example, there is a counterclockwise net flow of water of strength 2 around α, so the triangle will start rotating counterclockwise. Interpreting φ as a vector field, this number $d\varphi(\alpha)$ corresponds to what is called the *curl* of the vector field in analysis. Adding this up for an object c made up from several simplices, the equation $\int_c d\varphi = \int_{\partial c} \varphi$ thus says that the rotation speed picked up by the object is obtained by summing up the vector field φ around the boundary of the object. This is a discrete version of what is commonly called the *curl theorem of Stokes*.



$\varphi \in C^1(S)$



$\psi \in C^2(S)$

(b) Another interpretation that works well if $p = \dim S - 1$ is shown in the picture above on the right in the case $p = 2$. We can then describe the orientation of a p-simplex by an outward-pointing normal vector as in the picture, and similarly to (a) the expression $d\psi(\beta)$ for $\psi \in C^p(S)$ and a $(p+1)$-simplex β adds up the outward-pointing values of ψ at the boundary of β – so in the picture we have $d\psi(\beta) = -3+2+1+1 = 1$. To interpret this number geometrically, imagine again that ψ describes the direction and strength of a water flow (in this case three-dimensional) in these normal directions. Then the number $d\psi(\beta)$ determines if there is a water source or sink inside β: In this case, we have a net flow of water of strength 1 coming out of β, so there must be a source of water inside β. Interpreting ψ as a vector field again, this number $d\psi(\beta)$ corresponds to what is called in analysis the *divergence* of the vector field. Adding this up again over an object c made up from simplices, the equation $\int_c d\psi = \int_{\partial c} \psi$ can therefore be understood as saying that to check if there is a source of water within c we have to compute the net water flow through its boundary. This can be interpreted as a discrete version of the *divergence theorem of Gauss*.

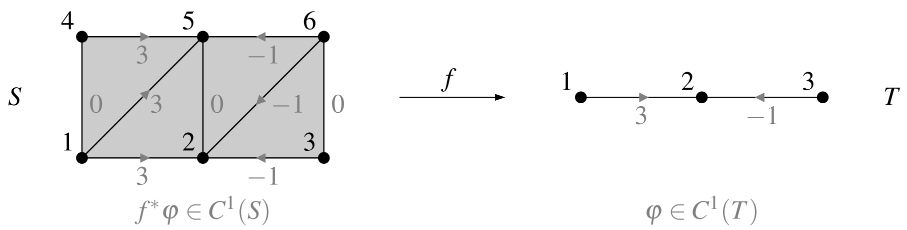
Construction 5.6 (Pullbacks and the simplicial cochain functor)**.** Note that the assignment of the simplicial cochain complex $C^\vee(S)$ to a simplicial complex S is actually a (contravariant) functor since it is the composition of the simplicial chain functor of Construction 2.9 with the dualization functor of Construction 5.2 (b). Hence, for a morphism $f\colon S \to T$ of simplicial complexes there is an associated morphism of cochain complexes from $C^\vee(T)$ to $C^\vee(S)$. In analogy to the pushforward $f_*\colon C(S) \to C(T)$ it is called the **pullback** by f and written as $f^*\colon C^\vee(T) \to C^\vee(S)$. According to the construction of the simplicial chain and dualization functors it is given by $f^*\varphi = \varphi \circ f_* \in C^p(S)$ for all $\varphi \in C^p(T)$, i. e. on p-simplices $[i_0,\dots,i_p] \in S$ by

$$(f^*\varphi)[i_0,\dots,i_p] = \varphi([f(i_0),\dots,f(i_p)]) \quad \in K.$$

Hence, intuitively it is just given by composition with f. Let us give two examples:

(a) If S is a subcomplex of a simplicial complex T, the pullback by the inclusion morphism is just the restriction $C^p(T) \to C^p(S)$, $\varphi \mapsto \varphi|_S := \varphi|_{C_p(S)}$ of linear maps $C_p(T) \to K$ to linear maps $C_p(S) \to K$.

(b) As a concrete example, the picture below shows the pullback $f^*\varphi$ of a 1-cocycle φ for the "vertical projection morphism" f of simplicial complexes given by $f(3i+j) = j$ for $i \in \{0,1\}$ and $j \in \{1,2,3\}$ (where the black numbers are the labels of the vertices and the gray numbers are the values of the cocycles on the edges).



After having understood the geometric meaning of simplicial cochain complexes in detail, let us now come back to cochain complexes in general. Of course, we also want to consider their homology, which in accordance with the language of Construction 5.2 will be called *cohomology*.

Construction 5.7 (Cohomology of cochain complexes)**.**

(a) For a cochain complex $C = ((C^p)_{p\in\mathbb{Z}}, (d_p)_{p\in\mathbb{Z}})$ as in Construction 5.2 (a), its p-th **cohomology group** is defined as the vector space

$$H^p(C) := \ker d_p / \operatorname{im} d_{p-1}$$

as in Definition 2.15, and its dimension $b^p(C) := \dim H^p(C)$ will be called the p-th **Betti number** of C. In the same way as in Definition 4.4, the cochain complex C is called **exact** if all these cohomology groups are trivial, i. e. if $\ker d_p = \operatorname{im} d_{p-1}$ for all $p \in \mathbb{Z}$.

(b) For a simplicial complex S, the **simplicial cohomology** of S is defined to be the cohomology of its simplicial cochain complex $C^\vee(S)$ of Construction 5.3, i. e. we set $H^p(S) := H^p(C^\vee(S))$ for all $p \in \mathbb{Z}$. As usual, if we want to indicate the ground field in the notation we will write it as $H^p(S,K) = H^p(C^\vee(S,K))$.

Remark 5.8. Note that apart from the labeling of its entries a cochain complex is really just the same as a chain complex, and cohomology is just the same as homology. In particular, all results that we have obtained for chain complexes and their homology hold in our new setting as well:

(a) Cohomology H^p is a functor $\mathsf{Ch}^\vee \to \mathsf{Vect}$ for all $p \in \mathbb{Z}$ as in Lemma 2.17. In particular, the simplicial cohomology groups are contravariant functors $\mathsf{Simp} \to \mathsf{Vect}$, so for a morphism $f\colon S \to T$ the pullback maps $f^*\colon C^p(T) \to C^p(S)$ of Construction 5.6 are also well-defined on simplicial cohomology as maps $f^*\colon H^p(T) \to H^p(S)$.

(b) If all cohomology groups of a cochain complex C are finite-dimensional with almost all of them being 0, there is an *Euler characteristic* $\chi(C) = \sum_{p\in\mathbb{Z}} (-1)^p\, b^p(C)$ as in Definition 2.24, and if the same holds for C^p as well we can compute it by $\chi(C) = \sum_{p\in\mathbb{Z}} (-1)^p \dim C^p$ as in Lemma 2.25.

(c) *Cochain homotopies* can be defined in the same way as in Definition 3.1 (now lowering the index by 1 instead of raising it), and the existence of such a cochain homotopy between two morphisms of cochain complexes implies that they induce the same map on cohomology as in Proposition 3.3.

(d) Any short exact sequence of cochain complexes gives rise to a long exact sequence in cohomology as in Proposition 4.8, and this assignment is functorial as in Construction 4.10.

08

In fact, at least in the simplicial case it is also immediate to see that all Betti numbers remain invariant under dualization:

Lemma 5.9 (Invariance of Betti numbers under dualization)**.** *For any chain complex C consisting only of finite-dimensional vector spaces we have $b^p(C^\vee) = b_p(C)$ for all $p \in \mathbb{Z}$.*

Proof. For a given $p \in \mathbb{Z}$ the relevant parts of the (co-)chain complexes C and $C^\vee$ are

$$\cdots \longrightarrow C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \longrightarrow \cdots \qquad \text{and} \qquad \cdots \longrightarrow C_{p-1}^\vee \xrightarrow{\partial_p^\vee} C_p^\vee \xrightarrow{\partial_{p+1}^\vee} C_{p+1}^\vee \longrightarrow \cdots.$$

Now by Remark 5.1 (d) we have $\dim C_p^\vee = \dim C_p$, and $\operatorname{rk} \partial_p^\vee = \operatorname{rk} \partial_p$ as well as $\operatorname{rk} \partial_{p+1}^\vee = \operatorname{rk} \partial_{p+1}$ since transposed matrices have the same rank. Hence by the dimension formula for linear maps we obtain

$$\begin{aligned} \dim H^p(C^\vee) &= \dim \ker \partial_{p+1}^\vee - \operatorname{rk} \partial_p^\vee \\ &= \dim C_p^\vee - \operatorname{rk} \partial_{p+1}^\vee - \operatorname{rk} \partial_p^\vee \\ &= \dim C_p - \operatorname{rk} \partial_{p+1} - \operatorname{rk} \partial_p \\ &= \dim \ker \partial_p - \operatorname{rk} \partial_{p+1} \\ &= \dim H_p(C). \end{aligned}$$

□

Remark 5.10.

(a) In particular, it follows from Lemma 5.9 that $\dim H^p(S) = \dim H_p(S) = b_p(S)$ for any simplicial complex S and $p \in \mathbb{Z}$.

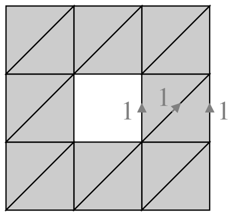
(b) Another easy consequence of Lemma 5.9 is that the dual of an exact sequence of finite-dimensional vector spaces (or chain complexes consisting of finite-dimensional vector spaces) is again exact, since exactness just means that the Betti numbers are zero. For example, for simplicial complexes S and T we obtain a *dual exact simplicial Mayer-Vietoris sequence* in cohomology

$$0 \longrightarrow C^\vee(S \cup T) \longrightarrow C^\vee(S) \times C^\vee(T) \longrightarrow C^\vee(S \cap T) \longrightarrow 0$$

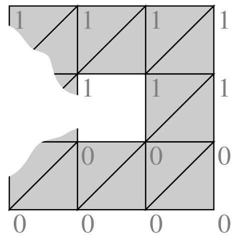
by dualizing Example 4.7.

Example 5.11. Remark 5.10 (a) means that we can detect holes in simplicial complexes using cohomology in the same way as with homology. But we have seen that the coboundary maps in cohomology look completely different from the boundary maps in homology. So let us quickly show geometrically why *cocycles* modulo *coboundaries* detect holes as well – which is probably less intuitive than considering cycles modulo boundaries. In fact, the following argument in simplicial cohomology is already a discrete version of the analytic method described in the introduction to these notes, as "functions satisfying some integrability conditions modulo functions that are derivatives".

Let S be the simplicial annulus shown below on the left, which has first Betti number $b_1(S) = 1$ by Construction 4.14 (b) due to its hole in the middle. To detect this hole using cohomology, consider the 1-cochain $\varphi \in C^1(S)$ shown in the picture (where the value of φ on all unmarked edges is zero). Note first of all that it is actually a cocycle, i. e. that $d\varphi = 0$: We could either check this by directly computing its coboundary using the formula of Construction 5.3, or – probably easier – by seeing that in the interesting right half of the annulus it is of the form $\varphi = d\psi$ for the 0-cochain ψ shown in the picture on the right, implying that $d\varphi = d^2\psi = 0$ there. In the spirit of Example 5.4 this just means that φ describes the elevation gain along the edges for the elevation function ψ.



$\varphi \in C^1(S)$



$\psi \in C^0(S)$ with $d\psi = \varphi$

However, this construction does not work globally due to the hole of S: To construct a global elevation function ψ whose elevation gain along edges is φ, the only possibility is the function ψ shown in the right picture, and this function would not be well-defined on S as the part cut out in the picture cannot be filled in consistently. In other words, the elevation gain function φ would require us to have an elevation profile in which we pick up a non-trivial elevation gain when going around the hole of the annulus, which is not possible. Hence, the cocycle φ is a non-zero element of $H^1(S)$, and thus a generator since $b_1(S) = 1$.

Hence, we can say that the cocycle φ is a coboundary locally, but not globally – which is exactly the discrete version of a function satisfying a local integrability condition $d\varphi = 0$, yet not being a global "derivative" of the form $\varphi = d\psi$.

As you might imagine, the statement of Lemma 5.9 (and also its consequences such as duals of exact sequences being exact in Remark 5.10 (b)) are absolutely crucial for homology and cohomology to interplay well, and thus for the rest of these notes. Unfortunately, there is a small technical issue here: We have used dimension arguments to prove Lemma 5.9, but starting from the next chapter most of the vector spaces in our chain complexes will actually consist of (highly) infinite-dimensional vector spaces – although usually their homology still has finite dimension. Hence, we need a replacement for the proof of this lemma that works in this more general situation as well. To prepare for this, let us first recall some properties of general (i. e. possibly infinite-dimensional) vector spaces from linear algebra to make sure that we are all on the same page.

Remark 5.12 (Infinite-dimensional linear algebra)**.** Let V be vector space, possibly infinite-dimensional. The following statements should all be well-known for finite-dimensional vector spaces; for proofs in the general case see e. g. [R]. The main guideline is that linear combinations of vectors in V are always finite by definition (after all, infinite sums would require some notion of convergence, which is not available for arbitrary ground fields). So, for example, by definition a family of vectors in V is a basis if and only if every element of V is a linear combination of *finitely many* vectors of the family in a unique way.

(a) Every linearly independent family of vectors in V can be extended to a basis of V, so in particular V always has a basis – although the proof of this statement in the infinite-dimensional case uses the Axiom of Choice and is thus not constructive. In other words, while there are some infinite-dimensional vector spaces with known bases (e. g. the vector space of polynomials in a variable x with basis $(x^k)_{k\in\mathbb{N}}$), there is in general no algorithm to compute a basis.

(b) A linear map from V to another vector space W can be constructed by picking a basis of V and specifying arbitrary image vectors in W for it; by linear extension there will then always be a unique linear map $V \to W$ realizing these values on the basis. This implies that every linear map $U \to W$ on a subspace U of V to another vector space W can be extended to a linear map $V \to W$, by extending a basis of U to one of V and setting the map to 0 on the additional basis vectors.

(c) The dimension $\dim V$ of V is well-defined as a cardinality, so e. g. a vector space with a countable basis cannot be isomorphic to a vector space with an uncountable basis.

Note that all statements of Remark 5.12 are of the form that infinite-dimensional vector spaces behave to a large extent just like one would expect from the finite-dimensional case. There is one big difference however regarding dualization (which is why this topic comes up in the current chapter): In contrast to Remark 5.1 (d), the dual $V^\vee$ of an infinite-dimensional vector space V is no longer isomorphic to V. In fact, it *never* is; one can show that its dimension is always larger than that of V in the sense of cardinality as in Remark 5.12 (c). (In case you have been exposed to some functional analysis and are surprised about this statement, let me clarify that we are talking about the *algebraic dual* $V^\vee$ of *all* linear functions on V here – although we are discussing topology, the vector spaces used for homology will actually never carry a norm or topology, so there is no notion of a *continuous dual* of all *continuous* linear functions.) We will not need this general statement in these notes, but it should warn us that the process of dualizing is not quite as harmless as it might have seemed in this chapter so far. To illustrate this fact let us consider one example here that will show up again in Corollary 16.5.

Example 5.13 (Dualizing infinite-dimensional vector spaces)**.** Let V be a real vector space with a countably infinite basis $(v_k)_{k\in\mathbb{N}}$. By Remark 5.12 (b) the elements of $V^\vee$ are uniquely given by specifying arbitrary values on these basis elements, so

$$V^\vee \cong \{(x_k)_{k\in\mathbb{N}} : x_k \in \mathbb{R}\}$$

is just the space of all real sequences (where x_k denotes the value of the element of $V^\vee$ on v_k for all $k\in\mathbb{N}$). We claim that this space $V^\vee$ does *not* have a countable basis, so that it cannot be isomorphic to V by Remark 5.12 (c). To see this, it suffices to prove that the uncountable family $(\varphi_z)_{z\in\mathbb{R}}$ in $V^\vee$ given by

$$\varphi_z = (z^k)_{k\in\mathbb{N}} = (z^0, z^1, z^2, \dots)$$

is linearly independent. So let $\lambda_1,\dots,\lambda_n \in \mathbb{R}$ for $n\in\mathbb{N}$ and $z_1,\dots,z_n\in\mathbb{R}$ be distinct with $\sum_{k=1}^n \lambda_k \varphi_{z_k} = 0 \in V^\vee$. Looking only at the first $n-1$ components of this equation gives

$$\sum_{k=1}^n \lambda_k z_k^l \quad \text{for } l = 0,\dots,n-1. \tag{$*$}$$

But this is an $n\times n$ linear system of equations for the variables $\lambda_1,\dots,\lambda_n$ with coefficient matrix $(z_k^l)_{1\le k\le n, 0\le l\le n-1}$. This matrix is known as a *Vandermonde matrix*, and it is a standard linear algebra result that it is invertible. Hence, the system $(*)$ has only the trivial solution $\lambda_1 = \cdots = \lambda_n = 0$, showing that $(\varphi_z)_{z\in\mathbb{R}}$ is an uncountable linearly independent family in $V^\vee$, and thus that $V^\vee$ is not isomorphic to V.

Fortunately however, dualization still behaves well with respect to homology, so that we obtain the following result analogous to Lemma 5.9.

Proposition 5.14. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be two linear maps of vector spaces, with corresponding dual maps $C^\vee \xrightarrow{g^\vee} B^\vee \xrightarrow{f^\vee} A^\vee$. Then:*

(a) $\ker(f^\vee) = \{\varphi\in B^\vee : \varphi|_{\operatorname{im} f} = 0\}$.

(b) $\operatorname{im}(g^\vee) = \{\varphi\in B^\vee : \varphi|_{\ker g} = 0\}$.

(c) *If $g\circ f = 0$ (and thus also $f^\vee\circ g^\vee = 0$) there is a natural isomorphism*

$$\ker(f^\vee)/\operatorname{im}(g^\vee) \cong (\ker g/\operatorname{im} f)^\vee.$$

Proof.

(a) This follows directly from the definition:

$$\ker(f^\vee) = \{\varphi\in B^\vee : \varphi\circ f = 0\} = \{\varphi\in B^\vee : \varphi|_{\operatorname{im} f} = 0\}.$$

(b) We need to show that

$$\operatorname{im}(g^\vee) = \{\varphi\in B^\vee : \varphi = \psi\circ g \text{ for some } \psi\in C^\vee\} \stackrel{!}{=} \{\varphi\in B^\vee : \varphi|_{\ker g} = 0\}.$$

"$\subset$" is obvious since any function of the form $\psi\circ g$ vanishes on $\ker g$.

"⊃": Let $\varphi \in B^\vee$ with $\varphi|_{\ker g} = 0$. We want to find $\psi \in C^\vee$ such that $\varphi = \psi \circ g$ to make the diagram on the right commutative. First define ψ on $\operatorname{im} g$: For any $y \in \operatorname{im} g$ with $y = g(x)$ set $\psi(y) := \varphi(x)$; this is actually well-defined since picking a different inverse image of y would change x by an element of $\ker g$, on which φ is zero.

$$\begin{tikzcd} B \arrow[rr, "\varphi"] \arrow[dr, "g"'] & & K \\ & C \arrow[ur, "\psi"'] & \end{tikzcd}$$

Hence we already have the required equation $\psi(g(x)) = \varphi(x)$ for all $x \in B$. By Remark 5.12 (b) we can now extend ψ – which so far is only defined on $\operatorname{im} g$ – to a linear function on all of C.

(c) Let us start with with the natural map $B^\vee \to (\ker g)^\vee$ that just restricts a linear function on B to one on $\ker g \subset B$. In the hope that it does not confuse you we will now restrict this restriction map to the subspace $\ker(f^\vee) \subset B^\vee$; by (a) the resulting linear functions in $(\ker g)^\vee$ will then vanish on $\operatorname{im} f$ and can thus be considered as well-defined functions in $(\ker g/\operatorname{im} f)^\vee$. We have thus obtained a natural linear map

$$\Phi\colon \ker(f^\vee) \to (\ker g/\operatorname{im} f)^\vee$$

and want to study its image and kernel:

- Φ is surjective: Any linear function in $(\ker g/\operatorname{im} f)^\vee$ comes (by composition with the quotient map) from a linear function in $(\ker g)^\vee$ that is zero on $\operatorname{im} f$, hence by extension as in Remark 5.12 (b) from a linear function in $B^\vee$ that is zero on $\operatorname{im} f$. By (a), this is just an element of $\ker(f^\vee)$.
- The kernel of Φ consists of all linear functions that are zero on $\ker g$, which by (b) is just $\operatorname{im}(g^\vee)$.

The statement thus follows from the homomorphism theorem applied to Φ. □

Remark 5.15.

(a) Applying Proposition 5.14 (c) to the linear maps in a chain complex C and its dual cochain complex $C^\vee$, we see that

$$H^p(C^\vee) \cong (H_p(C))^\vee,$$

i.e. that "(co-)homology commutes with dualization". This reproves Lemma 5.9 and establishes it also in the case when the vector spaces C_p are infinite-dimensional. If the homology groups are infinite-dimensional as well, note however that the statement $\dim H^p(C^\vee) = \dim H_p(C)$ is still correct in the sense that both dimensions are infinite, but that in this case $H^p(C^\vee)$ is not isomorphic to $H_p(C)$ by Example 5.13.

(b) As in Remark 5.10 (b), Proposition 5.14 (c) clearly implies that the dual of any exact sequence of vector spaces or chain complexes is still exact.

(c) Actually, even in the finite-dimensional case Proposition 5.14 (c) is stronger than Lemma 5.9 as it is not just a dimension statement, but also establishes $H^p(C^\vee)$ as canonically dual to $H^p(C)$. Hence, the pairing of Remark 5.1 (b) also works between $H^p(C^\vee)$ and $H_p(C)$ for all $p \in \mathbb{Z}$, so e.g. the "integral" $\int_c \varphi$ of Example 5.4 is well-defined for (co-)homology classes $c \in H_p(S)$ and $\psi \in H^p(S)$ of a simplicial complex S and $p \in \mathbb{Z}$ as well. Moreover, by Remark 5.1 (d) for any finite basis of $H_p(C)$ there is an associated dual basis of $H^p(C^\vee)$.

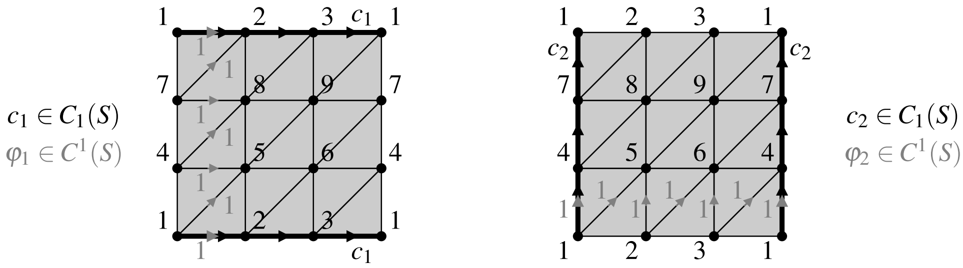
Example 5.16. Let S be a simplicial torus as in Example 1.17 (d). We have already computed its homology in Example 2.27 and found out that $b_1(S) = 2$, with a basis of $H_1(S)$ given by a horizontal circle c_1 and vertical circle c_2 as drawn in bold in the pictures below.

In the picture we have also drawn two 1-cochains φ_1 and φ_2, constructed in a similar way to Example 5.11, and with their values again given by the gray numbers and arrows. We claim that this is in fact the dual basis to c_1 and c_2 in $H^1(S)$ in the sense of Remark 5.15 (c):

- It is checked immediately that $d\varphi_1 = d\varphi_2 = 0$.
- We have $\int_{c_1} \varphi_1 = \int_{c_2} \varphi_2 = 1$: For both $i \in \{1,2\}$ the cycle c_i and the cocycle φ_i have exactly one edge in common, and they both have multiplicity 1 on it.

- In contrast, c_1 has no edge in common with φ_2, so $\int_{c_1} \varphi_2 = 0$. In the same way, we obtain $\int_{c_2} \varphi_1 = 0$.

Hence, φ_1 and φ_2 form the dual basis to c_1 and c_2 – note again as in Example 5.11 that they look completely different although they describe the same topological properties of the torus.



Exercise 5.17. Consider the simplicial sphere $S^{\underline{4}}$ for the set $\underline{4} := \{0,1,2,3,4\}$.

(a) Let
$$\varphi = [0,1,2]^\vee + [0,1,4]^\vee - [0,2,3]^\vee + [0,3,4]^\vee \quad \in C^2(S^{\underline{4}}),$$
where $[i_0,i_1,i_2]^\vee$ denotes the unique linear form $C_2(S^{\underline{4}}) \to K$ with value 1 on $[i_0,i_1,i_2]$ and 0 on all $[j_0,j_1,j_2]$ with $\{j_0,j_1,j_2\} \neq \{i_0,i_1,i_2\}$.

Show that there exists a cochain $\psi \in C^1(S^{\underline{4}})$ with $d\psi = \varphi$.

(b) For a 3-cochain $\varphi \in C^3(S^{\underline{4}})$, show that there exists a cochain $\psi \in C^2(S^{\underline{4}})$ with $d\psi = \varphi$ if and only if $\int_{\partial[0,1,2,3,4]} \varphi = 0$.

Singular Homology

6. The Construction of Singular Homology

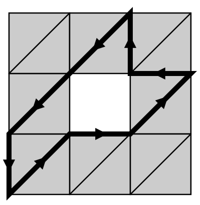
In the last chapters we have studied simplicial complexes in detail from a topological point of view. It is certainly not obvious however that the properties of simplicial homology that we obtained are actually topological in the sense that two simplicial complexes with homeomorphic topological realizations have isomorphic homology groups. In fact, although this statement is true it would be very hard to prove it using simplicial methods since the combinatorics of such simplicial complexes can be completely different.

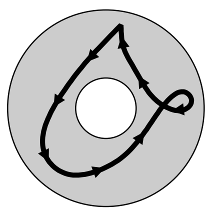
Instead, it turns out to be easier to prove this statement by constructing an entirely new homology theory called *singular homology* that is visibly topological, and then showing that singular homology actually coincides with simplicial homology when both are defined. This has the additional advantage that it also generalizes our old theory substantially since it is applicable to *all* topological spaces, and not just to topological spaces obtained from realizations of simplicial complexes. Most notably, this will then also construct homology groups of topological spaces that are not necessarily compact. It should be said clearly however that the spirit of singular homology is still the study of topological spaces that can be glued from nice pieces of $\mathbb{R}^n$ with the standard topology – so while you are able to apply singular homology to completely arbitrary topological spaces with weird topologies, this is not likely to give you anything useful.

It is the goal of the second part of these notes up to Chapter 9 to construct singular homology, show its properties, compare it to simplicial homology, and study several of its applications.

The idea to construct the singular homology of a topological space X is to replace abstract simplices by geometric simplices that are mapped continuously to X. For example, consider the "loop around an annulus" from the introduction to Chapter 2 whose simplicial version is shown again in the picture below on the left. In the singular setting in the right picture, the space X is now an actual annulus without a simplicial structure, and the 1-cycle in the picture is now a continuous map from an interval to X (of which we have only drawn its image in X). Note that in this 1-dimensional case this is the same approach as for the construction of the fundamental group [G3, Chapter 7]. In particular, the map to X really does not have to satisfy any other condition besides continuity, so it need not be injective or "smooth" in any way – which by the way is the reason for the name "singular homology". In contrast to the construction of the fundamental group however, as in the simplicial case the equivalence relation on singular 1-cycles will now be given by boundaries of 2-chains instead of homotopy, and we do not require the loops to be anchored at a given base point of X.

simplicial homology
(of a simplicial complex)





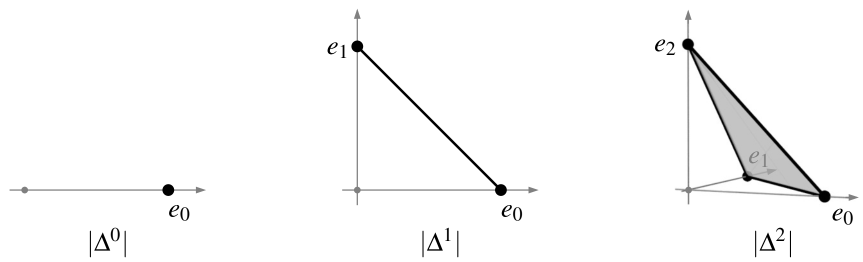
singular homology
(of a topological space)

To construct p-dimensional singular chains, we will of course just consider continuous maps from a geometric p-simplex to X. As these maps are arbitrary, we can choose a fixed p-simplex here in the same way as one considers continuous maps from the unit interval when constructing the fundamental group. With the background of Proposition 4.25, it is customary to work with ordered instead of oriented simplices here as they are simpler from an algebraic point of view. 09

Notation 6.1 (Standard simplices)**.** For $p \in \mathbb{N}$, we call $\Delta^p := (0,\dots,p)$ the **(ordered) standard** p**-simplex** with standard geometric realization

$$|\Delta^p| = \langle e_0,\dots,e_p \rangle = \{(x_0,\dots,x_p) : x_0,\dots,x_p \in \mathbb{R}_{\geq 0} \text{ with } x_0+\dots+x_p = 1\} \quad \subset \mathbb{R}^{\{0,\dots,p\}}$$

as in Construction 1.12 (a).



Definition 6.2 (Singular simplices and chains)**.** Let X be a topological space, and let $p \in \mathbb{N}$.

(a) A **singular** p**-simplex** in X is a continuous map $\sigma\colon |\Delta^p| \to X$.

(b) We denote by $C_p(X)$ the vector space with basis given by all singular p-simplices in X. The elements of $C_p(X)$ are called **singular** p**-chains** in X.

As in the simplicial case, we set $C_p(X) = 0$ for $p \in \mathbb{Z}_{<0}$, and write $C_p(X,K)$ for $C_p(X)$ if we want to indicate the dependence on the ground field K.

Remark 6.3.

(a) To keep the notations concise, we have chosen to denote the groups $C_p(X)$ of singular chains of a topological space X by the same letter as the groups $C_p(S)$ of simplicial chains of a simplicial complex S in Definition 2.2. This should not lead to any confusion as the type of space in the brackets always indicates which chains are meant.

(b) Note that $C_p(X)$ is almost always highly infinite-dimensional, so it is next to impossible to do any explicit computations with it. In order to work with these spaces, it is therefore much more important than in the simplicial case to establish their general properties such as a homotopy invariance and the Mayer-Vietoris sequence. We will do this in the current and the next chapter.

(c) Although we will usually visualize a singular simplex $\sigma\colon |\Delta^p| \to X$ by drawing its image $\sigma(|\Delta^p|) \subset X$, it is important to realize that the singular simplex is actually the map itself and not just its image as a subset of X. In particular, a reparametrization of σ by precomposing it with a non-trivial homeomorphism of $|\Delta^p|$ will give rise to a different singular simplex (see however Exercise 6.17). As we are considering ordered instead of oriented simplices, this also applies if the homeomorphism is only a relabeling of the vertices. Also, note that even if $\sigma\colon |\Delta^p| \to X$ is a constant map it is a perfectly valid and non-trivial singular p-simplex that cannot be neglected in computations.

It is often useful however to allow singular simplices whose source is an ordered simplex other than Δ^p:

Construction 6.4 (Singular simplices with arbitrary domains)**.** Let X be a topological space, and let $\alpha = (i_0,\dots,i_p)$ be an abstract ordered p-simplex. Then we will consider any continuous map $\sigma\colon |\alpha| \to X$ with $|\alpha| := |\{i_0,\dots,i_p\}|$ as a singular p-simplex in X by precomposing it with the unique affine-linear map $|\Delta^p| \to |\alpha|$ that sends e_k to e_{i_k} for all $k = 0,\dots,p$. Note that this affine-linear map is a homeomorphism if all vertices of α are distinct. Otherwise, the dimension of the geometric simplex $|\alpha|$ is strictly less than p, but by precomposing σ with the map $|\Delta^p| \to |\alpha|$ we still get a singular p-simplex.

By Remark 6.3 (c), this construction depends on the ordering (and not just the orientation) of α. For simplicity, we will not indicate this dependence in the notation.

In fact, especially with our strong background in simplicial geometry that we have by now it is much more natural to not fix the domain of singular simplices to be $|\Delta^p|$, but rather allow continuous maps $\sigma\colon |\alpha| \to X$ for an arbitrary ordered simplex α as above. Consequently, in what follows we will always write singular simplices in this form, using the construction above. The only reason why we did not do so in the first place in Definition 6.2 is a very subtle set-theoretic one: Allowing maps $\sigma\colon |\alpha| \to X$ for an arbitrary ordered simplex α (hence in particular for an arbitrary underlying set for α) as a basis for $C_p(X)$ would require us to talk about the set of all sets, which unfortunately does not exist due to Russell's paradox [G1, Remark 1.13].

Having settled this issue, it is now completely straightforward to construct boundary maps for singular simplices, and thus singular homology as in Chapter 2:

Construction 6.5 (Boundary map for singular chains)**.** Let X be a topological space.

(a) For any singular simplex $\sigma\colon |\alpha| \to X$ as in Construction 6.4 and any ordered simplex β in D^α, i. e. with $\beta \subset \alpha$ as sets and thus $|\beta| \subset |\alpha|$, we set $\sigma|_\beta := \sigma|_{|\beta|}\colon |\beta| \to X$. (In our applications, β will always be obtained from α by leaving out some vertices and keeping the order of the remaining vertices, but we do not require this to be the case.) By linearity, we extend this construction to obtain a restriction map $C_p^{\mathrm{ord}}(D^\alpha) \to C_p(X)$, $c \mapsto \sigma|_c$.

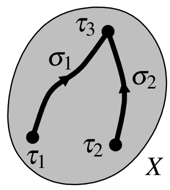
(b) For $p > 0$, the **boundary** of a singular p-simplex $\sigma\colon |\alpha| \to X$ for an ordered p-simplex $\alpha = (i_0, \dots, i_p)$ as in Construction 6.4 is the singular $(p-1)$-chain

$$\partial\sigma := \sigma|_{\partial\alpha} \overset{2.5\,(a)}{=} \sum_{k=0}^{p} (-1)^k \cdot \sigma|_{(i_0,\dots,\widehat{i_k},\dots,i_p)}.$$

Again, we extend this to linear **boundary maps** $\partial\colon C_p(X) \to C_{p-1}(X)$, and set this map to zero if $p \le 0$. Note that for this extension we need Remark 5.12 (b) stating that specifying a linear map on a basis also works for infinite-dimensional vector spaces.

(c) Following Definition 2.5 (b), a singular chain in the kernel of the boundary map is called **closed** or a **cycle**, and a singular chain in its image is called a **boundary**.

Example 6.6. The picture on the right shows the singular analogue of Examples 2.4 and 2.7: The paths σ_1 and σ_2 are (the images of) two singular 1-simplices in a topological space X, where the arrows indicate the order of their vertices. If we form from this the singular 1-chain $c = \sigma_1 + \sigma_2 \in C_1(X)$, its boundary is $\partial c = 2\tau_3 - \tau_1 - \tau_2 \in C_0(X)$, where τ_i for $i \in \{1,2,3\}$ denotes the map that sends the only point of a geometric 0-simplex to the corresponding marked point in the picture.



Remark 6.7. Let $\sigma\colon |\alpha| \to X$ be a singular simplex in a topological space X.

(a) It is obvious that $(\sigma|_\beta)|_\gamma = \sigma|_\gamma$ whenever $\gamma \subset \beta \subset \alpha$.

(b) By construction, it follows immediately from Lemma 2.12 (in its ordered version, see Construction 4.23) that

$$\partial^2\sigma = \partial\sigma|_{\partial\alpha} \overset{(a)}{=} \sigma|_{\partial^2\alpha} = 0.$$

Hence, the singular chain groups of Definition 6.2 with the boundary maps of Construction 6.5 (b) form a chain complex $C(X)$. We call it the **singular chain complex** of X.

Definition 6.8 (Singular homology)**.** Let X be a topological space. For any $p \in \mathbb{Z}$, the p-th **singular homology group** $H_p(X)$ is defined to be the p-th homology group $H_p(C(X)) := H_p(C(X,K))$ of the singular chain complex $C(X)$ of X. The corresponding Betti numbers are called the **Betti numbers** of X and denoted by

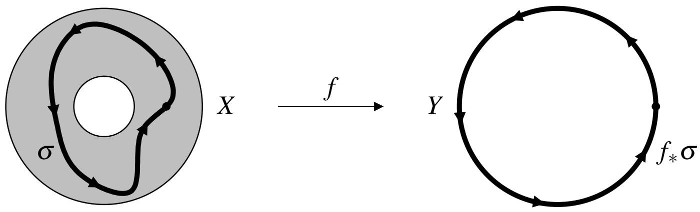
$$b_p(X) := b_p(X,K) = \dim H_p(X,K) \quad \in \mathbb{N} \cup \{\infty\}.$$

Construction 6.9 (Pushforwards and the singular chain functor)**.** In the same way as for simplicial homology in Construction 2.9 and Notation 2.19, we can make the assignment of the singular chain complex $C(X)$ to a topological space X into a functor by constructing a singular **pushforward map** f_* to any continuous map $f\colon X \to Y$ between two topological spaces X and Y. In fact, the idea for its construction is the same as well, namely just a composition with f: For a singular simplex $\sigma\colon |\alpha| \to X$ with α an ordered p-simplex, we set

$$f_*\sigma := f \circ \sigma\colon |\alpha| \to Y. \qquad (*)$$

It is clear that this can be extended to linear maps $f_*\colon C_p(X) \to C_p(Y)$ for all $p \in \mathbb{Z}$ by linearity, giving a morphism of chain complexes $f_*\colon C(X) \to C(Y)$, and that this makes the construction of singular chains into a functor $\mathsf{Top} \to \mathsf{Ch}$. Composing with the homology functor of Lemma 2.17, we thus see that taking the p-th singular homology group $H_p(X)$ of a topological space X is a functor $\mathsf{Top} \to \mathsf{Vect}$ for all $p \in \mathbb{Z}$ as well, i. e. that the pushforward is also well-defined as a map $f_*\colon H_p(X) \to H_p(X)$ on homology. In particular, this means that homeomorphic topological spaces indeed have naturally isomorphic singular homology groups – which was one of the goals of our new homology theory.

Intuitively, the formula $(*)$ just means that the pushforward is obtained by taking images under f. Consider for example the radial projection f from the annulus X to its outer circle Y in the picture below. In this case, the pushforward of the singular simplex $\sigma \in H_1(X)$ describing a path around the annulus is just a path $f_*\sigma \in H_1(Y)$ running once around the circle.



Remark 6.10 (First properties of singular homology)**.** Let X be a topological space. The following properties of simplicial homology carry over with essentially the same proof to the singular setting:

(a) If X has finitely many path-connected components $X_1, \dots, X_n$, we have as in Lemma 2.21

$$H_p(X) \cong H_p(X_1) \times \cdots \times H_p(X_n)$$

since every singular simplex must lie in one component, and its boundary will then be in the same component.

(b) A singular 0-simplex in X is just a map from a geometric 0-simplex (i. e. a point) to X, which we can identify with its image point. As in Lemma 2.22, for $X \neq \emptyset$ there is thus a well-defined surjective *degree map* $\deg\colon H_0(X) \to K$ sending each such point in X to 1, and this is an isomorphism if X is path-connected since then the difference of any two points is the boundary of a path (i. e. of a singular 1-simplex), and hence zero in homology.

(c) Consequently, if X has finitely many path-connected components, their number is equal to $b_0(X)$ as in Corollary 2.23. However, if X has infinitely many path-connected components (such as e. g. for $X = \mathbb{Z}$) then $b_0(X)$ will be infinite – showing in contrast to the simplicial situation that Betti numbers no longer need to be finite. This also means that we cannot immediately write down Euler characteristic computations unless we know beforehand that the numbers involved in it are finite.

(d) If X is the topological space with exactly one point, there is exactly one singular p-simplex in X for any $p \in \mathbb{N}$, namely the constant map. Hence, the singular chain complex $C(X)$ is exactly the same as in Example 4.24, showing that the singular homology of X is trivial: We have

$$b_p(X) = \begin{cases} 1 & \text{if } p = 0, \\ 0 & \text{if } p \neq 0. \end{cases}$$

As already mentioned, one of our main next goals is to show that, for any simplicial complex S, the singular homology groups of $|S|$ are indeed naturally isomorphic to the simplicial homology groups for of S – as one would hope (see Proposition 8.1). We will need several preparations before we can actually show this, but it is already obvious now that there is a natural map between the two theories by reinterpreting an ordered simplex in a simplicial complex as a singular simplex in its topological realization:

Construction 6.11 (Singular from simplicial homology)**.** Let S be a simplicial complex with standard geometric realization $|S|$. Note that any ordered p-simplex α in S gives rise to a singular p-simplex

$$\rho(\alpha)\colon |\alpha| \to |S|$$

in $|S|$, which is just the inclusion map. Extending by linearity, we thus obtain a linear map $\rho\colon C_p^{\text{ord}}(S) \to C_p(|S|)$, which commutes with the boundary maps since Definition 2.5 and Construction 6.5 are clearly compatible. Hence, $\rho\colon C^{\text{ord}}(S) \to C(|S|)$ is a morphism of chain complexes; we will call it the **topological realization morphism**. By Lemma 2.17 it gives rise to linear maps $H_p^{\text{ord}}(S) \to H_p(|S|)$ between the homology groups. As we have seen already in Proposition 4.25 that ordered simplicial homology agrees with ordinary simplicial homology, we can also interpret this as natural linear maps $H_p(S) \to H_p(|S|)$.

Moreover, it is checked immediately that this construction commutes with morphisms, i. e. that a morphism $f\colon S \to T$ of simplicial complexes gives rise to a commutative diagram

$$\begin{tikzcd} H_p^{\text{ord}}(S) \arrow[r] \arrow[d] & H_p(|S|) \arrow[d] \\ H_p^{\text{ord}}(T) \arrow[r] & H_p(|T|), \end{tikzcd}$$

in which the vertical maps are the (ordered) simplicial and singular pushforwards by f and $|f|$, respectively.

Note that the map ρ only uses the structure of $|S|$ as a topological space, so that we can in fact use any geometric realization $r\colon V(S) \to W$ as in Construction 1.13 to define it. Actually, the construction even works for an arbitrary map r to give a morphism of chain complexes from $C(S)$ to $C(r(S))$, or to $C(X)$ for any other topological space X that contains $r(S)$. We will use this in the following for prisms $P(S)$ as in Definition 3.7 to consider ρ as a morphism from $C(P(S))$ to $C(|S| \times [0,1])$ instead of to $C(|P(S)|)$, choosing the map r that sends a vertex i to $(e_i, 0)$ and i' to $(e_i, 1)$ for $i \in V(S)$. As we had mentioned in Construction 3.6, this is in fact a geometric realization of the prism $P(S)$, but we did not bother to prove this since as remarked above the definition of ρ also works without this assertion.

Construction 6.11 allows us already to carry over results from the simplicial to the singular world. Let us do this for the homotopy theory of Chapter 3 to derive that homotopic continuous maps between topological spaces induce the same maps on singular homology – using the corresponding statement in the simplicial setting of Lemma 3.8.

A visualization of this situation is given in the picture below, where $X = S^1$ is a circle, embedded as the inner and outer boundary of an annulus Y by the two maps $f, g\colon X \to Y$, respectively. Of course, f and g are homotopic maps given by moving radially away from the center. Now let us consider the singular 1-chain $c = \sigma_1 + \sigma_2 + \sigma_3$ running once around the circle, made up from three singular 1-simplices defined on a geometric 1-simplex $|\alpha|$.

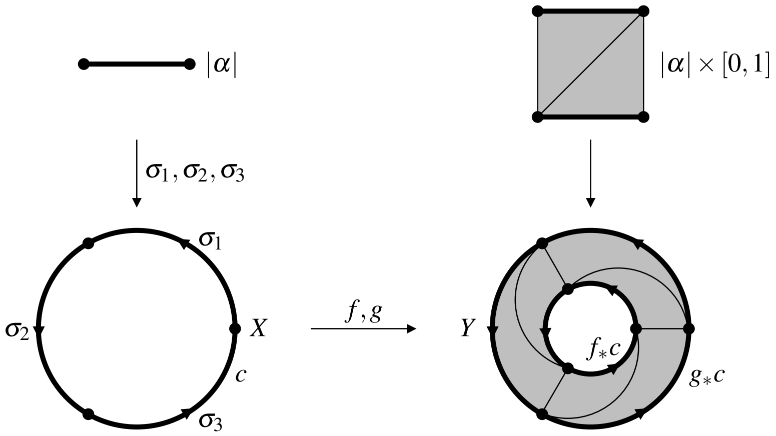
Of course, due to f and g being homotopic the pushforwards f_*c and g_*c should now define the same singular homology class in $H_1(Y)$. To prove this using our results from simplicial geometry, we can proceed for each singular simplex $\sigma_i\colon |\alpha| \to X$ in three steps:

- Consider the (simplicial) prism $P(D^\alpha)$ over D^α (resp. over α) as in Construction 3.6 and Definition 3.7. In the following picture, it is shown on the upper right.

- Pass from the simplicial to the singular world by considering this prism as a topological space rather than a simplicial complex, i. e. apply the map ρ from Construction 6.11.
- Map the prism (as a topological space) by the given homotopy between f and g continuously to the space X, shown as the "curved prisms" on the bottom right of the picture.

You can see from the picture that the three curved prisms together form a singular 2-chain whose boundary is $g_*c - f_*c$, showing that the two pushforwards agree in homology.

In the following proof, the three steps above are clearly visible when constructing the required chain homotopy between the singular homology groups.



Proposition 6.12 (Singular homotopies)**.** *Let $f,g\colon X \to Y$ be two homotopic maps between the topological spaces X and Y. Then the pushforwards f_* and g_* from $C(X)$ to $C(Y)$ are chain homotopic.*

In particular, by Proposition 3.3 these pushforwards induce the same map from $H_p(X)$ to $H_p(Y)$ for all $p \in \mathbb{Z}$.

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Proof. Let $R\colon X \times [0,1] \to Y$ be a homotopy between f and g, so that $R(\cdot,0) = f$ and $R(\cdot,1) = g$. We will construct from this a chain homotopy $h\colon C_p(X) \to C_{p+1}(Y)$ between f_* and g_*.

To do so, consider first for a fixed singular simplex $\sigma\colon |\alpha| \to X$ and $p \in \mathbb{Z}$ the composed linear map

$$h_\sigma\colon C_p^{\text{ord}}(D^\alpha) \xrightarrow{\tilde{h}} C_{p+1}^{\text{ord}}(P(D^\alpha)) \xrightarrow{\rho} C_{p+1}(|\alpha| \times [0,1]) \xrightarrow{R_*^\sigma} C_{p+1}(Y), \tag{1}$$

where $\tilde{h}$ is the simplicial prism map of Definition 3.7, ρ is the topological realization morphism of the prism in $|\alpha| \times [0,1]$ as in Construction 6.11, and R_*^σ is the singular pushforward by the continuous map $R^\sigma\colon |\alpha| \times [0,1] \to Y$, $(x,t) \mapsto R(\sigma(x),t)$ as in Construction 6.9. We set $h(\sigma) := h_\sigma(\alpha)$, and extend this by linearity to a linear map $h\colon C_p(X) \to C_{p+1}(Y)$.

Note that, if β is an ordered simplex with $\beta \subset \alpha$, to construct $h(\sigma|_\beta)$ we can still use σ in the last map of (1), i. e. we have $h(\sigma|_\beta) = h_\sigma(\beta)$. This means that

$$h(\partial\sigma) = h(\sigma|_{\partial\alpha}) = h_\sigma(\partial\alpha). \tag{2}$$

Now we know by (the ordered version of) Lemma 3.8 that $\tilde{h}$ is a chain homotopy between $\tilde{f}_*$ and $\tilde{g}_*$, where $\tilde{f},\tilde{g}\colon D^\alpha \to P(D^\alpha)$ are the inclusions of the bottom and top face, respectively, i. e. we have

$$\partial \circ \tilde{h} + \tilde{h} \circ \partial = \tilde{g}_* - \tilde{f}_* \quad \text{as morphisms } C^{\text{ord}}(D^\alpha) \to C^{\text{ord}}(P(D^\alpha)).$$

Composing this with the morphisms ρ and R_*^σ (that commute with ∂) thus gives

$$\partial \circ h_\sigma + h_\sigma \circ \partial = R_*^\sigma \circ \rho \circ \tilde{g}_* - R_*^\sigma \circ \rho \circ \tilde{f}_*,$$

and applying this to α yields by (2) the desired relation

$$\partial h(\sigma) + h(\partial\sigma) = g_*\sigma - f_*\sigma. \qquad \square$$

Corollary 6.13 (Singular homology is a homotopy invariant)**.** *If $f\colon X \to Y$ is a homotopy equivalence between two topological spaces X and Y, the pushforward $f_*\colon H_p(X) \to H_p(Y)$ is an isomorphism for all $p \in \mathbb{Z}$.*

Proof. Let $g\colon Y \to X$ be a homotopy inverse of f. Then $g \circ f\colon X \to X$ and $f \circ g\colon Y \to Y$ are homotopic to the identity, so $(g \circ f)_* = g_* \circ f_*\colon C(X) \to C(X)$ and $(f \circ g)_* = f_* \circ g_*\colon C(Y) \to C(Y)$ are chain homotopic to the identity by 6.12. Hence, $f_*\colon C(X) \to C(Y)$ is a chain homotopy equivalence, and thus by Corollary 3.5 induces an isomorphism on homology. □

Example 6.14 (Singular homology of contractible spaces)**.** By Corollary 6.13, any contractible topological space X has the same Betti numbers as a one-pointed space, i. e. $b_0(X) = 1$ with all other Betti numbers being zero by Remark 6.10 (d).

For example, this applies to all star-shaped subsets X of $\mathbb{R}^n$ for $n \in \mathbb{N}$, in particular to all geometric simplices or to $X = \mathbb{R}^n$ itself.

Exercise 6.15. Compute the singular homology of the following topological spaces:

(a) the rational numbers $\mathbb{Q}$ (with the subspace topology of $\mathbb{R}$);

(b) the space X_n of all positive definite symmetric real matrices of a given size $n \in \mathbb{N}_{>0}$ (with the subspace topology of the space $\mathbb{R}^{n\times n}$ of all $n \times n$-matrices).

Exercise 6.16. Let $X \subset \mathbb{R}^n$ be a non-empty convex subspace with $n \in \mathbb{N}$. We have seen that there is a linear degree map $\deg\colon C_0(X) \to K$ mapping any singular 0-simplex to $1 \in K$. Regarding K as chain complex, where K is the only non-zero term and sits in degree 0, construct an explicit chain homotopy showing that deg defines a chain homotopy equivalence $C(X) \to K$.

Exercise 6.17 (Invariance of singular homology under reparametrizations)**.** Let X be a topological space. For all $p \in \mathbb{N}$ and every singular simplex $\sigma\colon |\Delta^p| \to X$ we choose a homeomorphism $f_\sigma\colon |\Delta^p| \to |\Delta^p|$ compatible with taking boundaries, i. e. such that the restriction of f_σ to $|\beta|$ for any ordered simplex β in $D^{\{0,\dots,p\}}$ is $f_{(\sigma|_\beta)}$, and set $\tilde{\sigma} := \sigma \circ f_\sigma$.

Show that the linear maps $C_p(X) \to C_p(X)$ for $p \in \mathbb{Z}$ obtained by mapping any p-simplex σ to $\tilde{\sigma}$ induce isomorphisms in homology. This justifies that we usually only draw the image of singular simplices in X when visualizing homology classes.

Exercise 6.18. In this exercise we will see why singular homology is constructed with simplices and not with cubes.

A definition with cubes would be: For any topological space X and $p \in \mathbb{N}$, let $C_p^\square(X)$ be the vector space with basis given by all continuous maps $\sigma\colon [0,1]^p \to X$. We define a boundary map by

$$\partial\colon C_p^\square(X) \to C_{p-1}^\square(X),\ \sigma \mapsto \sum_{k=1}^{p} (-1)^k \left(\sigma|_{x_k=1} - \sigma|_{x_k=0}\right) \quad \text{for } p > 0,$$

where each face $\{x \in [0,1]^p : x_k = a\}$ for $a \in \{0,1\}$ is identified with $[0,1]^{p-1}$, keeping the order of the remaining coordinates. Moreover, we construct a subdivision map

$$\mathrm{sd}\colon C_p^\square(X) \to C_p^\square(X),\ \sigma \mapsto \sum_Q \sigma|_Q,$$

where the sum is taken over all 2^p subcubes $Q \subset [0,1]^p$ of side length $\frac{1}{2}$ obtained by subdividing each interval $[0,1]$ at its midpoint, and all these subcubes are rescaled to $[0,1]^p$, so that $\sigma|_Q \in C_p^\square(X)$.

It is easy to check that this makes $C^\square(X)$ into a chain complex and $\mathrm{sd}\colon C^\square(X) \to C^\square(X)$ into a morphism of chain complexes.

(a) Compute the "cubical homology of a point", i. e. the Betti numbers of the chain complex $C^\square(X)$ for the topological space X with only one point.

(b) For an arbitrary non-empty topological space X and any ground field K, show that sd is never chain homotopic to the identity.

7. The Mayer-Vietoris Sequence in Singular Homology

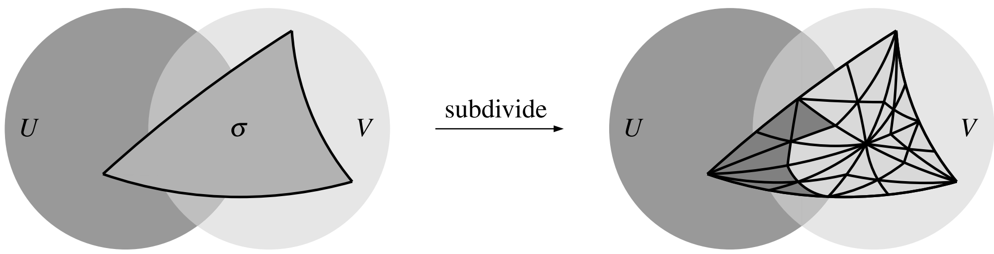
In simplicial geometry, one of the main tools both for computing and for proving general statements on homology was clearly the Mayer-Vietoris sequence of Corollary 4.12. We therefore want to investigate now if a corresponding result holds in singular homology as well, i. e. if for two subspaces U and V of a topological space we have a long exact sequence

$$\cdots \longrightarrow H_p(U\cap V) \longrightarrow H_p(U)\times H_p(V) \longrightarrow H_p(U\cup V) \longrightarrow H_{p-1}(U\cap V) \longrightarrow \cdots. \qquad (1)$$

As in simplicial geometry, we would expect this sequence to come from a short exact sequence of singular chain complexes

$$0 \longrightarrow C(U\cap V) \longrightarrow C(U)\times C(V) \longrightarrow C(U\cup V) \longrightarrow 0, \qquad (2)$$

with morphisms constructed from the pushforwards by the various inclusions. However, there is a big difference compared to the simplicial setting here: Whereas an abstract simplex in the union of two simplicial complexes must always lie in one of these simplicial complexes, the corresponding statement is no longer true in the singular setting as there will be singular simplices in $U\cup V$ that do not lie in U or V alone, such as σ in the picture below on the left. This means that the last map $C(U)\times C(V)\to C(U\cup V)$ in (2) is no longer surjective. In other words, the sequence (2) fails to be exact, so that we cannot use it to derive the long exact sequence (1) from it.



Fortunately, it turns out however that the long exact sequence (1) still exists – we just have to adapt our proof. The idea is that it should be possible – at least for open subsets U and V that overlap a bit – to subdivide σ fine enough so that each singular simplex in the subdivision lies in U or V, restoring the exactness of (2). We have shown this situation on the right side of the picture, where we took a double barycentric subdivision of σ and shaded the singular subsimplices of σ in dark or light color depending on whether they lie in U or V.

It is the main goal of this chapter to prove these statements. Let us start by formalizing the observation that (2) would be exact – and hence we could derive the long exact sequence (1) from it – if we could restrict our attention to singular simplices in $U\cup V$ that lie completely in U or V.

Construction 7.1 (Singular chains on two open subsets)**.** Let U and V be subsets of a topological space X. For all $p\in\mathbb{Z}$ we denote by $C_p(U\,|\,V)$ the vector subspace generated by all singular simplices $\sigma\colon |\alpha|\to U\cup V$ whose image lies completely in U or in V. Note that the singular boundary map sends $C_p(U\,|\,V)$ to $C_{p-1}(U\,|\,V)$, and hence we obtain a chain complex $C(U\,|\,V)$ whose homology groups we denote by $H_p(U\,|\,V)$. By construction, the sequence of chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(U\cap V) & \longrightarrow & C(U)\times C(V) & \longrightarrow & C(U\,|\,V) & \longrightarrow & 0 \\ & & c & \longmapsto & (c,-c) & & & & \\ & & & & (c,c') & \longmapsto & c+c' & & \end{array}$$

is then exact just as in Example 4.7, so that we obtain a long exact homology sequence

$$\cdots \longrightarrow H_p(U\cap V) \longrightarrow H_p(U)\times H_p(V) \longrightarrow H_p(U\,|\,V) \longrightarrow H_{p-1}(U\cap V) \longrightarrow \cdots$$

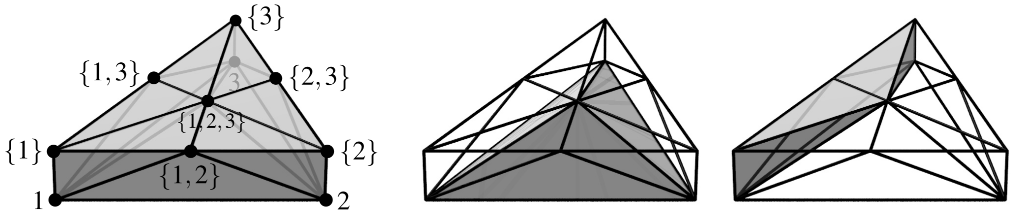
by Proposition 4.8. Hence, to obtain the desired result we only have to show that the inclusion morphism $C(U\,|\,V) \to C(U\cup V)$ induces an isomorphism in homology $H_p(U\,|\,V) \to H_p(U\cup V)$ for all $p\in\mathbb{Z}$ if U and V are open. As explained above, it is the main task of this chapter to prove this, with an inverse morphism given by sufficiently fine subdivisions.

To study these singular subdivisions and show that they give rise to an isomorphism in homology, let us first go back to simplicial geometry and construct a homotopy between a simplicial complex and its barycentric subdivision. For this we need an alternative version of the prism of Construction 3.6.

Construction 7.2 (Subdivision prism construction)**.** Let S be a simplicial complex. We want to construct a simplicial complex $Q(S)$ realizing the prism $|S|\times[0,1]$ by taking S for the bottom face and its barycentric subdivision $\mathrm{Sd}(S)$ for the top face. To do so, we connect every vertex $\alpha\in V(\mathrm{Sd}(S))$ to all subsets of α (considered as vertices of $\mathrm{Sd}(S)$ in the top face) and all elements of α (considered as vertices of S in the bottom face), i. e. we set

$$Q(S) := \big\{\{i_0,\dots,i_{k-1},\alpha_k,\dots,\alpha_p\} : k\le p \text{ and } \{i_0,\dots,i_{k-1}\}\subset\alpha_k\subset\cdots\subset\alpha_p\in S\big\}.$$

The following picture on the left shows this construction for $S = D^{\{1,2,3\}}$.



As in our original prism construction, we do not need an exact proof that $Q(S)$ realizes $|S|\times[0,1]$ as we will just define the corresponding homotopy algebraically and thus also give an algebraic proof that it works. For this we will proceed by induction on the dimension rather than writing down an explicit formula as in Definition 3.7 (b) because $Q(S)$ is a little more complicated than $P(S)$ in Construction 3.6. First define a "connecting to α map" b_α for any oriented simplex of $Q(D^\alpha)$ by

$$b_\alpha[i_0,\dots,i_{k-1},\alpha_k,\dots,\alpha_p] := [\alpha,i_0,\dots,i_{k-1},\alpha_k,\dots,\alpha_p],$$

extended to a linear map $b_\alpha\colon C_p(Q(D^\alpha))\to C_{p+1}(Q(D^\alpha))$ for all p. Note that then

$$\partial b_\alpha[i_0,\dots,i_{k-1},\alpha_k,\dots,\alpha_p] = [i_0,\dots,i_{k-1},\alpha_k,\dots,\alpha_p] - b_\alpha(\partial[i_0,\dots,i_{k-1},\alpha_k,\dots,\alpha_p])$$

by Definition 2.5 (a), so we have

$$\partial b_\alpha + b_\alpha\partial = \mathrm{id} \tag{1}$$

on $C(Q(D^\alpha))$. We can use this to define the *subdivision map* $\mathrm{sd}\colon C(S)\to C(\mathrm{Sd}(S))$ (that intuitively sends a simplex at the bottom of $Q(S)$ to its subdivision at the top) inductively by $\mathrm{sd}([i]) := [\{i\}]$ for a 0-simplex, and

$$\mathrm{sd}(\alpha) := b_\alpha(\mathrm{sd}(\partial\alpha)) \tag{2}$$

otherwise: To construct the subdivision of α, we subdivide its boundary and connect every simplex in this subdivision of $\partial\alpha$ to the barycenter. Let us also check inductively that sd is a morphism of chain complexes, i. e. commutes with the boundary map: This is obvious in dimension 0, and for $\alpha\in C_p(S)$ with $p>0$ we have

$$\partial\,\mathrm{sd}(\alpha) \overset{(2)}{=} \partial b_\alpha(\mathrm{sd}(\partial\alpha)) \overset{(1)}{=} \mathrm{sd}(\partial\alpha) - b_\alpha(\partial\,\mathrm{sd}(\partial\alpha)) \overset{\text{ind.}}{=} \mathrm{sd}(\partial\alpha) - b_\alpha(\mathrm{sd}(\partial^2\alpha)) = \mathrm{sd}(\partial\alpha). \tag{3}$$

Finally, we have to construct our homotopy $h\colon C_p(S)\to C_{p+1}(Q(S))$ that assigns to every simplex $\alpha\in C(S)$ the prism over it. If you believe in the left picture above you may want to directly accept the following Lemma 7.3 – which is just the algebraic version of the picture – without going through the following construction and computations. Otherwise, as shown in the other pictures above in the case $\alpha = \{1,2,3\}$, this prism can be assembled recursively from two components: a pyramid over α with apex the vertex $\alpha \in V(\mathrm{Sd}(S))$ (as in the middle picture), and for every face of α the prism over it, connected again to $\alpha \in V(\mathrm{Sd}(S))$ (as in the right picture). In formulas (and getting the signs right) this means that we set $h([i]) := [i,\{i\}]$ in dimension 0, and

$$h(\alpha) := -b_\alpha(\alpha + h(\partial\alpha)) \tag{4}$$

otherwise. Indeed, we can then check that h is a homotopy resp. prism map between the top and bottom face, i.e. that $\partial h(\alpha) + h(\partial\alpha) = \mathrm{sd}(\alpha) - \alpha$ for all $\alpha \in C_p(S)$: Again this is obvious for $p = 0$, and for $p > 0$ we have

$$\begin{aligned}
\partial h(\alpha) + h(\partial\alpha) &\overset{(4)}{=} -\partial b_\alpha(\alpha + h(\partial\alpha)) + h(\partial\alpha) \\
&\overset{(1)}{=} -\alpha - h(\partial\alpha) + b_\alpha(\partial\alpha + \partial h(\partial\alpha)) + h(\partial\alpha) \\
&= -\alpha + b_\alpha(\partial\alpha + \mathrm{sd}(\partial\alpha) - \partial\alpha - h(\partial^2\alpha)) \qquad \text{(induction applied to } \partial h(\partial\alpha)) \\
&\overset{(2)}{=} \mathrm{sd}(\alpha) - \alpha.
\end{aligned}$$

Summarizing, we have thus shown the following statement analogous to Lemma 3.8:

Lemma 7.3 (Subdivision homotopy)**.** *Let S be a simplicial complex. With notations as in Construction 7.2, the prism map h is a homotopy between the top and bottom face maps of $Q(S)$, i.e. we have*

$$\partial \circ h + h \circ \partial = g - f$$

as morphisms $C(S) \to C(Q(S))$ of chain complexes, where $f\colon C(S) \to C(Q(S))$, $\alpha \mapsto \alpha$ is the push-forward by the bottom face inclusion and $g\colon C(S) \to C(Q(S))$, $\alpha \mapsto \mathrm{sd}(\alpha)$ subdivides a simplex and embeds it in the top face.

As in Proposition 6.12, let us now push this homotopy to the singular setting by the topological realization morphism, giving a homotopy between singular simplices and their barycentric subdivisions. In fact, the proof of Proposition 7.5 below is almost identical to that of Proposition 6.12, just using our new prism construction and forgetting about the given topological homotopy.

Construction 7.4 (Barycentric subdivision of singular simplices)**.** Let $\sigma\colon |\alpha| \to X$ be a singular simplex in a topological space X, with α an ordered simplex as in Construction 6.4.

(a) As in Construction 6.5 (a), for every ordered simplex β in the barycentric subdivision of α we have $r(\beta) \subset |\alpha|$ for the barycentric realization r, and set $\sigma|_\beta := \sigma|_{r(\beta)}\colon r(\beta) \to X$. We extend this by linearity to a restriction map $C_p^{\mathrm{ord}}(\mathrm{Sd}(D^\alpha)) \to C_p(X)$, $c \mapsto \sigma|_c$ for all $p \in \mathbb{Z}$.

(b) We define the **barycentric subdivision** of σ to be $\mathrm{sd}(\sigma) := \sigma|_{\mathrm{sd}(\alpha)}$ as in (a), with $\mathrm{sd}(\alpha)$ defined in Construction 7.2. Again, by linearity we extend this to a singular *subdivision map* $\mathrm{sd}\colon C_p(X) \to C_p(X)$ for all $p \in \mathbb{Z}$. Note that

$$\partial\,\mathrm{sd}(\sigma) \overset{6.5\,(b)}{=} \sigma|_{\partial\,\mathrm{sd}(\alpha)} \overset{7.2}{=} \sigma|_{\mathrm{sd}(\partial\alpha)} = \mathrm{sd}(\partial\sigma),$$

so $\mathrm{sd}\colon C(X) \to C(X)$ is a morphism of chain complexes, and thus induces maps on homology $H_p(X) \to H_p(X)$ by Lemma 2.17.

Proposition 7.5 (Singular subdivisions)**.** *For any topological space X, the subdivision morphism* $\mathrm{sd}\colon C(X) \to C(X)$ *of Construction 7.4 (b) is chain homotopic to the identity. In particular, by Proposition 3.3 the corresponding maps* $\mathrm{sd}\colon H_p(X) \to H_p(X)$ *on homology are the identity for all $p \in \mathbb{Z}$.*

Proof. To construct a chain homotopy between sd and the identity, consider first for a fixed singular simplex $\sigma\colon |\alpha| \to X$ and $p \in \mathbb{Z}$ the composed linear map

$$h_\sigma\colon C_p^{\mathrm{ord}}(D^\alpha) \xrightarrow{\tilde h} C_{p+1}^{\mathrm{ord}}(Q(D^\alpha)) \xrightarrow{\rho} C_{p+1}(|\alpha| \times [0,1]) \xrightarrow{R_*^\sigma} C_{p+1}(Y), \tag{1}$$

where $\tilde{h}$ is the ordered version of the simplicial homotopy of Construction 7.2, ρ is the topological realization morphism of the prism in $|\alpha| \times [0,1]$ as in Construction 6.11, and R_*^σ is the singular pushforward by the continuous map $R^\sigma \colon |\alpha| \times [0,1] \to X,\ (x,t) \mapsto \sigma(x)$ as in Construction 6.9. We set $h(\sigma) := h_\sigma(\alpha)$ and extend this by linearity to a linear map $h \colon C_p(X) \to C_{p+1}(X)$.

As in the proof of Proposition 6.12, if β is an ordered simplex with $\beta \subset \alpha$, to construct $h(\sigma|_\beta)$ we can still use σ in the last map of (1), i. e. we have $h(\sigma|_\beta) = h_\sigma(\beta)$. Applying this to the boundary faces of α, we thus see that

$$h(\partial\sigma) = h(\sigma|_{\partial\alpha}) = h_\sigma(\partial\alpha). \tag{2}$$

Now we know by (the ordered version of) Lemma 7.3 that

$$\partial \circ \tilde{h} + \tilde{h} \circ \partial = \tilde{g} - \tilde{f} \quad \text{as morphisms } C^{\text{ord}}(D^\alpha) \to C^{\text{ord}}(Q(D^\alpha)),$$

where $\tilde{f}$ is the bottom face inclusion and $\tilde{g}$ the top face subdivision map. Composing this with the morphisms ρ and R_*^σ thus gives

$$\partial \circ h_\sigma + h_\sigma \circ \partial = R_*^\sigma \circ \rho \circ \tilde{g} - R_*^\sigma \circ \rho \circ \tilde{f},$$

and applying this to α yields by (2) the desired equation

$$\partial h(\sigma) + h(\partial\sigma) = g(\sigma) - f(\sigma). \qquad \square$$

Remark 7.6. Assume that the topological space X in Proposition 7.5 is the union of two subspaces U and V. Then the subdivision map sd as well as all other maps involved in (the proof of) the proposition clearly map simplices in U or V again to simplices in U or V, respectively. Hence, literally the same proof shows that sd is also chain homotopic to the identity as morphisms of chain complexes $C(U\,|\,V) \to C(U\,|\,V)$, and thus induces the identity $H_p(U\,|\,V) \to H_p(U\,|\,V)$ for all $p \in \mathbb{Z}$ as well.

We have thus seen now that subdivisions of singular simplices do not affect homology. To carry out our idea to establish the singular Mayer-Vietoris sequence, it therefore only remains to show that sufficiently fine subdivisions of singular simplices in a union $U \cup V$ of two open subsets U and V of a topological space consist only of simplices in U or V. Again, let us start with a simplicial preparation proving the geometrically obvious fact that repeated barycentric subdivisions of a geometric simplex will lead to arbitrarily small subsimplices.

Lemma 7.7. *Let S be a simplicial complex. For any $\varepsilon \in \mathbb{R}_{>0}$ there is a number $n \in \mathbb{N}$ such that every edge of the n-fold barycentric subdivision $\mathrm{Sd}^n(S)$ has length smaller than ε (in its corresponding barycentric subdivision, and with respect to any fixed norm).*

Proof. For $n \in \mathbb{N}$ and any vertex $i \in V(\mathrm{Sd}^n(S))$ let us denote by $r(i) \in \mathbb{R}^{V(S)}$ the position of this vertex in its barycentric realization. Moreover, let

$$a_n := \max\{\|r(i_0) - r(i_1)\| : \{i_0, i_1\} \in \mathrm{Sd}^n(S)\}$$

the maximum length of an edge in $\mathrm{Sd}^n(S)$ in this realization. To see what happens to this number when we pass from $\mathrm{Sd}^n(S)$ to $\mathrm{Sd}^{n+1}(S)$, consider any edge between two vertices of $\mathrm{Sd}^{n+1}(S)$, which by Definition 1.20 we can write as $\{i_0,\dots,i_l\}$ and $\{i_0,\dots,i_p\}$ with $0 \le l < p$ for a p-simplex $\{i_0,\dots,i_p\} \in \mathrm{Sd}^n(S)$. By Proposition 1.21 we then have

$$r(\{i_0,\dots,i_p\}) - r(\{i_0,\dots,i_l\}) = \frac{r(i_0)+\cdots+r(i_p)}{p+1} - \frac{r(i_0)+\cdots+r(i_l)}{l+1} = \sum_{j=0}^{l} \sum_{k=l+1}^{p} \frac{r(i_k)-r(i_j)}{(l+1)(p+1)},$$

and thus by taking norms

$$\|r(\{i_0,\dots,i_p\}) - r(\{i_0,\dots,i_l\})\| \le \sum_{j=0}^{l} \sum_{k=l+1}^{p} \frac{a_n}{(l+1)(p+1)} = \frac{p-l}{p+1}\,a_n \le \frac{p}{p+1}\,a_n \le \frac{\dim S}{\dim S+1}\,a_n.$$

Hence, we obtain

$$a_{n+1} \le \frac{\dim S}{\dim S+1}\,a_n, \quad \text{and thus by induction} \quad a_n \le \left(\frac{\dim S}{\dim S+1}\right)^n a_0.$$

11 As this bound tends to 0 for $n \to \infty$, the result follows. $\square$

Corollary 7.8. *Let $c \in C_p(U \cup V)$ be a singular p-chain for some $p \in \mathbb{N}$ in the union of two open subsets U and V of a topological space. Then for some $n \in \mathbb{N}$ we have $\mathrm{sd}^n(c) \in C_p(U\,|\,V)$, i. e. the n-fold barycentric subdivision of c contains only singular simplices lying entirely in U or V.*

Proof. It suffices to consider a single singular simplex $\sigma\colon |\alpha| \to U \cup V$, since c contains only finitely many of such simplices, and we can take the maximum n needed for one of them.

Note that then the geometric simplex $|\alpha|$ is compact and covered by the two open subsets $\sigma^{-1}(U)$ and $\sigma^{-1}(V)$. It is a result from topology that in this situation there is an $\varepsilon \in \mathbb{R}_{>0}$ (a so-called *Lebesgue number*) such that any open ε-ball in $|\alpha|$ (i. e. with any center point) lies entirely in $\sigma^{-1}(U)$ or $\sigma^{-1}(V)$ [G3, Lemma 8.10]. By Lemma 7.7 applied to D^α there is now a number $n \in \mathbb{N}$ such that the barycentric realization $r(\beta) \subset |\alpha|$ of any simplex β in the n-fold subdivision of α has all edge lengths smaller than ε. Hence $r(\beta)$ lies within a ball of radius less than ε, and must therefore be contained in $\sigma^{-1}(U)$ or $\sigma^{-1}(V)$; which means that $\sigma|_\beta$ lies in U or V. $\square$

Corollary 7.9. *Let U and V be open subsets of a topological space. Then for all $p \in \mathbb{Z}$ the map $H_p(U\,|\,V) \to H_p(U \cup V)$ in homology induced by the inclusion morphism of chain complexes $C(U\,|\,V) \to C(U \cup V)$ is an isomorphism.*

Proof. We will show that the map is injective and surjective.

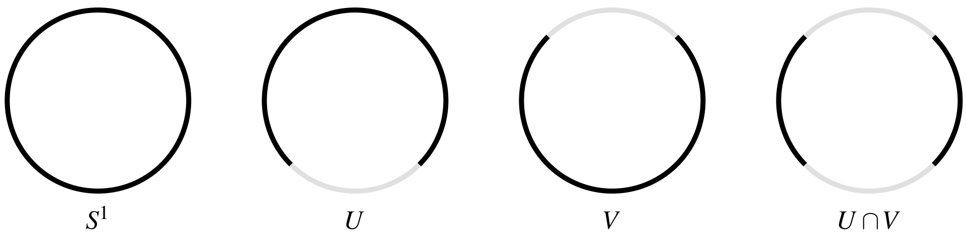
- Injectivity: Let $c \in H_p(U\,|\,V)$ with $c = 0 \in H_p(U \cup V)$, i. e. we have $c = \partial c'$ for a chain $c' \in C_{p+1}(U \cup V)$. By Corollary 7.8 there is some $n \in \mathbb{N}$ with $\mathrm{sd}^n(c') \in C_{p+1}(U\,|\,V)$. Then $\partial\,\mathrm{sd}^n(c') = \mathrm{sd}^n(\partial c') = \mathrm{sd}^n(c)$, so $\mathrm{sd}^n(c) = 0 \in H_p(U\,|\,V)$. By Remark 7.6 this means that $c = 0 \in H_p(U\,|\,V)$.
- Surjectivity: Let $c \in H_p(U \cup V)$, i. e. $\partial c = 0$. By Corollary 7.8 we have $\mathrm{sd}^n(c) \in H_p(U\,|\,V)$. It is mapped to $\mathrm{sd}^n(c) \in H_p(U \cup V)$, which agrees with c by Proposition 7.5. $\square$

Corollary 7.10 (The singular Mayer-Vietoris homology sequence)**.** *For any two open subsets U and V of a topological space, there is an exact **Mayer-Vietoris sequence***

$$\cdots \longrightarrow H_p(U \cap V) \longrightarrow H_p(U) \times H_p(V) \longrightarrow H_p(U \cup V) \longrightarrow H_{p-1}(U \cap V) \longrightarrow \cdots.$$

Proof. This follows immediately by combining Construction 7.1 with Corollary 7.9. $\square$

Example 7.11 (Homology of a circle)**.** As a simple concrete example, let us compute the singular homology of a circle S^1 by writing it as $S^1 = U \cup V$ as in the picture below, where U, V, and $U \cap V$ are drawn in black.



As U and V are contractible, their Betti numbers are

$$b_p(U) = b_p(V) = \begin{cases} 1 & \text{if } p = 0, \\ 0 & \text{if } p \neq 0 \end{cases}$$

by Example 6.14. Moreover, $U \cap V$ is a disjoint union of two two contractible spaces, hence by Remark 6.10 (a) we have $b_0(U \cap V) = 2$ and $b_p(U \cap V) = 0$ for all $p \neq 0$. The Mayer-Vietoris sequence of Corollary 7.10 thus splits for $p > 1$ as

$$\underbrace{H_p(U) \times H_p(V)}_{=0} \longrightarrow H_p(S^1) \longrightarrow \underbrace{H_{p-1}(U \cap V)}_{=0},$$

showing that in this case $b_p(S^1) = 0$, and in its last part as

$$\underbrace{H_1(U)\times H_1(V)}_{=0} \longrightarrow H_1(S^1) \longrightarrow \underbrace{H_0(U\cap V)}_{\cong K^2} \longrightarrow \underbrace{H_0(U)\times H_0(V)}_{\cong K^2} \longrightarrow H_0(S^1) \longrightarrow 0.$$

But S^1 is path-connected, so we have $b_0(S^1) = 1$ by Remark 6.10 (b), and thus it also follows from this sequence that $b_1(S^1) = 1$ by Remark 4.5 (b).

Note that this is clearly what we would have expected from geometry: firstly, since S^1 is just the topological realization of the simplicial 1-sphere $S^{\{1,2,3\}}$, which has the same Betti numbers by Example 2.26; and secondly, since S^1 has just one “one-dimensional hole”.

Remark 7.12. Due to the need for subdivisions to pass from $C(U\,|\,V)$ to $C(U\cup V)$ in Construction 7.1, the connecting map $H_p(U\cup V) \to H_{p-1}(U\cap V)$ for $p\in\mathbb{Z}$ in the singular Mayer-Vietoris homology sequence of Corollary 7.10 is a little more complicated than in the simplicial case: As in (1) in the introduction to Chapter 4, the first step in this map is to write a closed chain $c\in H_p(U\cup V)$ as a sum $c = c_1 + c_2$ of two chains in U and V, respectively – but for singular chains, we will in general need to apply a suitable subdivision first to make this splitting possible.

Clearly, we could go on now and use the Mayer-Vietoris sequence of Corollary 7.10 to compute the singular homology of many other topological spaces. For example, it would be just as easy to determine the singular homology of the other spheres S^n in the same way as well – to obtain the same result as in the simplicial case in Example 3.13. However, our very next step will be to prove that the simplicial homology of a simplicial complex always agrees with the singular homology of its topological realization anyway. So let us not waste any time now with concrete examples and go straight to the proof of this general statement in the next chapter, which will then immediately give us the singular homology of most spaces we are interested in.

Exercise 7.13 (Relative homology)**.** For any subset Y of a topological space X and $p\in\mathbb{N}$, let $C_p(X;Y) := C_p(X)/C_p(Y)$. The boundary maps of $C(X)$ descend to these quotients and make $C(X;Y)$ into a chain complex. The homology groups of this complex are called the *relative homology groups* of X and Y and denoted by $H_p(X;Y)$. Of course, by definition there is then a short exact sequence of chain complexes

$$0 \longrightarrow C(Y) \longrightarrow C(X) \longrightarrow C(X;Y) \longrightarrow 0,$$

which induces a corresponding long exact sequence in homology.

Now let $A\subset Y\subset X$ such that A is closed and Y is open in X. Show that the inclusion maps induce isomorphisms on relative homology groups $H_p(X\backslash A; Y\backslash A) \to H_p(X;Y)$.

(Hint: It is useful to express $C(X\backslash A; Y\backslash A)$ in terms of the complex $C(U\,|\,V)$ for suitable U and V.)

Exercise 7.14 (Local homology)**.** Let X be a topological space and $x\in X$ such that $\{x\}\subset X$ is closed.

(a) Show that the relative homology groups $H_p(U;U\backslash\{x\})$ of Exercise 7.13, where U is any open neighborhood of x, are independent of U. They are called the *local homology groups* of X at x.

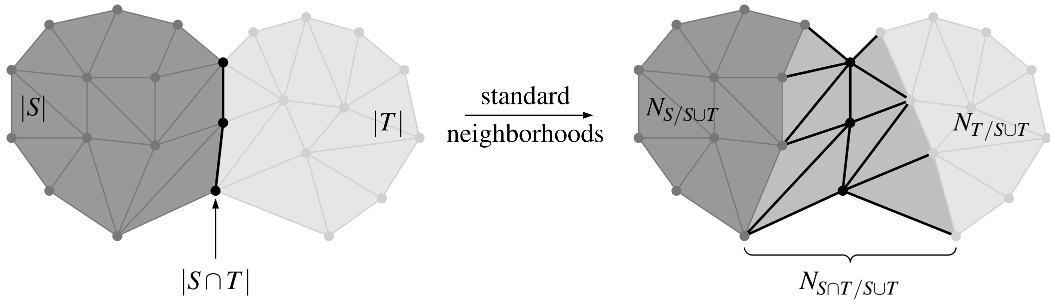
(b) Compute these local homology groups for the half-space $X = \{(x_1,\dots,x_n)\in\mathbb{R}^n : x_n \geq 0\}$ with $n\in\mathbb{N}_{>0}$, and any $x\in X$.

8. Comparison of Simplicial and Singular Homology

Having introduced and studied both simplicial and singular homology, we are now ready to compare these two theories. In this chapter, we will thus construct a way to pass from simplicial to singular homology, and vice versa from singular to simplicial homology.

Going from the simplicial to the singular setting seems to be straightforward: Every simplicial complex S has a topological realization $|S|$ for which singular homology is defined, and we have already seen in Construction 6.11 that there are natural topological realization morphisms $H_p(S) \to H_p(|S|)$ for all $p \in \mathbb{Z}$. We are going to show now that these maps are isomorphisms, i. e. that singular homology agrees with simplicial homology in this sense. In fact, the idea behind this statement is quite simple: We know already that both theories are trivial on simplicial disks and satisfy Mayer-Vietoris, so they should just be the same by Mayer-Vietoris induction.

Unfortunately, as it happens quite often in topology, there are some technical complications with this argument. Namely, the singular Mayer-Vietoris statement of Corollary 7.10 requires the two subspaces to be open, but (the topological realizations of) simplicial subcomplexes are not. For example, consider the picture below on the left, which is completely analogous to the situation in the introduction to Chapter 7, but in which U and V are replaced by the topological realizations of two simplicial complexes S and T. In this case the topological realization of $S \cap T$ is just a line, and thus it is clear that no subdivision of a singular simplex covering both spaces can help us to make each part of it lie in $|S|$ or $|T|$.



From the picture, the obvious way around this problem seems to be to replace $|S|$ and $|T|$ by open neighborhoods, such as the standard neighborhoods $N_{S/S\cup T}$ and $N_{T/S\cup T}$ (see Proposition 1.24) in the picture above on the right. However, we have to make sure of course that this does not change the homology of these spaces, i. e. that $|S|$ and $|T|$ are deformation retracts of these neighborhoods. By Proposition 1.24 (b) this is guaranteed if all subcomplexes are full, which we can achieve by a barycentric subdivision due to Lemma 1.23. Luckily, this barycentric subdivision changes neither the simplicial nor the singular homology of our space, so that our argument finally works out. Let us formulate this proof rigorously:

Proposition 8.1 (Singular homology of simplicial complexes)**.** *For any simplicial complex S, the topological realization morphism of Construction 6.11 induces isomorphisms between the simplicial homology $H_p(S)$ and the singular homology $H_p(|S|)$ for all $p \in \mathbb{Z}$.*

Proof. We will prove the statement by Mayer-Vietoris induction as in Lemma 4.18. For a simplicial disk D^α, both homologies are trivial: the simplicial homology of D^α by Example 3.12, and the singular homology of $|D^\alpha|$ by Example 6.14.

For the induction step, let S and T be two simplicial complexes such that the statement of the proposition holds for S, T, and $S \cap T$. By replacing these simplicial complexes by their barycentric subdivisions (which changes neither their simplicial homology nor their topological realizations by

Proposition 4.22 and Proposition 1.21), we can assume by Lemma 1.23 that $S \cap T$ is a full subcomplex of S and T, and that S and T are full subcomplexes of $S \cup T$. On the singular homology side, we can then replace $|S|$, $|T|$, and $|S \cap T|$ by their standard neighborhoods in $|S \cup T|$ as in Proposition 1.24 as they are deformation retracts of these neighborhoods and thus have the same homology by Corollary 6.13. Now consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^{\mathrm{ord}}(S\cap T) & \longrightarrow & C^{\mathrm{ord}}(S)\times C^{\mathrm{ord}}(T) & \longrightarrow & C^{\mathrm{ord}}(S\cup T) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C(N_{S\cap T/S\cup T}) & \longrightarrow & C(N_{S/S\cup T})\times C(N_{T/S\cup T}) & \longrightarrow & C(N_{S/S\cup T}\,|\,N_{T/S\cup T}) & \longrightarrow & 0, \end{array}$$

where the maps are as follows:

- The top row is just the simplicial short exact Mayer-Vietoris sequence of Example 4.7.
- The bottom row is the singular short exact Mayer-Vietoris sequence of Construction 7.1, noting that $N_{S\cap T/S\cup T} = N_{S/S\cup T} \cap N_{T/S\cup T}$ by Exercise 1.26 (b).
- The columns are the topological realization morphisms of Construction 6.11, followed by the pushforwards by the inclusions of $|S|$, $|T|$, and $|S \cap T|$ in their standard neighborhoods. On the right side of the diagram, note that any ordered simplex in $S \cup T$ actually maps to $N_{S/S\cup T}$ or $N_{T/S\cup T}$, so we can consider the vertical map as going to $C(N_{S/S\cup T}\,|\,N_{T/S\cup T})$ instead of $C(N_{S/S\cup T} \cup N_{T/S\cup T})$.

Let us now take the homology in this diagram as in Construction 4.10. In the top row we just get the ordinary simplicial homology (strictly speaking in the ordered version, but this does not make a difference by Proposition 4.25). In the bottom row we get the singular homology of the topological realizations (again, strictly speaking, in the right term we get $H_p(N_{S/S\cup T}\,|\,N_{T/S\cup T})$ instead of $H_p(N_{S/S\cup T} \cup N_{T/S\cup T}) \overset{1.26\,(a)}{=} H_p(|S \cup T|)$, but this does not make a difference by Corollary 7.9). Hence, we obtain a commutative diagram with exact rows for all $p \in \mathbb{Z}$

$$\begin{array}{ccccccccc} H_p(S\cap T) & \longrightarrow & H_p(S)\times H_p(T) & \longrightarrow & H_p(S\cup T) & \longrightarrow & H_{p-1}(S\cap T) & \longrightarrow & H_{p-1}(S)\times H_{p-1}(T) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_p(|S\cap T|) & \longrightarrow & H_p(|S|)\times H_p(|T|) & \longrightarrow & H_p(|S\cup T|) & \longrightarrow & H_{p-1}(|S\cap T|) & \longrightarrow & H_{p-1}(|S|)\times H_{p-1}(|T|). \end{array}$$

As the outer four columns are isomorphisms are isomorphisms by the induction assumption, we therefore conclude by the 5-Lemma 4.19 that the middle vertical map $H_p(S \cup T) \to H_p(|S \cup T|)$ is an isomorphism as well. This finishes the induction step for the Mayer-Vietoris induction. □

Example 8.2.

(a) If S and T are simplicial complexes with homeomorphic topological realizations $|S|$ and $|T|$, it follows from Proposition 8.1 that the simplicial homology groups $H_p(S)$ and $H_p(T)$ are isomorphic for all $p \in \mathbb{Z}$. Note however that it is in general hard to write down an explicit isomorphism without leaving the simplicial world.

(b) For an $(n+1)$-simplex α, the topological realization of the simplicial sphere S^α is just the topological n-sphere S^n. Our computation of the simplicial homology of S^α in Example 3.13 thus tells us by Proposition 8.1 that the singular Betti numbers of the sphere S^n are

$$b_p(S^n) = \begin{cases} 1 & \text{for } p = 0 \text{ or } p = n, \\ 0 & \text{otherwise.} \end{cases}$$

Geometrically, this just means that S^n has "one n-dimensional hole".

(c) The complement $\mathbb{R}^{n+1}\backslash\{0\}$ of a point in $\mathbb{R}^{n+1}$ is homotopy equivalent to S^n for all $n \in \mathbb{N}$, hence by (b) and Corollary 6.13 we also have

$$b_p(\mathbb{R}^{n+1}\backslash\{0\}) = \begin{cases} 1 & \text{for } p = 0 \text{ or } p = n, \\ 0 & \text{otherwise.} \end{cases}$$

(d) By the calculation of Example 2.27, the non-zero singular Betti numbers of the torus $S^1 \times S^1$ are $b_0(S^1 \times S^1) = b_2(S^1 \times S^1) = 1$ and $b_1(S^1 \times S^1) = 2$, with a basis of $H_1(S^1 \times S^1)$ given by circles in each of the two factors. Similarly, Example 2.28 gives the singular Betti numbers of the real projective plane and shows that they depend on the chosen ground field.

Actually, to study some concrete examples with a little more interesting homology groups, let us consider general projective spaces in more detail and start by quickly recalling their definition from [G3, Construction 5.24] or [G5, Definition 3.2]. Initially, this is a pure linear algebra construction that works over any ground field, but to make projective spaces into topological spaces interesting for our purposes we will consider them only over the real or complex numbers.

Construction 8.3 (Projective spaces)**.** Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and let $n \in \mathbb{N}$. We define the **projective n-space** over $\mathbb{K}$ to be the set $\mathbb{P}^n_{\mathbb{K}}$ of all 1-dimensional linear subspaces of $\mathbb{K}^{\{0,\dots,n\}} \cong \mathbb{K}^{n+1}$.

Clearly, such a 1-dimensional linear subspace is uniquely determined by a spanning non-zero vector in $\mathbb{K}^{\{0,\dots,n\}}$, with two such vectors giving the same linear subspace if and only if they are scalar multiples of each other. In other words, we have

$$\mathbb{P}^n_{\mathbb{K}} = (\mathbb{K}^{\{0,\dots,n\}} \backslash \{0\}) / \sim$$

with the equivalence relation

$$(x_0,\dots,x_n) \sim (x'_0,\dots,x'_n) \quad :\Leftrightarrow \quad x_i = \lambda x'_i \text{ for some } \lambda \in \mathbb{K}^* \text{ and all } i.$$

The equivalence class of $(x_0,\dots,x_n)$ in $\mathbb{P}^n_{\mathbb{K}}$ is usually denoted by $(x_0 : \dots : x_n)$. We make $\mathbb{P}^n_{\mathbb{K}}$ into a topological space by giving it the quotient topology inherited from $\mathbb{K}^{\{0,\dots,n\}} \backslash \{0\}$. Its main properties are:

(a) The subspaces $U_i := \{(x_0 : \dots : x_n) \in \mathbb{P}^n_{\mathbb{K}} : x_i \neq 0\}$ for $i = 0,\dots,n$ are homeomorphic to $\mathbb{K}^n$ by the map

$$U_i \to \mathbb{K}^n,\ (x_0 : \dots : x_n) \mapsto \left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i} \right),$$

giving rise to an open cover $\mathbb{P}^n_{\mathbb{K}} = U_0 \cup \dots \cup U_n$ by n-dimensional $\mathbb{K}$-vector spaces.

(b) $\mathbb{P}^n_{\mathbb{K}}$ is compact, as it is the image of a compact sphere under the continuous map

$$\{x \in K^{\{0,\dots,n\}} : ||x|| = 1\} \to \mathbb{P}^n,\ (x_0,\dots,x_n) \mapsto (x_0 : \dots : x_n).$$

(c) For $n = 1$, we have $\mathbb{P}^1_{\mathbb{K}} = U_0 \cup \{(0:1)\}$, so by (a) and (b) it is a compact space obtained from $\mathbb{K}^n$ by adding a single point. We can therefore think of $\mathbb{P}^1_{\mathbb{K}}$ as the one-point compactification of $\mathbb{K}$, adding a "point at infinity". This makes $\mathbb{P}^1_{\mathbb{R}}$ homeomorphic to $\mathbb{R} \cup \{\infty\} \cong S^1$ and $\mathbb{P}^1_{\mathbb{C}}$ homeomorphic to $\mathbb{C} \cup \{\infty\} \cong S^2$. It can be shown that $\mathbb{P}^2_{\mathbb{R}}$ is just the real projective plane of Example 1.17 (d) [G5, Remark 3.6 (a)]; the other projective spaces for higher values of n are new spaces that we have not met so far in this course. They do not come with a natural embedding in a real vector space, but will be manifolds in the sense of Example 10.17 (a). To compute their homology, they could be written as topological realizations of simplicial complexes, but this is not the easiest way to proceed. Instead, let us now use singular Mayer-Vietoris for this purpose.

Example 8.4 (Homology of complex projective spaces)**.** For projective spaces, the computation of their singular homology is actually easier in the complex case than in the real case: Let us show inductively that

$$b_p(\mathbb{P}^n_{\mathbb{C}}) = \begin{cases} 1 & \text{for } p = 0,2,4,\dots,2n, \\ 0 & \text{otherwise} \end{cases}$$

for all $n \in \mathbb{N}$, with the case $n = 0$ being trivial since $\mathbb{P}^0_{\mathbb{C}}$ is just a point. For the other cases, we will use Mayer-Vietoris for the open subsets

$$U = \{(x_0 : \dots : x_n) \in \mathbb{P}^n_{\mathbb{C}} : x_0 \neq 0\} \quad \text{and} \quad V = \{(x_0 : \dots : x_n) \in \mathbb{P}^n_{\mathbb{C}} : (x_1,\dots,x_n) \neq (0,\dots,0)\},$$

which clearly cover $\mathbb{P}^n_{\mathbb{C}}$. Note first that U is just $\mathbb{C}^n$ by Construction 8.3 (a). In particular, $U \cap V$ is the complement of the origin in $\mathbb{C}^n = \mathbb{R}^{2n}$, which deformation retracts to the sphere S^{2n-1}. Finally, by moving the 0-th coordinate to zero V deformation retracts to

$$\{(0:x_1:\cdots:x_n) \in \mathbb{P}^n_{\mathbb{C}} : (x_1,\dots,x_n) \neq (0,\dots,0)\} \cong \mathbb{P}^{n-1}_{\mathbb{C}},$$

for which we can assume the homology to be known by induction. In particular, all odd homology of U and V vanishes, so the sequence of Corollary 7.10 splits into the parts

$$0 \longrightarrow H_{2k+1}(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow H_{2k}(S^{2n-1}) \longrightarrow H_{2k}(\mathbb{C}^n) \times H_{2k}(\mathbb{P}^{n-1}_{\mathbb{C}}) \longrightarrow H_{2k}(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow H_{2k-1}(S^{2n-1}) \longrightarrow 0$$

for all $k \in \mathbb{N}$. From this we can get all the information we need:

- For $k = 0$, the map $H_{2k}(S^{2n-1}) \to H_{2k}(\mathbb{C}^n) \times H_{2k}(\mathbb{P}^{n-1}_{\mathbb{C}})$ is injective since S^{2n-1} is path-connected. Hence, we must have $b_1(\mathbb{P}^n_{\mathbb{C}}) = 0$, and thus $b_0(\mathbb{P}^n_{\mathbb{C}}) = 1$ by Remark 4.5 (b).
- For $k > 0$ we have $H_{2k}(S^{2n-1}) = 0$ by Example 8.2 (b), so again looking at the beginning of our exact sequence we see that in fact all odd homology of $\mathbb{P}^n_{\mathbb{C}}$ vanishes. Looking at the end, we obtain the exact sequence

  $$0 \longrightarrow H_{2k}(\mathbb{P}^{n-1}_{\mathbb{C}}) \longrightarrow H_{2k}(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow H_{2k-1}(S^{2n-1}) \longrightarrow 0.$$

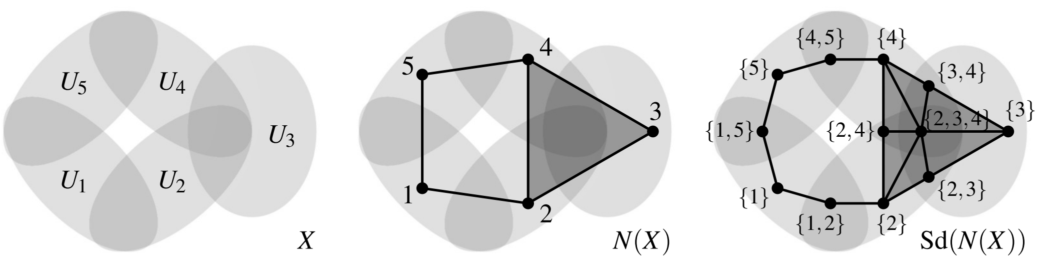
  Now for $k < n$ the first term has Betti number 1 by induction, for $k = n$ the last term has Betti number 1 by Example 8.2 (b), and all other outer Betti numbers are 0.

As we have claimed, this yields $b_p(\mathbb{P}^n_{\mathbb{C}}) = 1$ for $p = 0,2,4,\dots,2n$, and $b_p(\mathbb{P}^n_{\mathbb{C}}) = 0$ in all other cases. 12

Having seen how the simplicial homology of a simplicial complex can be reinterpreted as singular homology, let us now go the other way around: start with the singular homology of an arbitrary topological space and see if we can express it in terms of simplicial homology. This would e. g. be very useful for calculations, as simplicial homology is algorithmically computable. However, due to the notion of a topological space being extremely general, it should be clear that this will not work without some assumptions. Let us start with an example showing the idea of these assumptions.

Example 8.5. Let X be a topological space with a given finite open cover $(U_i)_{i \in I}$ such that all these open subsets, as well as all their possible intersections, are contractible. An example of this situation is shown in the picture below on the left, where $X = U_1 \cup \cdots \cup U_5 \subset \mathbb{R}^2$ is a union of ellipses. As in the picture, note that the contractibility assumption is clearly always satisfied if all U_i are convex sets in some vector space, since convex sets are contractible and intersections of convex sets are again convex.

In this situation, it should be geometrically clear that the homology of X – e. g. the hole in X in the picture below – does not come from the individual subsets U_i themselves, but only from the way they are glued together globally. But this information can be encoded neatly in a simplicial complex: Let us consider the simplicial complex $N(X)$ (shown in the middle picture below) whose vertex set is I and in which a subset $\{i_0,\dots,i_p\} \subset I$ forms a simplex if the intersection $U_{i_0} \cap \cdots \cap U_{i_p}$ is non-empty. It is called the *nerve* of X, and in the picture we can see that it can be viewed as some kind of simplicial approximation of X, recovering e. g. the hole in X.



We want to show now that this is a general statement: Under the given assumptions, the simplicial homology of the nerve $N(X)$ is always isomorphic to the singular homology of X. To prove this, we will first need a map in one direction, and the easiest way to set this up is to subdivide the nerve as in the picture above on the right. We can then place each vertex $\{i_0,\dots,i_p\} \in V(\mathrm{Sd}(N(X))) = N(X)$ at any point in the intersection $U_{i_0}\cap\cdots\cap U_{i_p}$, and fill in the higher-dimensional simplices by the contractibility assumption. Let us now carry out this construction rigorously.

Definition 8.6 (Nerves)**.** Let X be a topological space with a fixed finite open cover $(U_i)_{i\in I}$. For any non-empty $\alpha = \{i_0,\dots,i_p\} \subset I$ we set $U_\alpha := U_{i_0}\cap\cdots\cap U_{i_p}$ and assume that all non-empty U_α are contractible – we will call this a **contractible open cover**. Then the abstract simplicial complex

$$N(X) := \{\alpha \subset I : \alpha \neq \emptyset \text{ and } U_\alpha \neq \emptyset\}$$

is called the **nerve** of X (with respect to the given open cover).

Construction 8.7 (Mapping a nerve to its topological space)**.** Let $N(X)$ be the nerve of a topological space with respect to some finite contractible open cover $(U_i)_{i\in I}$ as in Definition 8.6. We want to construct a corresponding continuous map $f\colon |N(X)| \to X$. To do so, we can replace $|N(X)|$ by $|\mathrm{Sd}(N(X))|$ since these two spaces are homeomorphic by Proposition 1.21. We proceed by induction on the dimension of the simplices in $\mathrm{Sd}(N(X))$, mapping the geometric realization of every p-simplex $\{\alpha_0,\dots,\alpha_p\} \in \mathrm{Sd}(N(X))$ with $\alpha_0 \subsetneq \cdots \subsetneq \alpha_p \in N(X)$ to $U_{\alpha_0} \subset X$:

(a) For $p = 0$, we pick an arbitrary point in U_{α_0} for the image of f on the geometric 0-simplex $|\{\alpha_0\}|$ (which is just a point).

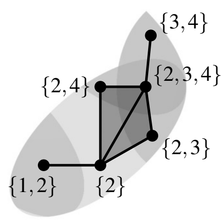
(b) For $p > 0$, by induction we have already constructed f on the boundary of $|\{\alpha_0,\dots,\alpha_p\}|$ (which is homeomorphic to S^{p-1}), with image in U_{α_0}. But U_{α_0} is contractible by assumption, so every continuous map to U_{α_0} from a sphere S^{p-1} can be extended to a continuous map from the disk D^p by contracting to the center. Hence, we can extend f to all of $|\{\alpha_0,\dots,\alpha_p\}|$, again with image in U_{α_0}; we do so in an arbitrary way.

Actually, part (b) of the construction simplifies if all U_i are convex sets in some vector space: In this case we can always pick the unique affine-linear extension; the convexity assumption then ensures that this extension really lies in U_{α_0}. The map f will then be affine-linear on each simplex of $\mathrm{Sd}(N(X))$, as e. g. in the right picture in Example 8.5. Note however that, in contrast to that picture, the map f need not be a geometric realization of $\mathrm{Sd}(N(X))$ in general (see e. g. Remark 8.9 (b)).

Proposition 8.8 (**Nerve Theorem**)**.** *Let X be a topological space with a finite contractible open cover as in Definition 8.6. Then the simplicial homology of the nerve $N(X)$ is isomorphic to the singular homology of X (with an isomorphism given by combining the topological realization morphism of Construction 6.11 with the pushforward by the map $|N(X)| \to X$ of Construction 8.7).*

Proof. For a fixed contractible open cover $(U_i)_{i\in I}$ of a given space X, we will prove the statement by a Mayer-Vietoris type induction argument on subspaces of X.

Namely, for all subsets $A = \{\alpha_1,\dots,\alpha_n\} \subset N(X)$ with $n \in \mathbb{N}$ we set, as shown in the picture for $A = \{\{2\},\{3,4\}\}$ in Example 8.5:



- Let $X(A) \subset X$ be the union of all U_α with $\alpha \in A$. Equivalently, this is the union of all U_β with $\alpha \subset \beta \in N(X)$ for some $\alpha \in A$.
- Let $S(A) \subset \mathrm{Sd}(N(X))$ be the full subcomplex with vertices given by all β with $\alpha \subset \beta \in N(X)$ for some $\alpha \in A$.

We will show by induction on n that the topological realization morphism of Construction 6.11 with the pushforward of Construction 8.7 induces isomorphisms $H_p(S(A)) \to H_p(|S(A)|) \to H_p(X(A))$ for all $p \in \mathbb{Z}$ and A as above. As $X(A) = X$ and $S(A) = \mathrm{Sd}(N(X))$ for $A = \{\{i\} : i \in I\}$, this shows that $H_p(\mathrm{Sd}(N(X))) \cong H_p(X)$ in this way, thus proving the proposition since $H_p(N(X)) \cong H_p(\mathrm{Sd}(N(X)))$ by Proposition 4.22.

The induction start for $n = 1$ is easy: For $A = \{\alpha\}$ we know that $X(A) = U_\alpha$ is contractible by assumption, and the simplicial complex $S(A)$ is star-shaped with respect to α. Both homologies are therefore trivial (see Example 6.14 and Proposition 3.11).

For the induction step let $A = \{\alpha_1, \dots, \alpha_n\}$ be a set with $n > 1$ elements. Note that by construction we have for $A_1 := \{\alpha_1, \dots, \alpha_{n-1}\}$ and $A_2 := \{\alpha_1 \cup \alpha_n, \dots, \alpha_{n-1} \cup \alpha_n\}$

$$\begin{aligned} & X(A_1) \cup X(\{\alpha_n\}) = X(A) \quad \text{and} \quad S(A_1) \cup S(\{\alpha_n\}) = S(A), \\ \text{as well as} \quad & X(A_1) \cap X(\{\alpha_n\}) = X(A_2) \quad \text{and} \quad S(A_1) \cap S(\{\alpha_n\}) = S(A_2). \end{aligned}$$

The simplicial and singular Mayer-Vietoris sequences of Example 4.7 and Construction 7.1 thus give a morphism of short exact sequences of chain complexes

$$\begin{tikzcd} 0 \arrow[r] & C^{\text{ord}}(S(A_2)) \arrow[r] \arrow[d] & C^{\text{ord}}(S(A_1)) \times C^{\text{ord}}(S(\{\alpha_n\})) \arrow[r] \arrow[d] & C^{\text{ord}}(S(A)) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C(X(A_2)) \arrow[r] & C(X(A_1)) \times C(X(\{\alpha_n\})) \arrow[r] & C(X(A_1) \,|\, X(\{\alpha_n\})) \arrow[r] & 0, \end{tikzcd}$$

with the vertical maps given by topological realization and Construction 8.7. The proof now continues in the same way as in Proposition 8.1: With Proposition 4.25 and Corollary 7.9, this diagram gives rise by Construction 4.10 to a commutative homology diagram with exact rows

$$\begin{tikzcd} H_p(S(A_2)) \arrow[r] \arrow[d] & H_p(S(A_1)) \times H_p(S(\{\alpha_n\})) \arrow[r] \arrow[d] & H_p(S(A)) \arrow[r] \arrow[d] & H_{p-1}(S(A_2)) \arrow[r] \arrow[d] & H_{p-1}(S(A_1)) \times H_{p-1}(S(\{\alpha_n\})) \arrow[d] \\ H_p(X(A_2)) \arrow[r] & H_p(X(A_1)) \times H_p(X(\{\alpha_n\})) \arrow[r] & H_p(X(A)) \arrow[r] & H_{p-1}(X(A_2)) \arrow[r] & H_{p-1}(X(A_1)) \times H_{p-1}(X(\{\alpha_n\})). \end{tikzcd}$$

As the outer four columns are isomorphisms by induction, the isomorphism $H_p(S(A)) \cong H_p(X(A))$ thus follows from the 5-Lemma 4.19. □

Remark 8.9.

(a) As we have already remarked in Example 8.5 and Construction 8.7, a common case of the Nerve Theorem occurs if a topological space X is a union of finitely many convex open sets in a real vector space: In this case, the contractibility condition is automatic, and the map from $|N(X)| \cong |\mathrm{Sd}(N(X))|$ to X inducing an isomorphism in homology can be obtained by choosing arbitrary points for the vertices of $\mathrm{Sd}(N(X))$ in their respective intersections and affine-linear interpolation.

This situation occurs e. g. in topological data analysis in Remark 9.19 (a).

(b) Under some mild additional assumptions, it can be shown that the map of Construction 8.7 does not only preserve homology, but is actually also a homotopy equivalence [H, Corollary 4G.3]. Note however that it is clearly not a homeomorphism, as we can already see from Example 8.5. In fact, the nerve $N(X)$ can easily acquire higher dimension as expected or desired: If we add another open subset U_6 to Example 8.5 which is just equal to U_3, this would not change the topological space X at all, but it would make the nerve 3-dimensional due to the intersection $U_2 \cap U_3 \cap U_4 \cap U_6$ now being non-empty – although X was given as a subset of the plane.

To finish this chapter, let us quickly also introduce singular cohomology and compare it to the simplicial setting. As you would probably expect from Chapter 5, not much is happening here as we are just talking about dualizing all our vector spaces. We have already discussed the geometric meaning of cohomology when discussing its simplicial version, so let us restrict ourselves now to its construction and properties.

Construction 8.10 (Singular cohomology)**.** Let X be a topological space. Analogously to Construction 5.3, the **singular cochain complex** of X is defined to be the dual cochain complex $C^\vee(X)$ of the singular chain complex $C(X)$ of Remark 6.7 (b) (see Construction 5.2), consisting of the **singular cochain** groups $C^p(X) = (C_p(X))^\vee$ and **singular coboundary maps** $d_p \colon C^p(X) \to C^{p+1}(X)$ of X. Elements of $C^p(X)$ are called **singular p-cocycles** or **singular p-coboundaries** of X if they are in the kernel or image of the singular coboundary map, respectively.

As in Construction 5.6, assigning to a topological space X its singular cochain complex $C^\vee(X)$ is a contravariant functor, with the morphism $C^\vee(Y) \to C^\vee(X)$ associated to a continuous map $f\colon X \to Y$ called the **pullback** f^*.

Applying the cohomology functor to $C^\vee(X)$ as in Construction 5.7, we obtain the **singular cohomology** $H^p(X,K) := H_p(X) = \ker d_p / \operatorname{im} d_{p-1}$ of X for all $p \in \mathbb{Z}$. In particular, the pullback maps for a continuous map $f\colon X \to Y$ pass to cohomology $f^*\colon H^p(Y) \to H^p(X)$.

By Proposition 5.14, homology commutes with dualization (also for infinite-dimensional spaces, which is important here!), so that we have natural isomorphisms $H^p(X) \cong (H_p(X))^\vee$ – telling us that singular cohomology does not contain any new information compared to homology, and that the dimensions of $H^p(X)$ are just the usual Betti numbers $b_p(X)$. As for our results on singular homology, this means:

(a) Homotopic continuous maps induce with their pullbacks the same map on cohomology (Proposition 6.12), and homotopy equivalent topological spaces have isomorphic singular cohomology (Corollary 6.13).

(b) For any two open subsets U and V of X there is a long exact Mayer-Vietoris sequence

$$\cdots \longrightarrow H^p(U \cup V) \longrightarrow H^p(U) \times H^p(V) \longrightarrow H^p(U \cap V) \longrightarrow H^{p+1}(U \cup V) \longrightarrow \cdots$$

obtained by dualizing Corollary 7.10.

(c) For any simplicial complex S, the simplicial cohomology $H^p(S)$ is naturally isomorphic to the singular homology $H^p(|S|)$ for all $p \in \mathbb{Z}$ by Proposition 8.1. For any contractible finite open cover of a topological space X, the Nerve Theorem of Proposition 8.8 implies that the simplicial cohomology $H^p(N(X))$ is naturally isomorphic to the singular cohomology $H^p(X)$.

9. Applications of Homology

In this last chapter on the singular homology of topological spaces we now want to study some interesting applications of our theory. Let us start with a few classical applications, some of which are just higher-dimensional generalizations of statements that can be obtained using the fundamental group [G3, Chapter 9], and some are new. As a first very simple statement, we will show that topology is able to distinguish dimensions of real vector spaces in the following sense.

Proposition 9.1 (**Invariance of dimension**, see [G3, Example 8.18]). *$\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$ if $m,n \in \mathbb{N}$ with $m \neq n$.*

Proof. Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be a homeomorphism with $m,n \in \mathbb{N}$. Removing one point, we obtain from this a restriction $f|_{\mathbb{R}^m\backslash\{0\}}\colon \mathbb{R}^m\backslash\{0\} \to \mathbb{R}^n\backslash\{f(0)\}$ that must be a homeomorphism as well. Hence the source and target must have the same Betti numbers, which by Example 8.2 (c) requires $m = n$. □

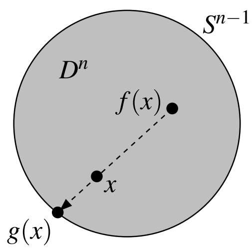
Remark 9.2. Note that Proposition 9.1 is local in the sense that the same statement also holds for open balls: As any open ball in $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$ itself, two open balls in finite-dimensional real vector spaces will be homeomorphic if and only if these vector spaces have the same dimension.

The following fixed point theorem can also be proven in low dimension using the fundamental group already. In fact, the proof given below is almost the same, just replacing the fundamental group with the appropriate homology group.

Proposition 9.3 (**Brouwer's Fixed Point Theorem**, see [G3, Proposition 9.1]). *Every continuous map $f\colon D^n \to D^n$ for $n \in \mathbb{N}$ has a fixed point.*

Proof. Assume that $f\colon D^n \to D^n$ is a continuous map without a fixed point.

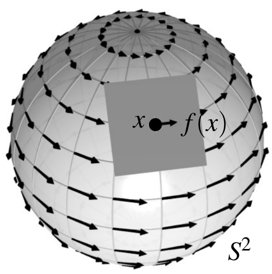
As in the picture on the right, we can then define a map $g\colon D^n \to S^{n-1}$ by sending a point $x \in D^n$ to the unique intersection point of the boundary sphere S^{n-1} with the ray from $f(x)$ that passes through x. This is well-defined since $f(x)$ and x are not the same point, and of course the point $g(x)$ could be computed explicitly to see that g is actually a continuous map. Moreover, as D^n is convex we can move every point x on a straight line to its image point $g(x)$ in D^n, showing since $g|_{S^{n-1}} = \mathrm{id}$ that g is actually a deformation retraction from D^n to S^{n-1}.



But this would imply by Corollary 6.13 that D^n has the same Betti numbers as S^{n-1}, which is not the case by Example 8.2 (b) as D^n is contractible. This contradiction shows that f cannot exist, i. e. that every continuous map from a disk to itself must have a fixed point. □

Example 9.4. Of course, the same fixed point theorem then holds for any topological space homeomorphic to D^n for some n. A common reformulation of Proposition 9.3 in non-mathematical terms is then that any geographic map of a city (which is assumed to be homeomorphic to D^2 here) placed anywhere in this city contains at least one point that lies at the exact point that it represents.

Next, let us consider a construction that we will study in much more detail later on when discussing analysis, namely that of so-called *tangent vector fields* (see Construction 12.13). Currently, we will only look at this situation for a (unit) sphere $S^n \subset \mathbb{R}^{n+1}$ as in the picture on the right, when a tangent vector field is just a continuous map $f\colon S^n \to \mathbb{R}^{n+1}$ such that for all $x \in S^n$ the vector $f(x)$ is orthogonal to x (in the Euclidean norm), i. e. intuitively that $f(x)$ is tangent to the sphere at x We will show now that for even n such a vector field must always have a zero – such as the north and south pole in the picture.



Proposition 9.5 (**Hairy Ball Theorem**). *If $n \in \mathbb{N}$ is even, every continuous map $f\colon S^n \to \mathbb{R}^{n+1}$ with*
13 *$f(x) \perp x$ for all $x \in S^n$ has a zero.*

Proof. Assume that there is a tangent vector field $f\colon S^n \to \mathbb{R}^{n+1}$ without a zero. By normalizing, we may then assume that $\|f(x)\| = 1$ for all $x \in S^n$, i. e. that we have a continuous map $f\colon S^n \to S^n$ with $f(x) \perp x$ for all $x \in S^n$. Now consider the continuous map

$$h\colon S^n \times [0,1] \to S^n,\ x \mapsto \cos(\pi t)\,x + \sin(\pi t)\,f(x)$$

(note that its image actually lies in S^n since $f(x) \perp x$). It is a homotopy between $h(\,\cdot\,,0) = \mathrm{id}$ and $h(\,\cdot\,,1) = -\mathrm{id}$ which moves each point $x \in S^n$ to its opposite point $-x$ along a half-circle in S^n with starting tangent direction $f(x)$. By Proposition 6.12, this means that the map $-\mathrm{id}\colon S^n \to S^n$ induces the identity map $H_n(S^n) \to H_n(S^n)$ on homology, i. e. by Example 8.2 (b) the identity map $K \to K$.

But changing the sign of one coordinate of $S^n \subset \mathbb{R}^{n+1}$ (i. e. a reflection at a hyperplane) induces the negative identity on homology by our simplicial computation of Example 3.13, so the map $-\mathrm{id}\colon S^n \to S^n$ which changes the signs of all $n+1$ coordinates is a multiplication by $(-1)^{n+1}$ in homology. As we have just seen that this map must be the identity on a non-zero vector space, we conclude (by choosing a ground field in which $-1 \neq 1$) that n must be odd. □

Remark 9.6.

(a) The reason for the name “Hairy Ball Theorem“ should readily be visible from the picture above. The statement is also often quoted by saying that a hedgehog cannot be combed without a bald spot.

(b) If $n = 2k+1$ is odd (with $k \in \mathbb{N}$) it is easy to write down a continuous map $f\colon S^n \to \mathbb{R}^{n+1}$ without zeros with $f(x) \perp x$ for all $x \in S^n$, e. g.

$$f\colon S^n \to \mathbb{R}^{n+1},\ (x_1, x_2, \dots, x_{2k+1}, x_{2k+2}) \mapsto (-x_2, x_1, \dots, -x_{2k+2}, x_{2k+1}).$$

Our last classical application of singular homology is a proof of the seemingly obvious statement that a closed non-self-intersecting curve in $\mathbb{R}^2$ divides the plane into exactly two regions (an interior and an exterior). Maybe this assertion seems so clear to you that you would not have expected that it needs a proof at all, but recalling the existence of surface-filling Peano curves might change your mind [G1, Chapter 24.D]. In fact, the proof of this statement is significantly harder than the three previous ones, and clearly depends crucially on what we mean by a “closed non-self-intersecting curve”.

To simplify the following proof a bit, let us actually consider curves in S^2 instead of $\mathbb{R}^2$ – which does not make a difference since we can think of S^2 as $\mathbb{R}^2$ compactified by a “point at infinity“. We will consider a so-called *Jordan curve*, which is just an injective continuous map $f\colon S^1 \to S^2$, and show that the complement of its image $f(S^1)$ has exactly two path-connected components, i. e. that its zeroth Betti number equals 2. Note that this complement is clearly open since $f(S^1)$ is compact and thus closed, and that the requirement of f being injective already rules out many pathological examples: For example, a surjective Peano curve $f\colon S^1 \to S^2$ is no longer possible since it would then be bijective and thus a homeomorphism by [G1, Proposition 24.32], in contradiction to Proposition 9.1 since taking away a point from f would then yield a homeomorphism between $\mathbb{R}$ and $\mathbb{R}^2$.

Proposition 9.7 (**Jordan Curve Theorem**).

(a) *For any injective continuous map $f\colon [0,1] \to S^2$ from the interval $[0,1]$ to the 2-sphere S^2 we have $b_0(S^2 \backslash f([0,1])) = 1$ and $b_1(S^2 \backslash f([0,1])) = 0$.*

(b) *For any injective continuous map $f\colon S^1 \to S^2$ we have $b_0(S^2 \backslash f(S^1)) = 2$.*

Proof.

(a) We have to show that the homology group $H_1(S^2 \backslash f([0,1]))$ is zero, and that the degree map $\deg\colon H_0(S^2 \backslash f([0,1])) \to K$ is an isomorphism. We will prove this by contradiction, so let us assume that for $p = 0$ or $p = 1$ there is a non-zero homology class $c \in H_p(S^2 \backslash f([0,1]))$, with the additional condition $\deg c = 0$ if $p = 0$.

Starting with the interval $I_0 := [0,1]$, we will construct from this a descending sequence of intervals $I_0 \supset I_1 \supset I_2 \supset \cdots$ such that c is non-zero in $H_p(S^2 \backslash f(I_n))$ for all n as well (where we consider c to be a homology class in $S^2 \backslash f(I_n)$ using the pushforward by the inclusion). To do this, assume that I_n is already constructed, and split it into two intervals $I_n = I'_n \cup I''_n$ intersecting at the midpoint P_n. The Mayer-Vietoris sequence of Corollary 7.10 for the open cover of $S^2 \backslash \{f(P_n)\}$ by $S^2 \backslash f(I'_n)$ and $S^2 \backslash f(I''_n)$ then gives as a part the exact sequence

$$H_{p+1}(S^2 \backslash \{f(P_n)\}) \longrightarrow H_p(S^2 \backslash f(I_n)) \longrightarrow H_p(S^2 \backslash f(I'_n)) \times H_p(S^2 \backslash f(I''_n)),$$

of which the left term is zero since $S^2 \backslash \{f(P_n)\}$ is homeomorphic to $\mathbb{R}^2$. Hence the second map in the sequence is injective. As c is non-zero in $H_p(S^2 \backslash f(I_n))$ we can therefore choose $I_{n+1} = I'_n$ or $I_{n+1} = I''_n$ such that c is non-zero in $H_p(S^2 \backslash f(I_{n+1})) > 0$ as well. This completes the construction of the sequence $I_0 \supset I_1 \supset I_2 \supset \cdots$.

But if $P \in [0,1]$ denotes the convergence point of this sequence of intervals, the class c must of course be zero in the homology of $S^2 \backslash \{f(P)\}$ as this space is homeomorphic to $\mathbb{R}^2$ again. In other words, we have $c = \partial c'$ for some $(p+1)$-chain c' in $S^2 \backslash \{f(P)\}$. But the image of all singular simplices in c' in $S^2 \backslash \{f(P)\}$ is a compact set, and this set is covered by the open subsets $S^2 \backslash f(I_n)$ for $n \in \mathbb{N}$. As this is an increasing sequence of open sets, it follows by compactness that c' must be a singular simplex in some $S^2 \backslash f(I_n)$ already. But this means that $c = \partial c'$ is trivial in the homology of $S^2 \backslash f(I_n)$ as well, in contradiction to our construction.

(b) We can obtain the circle S^1 by gluing two intervals I and J at their ends, giving rise to gluing points P and Q. By (a) we have $H_1(S^2 \backslash f(I)) \times H_1(S^2 \backslash f(J)) = 0$, so starting the Mayer-Vietoris sequence for covering $S^2 \backslash f(\{P,Q\})$ with $S^2 \backslash f(I)$ and $S^2 \backslash f(J)$ at this term we obtain the exact sequence

$$0 \to H_1(S^2 \backslash f(\{P,Q\})) \to H_0(S^2 \backslash f(S^1)) \to H_0(S^2 \backslash f(I)) \times H_0(S^2 \backslash f(J)) \to H_0(S^2 \backslash f(\{P,Q\})) \to 0.$$

From this we can now easily obtain the desired result by an Euler characteristic computation: The space $S^2 \backslash f(\{P,Q\})$ is homeomorphic to the punctured plane $\mathbb{R}^2 \backslash \{0\}$ and thus has Betti numbers $b_0(S^2 \backslash f(\{P,Q\})) = b_1(S^2 \backslash f(\{P,Q\})) = 1$ by Example 8.2 (c), and by (a) we have $b_0(S^2 \backslash f(I)) = b_0(S^2 \backslash f(J)) = 1$. Hence, by Remark 4.5 (b) it follows that $1 - b_0(S^2 \backslash f(S^1)) + (1+1) - 1 = 0$, which means that $b_0(S^2 \backslash f(S^1)) = 2$. □

Exercise 9.8. Prove for every $n \in \mathbb{N}_{>0}$:

(a) If $f\colon S^n \to S^n$ is a continuous map with $n \in \mathbb{N}$ even, there is a point $x \in S^n$ with $f(x) = x$ or $f(x) = -x$.

(b) Every real matrix $A \in \mathbb{R}^{n \times n}$ with positive entries has an eigenvector with positive entries.

Exercise 9.9. Let S be a connected graph (i. e. a 1-dimensional simplicial complex) with v vertices and e edges. By an embedding of S into a topological space X we mean a continuous injective map $f\colon |S| \to X$.

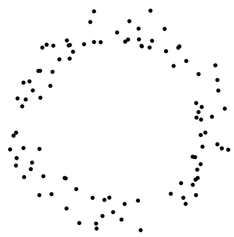
(a) For any embedding of S into the sphere S^2, show that $b_0(S^2 \backslash f(|S|)) = e - v + 2$.

(b) Now let $S = \{\{i,j\} : i,j \in \{1,2,3,4,5\}\}$ be the graph with five vertices and every possible edge between them. Show that f cannot be embedded in S^2, but that it can be embedded in the torus $S^1 \times S^1$.

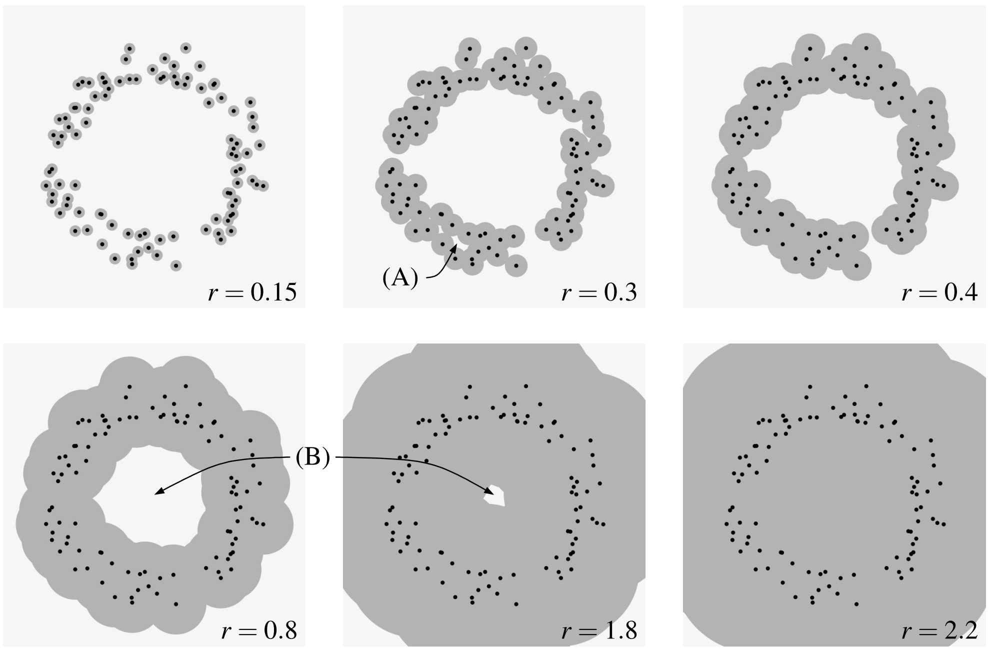
Exercise 9.10. Generalize the proof of Proposition 9.7 to show by induction on $n \in \mathbb{N}_{\geq 2}$ that $b_0(S^n \backslash f(S^{n-1})) = 2$ for any injective continuous map $f\colon S^{n-1} \to S^n$.

To conclude our algebraic study of topology, we now want to discuss a modern application of homology to a branch of mathematics called *topological data analysis*, whose goal is to analyze the geometry of large point cloud data. Let us start with an example explaining the idea behind this technique.

Example 9.11 (Noisy annulus)**.** You would probably agree that the picture on the right shows a noisy image of an annulus. But of course, in reality the picture just consists of a large number of discrete points – in fact, we have generated it by plotting random points in the plane with distance between 2 and 3 from the origin. So what does it mean exactly that the points are arranged in the shape of an annulus, and how can we figure this out? In other words, from an application point of view: If a computer acquired the data of these points, how can it detect the underlying shape?



One possible idea is to replace the points by circles around them with increasing radius r, as shown in the pictures below. Considering the unions of these circles, we obtain subspaces of $\mathbb{R}^2$ of which we can compute and compare the singular homology. As we can see in the pictures, there are sometimes a few spurious holes such as (A) that do not represent an actual feature of the data and vanish again for a slightly bigger radius. In contrast, the main hole (B) persists in a large range of the radius (at least from $r = 0.80$ to $r = 1.80$), indicating that this is an important feature of the image.



To make such statements precise, we have to compare homology groups for different values of the radius. More precisely, as in the picture above we will pick a sequence $r_1 < \cdots < r_k$ of radii, denote by $U^{(i)}$ the union of the circles of radius r_i around the given points (i. e. the gray subspaces of $\mathbb{R}^2$ in the picture), and compute the homology groups $H_p(U^{(i)})$ for all $p \in \mathbb{Z}$. As $U^{(1)} \subset U^{(2)} \subset \cdots \subset U^{(k)}$, the pushforwards by the inclusion maps will induce linear maps

$$H_p(U^{(1)}) \longrightarrow H_p(U^{(2)}) \longrightarrow \cdots \longrightarrow H_p(U^{(k)}),$$

and we want to study how long elements can survive in this sequence. To tackle such questions, let us first consider sequences of vector spaces in general and show that they can always be decomposed uniquely into simple parts, each corresponding to a single element that is created and destroyed at specific stages of the sequence.

Construction 9.12.

(a) For a sequence of vector spaces $C = ((C^{(k)})_{k\in\mathbb{Z}}, (f^{(k)})_{k\in\mathbb{Z}})$, written with ascending upper indices as

$$\cdots \longrightarrow C^{(k-1)} \xrightarrow{f^{(k-1)}} C^{(k)} \xrightarrow{f^{(k)}} C^{(k+1)} \longrightarrow \cdots,$$

we denote by $f^{(k,l)} \colon C^{(k)} \to C^{(l)}$ for $k,l \in \mathbb{Z}$ with $k \le l$ the composition of the given maps starting at position k and ending at position l, i. e.

$$f^{(k,l)} = f^{(l-1)} \circ \cdots \circ f^{(k+1)} \circ f^{(k)},$$

and set $C^{(k,l)} := \operatorname{im} f^{(k,l)} \subset C^{(l)}$. Note that $f^{(k,k)} = \mathrm{id}_{C^{(k)}}$ and $f^{(k,k+1)} = f^{(k)}$ for all $k \in \mathbb{Z}$; in particular we have $C^{(k,k)} = C^{(k)}$.

(b) For any $i, j \in \mathbb{Z}$ with $i \le j$, the **interval sequence** $I_{i,j}$ is the sequence of vector spaces

$$\cdots \longrightarrow 0 \longrightarrow \underset{\substack{\uparrow \\ i}}{K} \xrightarrow{\mathrm{id}} K \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} K \xrightarrow{\mathrm{id}} \underset{\substack{\uparrow \\ j}}{K} \longrightarrow 0 \longrightarrow \cdots$$

corresponding to a single basis element that lives from position i to position j. We call the number $j-i+1$ of its non-zero entries the **length** of the sequence. Clearly, in the notation of (a) we have

$$\dim I_{i,j}^{(k,l)} = \begin{cases} 1 & \text{if } i \le k \le l \le j, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 9.13 (**Interval decomposition** of sequences)**.** *Let C be a sequence of finite-dimensional vector spaces with only finitely many non-zero entries. Then C is isomorphic to a product of finitely many interval sequences. Moreover, for any $i, j \in \mathbb{Z}$ with $i \le j$ the number of times $I_{i,j}$ occurs in this decomposition is uniquely determined by C.*

Proof. We prove the existence by induction on $\dim C := \sum_{p \in \mathbb{Z}} \dim C^{(p)}$, the case $\dim C = 0$ being trivial. For $\dim C > 0$, let $j \in \mathbb{Z}$ be the minimum such that $f^{(j)}$ is not injective, so that the sequence starts with inclusions $\cdots \subset C^{(j-2)} \subset C^{(j-1)} \subset C^{(j)}$. Pick any non-zero $x \in \ker f^{(j)} \subset C^{(j)}$, and denote by $i \in \mathbb{Z}$ the smallest integer with $x \in C^{(i)}$.

We can then split off $I_{i,j}$ from C: Pick ascending bases $\cdots \subset B^{(j-2)} \subset B^{(j-1)} \subset B^{(j)}$ of the corresponding vector spaces in C, taking x into the bases starting at $B^{(i)}$. If we now set $C'^{(k)}$ to be the vector space with basis $B^{(k)} \backslash \{x\}$ for $k \in \{i, \dots, j\}$ and $C'^{(k)} := C^{(k)}$ otherwise, we obtain a subsequence C' with $C \cong C' \times I_{i,j}$, where all non-zero vector spaces in $I_{i,j}$ are generated by x. As $\dim C' < \dim C$, the existence of a decomposition of C into interval sequences now follows by the induction assumption.

As for uniqueness, assume that we are given a decomposition of C into interval sequences, with $I_{i,j}$ occurring in the decomposition exactly $n_{i,j}$ times for $i \le j$. We will show by descending induction on $j-i$ that $n_{i,j}$ is uniquely determined by C: Clearly, $n_{i,j}$ must be zero for $j-i$ large enough since only finitely many vector spaces in C are non-zero. The induction step follows since by Construction 9.12 (b) we have

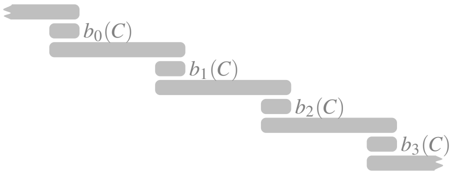
$$\dim C^{(i,j)} = \sum_{k,l:\, k \le i \le j \le l} n_{k,l} = n_{i,j} + \bigl(\text{terms } n_{k,l} \text{ with } l-k > j-i\bigr),$$

which determines $n_{i,j}$ in terms of $\dim C^{(i,j)}$ and inductively known terms. □ 14

Example 9.14 (Interval decomposition of chain complexes and exact sequences)**.**

(a) One of the easiest cases of an interval decomposition is obtained for a chain complex. Obviously, an interval sequence is a chain complex if and only if it has length at most 2, and consequently a sequence C of vector spaces satisfying the finiteness conditions of Proposition 9.13 is a chain complex if and only if all interval sequences in its interval decomposition have length at most 2. It is common to draw these interval sequences as actual intervals as indicated in gray in the picture below, resulting in the so-called **barcode** of the sequence. Note that for a general chain complex C each interval in this barcode can occur any number of times (including zero), and that the vertical order of the intervals is chosen arbitrarily.

$$\cdots \longrightarrow C^{(0)} \longrightarrow C^{(1)} \longrightarrow C^{(2)} \longrightarrow C^{(3)} \longrightarrow \cdots$$



(b) Continuing this example, the interval sequences of length 2 are clearly exact, whereas the ones $I_{i,i}$ of length 1 are not and have i-th Betti number equal to 1, with all others being zero. Consequently, the barcode of a chain complex C will contain the interval of length 1 at i exactly $b_i(C)$ times, as shown in the picture above. As a special case, C is an exact sequence if and only if all its Betti numbers are zero, i. e. if and only if its barcode consists only of intervals of length 2. This barcode can then also be interpreted by saying that each vector space $C^{(p)}$ in an exact sequence is made up from parts coming from $C^{(p-1)}$ and $C^{(p+1)}$, i. e. from the immediate left and right – which we had already seen e. g. in Example 4.6.

Remark 9.15 (Decomposition of short exact sequences of chain complexes)**.** In fact, a decomposition similar to Example 9.14 also exists for a short exact sequence $0 \to A \to B \to C \to 0$ of chain complexes, assuming again that all vector spaces in it are finite-dimensional and only finitely many of them are non-zero. This situation is described by a 2-dimensional commutative diagram of vector spaces as in Definition 4.1, with three non-zero columns and an arbitrary number of rows. As in Proposition 9.13, one can show inductively that such a short exact sequence can be decomposed as a product of simple building blocks, in which all non-zero vector spaces are just K and all non-zero maps are the identity. However, these building blocks are now of course no longer intervals, but rather two-dimensional shapes. By the argument of Example 9.14 it is easy to see which shapes are allowed: The rows are exact, hence each allowed shape must have horizontal length exactly 2 in each occurring row. The columns are complexes, hence each allowed shape must have vertical length at most 2 in each column. Moreover, squares as in the picture below on the right are generally forbidden since they violate commutativity. This leaves exactly the five shapes (1) to (5) for the building blocks, which might occur at any vertical position.

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
K & \to & K & \to & 0 \\
\vdots & & \vdots & & \vdots
\end{array}
\qquad
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
0 & \to & K & \to & K \\
\vdots & & \vdots & & \vdots
\end{array}
\qquad
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
K & \to & K & \to & 0 \\
\downarrow & & \downarrow & & \downarrow \\
K & \to & K & \to & 0 \\
\vdots & & \vdots & & \vdots
\end{array}
\qquad
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
0 & \to & K & \to & K \\
\downarrow & & \downarrow & & \downarrow \\
0 & \to & K & \to & K \\
\vdots & & \vdots & & \vdots
\end{array}
\qquad
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
0 & \to & K & \to & K \\
\downarrow & & \downarrow & & \downarrow \\
K & \to & K & \to & 0 \\
\vdots & & \vdots & & \vdots
\end{array}
\qquad\Bigg|\qquad
\begin{array}{ccc}
K & \to & K \\
\downarrow & & \downarrow \\
0 & \to & K \\
 & & \\
K & \to & 0 \\
\downarrow & & \downarrow \\
K & \to & K
\end{array}$$

(1) (2) (3) (4) (5)

Note that this gives a visual proof, and thus an intuitive explanation, for the long exact homology sequence of Proposition 4.8: As in Example 9.14, the homology of the complexes sees only the vertical parts of length 1, i. e. the entries marked in dark gray. Hence, the building blocks above give rise to a barcode of the sequence $\cdots \to H_p(A) \to H_p(B) \to H_p(C) \to H_{p-1}(A) \to \cdots$ consisting only of intervals of length 2, i. e. an exact sequence – with shape (5) corresponding to the connecting map $H_p(C) \to H_{p-1}(A)$. The only drawback of this alternative proof besides the finiteness assumption is that the maps in homology constructed in this way are not natural: Whereas the number of times a certain building block occurs in the decomposition is uniquely determined, the actual decomposition is not, corresponding to the choice of bases in the proof of Proposition 9.13.

But let us now apply Proposition 9.13 to sequences of homology groups, as already suggested in Example 9.11. This will lead to longer intervals in our sequences, and it will in fact be the particularly long ones that are important for our applications.

Construction 9.16 (Persistent homology)**.** Let

$$\cdots \longrightarrow C^{(k-1)} \xrightarrow{f^{(k-1)}} C^{(k)} \xrightarrow{f^{(k)}} C^{(k+1)} \longrightarrow \cdots$$

be a sequence of chain complexes as in Definition 4.1. By Lemma 2.17, this gives rise to sequences of vector spaces

$$\cdots \longrightarrow H_p(C^{(k-1)}) \xrightarrow{f_p^{(k-1)}} H_p(C^{(k)}) \xrightarrow{f_p^{(k)}} H_p(C^{(k+1)}) \longrightarrow \cdots \qquad (*)$$

in homology for all $p \in \mathbb{Z}$. As in Construction 9.12 (a), denote by $f_p^{(k,l)} : H_p(C^{(k)}) \to H_p(C^{(l)})$ the composition of these maps from k to l for $k \le l$. Then the images $\operatorname{im} f_p^{(k,l)}$ are called the **persistent homology groups** of the given sequence; their dimensions $b_p^{(k,l)} = \operatorname{rk} f_p^{(k,l)}$ are referred to as its **persistent Betti numbers**. In particular, the number $b_p^{(k,k)} = b_p(C^{(k)})$ is the ordinary p-th Betti number of the chain complex $C^{(k)}$.

Remark 9.17 (Visualization of persistent homology)**.** Consider a sequence of chain complexes as in Construction 9.16, and assume that all sequences $(*)$ of its homology groups for $p \in \mathbb{Z}$ consist of finite-dimensional vector spaces, of which only finitely many are non-zero. According to Proposition 9.13, we can then visualize its persistent homology by the interval decompositions of the homology sequences $(*)$. More precisely, as in the proof of the proposition the persistent Betti number $b_p^{(k,l)}$ (which is by definition the dimension of the space of homology classes surviving at least from position k to position l in the p-th homology sequence) is the same as the number of intervals in the decomposition of the p-th homology sequence containing the interval from k to l. The interval decomposition can then e. g. be drawn as a barcode as in Example 9.14.

In Example 9.11, our idea for persistent homology was to take the singular homology of unions of circles with increasing radius. There is also another way to obtain very similar homology groups; let us introduce both methods now and compare them.

Construction 9.18 (Čech and Vietoris-Rips complex)**.** Let M be a finite subset of $\mathbb{R}^n$ for some $n \in \mathbb{N}$, and let $r \in \mathbb{R}_{>0}$.

(a) The **Čech complex** $\check{C}(M,r)$ is defined to be the nerve of the union $\bigcup_{x\in M} U_r(x) \subset \mathbb{R}^n$ of the open balls with radius r and center points in M. By Definition 8.6, this is just the simplicial complex with vertex set M that contains a subset $\{x_0,\dots,x_p\} \subset M$ as a simplex if and only if $U_r(x_0) \cap \cdots \cap U_r(x_p) \neq \emptyset$.

(b) The **Vietoris-Rips complex** $\mathrm{VR}(M,r)$ is the simplicial complex with vertex set M that contains a subset $\{x_0,\dots,x_p\} \subset M$ as a simplex if and only if $\|x_i - x_j\| < 2r$ for all $i,j \in \{0,\dots,p\}$.

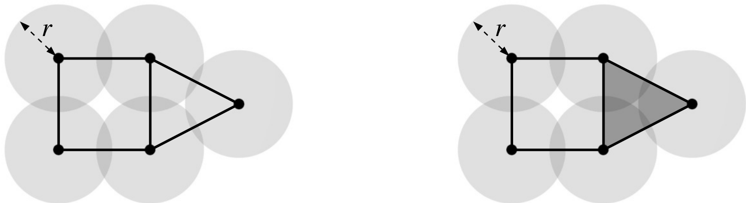
For both constructions, increasing the radius will lead to a bigger simplicial complex, so by picking a finite sequence of increasing radii we can produce a corresponding sequence of simplicial chain complexes (with the morphisms induced by inclusions) and study its persistent homology.

Remark 9.19. Let M be a finite subset of $\mathbb{R}^n$ as above, and let $r \in \mathbb{R}_{>0}$.

(a) By the Nerve Theorem of Proposition 8.8 (together with Remark 8.9 (a)), the simplicial homology of the Čech complex $\check{C}(M,r)$ as in Construction 9.18 (a) is naturally isomorphic to the singular homology of the union of balls $\bigcup_{x\in M} U_r(x) \subset \mathbb{R}^n$. The picture below on the left shows an example, where M consists of the five black points and it is clearly visible that the simplicial complex and the union of balls have the same Betti numbers (namely 2 in dimension 1 and 1 in dimension 0). Hence, studying the persistent simplicial homology of $\check{C}(M,r)$ for a given sequence of radii is exactly what we have done in Example 9.11.

(b) The construction of the Vietoris-Rips complex follows a similar idea. In fact, the edges of both complexes just agree since the conditions $U_r(x_0) \cap U_r(x_1) \neq \emptyset$ and $\|x_0 - x_1\| < 2r$ are equivalent. Moreover, the Čech complex $\check{C}(M,r)$ is clearly always a subcomplex of $\mathrm{VR}(M,r)$ since $U_r(x_0) \cap \cdots \cap U_r(x_p) \neq \emptyset$ requires $\|x_i - x_j\| < 2r$ for all $i,j \in \{0,\dots,p\}$ by the triangle inequality. However, the Vietoris-Rips complex only checks pairwise distances of points in M and thus cannot detect the full geometry of the union of balls: In the picture below, the three balls on the right have a pairwise non-empty intersection but not a common intersection point, hence the corresponding 2-simplex exists in the Vietoris-Rips complex, but not in the Čech complex. Hence, the simplicial homology of $\mathrm{VR}(M,r)$ does not have an immediate geometric interpretation, which might seem as a disadvantage. However, the Vietoris-Rips complex also has two major advantages:

- It is much faster to compute, as it is easier to check pairwise distances than to compute intersection points of finitely many balls.
- The Vietoris-Rips complex can be defined in exactly the same way for any finite metric space M, without referring to an embedding in $\mathbb{R}^n$.



Čech complex $\check{\mathrm{C}}(M,r)$ Vietoris-Rips complex $\mathrm{VR}(M,r)$

Moreover, from the point of view of persistent homology, it seems in the picture above that the difference between the Čech and Vietoris-Rips complex should not matter much: The hole in the middle of the three balls on the right is very small and will vanish fast when we increase the radius of the balls, hence it will not contribute significantly to the persistent homology. The following lemma and exercise give exact statements along these lines.

Lemma 9.20. *Let M be a finite subset of $\mathbb{R}^n$ for some $n \in \mathbb{N}$, and let $r \in \mathbb{R}_{>0}$. Then*

$$\check{\mathrm{C}}(M,r) \subset \mathrm{VR}(M,r) \subset \check{\mathrm{C}}(M,2r).$$

Proof. We have already observed the first inclusion in Remark 9.19 (b). The second inclusion is equally simple: If $x_0,\dots,x_p \in \mathbb{R}^n$ with $\|x_i - x_j\| < 2r$ for all $i,j \in \{0,\dots,p\}$, every ball $U_{2r}(x_i)$ clearly contains x_0, so $U_{2r}(x_0) \cap \cdots \cap U_{2r}(x_p) \neq \emptyset$. □

Exercise 9.21. Find the smallest constant $c \in \mathbb{R}_{>1}$ such that $\check{\mathrm{C}}(M,r) \subset \mathrm{VR}(M,r) \subset \check{\mathrm{C}}(M,cr)$ for any finite subset M of $\mathbb{R}^2$ and $r \in \mathbb{R}_{>0}$.

Example 9.22 (Noisy annulus revisited)**.** The picture below shows the barcodes for the persistent simplicial homology of the Čech and Vietoris-Rips complex for the noisy annulus of Example 9.11. In contrast to this old example, we have picked many more values of the radius r, ranging from 0 to about 2, to obtain a better resolution. The black bars are the intervals in the decomposition of the homology in dimension 1, and the gray bars correspond to dimension 0. As usual, the vertical order of the intervals in the barcode has been chosen arbitrarily.

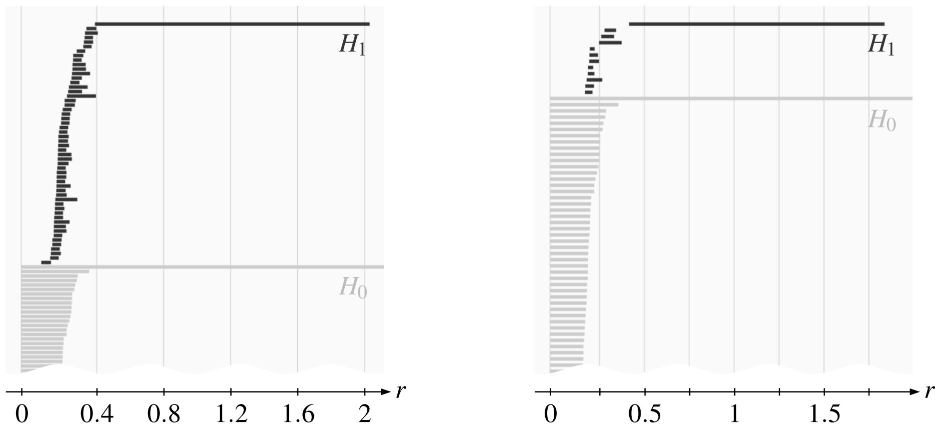
Many features of these barcodes can in fact be explained by the pictures in Example 9.11:

(a) The most important feature of the barcodes is clearly the one very long interval in dimension 0 and 1 each, indicating that the original noisy point cloud looks connected (there is exactly one long-lived homology class in dimension 0) and has one 1-dimensional hole (corresponding to the one long-lived class in dimension 1). This is the main information we are interested in when studying persistent homology.

(b) Looking at the long interval in homology dimension 1 in the Čech barcode more closely, we can see its bounds already from the pictures in Example 9.11 since the 1-dimensional big hole is present from about $r \approx 0.4$ to $r \approx 2$. In homology dimension 0, the $r = 0.3$ picture in Example 9.11 is still disconnected whereas the $r = 0.4$ picture is not – corresponding to the second longest interval in the barcode vanishing between these values.

(c) For small values of r the balls around the given points barely intersect, leading to very many connected components. In the barcodes, this corresponds to many intervals in homology dimension 0 vanishing very fast. In fact, we have cut off several of these intervals in the picture below as they do not contain relevant information.

(d) The Vietoris-Rips barcode has far fewer short-lived intervals in homology dimension 1 than the Čech barcode. This is due to the phenomenon as in the picture of Remark 9.19: Many spurious 1-dimensional holes in the Čech complex are simplicial 1-spheres coming from three balls that intersect pairwise but do not have a common intersection point. By construction, such holes can never appear in the Vietoris-Rips complex as they are always filled in by a simplicial 2-disk. Hence, there are no corresponding intervals in the Vietoris-Rips barcode.



Čech barcode

Vietoris-Rips barcode

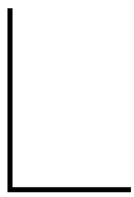
Smooth Manifolds

10. Manifolds

Up to now in these notes we have studied the geometry of topological spaces using mainly algebraic methods. As already announced in the introduction, let us now change our point of view entirely and switch to analysis. We have already seen in Example 5.4 and Remark 5.5 that the simplicial coboundary map can be interpreted as a discrete version of differentiation, and we now want to set up an analytic theory of cohomology in which the coboundary map actually takes derivatives of functions in the usual sense. The so-called *de Rham cohomology* obtained in this way will again turn out to be compatible with our previous constructions, i. e. it will be naturally isomorphic to singular cohomology whenever both are defined.

But before we can construct this analytic cohomology in Chapter 13, we have to face a fundamental problem: Which geometric spaces do we want to consider? So far you probably only know how to differentiate functions on open subsets of $\mathbb{R}^n$, but this class of spaces looks very restrictive for our purposes. On the other hand, arbitrary topological spaces (as in the singular theory) clearly do not carry enough structure to allow a definition of derivatives. The key observation to solve this problem is that differentiation is a local construction, and thus it should be fine to consider topological spaces that locally look like $\mathbb{R}^n$. Such spaces will be called *manifolds*, and we will introduce and study them in the next three chapters – completely independently of (co-)homology, for the time being.

So what exactly does it mean to "locally look like $\mathbb{R}^n$"? Does the L-shaped subspace of $\mathbb{R}^2$ shown in the picture on the right look like an interval, i. e. locally like $\mathbb{R}$? It is clearly *homeomorphic* to an interval, so from a topological point of view the answer would be yes. However, from a differentiable point of view, the corner point seems to make a difference. Hence it will not be sufficient for our purposes to define a manifold to be just a topological space locally homeomorphic to $\mathbb{R}^n$, i. e. such that every point has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.



However, the idea of homeomorphisms $\varphi: U \to V$ from open subsets U of our topological space X that we want to consider to open subsets V of $\mathbb{R}^n$ is still useful: Choosing such a homeomorphism φ allows us to associate coordinates in $\mathbb{R}^n$ to points of U, and thus to define e. g. the differentiability of functions on U with respect to these coordinates. We will call φ a *(coordinate) chart* for that reason. But these charts will be additional given data, and of course we will need several of them to cover all points of X. Hence, we have to make sure that this definition of differentiability does not depend on the chosen chart, i. e. that our given charts are compatible in a certain sense. Let us now say exactly
15 what we mean by this.

Notation 10.1 (Smooth maps and diffeomorphisms)**.** Let $\varphi: U \to V$ be a map between open subsets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ for $m, n \in \mathbb{N}$.

(a) We say that φ is **smooth** if it is infinitely differentiable. In the case $V = \mathbb{R}$, the vector space of all smooth functions from U to $\mathbb{R}$ will be written as $\Omega^0(U)$. (This corresponds to such functions being "differential forms of degree 0", a concept to be introduced in Construction 13.1 (a)). Another standard notation in analysis for $\Omega^0(U)$ is clearly $C^\infty(U)$, but we will avoid this notation here since the letter C is ubiquitous in (co-)homology theory already.)

In particular, for an arbitrary open subset $V \subset \mathbb{R}^m$ the map $\varphi: U \to V$ is smooth if and only if all its component functions are smooth, i. e. lie in $\Omega^0(U)$.

(b) The map φ is called a **diffeomorphism** if it is bijective, and both φ and φ^{-1} are smooth. (Note that sometimes in the literature a diffeomorphism only requires φ and φ^{-1} to be continuously differentiable once. Also note that a diffeomorphism clearly requires $m = n$ as the Jacobian matrices of φ and φ^{-1} need to be inverses of each other.)

Definition 10.2 (Charts and atlases)**.** Let X be a topological space, and let $n \in \mathbb{N}$.

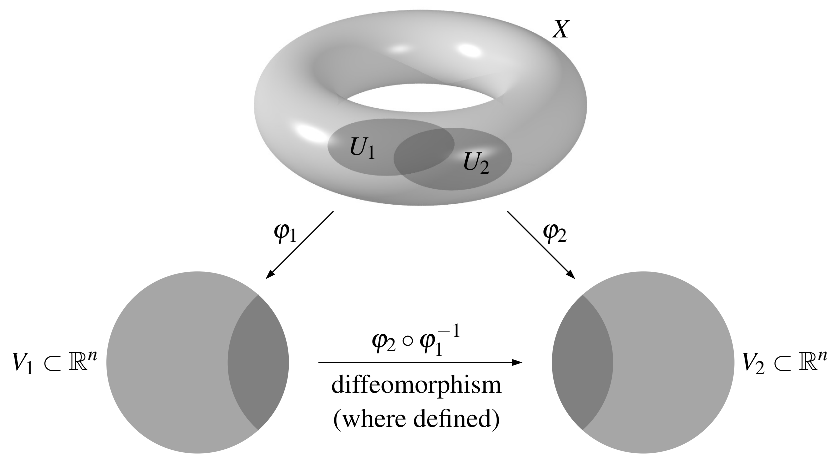
(a) As in the picture below, an (n-dimensional coordinate) **chart** on X is a homeomorphism $\varphi \colon U \to V$ from an open subset $U \subset X$ to an open subset $V \subset \mathbb{R}^n$. For a point $x \in U$, we call $\varphi(x) \in \mathbb{R}^n$ the **(local) coordinates** of x in this chart.

(b) We say that two n-dimensional charts $\varphi_1 \colon U_1 \to V_1$ and $\varphi_2 \colon U_2 \to V_2$ are **compatible** if the **transition map**

$$\varphi_2 \circ \varphi_1^{-1} \colon \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

(such as between the dark circle segments in the picture) is a diffeomorphism.

(c) An (n-dimensional) **atlas** on X is a family $(\varphi_i)_{i \in I}$ of pairwise compatible n-dimensional charts $\varphi_i \colon U_i \to V_i$ covering X, i. e. with $\bigcup_{i \in I} U_i = X$.



Remark 10.3.

(a) Note the subtle but crucial difference between homeomorphisms and diffeomorphisms in Definition 10.2: A priori, X is only a topological space, so it would not make sense to require the charts to be diffeomorphisms. Instead, as mentioned already in the introduction we will use these charts to define a notion of differentiability on X. In order for this notion to be well-defined, we will need that the transition maps (between open subsets of $\mathbb{R}^n$) are diffeomorphisms – homeomorphisms would not suffice here. In fact, the maps $\varphi_2 \circ \varphi_1^{-1} \colon \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ in Definition 10.2 (b) are already homeomorphisms by construction, so their smoothness is the only condition to be checked here.

(b) As you can see, our setup in these notes will be geared towards *infinite differentiability* only. This will allow us to define derivatives of any order on X, and has the additional advantage that differentiation does not degrade the order of differentiability. However, one could equally well fix a natural number $k \in \mathbb{N}$ and require the transition functions in Remark 10.2 (b) to only be k-fold differentiable, or k-fold continuously differentiable, leading to different notions of compatibility, atlases, and finally manifolds, that might be useful for other purposes. It would even be possible to choose $k = 0$, in which case condition (b) in Definition 10.2 is trivial as already remarked in (a), and we would obtain so-called *topological manifolds*.

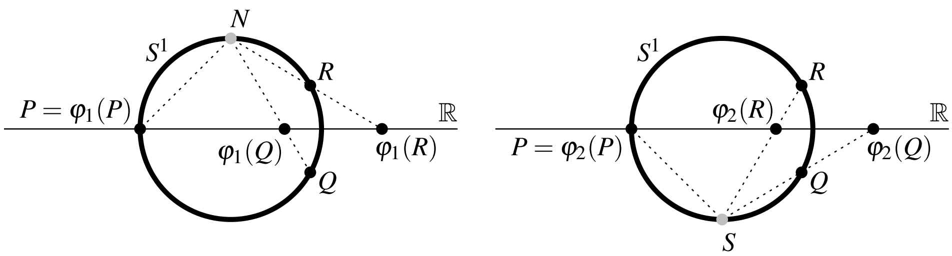
(c) If you know some complex analysis, another variant of Definition 10.2 would be to replace $\mathbb{R}$ by $\mathbb{C}$, and thus consider complex instead of real differentiability (i. e. transition functions that are holomorphic maps). The *complex manifolds* obtained in that way are clearly very important objects in mathematics, and share the same interesting flavor with one-dimensional complex analysis that their initial definitions are identical to the real case, yet their behavior very different. In these notes we will see a few complex examples such as e. g. the complex projective spaces of Construction 8.3, but in order not to assume any complex analysis we will always consider them over $\mathbb{R}$, so that they will become real $2n$-dimensional manifolds instead of complex n-dimensional ones.

Example 10.4.

(a) Every open subset $X \subset \mathbb{R}^n$ admits a natural n-dimensional atlas with the identity map id_X as the only chart.

(b) Consider the n-dimensional sphere $S^n = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : ||x||^2 + y^2 = 1\} \subset \mathbb{R}^{n+1}$. Removing the north pole $N := (0,1)$ or the south pole $S := (0,-1)$, we clearly obtain two open subsets homeomorphic to $\mathbb{R}^n$ that cover S^n. In fact, homeomorphisms $\varphi_1 \colon S^n\backslash\{N\} \to \mathbb{R}^n$ and $\varphi_2 \colon S^n\backslash\{S\} \to \mathbb{R}^n$ can be obtained by projecting from N or S onto $\mathbb{R}^n \times \{0\}$, respectively, as shown in the picture below for $n = 1$. The explicit formulas for these maps are

$$\varphi_1 \colon S^n\backslash\{N\} \to \mathbb{R}^n,\ (x,y) \mapsto \frac{x}{1-y} \quad \text{and} \quad \varphi_2 \colon S^n\backslash\{S\} \to \mathbb{R}^n,\ (x,y) \mapsto \frac{x}{1+y}.$$



From this it is a straightforward calculation to obtain the inverse of e. g. φ_1 as

$$\varphi_1^{-1} \colon \mathbb{R}^n \to S^n\backslash\{N\},\ x \mapsto \left(\frac{2x}{||x||^2+1}, \frac{||x||^2-1}{||x||^2+1} \right),$$

and thus the transition map between the two charts as

$$\varphi_2 \circ \varphi_1^{-1} \colon \mathbb{R}^n\backslash\{0\} \to \mathbb{R}^n\backslash\{0\},\ x \mapsto \frac{x}{||x||^2}.$$

This map is clearly smooth, so φ_1 and φ_2 form an n-dimensional atlas on S^n.

(c) We have already seen in Construction 8.3 that the (real or complex) projective spaces $\mathbb{P}^n_{\mathbb{K}}$ can be covered by the open subsets $U_i = \{(x_0 : \cdots : x_n) : x_i \neq 0\}$ for $i = 0, \dots, n$, which admit charts

$$\varphi_i \colon U_i \to \mathbb{K}^n,\ (x_0 : \cdots : x_n) \mapsto \left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i} \right).$$

A simple computation shows again that the transition maps between these charts are smooth, so $\varphi_0, \dots, \varphi_n$ form an atlas on $\mathbb{P}^n_{\mathbb{K}}$ (of real dimension n or $2n$ in the cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$, respectively).

(d) The (non-differentiable) function

$$\varphi \colon \mathbb{R} \to \mathbb{R},\ x \mapsto \begin{cases} 2x & \text{if } x \geq 0, \\ x & \text{if } x \leq 0 \end{cases}$$

is a homeomorphism, and thus it is a chart defining a 1-dimensional atlas on $\mathbb{R}$ – note that the condition of Definition 10.2 (b) is empty for atlases containing only one chart. However, this chart is incompatible with the natural identity chart of (a), so we will usually not consider it.

(e) Both the interval $X = [0,1] \subset \mathbb{R}$ and the union $Y = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 = 0 \text{ or } x_2 = 0\}$ of the two coordinate axes in $\mathbb{R}^2$ do not admit a 1-dimensional atlas: If we remove the zero point from any connected open neighborhood of it, it will remain connected for X and separate into four connected components for Y, whereas removing a point from a connected open subset of $\mathbb{R}$ yields exactly two connected components. Hence, there can be no homeomorphism between these spaces, i. e. no chart containing the zero point.

However, X will be a 1-dimensional *manifold with boundary*, a concept that we will introduce in Construction 14.3.

Let us now see how we can use atlases to define concepts of differentiability on topological spaces. The setting here is similar to matrix computations in linear algebra: When considering linear maps between vector spaces, computations are easily done in terms of matrices for maps $K^n \to K^m$ with $m,n \in \mathbb{N}$. To transfer these methods to a morphism $f\colon V \to W$ between arbitrary vector spaces with $\dim V = n$ and $\dim W = m$, one uses coordinate maps $\varphi\colon V \to K^n$ and $\psi\colon W \to K^m$ and studies the linear map $\psi \circ f \circ \varphi^{-1}\colon K^n \to K^m$ instead that sends the coordinate vector of $x \in V$ to the coordinate vector of $f(x) \in W$. As this is now a morphism between $K^n \to K^m$, one can use matrix techniques again for computations. We will apply the same idea now to carry over notions of differentiability from $\mathbb{R}^n$ to topological spaces with atlases, using charts as coordinate maps:

Construction 10.5 (Smooth maps between topological spaces with atlases)**.** Let X and Y be topological spaces with atlases $(\varphi_i)_{i\in I}$ and $(\psi_j)_{j\in J}$ of dimensions n and m, respectively. Moreover, let $f\colon X \to Y$ be an arbitrary map.

(a) For a given point $a \in X$ let $\varphi_i\colon U_i \to V_i$ with $i \in I$ be a chart with $a \in U_i$, and let $\psi_j\colon U'_j \to V'_j$ with $j \in J$ be a chart with $f(a) \in U'_j$. We can then form the composition

$$f_{i,j} := \psi_j \circ f \circ \varphi_i^{-1}\colon V_i \to V'_j$$

that locally around a sends the coordinates of a point $x \in U_i$ to the coordinates of $f(x) \in U'_j$ in these charts and is a map between open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. In this situation, we call f **smooth** at the point a if $f_{i,j}$ is smooth at its coordinate point $\varphi_i(a)$. Note that this is well-defined by the chart compatibility condition of Definition 10.2: If we pick different charts $\varphi_{\tilde{i}}$ and $\psi_{\tilde{j}}$ containing a and $f(a)$, respectively, we have

$$f_{\tilde{i},\tilde{j}} = \psi_{\tilde{j}} \circ f \circ \varphi_{\tilde{i}}^{-1} = \underbrace{\psi_{\tilde{j}} \circ \psi_j^{-1}}_{(2)} \circ \underbrace{\psi_j \circ f \circ \varphi_i^{-1}}_{=f_{i,j}} \circ \underbrace{\varphi_i \circ \varphi_{\tilde{i}}^{-1}}_{(2)} \tag{1}$$

where defined. As the transition maps (2) are smooth, the smoothness of $f_{\tilde{i},\tilde{j}}$ is therefore equivalent to the smoothness of $f_{i,j}$ at the point under consideration.

Of course, we say that f is a smooth map if it is smooth at every point $a \in X$: As expected from analysis, smoothness is a pointwise condition.

(b) If f is smooth at $a \in X$, we define the **rank** $\operatorname{rk}_a f$ of f at a to be the rank of the derivative $f'_{i,j}(\varphi_i(a))$, i. e. the rank of the corresponding Jacobian matrix. Again, this is well-defined by (1) since a different choice of charts will just multiply the Jacobian matrix of $f_{i,j}$ with an invertible matrix from the left and right, which does not change its rank.

Remark 10.6. In a similar way, we could define what it means that a map between topological spaces with atlases is k-fold (continuously) differentiable, but for our purposes we will not need this. Note however that Construction 10.5 only defines what it means that a map is differentiable (resp. smooth), but not what its derivative is in this case! In fact, constructing actual derivatives in a well-defined way is a bit more involved; we will do this in Construction 12.13.

Example 10.7. Consider the circle $S^1 \subset \mathbb{R}^2$ with the atlas of Example 10.4 (b). For simplicity, we will restrict our attention to the chart φ_1 on $S^1\backslash\{(0,1)\}$ as the computations for φ_2 are entirely analogous. Recall from the example above that its inverse $\varphi_1^{-1}\colon \mathbb{R} \to S^1\backslash\{(0,1)\}$ is given by

$$\varphi_1^{-1}(x) = \left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1}\right) \quad \text{with Jacobian matrix} \quad (\varphi_1^{-1})'(x) = \left(\frac{2-2x^2}{(x^2+1)^2}, \frac{4x}{(x^2+1)^2}\right)^{\mathsf{T}}.$$

(a) The embedding $f: S^1 \to \mathbb{R}^2$ (with the natural atlas on $\mathbb{R}^2$) is smooth, since $f \circ \varphi_1^{-1} = \varphi_1^{-1}$ is clearly a smooth map from $\mathbb{R}$ to $\mathbb{R}^2$. It has rank 1 everywhere as $(\varphi_1^{-1})'(x) \neq 0$ for all $x \in \mathbb{R}$.

(b) The vertical projection $f: S^1 \to \mathbb{R}$, $(x,y) \mapsto x$ is also smooth as $f \circ \varphi_1^{-1}$ is just the second coordinate of φ_1^{-1} in this case. Its derivative is $(f \circ \varphi_1^{-1})'(x) = \frac{2-2x^2}{(x^2+1)^2}$, so f has rank 1 everywhere except at the points $(\pm 1, 0)$, where the rank is 0. Geometrically, this means that the linear approximation of the vertical projection at these two points is a constant map, whereas it is non-constant at all other points.

Exercise 10.8. Let $f: X \to Y$ be a continuous surjective map between two topological spaces with the following property: For every $a \in Y$ there exists a open neighborhood U_a of a such that $f^{-1}(U_a)$ is a disjoint union of open subsets each homeomorphic to U_a through f.

(a) If Y has an atlas, show that there is an atlas on X such that f is smooth.

(b) On the other hand, show by example that if X has an atlas there need not be an atlas on Y such that f is smooth.

From the examples considered so far, it might be tempting to just define a manifold as a topological space X with an atlas. However, there are two slight technical problems with this that we still have to address. The first one is that we really use the atlas only to define what smoothness on X means, and thus the exact form of the charts is just an irrelevant choice of coordinates. For example, we could replace any chart $\varphi: U \to V \subset \mathbb{R}^n$ with $\psi \circ \varphi: U \to V'$ for an arbitrary diffeomorphism $\psi: V \to V' \subset \mathbb{R}^n$ without changing which maps on X are smooth. We could even keep the old chart and add $\psi \circ \varphi$ to the atlas as well, as these two charts are compatible. We could also throw in restrictions of already existing charts, or remove charts if the remaining ones already cover X. All these operations clearly change the atlas, but not the information we are actually interested in – and thus we want to say that all these atlases determine the same manifold, which we would not do if a manifold was just a topological space with an atlas. One way to achieve this is the following construction.

Construction 10.9 (Smooth structures)**.** An atlas $(\varphi_i)_{i \in I}$ on a topological space X is called **maximal** if every chart on X that is compatible with the charts in the atlas is already contained in the atlas, i. e. if the atlas cannot be enlarged without breaking compatibility. This ensures by definition that we cannot modify a maximal atlas in an irrelevant way by adding or removing compatible charts as above. Hence, a maximal atlas exactly captures the idea of encoding which maps on X are smooth, without any arbitrary choices. A maximal atlas on X is therefore also called a **smooth structure** on X, and this is what we will require to be given for manifolds.

Yes, a maximal atlas is huge, and we will most definitely never write one down. But every atlas clearly defines a unique maximal one (namely by adding all compatible charts to it), so it will be okay to give an atlas on X as we did e. g. in Example 10.4 to determine a smooth structure on X. We just have to remember that another atlas determines the same smooth structure if all charts in both atlases together are compatible. (In fact, an equivalent definition would be that a smooth structure is an equivalence class of atlases, where two atlases are called equivalent if their union is an atlas as well.)

Remark 10.10 (Charts to open balls or $\mathbb{R}^n$)**.** Assume that we have an n-dimensional smooth structure on a topological space X. The flexibility in the coordinate charts can actually be used to choose (non-maximal) atlases determining this smooth structure with additional properties, which might sometimes be useful:

(a) We can take a chart $\varphi_a: U_a \to V_a$ with $a \in U_a$ for every $a \in X$, and restrict it to an open ball around $\varphi_a(a) \in V_a$ in the target. These new charts will then still cover X, and thus they suffice to determine the smooth structure. In other words, we can assume that all charts have targets which are open balls in $\mathbb{R}^n$. By composing with a scalar multiplication and a translation, we might even assume that the target spaces of all charts are the open unit ball in $\mathbb{R}^n$.

(b) In fact, the open unit ball $U_1(0) \subset \mathbb{R}^n$ is diffeomorphic to all of $\mathbb{R}^n$ by the smooth map

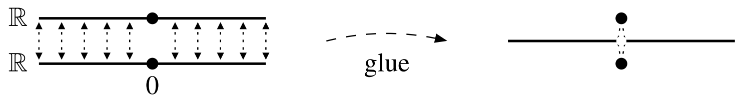
$$\varphi\colon U_1(0) \to \mathbb{R}^n,\ x \mapsto \frac{x}{\sqrt{1-\|x\|^2}} \quad \text{with inverse} \quad \varphi^{-1}\colon \mathbb{R}^n \to U_1(0),\ x \mapsto \frac{x}{\sqrt{1+\|x\|^2}},$$

so by composing charts as in (a) with φ we might even assume that all charts in an atlas determining the smooth structure are homeomorphisms whose target space is all of $\mathbb{R}^n$.

The second problem with defining a manifold as a topological space with a (maximal) atlas is simply that this would still allow some pathological examples that we rather do not want to admit. We therefore need two more technical conditions from point-set topology to make our spaces well-behaved (see e. g. Proposition 11.13 and Proposition 11.16): the Hausdorff property and the second countability axiom.

Remark 10.11 (Hausdorff spaces)**.** Recall that a topological space X is called a **Hausdorff space** if points can be separated by open sets, i. e. if for any two distinct points $a,b \in X$ there are open neighborhoods U of a and V of b in X with $U \cap V = \emptyset$. Almost by definition, the Hausdorff property means that limits of sequences in X or functions to X are unique if they exist [G3, Remark 4.6]. It is probably not hard to believe that analysis would break down badly without this property, so let us just assume this.

A standard counterexample (that we thus do not want to consider) is the "real line with a doubled origin" shown in the picture below on the right, which can be obtained by gluing two copies of $\mathbb{R}$ along the open subset $\mathbb{R}\backslash\{0\}$. This space admits a smooth structure determined by the two charts coming from the two copies of $\mathbb{R}$, but it is not a Hausdorff space as any two neighborhoods of the two origins intersect. Indeed, the sequence $(\frac{1}{n})_{n\in\mathbb{N}_{>0}}$ in this space converges to both origins simultaneously.



The main properties of Hausdorff spaces that we will need are the following:

(a) Every metric space (hence in particular every subset of $\mathbb{R}^n$) is a Hausdorff space, as well as every subset of a Hausdorff space [G3, Example 4.3]. In fact, all topological spaces occurring in these notes are easily seen to be Hausdorff spaces, so we will usually not mention this in what follows.

(b) Every compact subset of a Hausdorff space is closed [G3, Proposition 4.12].

Exercise 10.12. Show for any connected Hausdorff space X with an atlas:

(a) X is path-connected.

(b) For any two points $x,y \in X$ there is a homeomorphism $f\colon X \to X$ with $f(x) = y$.

Definition 10.13 (Second countability)**.** A topological space X is said to be **second countable** (or to satisfy the **second countability axiom**) if it has a countable basis in the sense of [G3, Definition 1.12], i. e. if there is a countable family $(U_i)_{i\in I}$ of open subsets of X such that every open subset of X is of the form $\bigcup_{i\in J} U_i$ for some $J \subset I$.

Example 10.14.

(a) $\mathbb{R}^n$ is second countable for all $n \in \mathbb{N}$: It is easy to check that a countable basis is given by all open balls $U_\varepsilon(x)$ for $\varepsilon \in \mathbb{Q}_{>0}$ and $x \in \mathbb{Q}^n$. 16

(b) Every subspace Y of a second countable topological space X is second countable: If $(U_i)_{i\in I}$ is a countable basis for X, then by definition of the subspace topology $(U_i \cap Y)_{i\in I}$ is a countable basis for Y.

(c) If a topological space X has a countable open cover by second countable spaces, then X itself is second countable: We can just take the union of countable bases for each of the open subsets to obtain a basis for the topology of X.

Taking (a), (b) and (c) together will readily imply that all topological spaces considered in these notes are second countable, hence we will usually not mention this any more in the future. However, it is also easy to come up with examples that do not satisfy this condition:

(d) $\mathbb{R}$ with the discrete topology is not second countable, since the only possible basis for this topology is the family of all singletons $\{x\}$ with $x \in \mathbb{R}$, which is not countable.

Remark 10.15 (First countability)**.** In case you wonder why the condition of Definition 10.13 is called *second* countable: There is also a weaker notion of a *first countable* topological space X only requiring for any point $a \in X$ a countable family $(U_i)_{i\in I}$ of open neighborhoods of a such that every open neighborhood of a is of the form $\bigcup_{i\in J} U_i$ for some $J \subset I$. Hence, first countability only requires locally that X is not "too big", whereas second countability also requires this globally.

An example of a topological space that is not even first countable is the cocountable topology on $\mathbb{R}$, i. e. where a non-empty subset $U \subset \mathbb{R}$ is open if and only if it is the complement of a countable set [G3, Example 1.5 (c)]: Assuming that this space had a countable family $(U_i)_{i\in I}$ of open neighborhoods of 0, their intersection $\bigcap_{i\in I} U_i$ would still be uncountable. So if we pick any non-zero $x \in \bigcap_{i\in I} U_i$ then $\mathbb{R}\backslash\{x\}$ would be an open neighborhood of 0 that is not a union of open subsets in the given family.

Finally, we are now ready to define manifolds, which will be our main objects of interest for the rest of this class.

Definition 10.16 (Manifolds)**.**

(a) For $n \in \mathbb{N}$, an n-dimensional (smooth) **manifold** is a second countable Hausdorff space $X \neq \emptyset$ together with a smooth n-dimensional structure $(\varphi_i)_{i\in I}$. We will denote it by $(X,(\varphi_i)_{i\in I})$, or just by X if the smooth structure is clear from the context. Its dimension will be written as $\dim X = n$.

(b) A **morphism** between two manifolds X and Y is just a smooth map $f\colon X \to Y$ in the sense of Construction 10.5.

This makes manifolds into a category, which we denote by Man.

Example 10.17.

(a) The topological spaces with atlases from Example 10.4 (a), (b), and (c) are manifolds: any open subset of $\mathbb{R}^n$, the sphere S^n and the real projective space $\mathbb{P}^n_{\mathbb{R}}$ are n-dimensional manifolds for all $n \in \mathbb{N}$, and the complex projective space $\mathbb{P}^n_{\mathbb{C}}$ is a $2n$-dimensional manifold.

In particular, we can see already that manifolds may or may not be compact, and they may or may not come with a natural embedding in a real vector space.

(b) Any non-empty open subset U of an n-dimensional manifold X is again an n-dimensional manifold, by restricting the charts $\varphi_i\colon U_i \to V_i$ to $U_i \cap U$ for all $i \in I$. In the following, we will always consider open subsets of a manifold as manifolds in this way.

(c) Manifolds form one of the categories in which bijective morphisms need not be isomorphisms: The morphism $f\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$ is bijective, but its inverse $f^{-1}\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto \sqrt[3]{x}$ is not differentiable, and thus not a morphism.

Remark 10.18.

(a) Let X be an n-dimensional manifold, and let $\varphi\colon U \to V$ be a chart with open subsets $U \subset X$ and $V \subset \mathbb{R}^n$. Reinterpreting this situation, we can then also consider φ as a diffeomorphism between the manifolds U (in the sense of Example 10.17 (b)) and V (in the sense of Example 10.17 (a)): The statement to check here is just that $\varphi \circ \varphi^{-1} = \mathrm{id}_V$ is a diffeomorphism, which is obvious.

In other words, we see that *every n-dimensional manifold is locally isomorphic to an open subset of $\mathbb{R}^n$* – which is the exact formulation of our original idea that a manifold should be a space that "locally looks like $\mathbb{R}^n$". In practice, this means that for any *local* property of a manifold we may assume that we are talking about an open subset of $\mathbb{R}^n$. We will often use this fact in the future to simplify our arguments.

(b) By definition, every manifold is a topological space, so there is a natural functor $\mathsf{Man} \to \mathsf{Top}$ that just forgets the additional conditions and the smooth structure. In particular, every manifold X has functorial singular homology and cohomology groups $H_p(X)$ resp. $H^p(X)$ for all $p \in \mathbb{Z}$.

(c) Having defined manifolds, we should probably mention the famous *Poincaré Conjecture*, which was one of the Millennium Problems with a one million dollar prize money, and has been solved by Grigori Perelman in 2010 in the positive [C]. It states that the only compact 3-dimensional topological manifold (in the sense of Remark 10.3 (b)) that is simply connected (i. e. path-connected with trivial fundamental group) is the sphere S^3.

Obviously, we are not going to prove this statement in these notes. For us, the Poincaré Conjecture should just show that even seemingly simple questions about the geometric shape of manifolds can be very hard and are still problems of current research.

Construction 10.19 (Disjoint unions and products of manifolds)**.** Let $(X,(\varphi_i)_{i\in I})$ with $\varphi_i \colon U_i \to V_i$ and $(Y,(\psi_j)_{j\in J})$ with $\psi_j \colon U'_j \to V'_j$ be two manifolds of dimensions $n = \dim X$ and $m = \dim Y$. The following constructions then work without any problems:

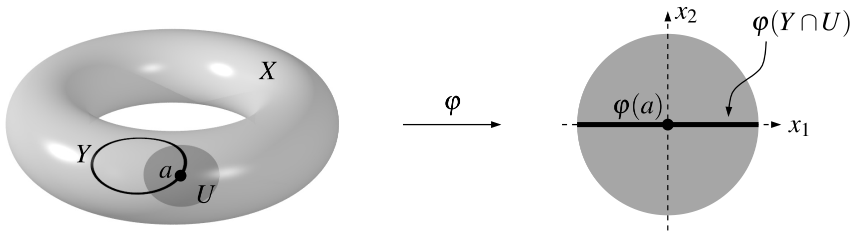
(a) If $m = n$ the *disjoint union* of X and Y is again an n-dimensional manifold: We can just take the charts of both parts together, and there are no compatibility issues since charts on X and Y will never intersect.

Note that this really requires $m = n$: By definition, a manifold does not have to be connected, but all its charts must have the same dimension.

(b) Taking the product maps $\varphi_i \times \psi_j \colon U_i \times U'_j \to V_i \times V'_j \subset \mathbb{R}^{n+m}$ for all pairs $(i, j) \in I \times J$, the *product* $X \times Y$ is a manifold of dimension $n+m$ in a natural way. In particular, the torus $S^1 \times S^1$ is a 2-dimensional manifold.

11. Submanifolds

Similarly to most other categories, having defined manifolds we should now study *submanifolds*, i. e. subsets of a manifold that are again manifolds in a natural way. Of course, one obvious case here is any open subset of a manifold, which we have seen in Example 10.17 (b) to be a manifold as well. But this situation is very special since both manifolds have the same dimension – one could clearly also imagine cases in which a manifold contains a smaller-dimensional one, such as the torus X containing the circle Y in the picture below on the left. Another example that you might know already from first-year mathematics is the case of submanifolds of $\mathbb{R}^n$ [G1, Definition 27.18]: Drawing the torus embedded in $\mathbb{R}^3$ as below means that we have exhibited it as a submanifold of $\mathbb{R}^3$.



The idea to define such submanifolds is very similar to the definition of manifolds themselves: Just as an n-dimensional manifold is a space that locally looks like $\mathbb{R}^n$, a k-dimensional submanifold of it is a space that locally looks like the standard embedding of $\mathbb{R}^k$ in $\mathbb{R}^n$ setting the last $n-k$ coordinates to zero. In the picture above, this means that the circle Y in the torus X is locally given by the equation $x_2 = 0$ in a suitable chart φ. Let us make this definition precise:

Definition 11.1 (Submanifolds)**.** For $k \le n$, a non-empty subset Y of an n-dimensional manifold X is called a k-dimensional **submanifold** of X if for any point $a \in Y$ there is a chart $\varphi\colon U \to V \subset \mathbb{R}^n$ for X with $a \in U$ such that $\varphi(U \cap Y) = \{x \in V : x_{k+1} = \cdots = x_n = 0\}$. Such a chart will be called **adapted to** Y. Clearly, by a translation in $\mathbb{R}^n$ we might assume that $\varphi(a) = 0$ if we wish.

Of course, the first thing to check is that submanifolds in this sense are actually manifolds themselves. Although this is expected from the geometric intuition, it is not obvious from the definition.

Construction 11.2 (Submanifolds as manifolds)**.** Let Y be a k-dimensional submanifold of an n-dimensional manifold X. We will construct from this a natural k-dimensional atlas on Y by restricting adapted charts on X to Y.

More precisely, for any $a \in Y$ let $\varphi\colon U \to V$ be a chart on X adapted to Y with $a \in U$. By restricting φ to Y in the source and to the first k coordinates in the target, this defines a chart

$$\tilde{\varphi}\colon \tilde{U} \to \tilde{V},\ x \mapsto (\varphi_1(x), \dots, \varphi_k(x))$$

containing a on Y, where $\tilde{U} = U \cap Y$ is open in Y, $\tilde{V} = \{(x_1,\dots,x_k) \in \mathbb{R}^k : (x_1,\dots,x_k,0,\dots,0) \in V\}$ is open in $\mathbb{R}^k$, and $\varphi_1,\dots,\varphi_k$ are first k component functions of φ.

Note that these charts are compatible: For any two charts φ_1, φ_2 on X the transition map $\tilde{\varphi}_2 \circ \tilde{\varphi}_1^{-1}$ is just a restriction of $\varphi_2 \circ \varphi_1^{-1}$, and thus smooth. Hence the charts $\tilde{\varphi}$ for all adapted charts φ form an atlas on Y. As the second countability and Hausdorff conditions pass to arbitrary subsets, this makes Y into a k-dimensional manifold.

Moreover, this manifold structure on Y has the property that it makes the embedding $Y \to X$ into a morphism: Taking an adapted chart φ on X and the corresponding chart $\tilde{\varphi}$ on Y, the inclusion is given by $(x_1,\dots,x_k) \mapsto (x_1,\dots,x_k,0,\dots,0)$, which is a smooth map between open subsets of $\mathbb{R}^k$ and $\mathbb{R}^n$.

We will see examples of submanifolds soon, but to simplify their study let us first discuss some general methods to construct them. Thinking of other categories, one would probably expect that new submanifolds can be obtained from old ones by taking inverse images or images under morphisms, so let us investigate this strategy. Unfortunately, without any further restrictions this fails badly, as the following examples show.

Example 11.3.

(a) Let $f\colon \mathbb{R}^2 \to \mathbb{R},\ (x,y) \mapsto xy$. Then the origin in $\mathbb{R}$ is clearly a 0-dimensional submanifold, but its inverse image $f^{-1}(\{0\}) = \{(x,y) \in \mathbb{R}^2 : xy = 0\}$ is not by Example 10.4 (e). Note that the derivative of f is $f' = (y,x)$, so the rank of f is 0 at the origin, whereas it is 1 everywhere else.

(b) Consider the vertical projection $f\colon S^1 \to \mathbb{R},\ (x,y) \mapsto x$. Then S^1 is clearly a manifold, but its image $f(S^1) = [-1,1] \subset \mathbb{R}$ is not, again by Example 10.4 (e). In this case as well, note by Example 10.7 (b) that f has rank 0 at the points $(\pm 1, 0)$ (which map to the critical boundary points of $[-1,1]$), whereas it has rank 1 everywhere else.

Hence, it seems that a rank condition on a morphism $f\colon X \to Y$ could solve our problems. In fact, we will show now that with such a rank condition one can bring f in suitable coordinates in a *normal form* which plays well with adapted charts. For simplicity, we will do this here only in the cases when the rank of f is maximal in the sense that it is either $\dim Y$ (in which the derivative f' in given charts is surjective at any point) or $\dim X$ (in which f' is injective), leading to good behavior with respect to inverse images or images of submanifolds, respectively. The key input to pass from the derivative f' to an actual local behavior of f is the inverse function theorem, which we briefly recall.

Remark 11.4 (Inverse function theorem)**.** In the following, we will need the *inverse function theorem* from analysis [G1, Proposition 27.6]: Any smooth map $f\colon U \to V$ with $f(0) = 0$ and invertible Jacobian matrix $f'(0)$ between two open neighborhoods U and V of 0 in $\mathbb{R}^n$ is a local diffeomorphism, i. e. after possibly shrinking U and V to smaller neighborhoods of the origin the map f will be bijective with a smooth inverse f^{-1}. Actually, this theorem is usually proven for continuously differentiable instead of smooth maps, but in fact it holds for smooth maps as well: As $(f^{-1})'(x) = (f'(f^{-1}(x)))^{-1}$ [G1, Remark 27.3], any order of differentiability that holds for f' and f^{-1} also holds for $(f^{-1})'$, hence by induction if f is smooth then so is f^{-1}.

Lemma 11.5 (Local normal form of smooth maps)**.** *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open neighborhoods of the origin, and let $f\colon U \to V$ be a smooth map with $f(0) = 0$.*

(a) *If $\operatorname{rk}_0 f = m$ (so in particular $m \le n$), after possibly shrinking U and V there is a diffeomorphism $\varphi\colon U \to W$ (i. e. a coordinate change) to an open neighborhood W of 0 in $\mathbb{R}^n$ such that $\varphi(0) = 0$ and $f \circ \varphi^{-1}\colon W \to V$ is the map $s\colon (x_1,\dots,x_n) \mapsto (x_1,\dots,x_m)$.*

(b) *If $\operatorname{rk}_0 f = n$ (so in particular $n \le m$), after possibly shrinking U and V there is a diffeomorphism $\varphi\colon V \to W$ to an open neighborhood W of 0 in $\mathbb{R}^m$ such that $\varphi(0) = 0$ and $\varphi \circ f\colon U \to W$ is the map $i\colon (x_1,\dots,x_n) \mapsto (x_1,\dots,x_n,0,\dots,0)$.*

Proof.

(a) By a permutation of the coordinates in $\mathbb{R}^n$ (which can be absorbed into φ) we may assume without loss of generality that $A := \frac{\partial f}{\partial(x_1,\dots,x_m)}(0)$ is invertible. We construct the auxiliary function

$$F\colon U \to V \times \mathbb{R}^{n-m},\ x \mapsto (f(x), x_{m+1},\dots,x_n) \quad \text{with} \quad F'(0) = \left(\begin{array}{c|c} A & * \\ \hline 0 & E \end{array}\right).$$

As $F'(0)$ is then invertible as well, the inverse function theorem of Remark 11.4 tells us that F is a local diffeomorphism of $\mathbb{R}^n$ around 0. As $f = s \circ F$, we then have $f \circ \varphi^{-1} = s$ with $\varphi := F$.

(b) This time we can permute the coordinates of $\mathbb{R}^m$ to assume that $A := \frac{\partial(f_1,\dots,f_n)}{\partial x}(0)$ is invertible, where $f_1,\dots,f_m$ are the component functions of f. We now construct the auxiliary function

$$F\colon U\times\mathbb{R}^{m-n}\to\mathbb{R}^m=\mathbb{R}^n\times\mathbb{R}^{m-n},\ (x,y)\mapsto f(x)+(0,y)\quad\text{with}\quad F'(0)=\left(\begin{array}{c|c} A & 0 \\ \hline * & E \end{array}\right).$$

Again, $F'(0)$ is invertible as well, so F is a local diffeomorphism of $\mathbb{R}^m$ around 0 by Remark 11.4. As $f=F\circ i$, we thus obtain $\varphi\circ f=i$ with $\varphi:=F^{-1}$. □

Exercise 11.6. As a generalization of Lemma 11.5, let $U\subset\mathbb{R}^n$ and $V\subset\mathbb{R}^m$ again be open neighborhoods of the origin, and let $f\colon U\to V$ be a smooth map with $f(0)=0$ such that $r:=\operatorname{rk}_a f$ is constant for $a\in U$.

(a) Show that after possibly shrinking U and V there are diffeomorphisms $\varphi\colon U\to U'$ and $\psi\colon V\to V'$ to open neighborhoods U' and V' of the origin in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, such that $\psi\circ f\circ\varphi^{-1}\colon U'\to V'$ is the map $(x_1,\dots,x_n)\mapsto(x_1,\dots,x_r,0,\dots,0)$.

(b) Deduce the two parts of Lemma 11.5 directly as special cases of (a).

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Let us now pass from this local the global setting.

Definition 11.7 (Submersions and immersions)**.** Let $f\colon X\to Y$ be a morphism of manifolds.

(a) The morphism f is called a **submersion** if $\operatorname{rk}_a f=\dim Y$ for all $a\in X$.

(b) The morphism f is called an **immersion** if $\operatorname{rk}_a f=\dim X$ for all $a\in X$.

Example 11.8. By the normal form of Lemma 11.5, any submersion is a locally surjective, and any immersion is locally injective. However, it is important to understand that the conditions of Definition 11.7 are not equivalent to local surjectivity or injectivity (intuitively since they require that surjectivity or injectivity are visible already on linear approximations, i. e. that we have "local linear surjectivity or injectivity"), and that they clearly do not imply global surjectivity or injectivity:

(a) The morphism $f\colon\mathbb{R}\to\mathbb{R},\ x\mapsto x^3$ of Example 10.17 (c) is globally and thus also locally bijective, but neither a submersion nor an immersion since its rank at the origin is 0.

(b) The inclusion $f\colon U\to X$ of any non-empty proper open subset U in a manifold X is a submersion and an immersion, but not surjective.

(c) The morphism $f\colon\mathbb{R}\to\mathbb{R}^2,\ t\mapsto(\cos t,\sin t)$ that covers the circle $S^1\subset\mathbb{R}^2$ infinitely often by the real line is an immersion, but not injective.

Proposition 11.9 (Inverse images and images of submanifolds)**.** *Let $f\colon X\to Y$ be a morphism of manifolds with $n:=\dim X$ and $m:=\dim Y$.*

(a) *If f is a submersion and Z is a k-dimensional submanifold of Y, then $f^{-1}(Z)$ is a submanifold of X of dimension $k+n-m$ if it is non-empty.*

(b) *If f is an immersion and Z is a k-dimensional submanifold of X such that $f|_Z\colon Z\to f(Z)$ is a homeomorphism onto its image, then $f(Z)$ is a submanifold of Y of dimension k, and this restriction $f|_Z$ is an isomorphism between Z and $f(Z)$.*

Proof.

(a) The submanifold condition is local, so we can restrict f to open neighborhoods $U\subset X$ of a point $a\in f^{-1}(Z)$ and $V\subset Y$ of $f(a)$ in the following way:

- By Remark 10.18 (a) we may assume that $U\subset\mathbb{R}^n$ and $V\subset\mathbb{R}^m$, and that a and $f(a)$ are the origin.
- On the target $V\subset Y$, we can pick adapted coordinates $x_1,\dots,x_m$ as in Definition 11.1 so that Z is given by the equations $x_{k+1}=\dots=x_m=0$.
- On the source $U\subset X$, we can then pick coordinates as in Lemma 11.5 (a) such that f is given by $(x_1,\dots,x_n)\mapsto(x_1,\dots,x_m)$.

Then $f^{-1}(Z) \cap U$ is given by the equations $x_{k+1} = \cdots = x_m = 0$ again (with the coordinates $x_1, \ldots, x_k, x_{m+1}, \ldots, x_n$ being unrestricted), hence it is a submanifold of dimension $k+n-m$.

(b) Now let $a \in Z$. We use the same strategy as in (a) to find open neighborhood charts U of $a = 0$ in $\mathbb{R}^n$ and V of $f(a) = 0$ in $\mathbb{R}^m$ such that Z is given by $x_{k+1} = \cdots = x_n = 0$, and f is given by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots, 0)$, this time using Lemma 11.5 (b). As $f|_Z$ is a homeomorphism onto its image, the open set $Z \cap U$ in Z is mapped to an open set $f(Z \cap U)$ in $f(Z)$ contained in $f(Z) \cap V$, so by shrinking V we may assume that $f(Z) \cap V = f(Z \cap U)$. Now the proof continues analogously to (a): The set $f(Z) \cap V$ is given by the equations $x_{k+1} = \cdots = x_n = x_{n+1} = \cdots = x_m = 0$, so it is a submanifold of dimension k.

Finally, note that the homeomorphism $f|_Z \colon Z \to f(Z)$ is given in these adapted coordinates by $(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0)$, so it is clearly an isomorphism onto its image. □

Remark 11.10.

(a) As one can see from the proof, the statement of Proposition 11.9 also holds under the weaker assumption that the rank of f is equal to $\dim Y$ only at all points of $f^{-1}(Z)$ in part (a), and equal to $\dim X$ only at all points of Z in part (b).

(b) Recall that any bijective continuous map from a compact space to a Hausdorff space is already a homeomorphism [G3, Corollary 4.14], so for a compact submanifold $Z \subset X$ the homeomorphism assumption for $f|_Z$ in Proposition 11.9 (b) can be weakened to injectivity. Without the compactness condition this is not possible however, as we will see in Example 11.11 (c).

Example 11.11.

(a) The most common example of Proposition 11.9 (a) is clearly a submersion $f \colon \mathbb{R}^n \to \mathbb{R}^k$ and the 0-dimensional submanifold $Z = \{0\}$ of $\mathbb{R}^k$, leading to an $(n-k)$-dimensional submanifold $f^{-1}(Z) = \{x \in \mathbb{R}^n : f(x) = 0\}$ of $\mathbb{R}^n$ (if it is non-empty). By Remark 11.10 (a) it suffices for this that $\mathrm{rk}_a f = k$ at all points $a \in \mathbb{R}^n$ with $f(a) = 0$, leading to the standard definition of a submanifold of $\mathbb{R}^n$ [G1, Definition 27.18].

For example, the sphere $S^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}$ is an n-dimensional submanifold of $\mathbb{R}^{n+1}$ as it is the inverse image of the origin under the morphism

$$f \colon \mathbb{R}^{n+1} \to \mathbb{R}, \; x \mapsto ||x||^2 - 1 = x_1^2 + \cdots + x_{n+1}^2 - 1,$$

which clearly has rank 1 at all points of the sphere.

(b) As a non-trivial example of Proposition 11.9 (b) we can take the morphism

$$f \colon \mathbb{R}^n \to \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}, \; x \mapsto \left(\frac{2x}{||x||^2+1}, \frac{||x||^2-1}{||x||^2+1} \right),$$

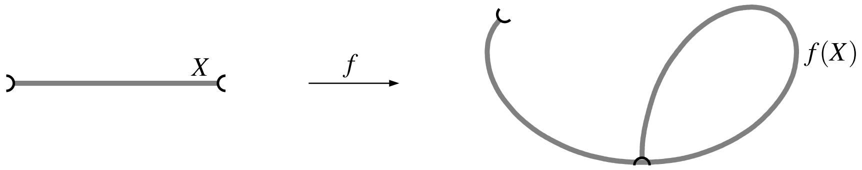
which we had seen already in Example 10.4 (b) as the inverse of the chart

$$\varphi \colon S^n \backslash \{N\} \to \mathbb{R}^n, \; (x,y) \mapsto \frac{x}{1-y}$$

on $S^n \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ obtained by projection from the north pole $N = (0,1)$. As $\varphi \circ f$ is the identity on $\mathbb{R}^n$, the rank of f must be n everywhere, i.e. f is an immersion. Moreover, we know that f is a homeomorphism onto its image $S^n \backslash \{N\}$ since φ is a chart for S^n. Hence, Proposition 11.9 (b) applies and shows that $\mathbb{R}^n$ is actually isomorphic to $S^n \backslash \{N\}$, i.e. (together with the corresponding statement for the other chart leaving out the south pole) that the manifold structure defined in Example 10.4 (b) is the same as the one obtained by Construction 11.2 for the submanifold S^n of $\mathbb{R}^{n+1}$.

(c) The picture below shows the morphism

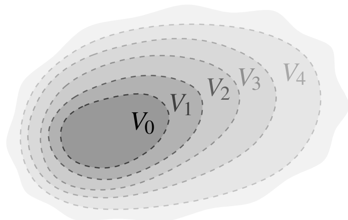
$$f \colon X = \left(0, \frac{3}{4}\pi\right) \to \mathbb{R}^2, \; t \mapsto (\sin 2t \sin t, \sin 2t \cos t).$$



Without even looking at the formula, one can see from the picture already why the condition of f being a homeomorphism between Z and $f(Z)$ in Proposition 11.9 (b) can in general not be replaced by the weaker assumption of injectivity: The map f is an immersion and injective, but its image $f(X)$ is not a 1-dimensional submanifold of $\mathbb{R}^2$ as there is no local chart to $\mathbb{R}$ around the origin for the same reason as in Example 10.4 (e).

Having introduced submanifolds, let us now pursue Example 11.11 (a) a little further: Probably, when visualizing manifolds you would want to think of them as submanifolds of $\mathbb{R}^n$ for some $n \in \mathbb{N}$, even if they have not been defined that way originally – such as e. g. for the Möbius strip or the torus in Example 1.17 where we have drawn pictures in 3-space to show their topology. In addition to visualization, this also has the advantage that it gives us a way to assign global and not just local coordinates to the points on a manifold. So let us ask the question: Can any manifold be embedded in some $\mathbb{R}^n$, i. e. is it always isomorphic to a submanifold of $\mathbb{R}^n$? It turns out that the answer to this question is yes, but given that the definition of a manifold contains almost no global conditions at all it should not come as a surprise that this statement is quite deep and requires some technical preparation.

In fact, the only global requirements for manifolds are the second countability and the Hausdorff axiom, so let us start by studying their implications. We want to show first that these conditions ensure that a manifold X is not too big in the sense that it has an "onion structure" as in the picture on the right: There is a sequence $V_0 \subset \overline{V_0} \subset V_1 \subset \overline{V_1} \subset V_2 \subset \cdots$ covering X with each V_k open and $\overline{V_k}$ compact, so in particular X can be exhausted by countably many compact subspaces.



Lemma 11.12. *Let X be a manifold.*

(a) *Every point of X has a compact neighborhood.*

(b) *There is a countable basis $(B_i)_{i \in I}$ for the topology of X such that every closure $\overline{B_i} \subset X$ for $i \in I$ is compact.*

Proof.

(a) Let $a \in X$ be any point. As the statement is local, we may assume that $X \subset \mathbb{R}^n$ is an open subset. Then $U_\varepsilon(a) \subset X$ for some $\varepsilon > 0$, and thus $\overline{U_{\varepsilon/2}(a)}$ is a compact neighborhood of a.

(b) As X is second countable, there is a countable basis $(B_j)_{j \in J}$ for the topology of X. We set $I := \{i \in J : \overline{B_i} \text{ is compact}\}$ and claim that $(B_i)_{i \in I}$ is a basis as well.

To prove this, we show that any B_j for $j \in J \backslash I$ can already be covered by suitable basis elements B_i for $i \in I$ and thus can be dropped from the basis. Consider an arbitrary point $a \in B_j$, and pick a compact neighborhood K of it by (a). Then a lies in the open set $B_j \cap \mathring{K}$, hence there is a basis element B_{i_a} for $i_a \in J$ with $a \in B_{i_a} \subset B_j \cap \mathring{K}$. In fact, we even have $i_a \in I$: Taking closures, we see that $\overline{B_{i_a}} \subset \overline{K} = K$ since K is closed by Remark 10.11 (b), hence $\overline{B_{i_a}}$ is compact as a closed subset of a compact space [G3, Proposition 4.10 (b)]. Taking these open sets for all $a \in B_j$ together, we obtain $B_j = \bigcup_{a \in B_j} B_{i_a}$ as claimed. □

Proposition 11.13 (Onion structure of manifolds)**.** *For any manifold X, there is a sequence $(V_k)_{k \in \mathbb{N}}$ of open subsets of X with*

$$V_0 \subset \overline{V_0} \subset V_1 \subset \overline{V_1} \subset V_2 \subset \cdots$$

such that each $\overline{V_k}$ is compact and $\bigcup_{k \in \mathbb{N}} V_k = X$.

Proof. By Lemma 11.12 (b) we can pick a countable basis $(B_i)_{i\in\mathbb{N}}$ for the topology of X such that every closure $\overline{B_i}$ is compact. We now construct the sequence $(V_k)_{k\in\mathbb{N}}$ recursively, starting with $V_0 := \emptyset$.

For the induction step, assume that V_k has already been constructed for some $k \in \mathbb{N}$. As $\overline{V_k}$ is compact, there is a finite set $I_k \subset \mathbb{N}$ with $\overline{V_k} \subset \bigcup_{i\in I_k} B_i$, for which we may assume that $\{0,\dots,k\} \subset I_k$. Set $V_{k+1} := \bigcup_{i\in I_k} B_i$. Then $\overline{V_k} \subset V_{k+1}$ by construction, V_{k+1} is open and $\overline{V_{k+1}} = \bigcup_{i\in I_k} \overline{B_i}$ as a finite union of compact spaces compact, and the condition $\{0,\dots,k\} \subset I_k$ ensures that $\bigcup_{k\in\mathbb{N}} V_k = \bigcup_{i\in\mathbb{N}} B_i = X$. □

Example 11.14.

(a) Of course, for a compact manifold X the statement of Proposition 11.13 is trivial since we can just take $V_k = X$ for all $k \in \mathbb{N}$.

(b) For $X = \mathbb{R}^n$, a possible choice of onion structure would be to take open balls $V_k = U_k(0)$ for all $k \in \mathbb{N}$. For $X = \mathbb{R}^n\backslash\{0\}$, we could take $V_0 = \emptyset$ and $V_k = \{x \in \mathbb{R}^n : \frac{1}{k} < \|x\| < k\}$ for $k > 0$.

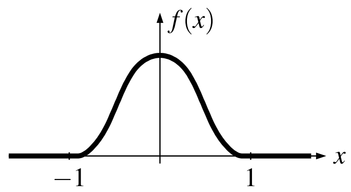
Onion structures serve several purposes; they will e. g. allow us to establish a manifold version of Mayer-Vietoris induction once we have studied de Rham cohomology (see Lemma 15.6). For the moment, we just need some part of the Mayer-Vietoris picture of Chapter 7, namely a dual analytic version of subdivisions. Recall that to study the singular homology of a union $U \cup V$ of open subsets in a topological space we had to subdivide singular simplices in $U \cap V$ fine enough so that each part of the subdivision lies in U or V. Let us now establish an analogous statement for functions: Roughly speaking, given a smooth function $f : U \cup V \to \mathbb{R}$ on a union of two open subsets of a manifold, we want to write it as $f = f_U + f_V$, where f_U and f_V are zero away from U and V, respectively. Of course, both f_U and f_V should still be smooth on $U \cup V$, so a naive solution along the lines of

$$f_U(x) := \begin{cases} f(x) & \text{if } x \in U, \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad f_V(x) := \begin{cases} f(x) & \text{if } x \in V\backslash U, \\ 0 & \text{otherwise} \end{cases}$$

is not what we are looking for. Instead, we need to construct a function f_U that interpolates smoothly between f on $U\backslash V$ and 0 on $V\backslash U$, and similarly for f_V, so that their sum still equals f. Note that it is sufficient to do this for the constant function 1, as for the general case we can then just multiply our interpolating functions with f. Such a decomposition of the constant function 1 into parts that are non-zero only at specific places is called a *partition of unity*, a crucial concept in manifold theory that we will introduce now – in fact, not just for unions of two subsets, but for arbitrary open covers. Its construction is based on the following so-called *bump functions*.

Construction 11.15 (Bump functions and functions with compact support)**.** Recall from analysis that there are non-zero smooth functions $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ that are identically zero outside a given range [G1, Exercise 11.12], such as e. g.

$$f : \mathbb{R} \to \mathbb{R},\ x \mapsto \begin{cases} \exp\left(\dfrac{1}{x^2-1}\right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1 \end{cases}$$



as shown in the picture on the right. In other words, this function approaches the horizontal axes infinitely flat to any order at the points $x = \pm 1$. It is usually given as an example for a *non-analytic* function, i. e. a function that cannot be represented by its Taylor series. 18

It is easy to extend this construction to manifolds. First of all, for a real-valued function $f : X \to \mathbb{R}$ on a manifold X we call

$$\operatorname{supp} f := \overline{f^{-1}(\mathbb{R}\backslash\{0\})} \quad \subset X$$

the **support** of f. Note that we are taking *the closure of* the set of all points with non-zero values here, so e. g. the function above has support $\operatorname{supp} f = [-1,1]$. The special feature of this function is that its support is actually compact – most "interesting" functions $f : \mathbb{R} \to \mathbb{R}$ that you would think of will probably have $\operatorname{supp} f = \mathbb{R}$ and thus non-compact support.

However, modeled on the function above we can easily construct smooth functions with compact support on arbitrary manifolds: Let X be an n-dimensional manifold, and let $a \in X$ be an arbitrary point with a given open neighborhood $U \subset X$. By a **bump function** for a in U we will mean a smooth function $f\colon X \to \mathbb{R}_{\geq 0}$ with $f(a) > 0$ and compact support $\operatorname{supp} f \subset U$. To obtain such a bump function, we can shrink U to a smaller open neighborhood of a such that there is a chart $\varphi\colon U \to V$ with V an open ball in $\mathbb{R}^n$; by a coordinate transformation on $\mathbb{R}^n$ we can furthermore assume that $V = U_2(0)$ is the ball around the origin with radius 2. Extending the function in the picture above to the multidimensional case and pulling it back by the chart φ then gives us a bump function

$$f\colon X \to \mathbb{R},\ x \mapsto \begin{cases} \exp\left(\dfrac{1}{\|\varphi(x)\|^2 - 1}\right) & \text{if } x \in U \text{ with } \|\varphi(x)\| < 1, \\ 0 & \text{otherwise} \end{cases}$$

as desired: It is smooth (first on U in the same way as the bump function in the picture above, and then also by extension by zero on all of X), and its support is $\varphi^{-1}(K_1(0))$, which is compact and contained in $\varphi^{-1}(U_2(0)) = U$.

Note again that taking the closure is important in this construction: The function f in the picture above is a bump function for 0 in $(-2,2)$, but not for 0 in $(-1,1)$ since its support $[-1,1]$ is not contained in $(-1,1)$. Intuitively, for a bump function for $a \in U$ it is not sufficient that f is zero outside U, it must in fact already be identically zero in a neighborhood of the boundary of U.

Proposition 11.16 (Partitions of unity)**.** *Let $(U_i)_{i \in I}$ be an open cover of a manifold X. Then there exists a (not necessarily finite) family $(f_j)_{j \in J}$ of smooth functions $f_j\colon X \to \mathbb{R}_{\geq 0}$ such that:*

(a) *The support of every function f_j for $j \in J$ is compact and contained in one of the open subsets U_i.*

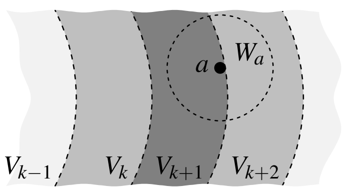
(b) *Every point $a \in X$ has an open neighborhood on which almost all f_j are identically zero – this ensures that the sum $\sum_{j \in J} f_j\colon X \to \mathbb{R}$ is locally finite, i. e. there are no convergence issues when studying local properties of this sum.*

(c) $\sum_{j \in J} f_j = 1$.

Such a family $(f_j)_{j \in J}$ is called a ***partition of unity*** *subordinate to the open cover $(U_i)_{i \in I}$.*

Proof. First of all note that we can replace (c) by the weaker condition that the sum $g := \sum_{j \in J} f_j$ is nowhere zero, i. e. that at least one f_j is non-zero at any point of X: We can then just normalize all functions by replacing each f_j by $\frac{f_j}{g}$.

To construct the family $(f_j)_{j \in J}$, pick an onion structure $(V_k)_{k \in \mathbb{N}}$ as in Proposition 11.13, setting $V_k = \emptyset$ for $k < 0$. For a fixed $k \in \mathbb{N}$ and any a in the compact set $\overline{V_{k+1}} \backslash V_k$ (shown in dark gray in the picture on the right), pick $i \in I$ with $a \in U_i$, and choose a bump function f_a for a in the open set $U_i \cap (V_{k+2} \backslash \overline{V_{k-1}})$ as in Construction 11.15. We denote by $W_a = f_a^{-1}(\mathbb{R} \backslash \{0\})$ the open set on which it is non-zero, contained in the dark and light gray region in the picture.



Note that these open sets W_a cover the compact set $\overline{V_{k+1}} \backslash V_k$, so we can pick a finite set $J_k \subset \overline{V_{k+1}} \backslash V_k$ such that $\overline{V_{k+1}} \backslash V_k \subset \bigcup_{a \in J_k} W_a$. Finally, set $J = \bigcup_{k \in \mathbb{N}} J_k$.

Then $(f_a)_{a \in J}$ satisfies the condition of the proposition: Property (a) holds for every f_a by construction. The sum $\sum_{a \in J} f_a$ is locally finite as required by (b) since every $x \in X$ is contained in some open set $V_{k+2} \backslash \overline{V_k}$, and only the functions f_a with a in the finite set $J_{k-1} \cup J_k \cup J_{k+1} \cup J_{k+2}$ can have non-zero values there. Finally, every $x \in X$ is contained in some compact set $\overline{V_{k+1}} \backslash V_k$, and by definition there will then be some $a \in J_k$ with $f_a(x) > 0$, showing (c) as remarked at the beginning of the proof. □

In fact, if one does not care about the support of the individual functions being compact, there is also a simpler version of partitions of unity that needs only one function for each open set in the given cover.

Corollary 11.17 (Alternative version of partitions of unity)**.** *Again let $(U_i)_{i\in I}$ be an open cover of a manifold X. Then there exists a family $(f_i)_{i\in I}$ of smooth functions $f_i\colon X\to\mathbb{R}_{\geq 0}$ labeled by the same index set I such that:*

(a) *The support of every function f_i for $i\in I$ is contained in U_i (however not necessarily compact).*

(b) *Every point $a\in X$ has an open neighborhood on which almost all f_i are identically zero.*

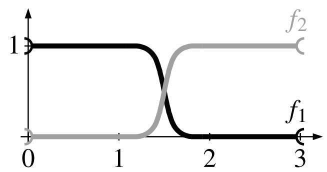
(c) $\sum_{i\in I} f_i = 1$.

Again, such a family $(f_i)_{i\in I}$ will be called a partition of unity subordinate to *$(U_i)_{i\in I}$.*

Proof. Take a partition of unity $(g_j)_{j\in J}$ as in Proposition 11.16, pick for each $j\in J$ an index $i_j\in I$ with $\operatorname{supp} g_j\subset U_{i_j}$, and set $f_i := \sum_{j\in J: i_j=i} g_j$ for all $i\in I$. (Note that there might and generally will be infinitely many j contributing to this sum, but not locally by Proposition 11.16 (b) – hence the local conditions (b) and (c) of the proposition carry over to the new family $(f_i)_{i\in I}$, whereas compactness of the support does not.) □

Example 11.18. Let $X=(0,3)\subset\mathbb{R}$, and consider the open cover of X by the two open subsets $U_1=(0,2)$ and $U_2=(1,3)$.

(a) A partition of unity for X subordinate to the given cover in the sense of Corollary 11.17, i. e. consisting of two non-negative smooth functions f_1 and f_2 with $f_1+f_2=1$, and having their support in U_1 and U_2, respectively, is easy to find without going through the construction in the proof; a graphical visualization is shown in the picture on the right.



(b) Note however that the functions of (a) do not have compact support due to the interval X being open. If we insist on a partition of unity consisting of functions with compact support as in Proposition 11.16, we have to replace e. g. f_1 by countably many functions – for example, by functions $f_{1,n}$ with $n\in\mathbb{N}_{>0}$ having support in the overlapping intervals $\left[\frac{c}{n+2},\frac{c}{n}\right]$ for a suitable $c\in(1,2)$, normalized so that their (locally finite) sum is 1.

Remark 11.19. Of course, dealing with infinite partitions of unity as in Example 11.18 (b) is much more uncomfortable than with finite ones as in (a). Note however that for compact manifolds we can achieve the best of both worlds: Even in Corollary 11.17 the supports of the functions f_j will then be compact since they are closed subsets of the compact space X [G3, Proposition 4.10 (b)].

Exercise 11.20. Let $U_i := \{(x_1,x_2)\in\mathbb{R}^2 : x_i>0\}\subset\mathbb{R}^2$ for $i\in\{1,2\}$.

Find an explicit partition of unity (f_1,f_2) for the submanifold $X:=U_1\cup U_2$ of $\mathbb{R}^2$ subordinate to the open cover (U_1,U_2) in the sense of Corollary 11.17.

We are now ready to use partitions of unity to prove our final result of this chapter:

Proposition 11.21 (**Whitney embedding**)**.** *Every compact manifold is isomorphic to a submanifold of $\mathbb{R}^N$ for some $N\in\mathbb{N}$.*

Proof. Let X be a compact n-dimensional manifold. Due to the compactness, we can cover X by finitely man charts $\varphi_i\colon U_i\to V_i$ for $i=1,\dots,k$. Moreover, we pick a partition of unity subordinate to this cover in the sense of Corollary 11.17, given by the smooth functions $f_i\colon X\to V_i\subset\mathbb{R}^n$ with support in U_i.

Now consider the map $F\colon X\to\mathbb{R}^{k(n+1)}=\mathbb{R}^{n+1}\times\cdots\times\mathbb{R}^{n+1}$ with the component functions

$$F_i\colon X\to\mathbb{R}^{n+1}=\mathbb{R}^n\times\mathbb{R},\ x\mapsto (f_i(x)\,\varphi_i(x),f_i(x)) := \begin{cases}(f_i(x)\,\varphi_i(x),f_i(x)) & \text{if } x\in U_i,\\ 0 & \text{otherwise}\end{cases}$$

for $i=1,\dots,k$. Note that despite their piecewise definition all F_i are smooth since they are identically zero in the open neighborhood $X\setminus\operatorname{supp} f_i$ of ∂U_i.

We claim that F is an isomorphism onto its image $F(X)$ as a submanifold of $\mathbb{R}^N := \mathbb{R}^{k(n+1)}$. By Proposition 11.9 (b) and Remark 11.10 (b), it suffices to prove that F is an injective immersion:

(a) F is injective: Let $x,y \in X$ with $F(x) = F(y)$; in particular, we then have $f_i(x) = f_i(y)$ for all $i = 1,\dots,k$. As $(f_i)_i$ is a partition of unity, this number must be non-zero for at least one i (which also requires $x,y \in U_i$). But for this i we then have $f_i(x)\,\varphi_i(x) = f_i(y)\,\varphi_i(y)$ as well, i. e. $\varphi_i(x) = \varphi_i(y)$. As $\varphi_i\colon U_i \to V_i$ is an isomorphism, this means that $x = y$.

(b) F is an immersion: Let $x \in X$, and again pick $i \in \{1,\dots,k\}$ with $f_i(x) > 0$, implying $x \in U_i$. Then with respect to the chart φ_i we have

$$F_i \circ \varphi_i^{-1}\colon V_i \to \mathbb{R}^{n+1},\ y \mapsto (\tilde{f}_i(y)\,y, \tilde{f}_i(y)) \quad \text{with } \tilde{f} := f \circ \varphi_i^{-1},$$

which has Jacobian matrix

$$(F_i \circ \varphi^{-1})'(y) = \begin{pmatrix} \tilde{f}_i'(y)\,y + \tilde{f}_i(y)\,E \\ \tilde{f}_i'(y) \end{pmatrix} \quad \in \mathbb{R}^{(n+1)\times n}$$

at $y = \varphi_i(x)$. By row transformations we can bring this into the form

$$\begin{pmatrix} \tilde{f}_i(y)\,E \\ \tilde{f}_i'(y) \end{pmatrix} \quad \in \mathbb{R}^{(n+1)\times n},$$

which has rank n since $\tilde{f}_i(y) = f_i(x) > 0$. Hence, F_i and thus also F has rank n at x, i. e. F is an immersion as required. $\square$

Remark 11.22.

(a) The Whitney embedding ensures that compact manifolds such as projective spaces that do not naturally come as submanifolds of some $\mathbb{R}^N$ can nevertheless be written in that way – which is helpful for our further study of compact manifolds since being able to assume an embedding in $\mathbb{R}^N$ will often simplify arguments (see e. g. Construction 12.13). However, the explicit construction of this embedding in the proof of Proposition 11.21 is rarely useful since it uses partitions of unity and a rather high-dimensional ambient space.

(b) Actually, the Whitney embedding statement is more general than what we have shown in Proposition 11.21. In addition, one can prove [B, Theorem II.10.8]:

- The statement also holds without the compactness assumption, i. e. any manifold can be embedded as a submanifold of $\mathbb{R}^N$ (in fact, even as a closed submanifold).
- There is a bound on the necessary ambient dimension in the sense that every manifold can be embedded in $\mathbb{R}^N$ with $N \le 2\dim X$ (see Exercise 1.35 for a simplicial version).

For us, Proposition 11.21 will suffice however, so we will not prove these generalizations.

(c) Even if it is not the topic of this class, let us quickly mention that some very interesting phenomena occur when considering the analogous situation over the complex numbers. Recall from Remark 10.3 (c) that complex manifolds could be defined in exactly the same way as real manifolds, replacing real by complex differentiability. In many aspects, they also behave in the same way: One could literally copy the definition of morphisms, ranks of maps, and submanifolds, and also Proposition 11.9 regarding inverse images and images of submanifolds under submersions and immersions, respectively, holds without change (basically because the inverse function theorem, which was the main ingredient for this statement, holds over $\mathbb{C}$ as well). However, as you might know from complex analysis, bump functions as in Construction 11.15 simply do not exist in the complex world since every holomorphic function is analytic [G4, Proposition 7.10]. For the same reason, partitions of unity are impossible over $\mathbb{C}$, and in fact, the Whitney embedding statement of 11.21 is false for complex manifolds. More drastically, a compact complex manifold (of positive dimension) can *never* be embedded into $\mathbb{C}^N$! A compelling reason for this if you know some basic complex analysis is the Maximum Modulus Principle [G4, Proposition 6.14], which also has a multivariate version: The coordinate functions of an embedding $f\colon X \to \mathbb{C}^N$ of a compact complex manifold X into $\mathbb{C}^N$ would clearly be bounded holomorphic functions due to the compactness of X, hence would have to have a local maximum somewhere – which is impossible unless f is constant, i. e. not an embedding.

Exercise 11.23.

(a) Let $(f_n)_{n\in\mathbb{N}}$ be a family of smooth functions $f_n\colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ with compact support. Show that there are constants $c_n \in \mathbb{R}_{>0}$ for $n \in \mathbb{N}$ such that the sum $f := \sum_{n=0}^{\infty} c_n f_n\colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ converges and is a smooth function as well.

(Hint: It is known from standard analysis that a sequence $g_n\colon \mathbb{R} \to \mathbb{R}$ of differentiable functions converges to a differentiable function with derivative $\lim\limits_{n\to\infty} g_n'$ if both $(g_n)_{n\in\mathbb{N}}$ and $(g_n')_{n\in\mathbb{N}}$ converge uniformly.)

(b) Show that every closed subset $A \subset \mathbb{R}$ is the zero locus of a smooth function, i. e. that there is a smooth function $f\colon \mathbb{R} \to \mathbb{R}$ with $A = \{x \in \mathbb{R} : f(x) = 0\}$.

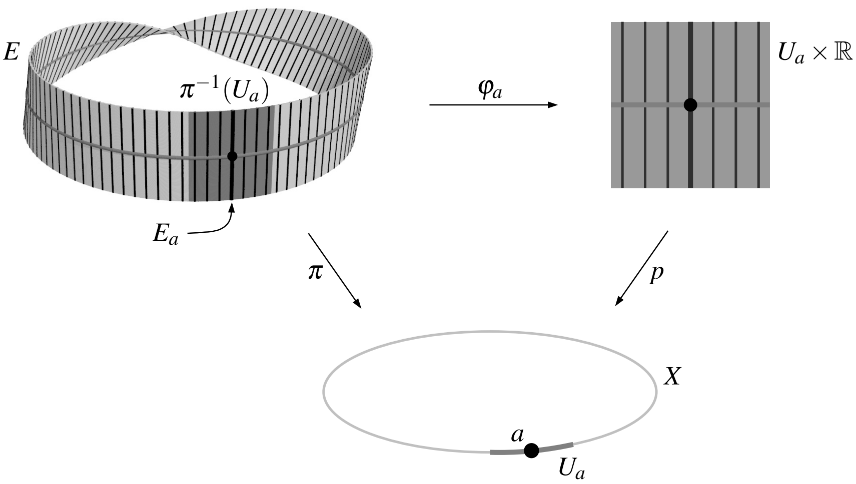
(Hint: Use second countability and suitable bump functions for the complement $\mathbb{R}\backslash A$.)

12. Vector Bundles

In this chapter we want to study so-called vector bundles, a certain kind of manifolds with an extra structure that often occurs in practice. The main motivation probably comes from the study of tangent vectors and tangent vector fields that we considered already in the Hairy Ball Theorem in Proposition 9.5: First of all, given an n-dimensional submanifold X of $\mathbb{R}^m$, we want to define the *tangent space* $T_{X,a}$ to X at any point $a \in X$ as the n-dimensional linear subspace of $\mathbb{R}^m$ which, when shifted to a, approximates X best around a. We then also want to consider *tangent vector fields*, i. e. maps that associate to any $a \in X$ a tangent vector at this point in a smooth way. Note that to be able to define what smoothness means in this situation, it is not sufficient to construct a tangent space separately at each point – instead, in Construction 12.13 we will make the union $T_X := \bigcup_{a \in X} \big(\{a\} \times T_{X,a}\big)$ of all tangent spaces into a global $2n$-dimensional manifold called the *tangent bundle*, so that a tangent vector field on X can be defined as a *smooth* map from X to T_X that assigns to every $a \in X$ a point in $T_{X,a}$.

It might be less geometrically obvious, but we will see later in Example 12.25 that linear approximations of a manifold and hence the tangent bundle can in fact also be defined abstractly for arbitrary manifolds, i. e. not only for submanifolds of $\mathbb{R}^m$. In fact, our setting of assigning smoothly varying tangent spaces to points of a manifold is just a special case of a very general concept called a *vector bundle*, which assigns to every point of a manifold a smoothly varying vector space.

Example 12.1 (The Möbius strip as a vector bundle)**.** To consider a completely different and easier example illustrating this concept, let E be the Möbius strip shown in the top left of the picture below. In contrast to Example 1.17 (c), let us remove its boundary points to make it into a 2-dimensional manifold, and let us consider the dark vertical lines to be the real line $\mathbb{R}$ instead of a bounded interval (which does not make a topological difference since these spaces are homeomorphic). The Möbius strip E then comes with a natural projection map $\pi\colon E \to X$ to its center line $X = S^1$, and the inverse image $E_a := \pi^{-1}(\{a\})$ of every point $a \in X$ is a 1-dimensional vector space (namely one of the vertical lines).



Note that this assignment of a 1-dimensional vector space E_a to every $a \in X$ is *locally trivial* in the sense that for every such point a there is an open neighborhood U_a (shown in dark gray in the picture) such that $\pi^{-1}(U_a)$ just looks like $U_a \times \mathbb{R}$, i. e. like a non-varying vector space $\mathbb{R}$ sitting at every point of U_a. It is only the *global structure* of E that is different, since running around the circle

X will exchange the positive and negative part of $E_a \cong \mathbb{R}$. We will see a precise definition of this bundle in Construction 12.15.

Actually, the same phenomenon occurs e. g. for the tangent bundle of the 2-sphere S^2 in the Hairy Ball Theorem of Proposition 9.5: Locally, we could flatten out the sphere so that its tangent spaces are the same at every point, but globally the tangent planes will wrap around the sphere and thus form a non-trivial varying family of 2-dimensional vector spaces. Let us now describe this situation formally, using the notations from the picture above.

Construction 12.2 (Vector bundles)**.** Let X be an n-dimensional manifold. A (real) **vector bundle** E of **rank** $\operatorname{rk} E = r \in \mathbb{N}$ on X is a manifold E of dimension $n+r$ together with a morphism $\pi\colon E \to X$ and a real vector space structure on the **fiber** $E_x := \pi^{-1}(\{x\}) \subset E$ for every $x \in X$, satisfying the following condition:

Any point $a \in X$ admits an open neighborhood $U_a \subset X$ and an isomorphism of manifolds $\varphi_a\colon \pi^{-1}(U_a) \to U_a \times \mathbb{R}^r$ such that the following two properties hold.

(a) On $\pi^{-1}(U_a)$ we have $\pi = p \circ \varphi_a$, where $p\colon U_a \times \mathbb{R}^r \to U_a$ is the projection onto the first factor (i. e. the diagram in the picture above commutes). In other words, φ_a maps the fiber $E_x \subset E$ to the fiber $\{x\} \times \mathbb{R}^r \subset U_a \times \mathbb{R}^r$, which is often expressed by saying that φ_a is a morphism **over** U_a and written as $\varphi_a|_{E_x}\colon E_x \to \mathbb{R}^r$, identifying $\{x\} \times \mathbb{R}^r$ with $\mathbb{R}^r$. Of course, as φ_a is an isomorphism, $\varphi_a|_{E_x}$ is bijective.

(b) This bijection $\varphi_a|_{E_x}\colon E_x \to \mathbb{R}^r$ is linear, i. e. an isomorphism of vector spaces, for all $x \in U_a$.

Instead of saying that E is a vector bundle on X with the data given above, one also often says that the morphism $\pi\colon E \to X$ is a vector bundle, and that X is the **base** of the bundle.

Example 12.3 (Trivial bundles)**.** For any manifold X and $r \in \mathbb{N}$, the manifold $E = X \times \mathbb{R}^r$ together with the morphism $\pi\colon E \to X$, $(x,v) \mapsto x$ and the identity maps $\varphi_a\colon X \times \mathbb{R}^r \to X \times \mathbb{R}^r$ for any $a \in X$ is a vector bundle of rank r over X. It is called the **trivial vector bundle** of rank r over X since it corresponds to a non-varying vector space $\mathbb{R}^r$ sitting at every point of X.

With this notation, one could say intuitively that the isomorphisms $\varphi_a\colon \pi^{-1}(U_a) \to U_a \times \mathbb{R}^r$ certify that E is locally isomorphic to the trivial vector bundle of rank r around each point $a \in X$. In fact, the maps φ_a are called **local trivializations** of E for this reason. Let us now make this into a precise statement by defining restrictions and morphisms of vector bundles in the expected way. 19

Construction 12.4 (Restrictions and morphisms of vector bundles)**.** Let $\pi\colon E \to X$ be a vector bundle with $\dim X = n$ and $\operatorname{rk} E = r$.

(a) Let $Y \subset X$ be a submanifold. Note that the map π is a submersion: This can be checked locally, so by Remark 10.18 (a) we may assume for any $a \in X$ that we have a local trivialization $\varphi_a\colon \pi^{-1}(U_a) \to U_a \times \mathbb{R}^r$ for an open subset $U_a \subset \mathbb{R}^n$, which we can then take as a chart for E. But in this chart, the map $\pi \circ \varphi_a^{-1}$ is just given by $U_a \times \mathbb{R}^r \to U_a$, $(x,v) \mapsto x$, which clearly has rank $n = \dim X$.

Hence, by Proposition 11.9 (a) the inverse image $E|_Y := \pi^{-1}(Y)$ is a submanifold of E. Note that the projection π restricts to a morphism $\pi|_{E|_Y}\colon E|_Y \to Y$ whose fiber over $a \in Y$ is just the vector space E_a. Moreover, for $a \in Y$ we can restrict the local trivialization maps $\varphi_a\colon \pi^{-1}(U_a) \to U_a \times \mathbb{R}^r$ of E to local trivializations $\pi^{-1}(U_a \cap Y) \to (U_a \cap Y) \times \mathbb{R}^r$ of $E|_Y$, making $E|_Y$ into a vector bundle of rank r on Y. We call it the **restriction** of E to Y.

(b) A **morphism** of E to another vector bundle $\psi\colon F \to X$ over the same base X is a morphism of manifolds $f\colon E \to F$ over X (i. e. with $\psi \circ f = \varphi$, such that f maps the fiber E_x to the fiber F_x for all $x \in X$) such that all maps $f|_{E_x}\colon E_x \to F_x$ are linear for the given vector space structures on the fibers. We denote the set of all such morphisms by $\operatorname{Hom}_X(E,F)$, or just $\operatorname{Hom}(E,F)$ if X is clear from the context.

In particular, this makes vector bundles over X into a category, and defines the notion of isomorphisms of vector bundles over X. Combining this with (a), Construction 12.2 just says that a vector bundle is a morphism $\pi\colon E \to X$, with a given vector space structure in each fiber, and the requirement that it is locally isomorphic to a trivial bundle.

Combining this with Remark 10.18 (a), this means in practice that for any purposes local on X we can always assume that a vector bundle is just the projection $\pi\colon U\times\mathbb{R}^r \to U$ of a trivial vector bundle of rank r over an open subset $U\subset\mathbb{R}^n$.

Remark 12.5.

(a) The definition of a vector bundle is very similar to that of a covering space in [G3, Chapter 8], with the only difference being that the fibers of a covering space are discrete topological spaces, whereas the fibers of a vector bundle are real vector spaces. Note that this vector space structure is actually important here, i. e. we do not only consider a vector bundle E over a base X as a topological space or manifold. In fact, from a purely topological point of view vector bundles are not very interesting: By linearly moving every point to the origin in each fiber, the base X is always a deformation retract of E, so that the singular homology of E is naturally isomorphic to that of X.

(b) Note that the vector space structure on the fibers of a vector bundle $\pi\colon E\to X$ of rank r is part of the fixed data that needs to be given to define a vector bundle, whereas the local trivializations $\varphi_a\colon \pi^{-1}(U_a)\to U_a\times\mathbb{R}^r$ are not – they just need to exist to guarantee that the vector space structure of the fibers varies smoothly, but similarly to charts of manifolds and bases of vector spaces they are not uniquely determined. The following construction shows what this means exactly.

Construction 12.6 ($\operatorname{Hom}(E,F)$ as a vector space)**.** Let $\pi\colon E\to X$ and $\psi\colon F\to X$ be vector bundles of rank r and k, respectively, on an n-dimensional manifold X.

(a) Clearly, for any two morphisms $f,g\in\operatorname{Hom}(E,F)$ and $\lambda\in\mathbb{R}$ there are linear pointwise addition and scalar multiplication maps in the fibers

$$E_x\to F_x,\ v\mapsto f|_{E_x}(v)+g|_{E_x}(v) \quad\text{and}\quad E_x\to F_x,\ v\mapsto \lambda f|_{E_x}(v)$$

that can be combined for all $x\in X$ into set-theoretic maps $f+g\colon E\to F$ and $\lambda f\colon E\to F$ over X. It now follows from the local triviality condition that these two maps are in fact smooth, hence also morphisms between E and F, and thus make $\operatorname{Hom}(E,F)$ into a real vector space: As smoothness is a local condition, we may assume by Construction 12.4 (b) that the bundles $\pi\colon U\times\mathbb{R}^r\to U$ and $\psi\colon U\times\mathbb{R}^k\to U$ are trivial with $U\subset\mathbb{R}^n$ open. The two morphisms of vector bundles are then given by $f\colon U\times\mathbb{R}^r\to U\times\mathbb{R}^k,\ (x,v)\mapsto(x,\tilde f(x,v))$ and $g\colon U\times\mathbb{R}^r\to U\times\mathbb{R}^k,\ (x,v)\mapsto(x,\tilde g(x,v))$ for smooth functions $\tilde f,\tilde g\colon U\times\mathbb{R}^r\to\mathbb{R}^k$. But then

$$\begin{aligned} f+g\colon U\times\mathbb{R}^r\to U\times\mathbb{R}^k,&\ (x,v)\mapsto(x,\tilde f(x,v)+\tilde g(x,v))\\ \text{and}\qquad \lambda f\colon U\times\mathbb{R}^r\to U\times\mathbb{R}^k,&\ (x,v)\mapsto(x,\lambda\tilde f(x,v))\end{aligned}$$

are also smooth since smooth $\mathbb{R}^k$-valued functions on open subsets of $\mathbb{R}^n\times\mathbb{R}^r$ form a vector space. This is exactly what it means that “the vector space structure in a bundle varies smoothly with the point”.

(b) Expanding the idea of (a), the map $\tilde f\colon U\times\mathbb{R}^r\to\mathbb{R}^k$ is smooth and linear in the second component, and thus by linear algebra must be of the form

$$\tilde f\colon U\times\mathbb{R}^r\to\mathbb{R}^k,\ (x,v)\mapsto A(x)\cdot v$$

for a unique smooth matrix-valued map $A\colon U\to\mathbb{R}^{k\times r},\ x\mapsto A(x)$. In other words, in given local trivializations a morphism $E\to F$ of vector bundles is just the same as a morphism of manifolds with target space $\mathbb{R}^{k\times r}$. In particular, a morphism between the trivial bundles $X\times\mathbb{R}^r$ and $X\times\mathbb{R}^k$ is globally the same as a morphism $X\to\mathbb{R}^{k\times r}$.

(c) A consequence of this is that – as for vector spaces, but unlike the case of manifolds – any bijective morphism of vector bundles is an isomorphism: If f is bijective then the matrix $A(x)$ of (b) must be invertible at any point, hence we must have $k=r$ and obtain an inverse morphism $A^{-1}: U \to \mathbb{R}^{r\times r}$, $x \mapsto A(x)^{-1}$ (which is smooth by the formula for the inverse matrix in [G1, Proposition 18.21]), showing that the inverse map $f^{-1}: F \to E$ is in fact a morphism of vector bundles.

Note that this shows again that vector bundles can be thought of as vector spaces varying smoothly with the point on a manifold: For (finite-dimensional) vector spaces, a linear map is given by a matrix after a choice of bases. For vector bundles, a morphism is given by a matrices varying smoothly with the point on a manifold after a choice of local trivializations.

Consequently, we would also expect that we can apply standard constructions for linear maps to morphisms of vector bundles fiber-by-fiber, e. g. taking their images and kernels to obtain new vector bundles that are subsets of old ones. There is a slight complication here however since vector bundles must have constant rank, i. e. we can only hope for a matrix-valued function to determine image and kernel vector bundles if all matrices have the same rank. As vector bundles are not the main focus of this class, let us not go into details here, but just consider injective morphisms leading to image subbundles of the target bundle in the following sense.

Definition 12.7 (Subbundles)**.** A **(vector) subbundle** of a vector bundle $\pi: E \to X$ is a vector bundle of the form $\pi|_F: F \to X$ for a submanifold $F \subset E$, with the vector space structure of F_a taken from E_a (so in particular, F_a has to be a vector subspace of F_a for all $a \in X$).

Lemma 12.8 (Image subbundles)**.** *Let X be an n-dimensional manifold, and consider a morphism $A: X \to \mathbb{R}^{k\times r}$, $x \mapsto A(x)$ for $k, r \in \mathbb{N}_{>0}$ with $\operatorname{rk} A(x) = r$ for all $x \in X$. Then the* ***image subbundle***

$$\operatorname{im} A := \bigcup_{x\in X} \big(\{x\} \times \operatorname{im} A(x)\big) = \{(x, A(x)\cdot v) : x \in X, v \in \mathbb{R}^r\} \qquad \subset X \times \mathbb{R}^k$$

is a subbundle of the trivial bundle $X \times \mathbb{R}^k$ which is isomorphic to the trivial bundle $X \times \mathbb{R}^r$.

Proof. As $\operatorname{rk} A(x) = r$ the matrix $A(x)^{\mathsf{T}} A(x) \in \mathbb{R}^{r\times r}$ is invertible for all $x \in X$, and thus we can construct the morphism

$$B: X \to \mathbb{R}^{r\times k},\ x \mapsto \big(A(x)^{\mathsf{T}} A(x)\big)^{-1} A(x)^{\mathsf{T}}$$

(note that the explicit formula for the inverse matrix [G1, Proposition 18.21] shows that B is in fact smooth); it is a left inverse for A in the sense that $B(x)A(x) = E$ for all $x \in X$. Now consider the corresponding morphisms of trivial vector bundles

$$f: X \times \mathbb{R}^r \to X \times \mathbb{R}^k,\ (x,v) \mapsto (x, A(x)\cdot v) \quad \text{and} \quad g: X \times \mathbb{R}^k \to X \times \mathbb{R}^r,\ (x,v) \mapsto (x, B(x)\cdot v)$$

as in Construction 12.6 (b). Clearly, the map f has image $\operatorname{im} A$, and as g is a left inverse we see that $f: X \times \mathbb{R}^r \to \operatorname{im} A$ must be a homeomorphism (with inverse $g|_{\operatorname{im} A}$). Moreover, $g \circ f = \mathrm{id}$ requires the rank of f in the sense of Construction 10.5 (b) to be $n + r$, so f is an immersion, and thus $f: X \times \mathbb{R}^r \to \operatorname{im} A$ is an isomorphism of manifolds by Proposition 11.9 (b). Finally, both f and g are linear in v, so g is a global trivialization of $\operatorname{im} A$ and thus makes it into a subbundle of $X \times \mathbb{R}^k$ isomorphic to $X \times \mathbb{R}^r$. □

Remark 12.9 (Local image subbundles)**.**

(a) The important point of Lemma 12.8 is not that $\operatorname{im} A$ is trivial, but that the smoothness of the morphism A ensures that the vector subspaces of the fibers determined by $\operatorname{im} A$ vary smoothly, making $\operatorname{im} A$ into a subbundle. Actually, being a subbundle is a local condition on the base, so if $\pi: E \to X$ is a vector bundle then any subset $F \subset E$ will also be a subbundle if there is an open cover $(U_i)_{i\in I}$ of X on which E is trivial and such that $\pi^{-1}(U_i) \cap F$ is an image subbundle of $\pi^{-1}(U_i)$ over U_i as in Lemma 12.8 in suitable local trivializations for all $i \in I$ – and due to the gluing process by the cover and the local trivializations the subbundle F of E obtained in this way will then in general be non-trivial.

(b) In particular, this situation occurs if we have an injective morphism $f: F \to E$ of vector bundles over a manifold X. Then the maps $f|_{E_x}: E_x \to F_x$ in the fibers over any $x \in X$ are injective as well, so in local trivializations of E and F over a cover $(U_i)_{i\in I}$ the morphism f

is given by functions $U_i \to \mathbb{R}^{k\times r}$ mapping into the space of matrices of rank r. Hence, the image of f is a subbundle of E (which can be thought of as a relative version of lemma 12.8.

Clearly, this applies if $F \subset E$ is already known to be a subbundle in the first place and f is the inclusion; hence we also see conversely that any subbundle of a vector bundle is always locally an image subbundle in the sense of Lemma 12.8.

Exercise 12.10 (Kernel subbundle)**.** Let $f\colon E \to F$ be a morphism between two vector bundles $\pi\colon E \to X$ and $\psi\colon F \to X$ over a manifold X. We denote by $\ker f := \bigcup_{x\in X} \ker(f|_{E_x}) \subset E$ the union of the kernels of the linear maps in the fibers given by f.

(a) Show that $\ker f$ is a subbundle of E if and only if the dimension of the vector spaces $\ker(f|_{E_x})$ for $x \in X$ is constant.

(b) Give an example of a morphism f for which $\ker f$ is not a subbundle of E.

Exercise 12.11 (Pullbacks of bundles)**.** Let $f\colon X \to Y$ be a morphism of manifolds, and let $\pi\colon E \to Y$ a vector bundle on Y of rank r. We set

$$f^*(E) := \{(x,v) \in X \times E : f(x) = \pi(v)\} \quad \subset X \times E.$$

(a) Show that $f^*(E)$ is a submanifold of $X \times E$, and that the natural projection $\pi\colon f^*(E) \to X$ onto the first factor is a smooth map with $\pi^{-1}(\{x\}) = E_{f(x)}$ for all $x \in X$.

(b) Show that $\pi\colon f^*(E) \to X$ is a rank-r vector bundle. It is called the *pullback bundle* of E by f.

Exercise 12.12. Show that every vector bundle on a compact manifold is a subbundle of a trivial bundle.

We can now use the idea of Remark 12.9 to give many interesting vector bundle constructions. Let us start with the tangent bundle already mentioned at the beginning of this chapter.

Construction 12.13 (Tangent bundles and derivatives of morphisms)**.** Let X be a k-dimensional manifold. For simplicity, let us assume for the moment that it comes as a submanifold $X \subset \mathbb{R}^n$ for some $n \in \mathbb{N}$. Recall that by the Whitney embedding of Proposition 11.21 and Remark 11.22 (b) this is not a restriction except that it makes a possibly unnatural choice. We will study how to avoid this choice in Example 12.25.

(a) For any $x \in X$ we have a chart $\varphi\colon U \to V \subset \mathbb{R}^k$ around x, whose inverse $\varphi^{-1}\colon V \to U \subset \mathbb{R}^n$ we can consider as a local parametrization for X. The linearization of this map is its Jacobian matrix $A_\varphi(x) := (\varphi^{-1})'(\varphi(x)) \in \mathbb{R}^{n\times k}$, hence we define the **tangent space** of X at x to be the k-dimensional vector space

$$T_{X,x} := \operatorname{im} A_\varphi(x) \quad \subset \mathbb{R}^n.$$

Note that this is independent of the chart φ: A different choice would just precompose the Jacobian matrix $A_\varphi(x)$ with the (invertible) Jacobian matrix of the transition map, which does not change its image. We now collect these tangent spaces in a global object

$$T_X := \bigcup_{x\in X} (\{x\} \times T_{X,x}) \quad \subset X \times \mathbb{R}^n.$$

Locally on a chart $\varphi\colon U \to V$, this is just the image subbundle $\operatorname{im} A_\varphi$ of Lemma 12.8, so Remark 12.9 (a) shows that T_X is a subbundle of rank k of $X \times \mathbb{R}^n$ with fibers $T_{X,x}$. We call
20 it the **tangent bundle** of X.

(b) Tangent spaces can be used to construct well-defined derivatives of a morphism $f\colon X \to Y$ between manifolds for any submanifold $Y \subset \mathbb{R}^m$: For a point $x \in X$, choose charts $\varphi\colon U \to V$ and $\psi\colon U' \to V'$ with $f(U) \subset U'$ around x and $y := f(x)$, respectively, and consider the linear map

$$f'(x)\colon T_{X,x} \to T_{Y,y},\ (\varphi^{-1})'(\varphi(x)) \cdot v \mapsto (f \circ \varphi^{-1})'(\varphi(x)) \cdot v.$$

Note that:

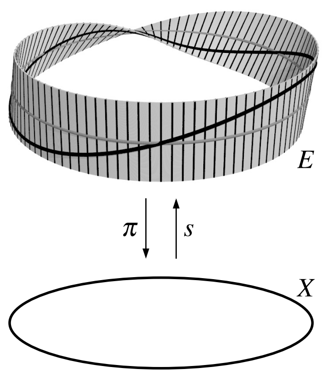
- Every vector in $T_{X,x}$ is of the form $(\varphi^{-1})'(\varphi(x)) \cdot v$ for a unique $v \in \mathbb{R}^k$, so $f'(x)$ is defined at every point of $T_{X,x}$. Moreover, the definition is independent of the choice of chart φ since a different choice would multiply both $(\varphi^{-1})'(\varphi(x))$ and $(f \circ \varphi^{-1})'(\varphi(x))$ with the same Jacobian matrix of the transition map.
- The definition is visibly independent of the chart ψ. However, note that the image of $T_{X,x}$ actually lies in $T_{Y,y}$ since

$$(f \circ \varphi^{-1})'(\varphi(x)) \cdot v = (\psi^{-1})'(\psi(y)) \cdot (\psi \circ f \circ \varphi^{-1})'(\varphi(x)) \cdot v \in T_{Y,y}.$$

We can think of $f'(x)$ as the linearization of f around x, mapping the linearization $T_{X,x}$ of X to the linearization $T_{Y,y}$ of Y. It is called the **derivative** of f at x. Varying the point, we can also consider it as a morphism of manifolds $f'\colon T_X \to T_Y$ (that sends a point in the fiber over x to the fiber over $f(x)$). In the rest of these notes, we will not need this construction however as we will mainly be interested in derivatives of real-valued functions (see Chapter 13).

We have also mentioned already in the introduction to this chapter that one often wants to consider functions on a manifold X that assign to any point $x \in X$ a vector in the fiber E_x of a vector bundle $\pi\colon E \to X$, such as e. g. tangent vector fields. Such functions are called sections of the vector bundle, and they are closely related to morphisms of vector bundles as in the following construction.

Construction 12.14 (Sections of vector bundles)**.** For a vector bundle $\pi\colon E \to X$, a smooth map $s\colon X \to E$ with $\pi \circ s = \mathrm{id}_X$ is called a **section** of E; we can think of it as a smooth assignment of a vector in the fiber E_x for any point $x \in X$. Hence, for example, the picture on the right shows a section of the Möbius strip vector bundle of Example 12.1 (where the image of s is the bold curve in E), and a section of the tangent bundle T_X can be thought of as a tangent vector field as in the Hairy Ball Theorem of Proposition 9.5. For any submanifold Y of X, the set of all sections of the restriction $E|_Y$ of Construction 12.4 (a) will be denoted by $E(Y)$. As in Construction 11.15, the **support** of a section $s \in E(Y)$ is defined to be the closure in Y of the set of all $x \in Y$ with $s(x) \neq 0$ in E_x.



There are a few important properties of sections that follow directly from the definitions:

(a) Any section $s\colon X \to E$ determines an isomorphism between X and its image $s(X)$: It is clear that the morphism $s\colon X \to s(X)$ is bijective, and $\pi|_{s(X)}\colon s(X) \to X$ is an inverse morphism.

In particular, there is always the *zero section* sending each point $x \in X$ to the origin in E_x, which allows us to consider X as being naturally embedded in E – given e. g. by the center line of the Möbius strip in the picture above.

(b) A section $s\colon X \to E$ is the same as a morphism of vector bundles $f\colon X \times \mathbb{R} \to E$ from the trivial rank-1 vector bundle on X to E: Given a section $s\colon X \to E$, there is an associated morphism of vector bundles $f\colon X \times \mathbb{R} \to E$, $(x, \lambda) \mapsto \lambda s(x)$ (with the scalar multiplication in the fiber E_x), and for any morphism $f\colon X \times \mathbb{R} \to E$ of vector bundles we can construct a section $s = f(\cdot, 1)\colon X \to E$. It is clear that these two constructions are inverse to each other. In particular, by Construction 12.6 (a) the spaces of sections $E(Y)$ for any submanifold $Y \subset X$ are vector spaces by pointwise addition and scalar multiplication in the fibers.

(c) A vector bundle $\pi\colon E \to X$ of rank 1 is isomorphic to a trivial bundle if and only if it has a section which is nowhere zero: Of course, any trivial bundle $\pi\colon X \times \mathbb{R}^r \to X$ of rank $r > 0$ has a section which is nowhere zero by taking $s\colon x \mapsto (x, v)$ for an arbitrary constant vector $v \in \mathbb{R}^r \backslash \{0\}$. Conversely, if E has rank 1 and $s\colon X \to E$ is nowhere zero, then the corresponding morphism of vector bundles $f\colon X \times \mathbb{R} \to E$ from (b) is bijective, hence an isomorphism by Construction 12.6 (c).

As an example, it seems intuitive that the rank-1 Möbius strip vector bundle of the picture above is non-trivial, and that in fact any section of it must cross the middle line of the strip. In order to be able to phrase this rigorously, let us now give an explicit construction of this vector bundle; it is a special case of the so-called *tautological bundle* of projective spaces.

Construction 12.15 (Tautological subbundles of projective spaces)**.** For $n \in \mathbb{N}$ and $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, consider the subset

$$E = \{((x_0 : \cdots : x_n),(v_0,\dots,v_n)) \in \mathbb{P}^n_{\mathbb{K}} \times \mathbb{K}^{n+1} : (x_0,\dots,x_n) \text{ and } (v_0,\dots,v_n) \text{ are linearly dependent}\}$$

of $\mathbb{P}^n_{\mathbb{K}} \times \mathbb{K}^{n+1}$ together with the projection $\pi\colon E \to \mathbb{P}^n_{\mathbb{K}}$ onto its first component. Note that on the open subset $U_0 = \{(x_0 : \cdots : x_n) : x_0 \neq 0\}$ of $\mathbb{P}^n_{\mathbb{K}}$ with coordinates $\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}$, it is the image subbundle for the matrix-valued function

$$A\colon U_0 \to \mathbb{K}^{(n+1)\times 1},\ (x_0 : \cdots : x_n) \mapsto \left(1,\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\right)^{\mathsf{T}}. \qquad (*)$$

The same holds over the other open subsets $U_i = \{(x_0 : \cdots : x_n) : x_i \neq 0\}$, so E is a subbundle of the trivial bundle $\mathbb{P}^n_{\mathbb{K}} \times \mathbb{K}^{n+1}$ on $\mathbb{P}^n_{\mathbb{K}}$ by Lemma 12.8 and Remark 12.9 (a). By construction, its fiber over a point $(x_0 : \cdots : x_n) \in \mathbb{P}^n_{\mathbb{K}}$ (which by definition is a 1-dimensional subspace of $\mathbb{K}^{n+1}$) is precisely this 1-dimensional subspace; hence it is called the **tautological subbundle** of $\mathbb{P}^n_{\mathbb{K}}$. As its fibers are all isomorphic to $\mathbb{K}$, it has rank 1 if $\mathbb{K} = \mathbb{R}$ and rank 2 if $\mathbb{K} = \mathbb{C}$.

In fact, we claim that the tautological subbundle of $\mathbb{P}^1_{\mathbb{R}} \cong S^1$ is just the Möbius strip bundle from the introduction to this chapter. To see this, consider first the open subset $U_0 \cong \mathbb{R}$ (on which E is trivial by Lemma 12.8), with the nowhere-zero section obtained by normalizing $(*)$, i. e.

$$s\colon U_0 \to U_0 \times \mathbb{R}^2,\ x \mapsto \left(x, \frac{A(x)}{\|A(x)\|}\right) = \left(x, \frac{(x_0,x_1)^{\mathsf{T}}}{\sqrt{x_0^2+x_1^2}}\right).$$

Now we go to the point at infinity $\infty = (0:1) \in \mathbb{P}^1_{\mathbb{R}} \backslash U_0$, where correspondingly the fiber of E is $\mathrm{Lin}((0,1)^{\mathsf{T}})$. We can approach this point from both sides by letting $\frac{x_1}{x_0}$ go to $\pm\infty$, obtaining

$$\lim_{x_1/x_0 \to \infty} s(x) = ((0:1),(0,1)^{\mathsf{T}}) \quad \text{and} \quad \lim_{x_1/x_0 \to -\infty} s(x) = ((0:1),(0,-1)^{\mathsf{T}}).$$

Hence, the two ends of the strip over U_0 are glued at this point at infinity in opposite directions, meaning that E over $\mathbb{P}^1_{\mathbb{R}}$ is topologically the Möbius strip (and that $s\colon U_0 \to U_0 \times \mathbb{R}^2$ does not extend to a smooth section of E over $\mathbb{P}^1_{\mathbb{R}}$).

The last example of a vector bundle that we want to consider in this chapter is the *normal bundle* of a submanifold Y of a manifold X. It is a subbundle of $Y \times \mathbb{R}^n$ whose fiber at a point $x \in Y$ is the space of all tangent vectors in $T_{X,x}$ orthogonal to $T_{Y,x}$, and it can be obtained analogously to Construction 12.13.

Construction 12.16 (Normal bundles)**.** Let Y be an r-dimensional submanifold of a k-dimensional manifold $X \subset \mathbb{R}^n$ (again we will see how to avoid a choice of embedding in $\mathbb{R}^n$ in Remark 12.29). At any point $x \in Y$, we define the **normal space** of Y in X at x to be

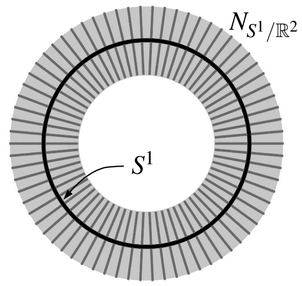
$$N_{Y/X,x} := \{v \in T_{X,x} \subset \mathbb{R}^n : v \perp T_{Y,x}\}, \quad \text{and set} \quad N_{Y/X} := \bigcup_{x\in Y} (\{x\} \times N_{Y/X,x}) \quad \subset Y \times \mathbb{R}^n.$$

To see that $N_{Y/X}$ is a vector bundle on Y, note first by Definition 11.1 that around any $x \in Y$ we have an adapted chart $\varphi\colon U \to V \subset \mathbb{R}^k$ in which Y is given by the vanishing of the last $k-r$ coordinates. Hence, if we form the Jacobian matrix $A_\varphi(x) := (\varphi^{-1})'(\varphi(x)) \in \mathbb{R}^{n\times k}$ exactly as in Construction 12.13, its first r columns will span $T_{Y,x}$, whereas all its columns span $T_{X,x}$.

Now recall the Gram-Schmidt orthonormalization process from linear algebra [G1, Remark 21.27 (c) and Proposition 21.31]: Given any k linearly independent vectors, it exhibits an explicit algorithm (in terms of smooth functions) to add to each of the vectors a certain linear combination of the previous ones so that the new vectors are orthonormal. In terms of matrices, this means that we can write $A_\varphi(x) = Q_\varphi(x)R_\varphi(x)$ where $Q_\varphi(x) \in \mathbb{R}^{n\times k}$ and $R_\varphi(x) \in \mathbb{R}^{k\times k}$ vary smoothly with x, the matrix

$Q_\varphi(x)$ has orthogonal columns, and $R_\varphi(x)$ is upper triangular and invertible. As the first r resp. all columns of $Q_\varphi(x)$ still span $T_{Y,x}$ resp. $T_{X,x}$, this means that in the given chart $N_{Y/X}$ is just the image subbundle $\operatorname{im}\tilde{Q}_\varphi$, where $\tilde{Q}_\varphi(x) \in \mathbb{R}^{n\times(k-r)}$ consists of the last $k-r$ columns of $Q_\varphi(x)$. Hence, by Lemma 12.8 and Remark 12.9 (a) we see that $N_{Y/X}$ is a vector bundle of rank $k-r$ on Y. It is called the **normal bundle** of Y in X.

Example 12.17. For the sphere $Y = S^n$ in $X = \mathbb{R}^{n+1}$ with $n \in \mathbb{N}$, the normal bundle $N_{S^n/\mathbb{R}^{n+1}}$ is a vector bundle of rank 1 on S^n. The picture on the right shows this situation in the case $n = 1$, where as before we have drawn the radial fibers $N_{S^1/\mathbb{R}^2,x}$ as bounded and attached to the corresponding point $x \in S^1$ (which gives a diffeomorphic manifold). Note that $N_{S^n/\mathbb{R}^{n+1}}$ has a nowhere vanishing section by mapping any point $x \in S^n$ to itself in the fiber $\mathbb{R}^{n+1}$, so the normal bundle $N_{S^n/\mathbb{R}^{n+1}}$ is in fact (isomorphic to) a trivial bundle by Construction 12.14 (c), although its embedding in $S^n \times \mathbb{R}^{n+1}$ is non-trivial.



By construction, note that we have $T_{Y,x} \oplus N_{Y/X,x} = T_{X,x}$ for any point $x \in Y$ if $Y \subset X \subset \mathbb{R}^n$ are submanifolds. As expected, we can make this into an equation of vector bundles by taking this equation fiber-by-fiber:

Construction 12.18 (Direct sums of subbundles)**.** Let F_1 and F_2 be two subbundles of a vector bundle $\pi\colon E \to X$, and assume that their fibers $F_{1,x}$ and $F_{2,x}$ form a direct sum in E_x for all $x \in X$, i. e. that $F_{1,x} \cap F_{2,x} = \{0\}$. Then we can form their fiberwise direct sum

$$F_1 \oplus F_2 := \bigcup_{x\in X} (F_{1,x} \oplus F_{2,x}),$$

which is again a vector subbundle of E on X, of rank $\operatorname{rk} F_1 + \operatorname{rk} F_2$: By Remark 12.9 (b) we may assume that F_1 and F_2 are image subbundles given by two matrix-valued functions with linearly independent columns at every point, and $F_1 \oplus F_2$ is then just the image subbundle given by joining these columns into one matrix.

Example 12.19.

(a) By construction, we have $T_Y \oplus N_{Y/X} = T_X|_Y$ as bundles on Y for any submanifold Y of a manifold $X \subset \mathbb{R}^n$.

(b) As a concrete case, consider $Y = S^2$ in $X = \mathbb{R}^3$, and note that $T_X|_Y = Y \times \mathbb{R}^3$ is trivial. Hence, by (a) we have

$$T_{S^2} \oplus N_{S^2/\mathbb{R}^3} = S^2 \times \mathbb{R}^3.$$

But $N_{S^2/\mathbb{R}^3}$ is trivial as well by Example 12.17, whereas T_{S^2} is not: By the Hairy Ball Theorem of Proposition 9.5, it does not have a nowhere vanishing section, and thus cannot be trivial by Construction 12.14 (c). Hence, a direct sum of a non-trivial and a trivial bundle can in fact be trivial – which might be counterintuitive, but it is of course the non-trivial embedding of $N_{S^2/\mathbb{R}^3}$ in $S^2 \times \mathbb{R}^3$ that makes this work.

Exercise 12.20.

(a) Show that the tangent bundle of the sphere S^1 is trivial.

(b) Note that the tangent bundle of S^2 is not trivial by the Hairy Ball Theorem of Proposition 9.5.

Is the tangent bundle of S^3 trivial?

An important application of the normal bundle $N_{Y/X}$ of a submanifold Y of a manifold X can already be seen from the picture in Example 12.17: Considered as a manifold itself, it seems that we can think of $N_{Y/X}$ as a homotopy equivalent open neighborhood of Y in X. Let us prove this statement now, which is usually called the *Tubular Neighborhood Theorem* since in the case $\dim Y = 1$ and

$\dim X = 3$ the (bounded) normal bundle $N_{Y/X}$ has 2-dimensional disks as fibers and thus looks like a tube around the 1-dimensional manifold Y.

Proposition 12.21 (Tubular Neighborhood Theorem). *For any compact submanifold X of $\mathbb{R}^n$, there is an $\varepsilon > 0$ such that the ε-neighborhood of the zero section of the normal bundle*

$$N^{\varepsilon}_{X/\mathbb{R}^n} := \{(x,v) \in N_{X/\mathbb{R}^n} : \|v\| < \varepsilon\} \quad \subset X \times \mathbb{R}^n$$

is mapped isomorphically by

$$f\colon N^{\varepsilon}_{X/\mathbb{R}^n} \to \mathbb{R}^n,\ (x,v) \mapsto x+v$$

21 *onto an open neighborhood of X in $\mathbb{R}^n$.*

Proof. Let us first consider the map $f\colon N_{X/\mathbb{R}^n} \to \mathbb{R}^n$, $(x,v) \mapsto x+v$ on the whole normal bundle and show that it is a local isomorphism around any point $(x,0)$ on the zero section. This is just a straightforward application of the inverse function theorem of Remark 11.4: Setting $k = \dim X$, consider a chart $\varphi\colon U \to V \subset \mathbb{R}^k$ for X around x and write $N_{X/\mathbb{R}^n}$ on this chart as an image subbundle $\operatorname{im} \tilde{Q}_\varphi$ with $\tilde{Q}_\varphi(x) \in \mathbb{R}^{n\times(n-k)}$ a matrix with orthonormal columns obtained by Gram-Schmidt normalization as in Construction 12.16. In the coordinates $(y,u) \in V \times \mathbb{R}^{n-k} \subset \mathbb{R}^n$ of the chart φ and the local trivialization of $N_{X/\mathbb{R}^n}$, the map f is then given by

$$\tilde{f}\colon V \times \mathbb{R}^{n-k} \to \mathbb{R}^n,\ (y,u) \mapsto \varphi^{-1}(y) + \tilde{Q}_\varphi(\varphi^{-1}(y)) \cdot u.$$

Its Jacobian matrix at a point $(y,0)$ is $\big((\varphi^{-1})'(y) \,\big|\, \tilde{Q}_\varphi(\varphi^{-1}(y))\big) \in \mathbb{R}^{n\times n}$, which is invertible since the first k columns span the tangent space and the last $n-k$ columns the normal space of X at that point. Hence, the map f is a local isomorphism around any point on the zero section of $N_{X/\mathbb{R}^n}$.

Hence, for any $x \in X$ we can choose an open neighborhood $U_x \subset X$ of x and $\varepsilon_x > 0$ such that f is a local isomorphism on $N^{\varepsilon_x}_{U_x/\mathbb{R}^n}$. But X is compact, so finitely many $U_{x_1}, \dots, U_{x_r}$ suffice to cover X. Choosing any $\varepsilon < \min(\varepsilon_{x_1}, \dots, \varepsilon_{x_r})$, the map f is thus an isomorphism from $N^{\varepsilon}_{X/\mathbb{R}^n}$ onto an open neighborhood of X.

It just remains to be shown that f is also a global isomorphism onto its image, i. e. that it is injective. We will do this by contradiction: Assume that this was false for any ε, then we can choose sequences $(x_i, v_i)_{i\in\mathbb{N}_{>0}}$ and $(\tilde{x}_i, \tilde{v}_i)_{i\in\mathbb{N}_{>0}}$ with $f(x_i,v_i) = f(\tilde{x}_i,\tilde{v}_i)$, i. e. $x_i + v_i = \tilde{x}_i + \tilde{v}_i$, and $\|v_i\| < \frac{1}{i}$ as well as $\|\tilde{v}_i\| < \frac{1}{i}$ for all i. For $i \to \infty$ we clearly have $v_i \to 0$ and $\tilde{v}_i \to 0$, and as X is compact we may assume after taking a subsequence that $x_i \to x$ and $\tilde{x}_i \to \tilde{x}$ converge in X. But since $x+v = \tilde{x}+\tilde{v}$, this requires $x = \tilde{x}$. Hence, the map f is actually also locally not injective around $(x,0)$, which is a contradiction to our construction. □

Remark 12.22.

(a) Note that the map f of the proof of Proposition 12.21 is just what we did in the picture of Example 12.17: We attached the normal spaces to their base point, i. e. moved $N_{X/\mathbb{R}^n,x}$ so that its origin is at $x \in X$.

(b) Rescaling the fibers as in Remark 10.10 (b), we see that in fact the normal bundle $N_{X/\mathbb{R}^n}$ itself is isomorphic to an open neighborhood of X in $\mathbb{R}^n$. Moreover, by Remark 12.5 (a) the submanifold X is a deformation retract of this neighborhood and thus has the same homology. We can thus think of the normal bundle $N_{X/\mathbb{R}^n}$ as a manifold analogue of the simplicial standard neighborhoods of Proposition 1.24.

(c) As you might expect, there are more general versions of the Tubular Neighborhood Theorem:

- The analogous statement also holds for a compact submanifold Y of a manifold $X \subset \mathbb{R}^n$, i. e. Y has a open neighborhood in X that is isomorphic to the normal bundle $N_{Y/X}$. The main change in the proof is then needed since the map $f\colon N_{Y/X} \to \mathbb{R}^n$, $(x,v) \mapsto x+v$ does not map to X, but only to $\mathbb{R}^n$. The solution to this problem is to use the Tubular Neighborhood Theorem for $X/\mathbb{R}^n$ as well to locally identify $\mathbb{R}^n$ with the normal bundle $N_{X/\mathbb{R}^n}$, hence we can consider f as locally mapping to $N_{X/\mathbb{R}^n}$, compose it with the bundle projection $\pi\colon N_{X/\mathbb{R}^n} \to X$ to map to X, and then apply an analogous proof as above to the map $\pi \circ f : N_{Y/X} \to X$ [B, Theorem II.11.14].

- Actually, there is also way to state and prove the theorem without using any embeddings in $\mathbb{R}^n$; we will briefly discuss this in Remark 12.29.
- The submanifold X does not have to be compact; in this case however one cannot choose a global ε but needs a neighborhood of the zero section in the normal bundle whose size varies with the point on the base.

Let us finish this chapter with an alternative construction of vector bundles. Note that all bundles on a manifold X that we have considered so far were subbundles of a trivial bundle $X \times \mathbb{R}^n$ for some $n \in \mathbb{N}$ – but this is clearly not required by definition, similarly to manifolds that are not required a priori to be submanifolds of some $\mathbb{R}^n$. In particular, considering only subbundles of $X \times \mathbb{R}^n$ allowed us to construct tangent and normal bundles only for submanifolds of $\mathbb{R}^n$ so far. The following construction using transition maps overcomes this problem and allows us to obtain more general vector bundles by a gluing process.

Construction 12.23 (Vector bundles from transition maps)**.**

(a) Let $\pi : E \to X$ be a vector bundle of rank r, and let $(U_i)_{i \in I}$ be an open cover of X over which π is trivial, i. e. such that there are local trivialization isomorphisms

$$\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^r$$

over U_i for all $i \in I$. Similarly to manifolds, for any $i, j \in I$ there are then **transition maps**

$$\varphi_j \circ \varphi_i^{-1} : (U_i \cap U_j) \times \mathbb{R}^r \to (U_i \cap U_j) \times \mathbb{R}^r$$

which are isomorphisms of trivial vector bundles over $U_i \cap U_j$, and thus by Construction 12.6 must be of the form

$$\varphi_j \circ \varphi_i^{-1} : (U_i \cap U_j) \times \mathbb{R}^r \to (U_i \cap U_j) \times \mathbb{R}^r, \ (x, v) \mapsto (x, A_{i,j}(x) \cdot v)$$

for smooth matrix-valued maps $A_{i,j} : U_i \cap U_j \to \mathrm{GL}(r, \mathbb{R})$, that by abuse of notation are also sometimes called the *transition maps* of the bundle. Note that for any $i, j, k \in I$ we clearly have $\varphi_i \circ \varphi_i^{-1} = \mathrm{id}$ and $\varphi_k \circ \varphi_i^{-1} = (\varphi_k \circ \varphi_j^{-1}) \circ (\varphi_j \circ \varphi_i^{-1})$ where defined, and thus there are compatibility relations between these maps

$$A_{i,i} = E \quad \text{and} \quad A_{i,k} = A_{j,k} \cdot A_{i,j} : U_i \cap U_j \cap U_k \to \mathrm{GL}(r, \mathbb{R}). \tag{$*$}$$

(b) The construction of (a) can actually be reversed: Let $(U_i)_{i \in I}$ be an open cover of a manifold X, and assume that for some $r \in \mathbb{N}$ we are given smooth maps $A_{i,j} : U_i \cap U_j \to \mathrm{GL}(r, \mathbb{R})$ for all $i, j \in I$ satisfying the compatibility relations $(*)$ for all $i, j, k \in I$. Then we can construct from these data a vector bundle E of rank r on X having these transition maps by taking the disjoint union of $U_i \times \mathbb{R}^r$ for all $i \in I$ modulo the equivalence relation

$$(x, v) \in U_i \times \mathbb{R}^r \quad \sim \quad (\tilde{x}, \tilde{v}) \in U_j \times \mathbb{R} \qquad :\Leftrightarrow \qquad x = \tilde{x} \text{ and } \tilde{v} = A_{i,j}(x) \cdot v,$$

giving the resulting space the quotient topology (note that this is actually an equivalence relation by $(*)$).

Exercise 12.24. Let $(U_i)_{i \in I}$ be an open cover of a manifold X, and let $A_{i,j}, B_{i,j} : U_i \cap U_j \to \mathrm{GL}(r, \mathbb{R})$ for $i, j \in I$ be two sets of transition maps for vector bundles, i. e. such that

$$A_{i,i} = B_{i,i} = E, \quad A_{i,k} = A_{j,k} \cdot A_{i,j}, \quad B_{i,k} = B_{j,k} \cdot B_{i,j}$$

(where defined) for all $i, j, k \in I$. Show that the vector bundle with transition maps $A_{i,j}$ as in Construction 12.23 (b) is isomorphic to the vector bundle with transition maps $B_{i,j}$ if and only if there exist smooth maps $\Lambda_i : U_i \to \mathrm{GL}(r, \mathbb{R})$ with

$$A_{i,j} = \Lambda_j \cdot B_{i,j} \cdot \Lambda_i^{-1} \quad \text{on } U_i \cap U_j$$

for all $i, j \in I$.

Example 12.25 (Tangent bundles revisited)**.** Let X be a k-dimensional manifold.

(a) Assume first that X is embedded as a submanifold of $\mathbb{R}^n$ for some n, with smooth structure $(\varphi_i)_{i\in I}$ of X for $\varphi_i\colon U_i \to V_i \subset \mathbb{R}^k$. Recall by Lemma 12.8 and Construction 12.13 (a) that on U_i a local trivialization ψ_i of the tangent bundle $\pi\colon T_X \to X$ is given by

$$\psi_i\colon \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k,\ (x,(\varphi_i^{-1})'(\varphi_i(x))\cdot v) \mapsto (x,v).$$

Forming the composition $\psi_j \circ \psi_i^{-1}$ on $(U_i \cap U_j) \times \mathbb{R}^k$ for another $j \in I$, we thus first send (x,v) by ψ_i^{-1} to

$$(x,(\varphi_i^{-1})'(\varphi_i(x))\cdot v) = (x,(\varphi_j^{-1})'(\varphi_j(x))\cdot(\varphi_j\circ\varphi_i^{-1})'(\varphi_i(x))\cdot v),$$

which is then sent by ψ_j to $(x,(\varphi_j \circ \varphi_i^{-1})'(\varphi_i(x))\cdot v)$. Hence, the transition maps for the tangent bundle are given by the matrix-valued functions

$$A_{i,j}\colon U_i \cap U_j \to \mathrm{GL}(k,\mathbb{R}),\ x \mapsto A_{i,j}(x) = (\varphi_j \circ \varphi_i^{-1})'(\varphi_i(x)), \qquad (*)$$

i. e. by the Jacobian matrices of the transition maps of the manifold.

(b) Using Construction 12.23 (b), we can now define the **tangent bundle** T_X of an arbitrary manifold X with smooth structure $(\varphi_i)_{i\in I}$ so that it coincides with our old definition by (a): It is simply the vector bundle of rank k on X given by the transition maps $(*)$. Note that if X does not have a natural embedding in $\mathbb{R}^n$ then T_X does not have a natural embedding in $X \times \mathbb{R}^n$ either, but it is still a natural vector bundle on X that can be defined without making any choices and has the usual geometric interpretation as collecting local linearizations of X.

Exercise 12.26. Let $f\colon X \to Y$ be a morphism of manifolds such that X is connected and $f'(x)\colon T_{X,x} \to T_{Y,f(x)}$ is the zero map for all $x \in X$. Show that f is a constant map.

Exercise 12.27. Let $X \subset \mathbb{R}^n$ be a submanifold.

(a) Show that $\Delta := \{(x,x) : x \in X\}$ is a submanifold of $X \times X$, and that

$$\delta\colon X \to \Delta,\ x \mapsto (x,x)$$

is an isomorphism.

(b) Show that the pullback bundle $\delta^* N_{\Delta/X\times X}$ as in Exercise 12.11 is isomorphic to the tangent bundle T_X.

Exercise 12.28. In this exercise we want to see an alternative construction of tangent spaces and tangent bundles that does not use embeddings in $\mathbb{R}^N$. Let X be an n-dimensional manifold, and let $\varphi\colon U \to V \subset \mathbb{R}^n$ be a chart. Recall that $\Omega^0(U)$ denotes the vector space of smooth real-valued functions on U.

(a) Show that the maps

$$\frac{\partial}{\partial \varphi_i}\colon \Omega^0(U) \to \Omega^0(U),\ f \mapsto \frac{\partial f}{\partial \varphi_i} := \frac{\partial (f\circ\varphi^{-1})}{\partial x_i} \circ \varphi$$

for $1 \le i \le n$ define linear homomorphisms verifying the Leibniz rule, i. e. such that

$$\frac{\partial (fg)}{\partial \varphi_i} = \frac{\partial f}{\partial \varphi_i}\cdot g + f \cdot \frac{\partial g}{\partial \varphi_i} \quad \text{for all } f,g \in \Omega^0(U).$$

(b) For all $x \in U$ and $i \in \{1,\dots,n\}$, consider the linear map

$$\left.\frac{\partial}{\partial \varphi_i}\right|_x\colon \Omega^0(X) \to \mathbb{R},\ f \mapsto \frac{\partial f|_U}{\partial \varphi_i}(x).$$

Show that $\left(\left.\frac{\partial}{\partial \varphi_i}\right|_x\right)_{1\le i\le n}$ is a basis of the vector space D_x of all linear maps $v\colon \Omega^0(X) \to \mathbb{R}$ satisfying $v(fg) = v(f)\cdot g(x) + f(x)\cdot v(g)$ for all $f,g \in \Omega^0(X)$.

(c) Show that the vector spaces D_x for $x \in X$ form a vector bundle over X isomorphic to the tangent bundle T_X.

Remark 12.29 (Outlook on Riemannian Manifolds)**.** In this chapter, we have often considered manifolds X embedded in $\mathbb{R}^n$, or vector bundles on a manifold X embedded as subbundles in the trivial bundle $X \times \mathbb{R}^n$.

Sometimes, the reason for this was just convenience: For example, we have a geometric intuition what tangent spaces to a submanifold X of $\mathbb{R}^n$ should be, and thus we initially defined the tangent bundle in this way in Construction 12.13. However, we now saw in Example 12.25 that this assumption of an embedding is not essential; we can define the tangent bundle of a manifold just as easily without it using transition maps.

In some of our constructions however the situation was more subtle: If Y is a submanifold of X, to define the normal subspace $N_{Y/X,x}$ of $T_{X,x}$ in Construction 12.16 we needed a notion of tangent vectors being orthogonal – which we obtained from the standard scalar product on $\mathbb{R}^n$ as the assumption $X \subset \mathbb{R}^n$ ensured that $T_{X,x} \subset \mathbb{R}^n$ as well. In the Tubular Neighborhood Theorem of Proposition 12.21 we used the metric induced by this scalar product again to restrict the normal bundle to a well-defined ε-neighborhood of its zero section, and more importantly to construct the map f that "adds a small normal vector to a point in the submanifold" to obtain a point in the ambient manifold – which only works well if the ambient manifold is $\mathbb{R}^n$. All these constructions relied essentially on our manifolds (or at least the fibers of our vector bundles) to live in a globally fixed $\mathbb{R}^n$; there is no analogous construction for arbitrary manifolds and vector bundles without such an embedding.

Actually, this is not quite true: While it is clearly correct that the constructions mentioned above do not work for arbitrary manifolds, we do not really need an embedding in $\mathbb{R}^n$ for them, but just some notion of scalar product resp. metric. This is provided by a so-called *Riemannian manifold*, which is defined to be a smooth manifold X together with a scalar product on each tangent space that varies smoothly with the point (in a sense that we could easily make precise with our current methods: Every point $a \in X$ must have a neighborhood U on which T_X is trivial, i. e. isomorphic to $U \times \mathbb{R}^k$ with $k = \dim X$, such that the scalar product on the tangent spaces over U is given in this local trivialization by a smooth function $U \to \mathbb{R}^{k \times k}$ into the space of all positive definite symmetric matrices). A smooth manifold embedded in $\mathbb{R}^n$ just inherits this scalar product from its ambient space, hence it is automatically a Riemannian manifold.

By construction, a Riemannian manifold enables us to define lengths and angles for tangent vectors at every point, in particular orthogonality. But in fact it allows much more: Recall from analysis that the length of a (smooth) path in $\mathbb{R}^n$ can be computed as the integral over the lengths of its tangent vectors, hence we can also define this notion on a Riemannian manifold. This makes every Riemannian manifold into a metric space by taking the shortest-path metric. Such shortest paths (also called geodesics) are roughly equivalent to straight lines in $\mathbb{R}^n$, and they allow us to define e. g. what it means to start from a point and move "in a straight way" into a certain tangent or normal direction for a certain length – which is needed for the map f in the Tubular Neighborhood Theorem of Proposition 12.21. Moreover, it can be seen that Riemannian manifolds also admit to measure the size of higher-dimensional submanifolds, to talk about curvature, and many other things.

The study of Riemannian manifolds is part of a field of mathematics called *differential geometry*. As you might expect, it would be easy to fill at least a one-semester course with only this. In these notes however we do not want to consider this topic any further as our goal is the *topological* and not the *metric* study of manifolds. For us, for example, the plane $\mathbb{R}^2$, an open disk in $\mathbb{R}^2$ of any size, an open ellipse in $\mathbb{R}^2$, and the open upper half sphere of S^2 are all the same smooth manifold, but they all have different size and curvature properties and thus are completely different Riemannian manifolds.

However, some statements on manifolds are purely topological, yet their most natural proof is done by making them into Riemannian manifolds in an arbitrary way (e. g. by using a Whitney embedding as in Proposition 11.21 or Remark 11.22 (b)) and applying techniques from differential geometry. Our current example for this is the Tubular Neighborhood Theorem as in Remark 12.22 (c): If Y is any submanifold of a manifold X, not embedded in any $\mathbb{R}^n$, one can in fact define its normal bundle $N_{Y/X}$ abstractly (i. e. without embedding it in $T_X|_Y$ or a trivial bundle and without using orthogonality) as a quotient bundle $T_X|_Y/T_Y$, making the varying quotient vector spaces $T_{X,x}/T_{Y,x}$

for $x \in Y$ into a vector bundle on Y. It is then a purely topological statement that this bundle is isomorphic to an open neighborhood of Y in X – but to prove it one has to use metric techniques as we did in Proposition 12.21.

De Rham Cohomology

13. Differential Forms and De Rham Cohomology

In the last part of these notes, we now want to study algebraic topology for manifolds, using analytic methods. With the idea of singular homology in mind, where elements of the p-th homology group of a topological space can be thought of as being represented by "closed p-dimensional objects" in this space, it might seem that the best starting point for an analytic version of the p-th homology of a manifold X would be to consider compact p-dimensional submanifolds of X. However, we had seen in Chapter 5 already that the more natural approach in analysis is to start with the dual notion of cohomology, in which we study functions on X and the coboundary operator is some kind of differentiation. This is the idea of the so-called *de Rham cohomology* of X that we are going to introduce in this chapter. Note that this does not mean that the idea of analytic homology being represented by compact submanifolds is wrong – actually, we will see in Example 16.9 that certain p-dimensional submanifolds of X do in fact determine "analytic p-dimensional homology classes" in X, it is just not the easiest way to start our journey.

It might look strange at first to try to use a differentiation operator as the coboundary operator for a cohomology theory: After all, we know that such a coboundary operator d must satisfy $d^2 = 0$, whereas second derivatives clearly do not vanish in general. However, we know that higher partial derivatives commute, i. e. that $\partial_i \partial_j f = \partial_j \partial_i f$ for all $i, j \in \{1, \dots, n\}$ and smooth functions $f : U \to \mathbb{R}$ on an open subset U of $\mathbb{R}^n$. Hence, if we build some antisymmetry into the theory that allows us to count $\partial_j \partial_i f$ with a different sign than $\partial_i \partial_j f$ in second derivatives (similarly to the factor $(-1)^k$ in Definition 2.5 for the simplicial boundary operator ∂, leading to $\partial^2 = 0$ in Lemma 2.12), we can make the equation $d^2 = 0$ work. This is achieved by the notion of *differential forms*, which we want to introduce first. We will start with their construction on open subsets of $\mathbb{R}^n$, and then carry them over to manifolds using the charts of an atlas.

Construction 13.1 (Differential forms on $\mathbb{R}^n$)**.** Let $U \subset \mathbb{R}^n$ be an open subset with $n \in \mathbb{N}$.

(a) For $p \in \mathbb{N}$ with $p \leq n$, a **differential form** of **degree** p or simply **p-form** on U is a family $(f_{i_1,\dots,i_p})_{i_1,\dots,i_p}$ of $\binom{n}{p}$ smooth functions $f_{i_1,\dots,i_p} : U \to \mathbb{R}$ with $1 \leq i_1 < \dots < i_p \leq n$. We will write it as

$$\omega = \sum_{1 \leq i_1 < \dots < i_p \leq n} f_{i_1,\dots,i_p} \, dx_{i_1} \wedge \dots \wedge dx_{i_p}$$

for formal symbols $dx_{i_1} \wedge \dots \wedge dx_{i_p}$. The vector space of all p-forms on U (with component-wise addition and scalar multiplication) will be denoted by $\Omega^p(U)$, where we understand $\Omega^0(U)$ to be the space of smooth functions on U as in Notation 10.1 (a). Moreover, as a notational convention, we set $\Omega^p(U) := \{0\}$ for all $p \in \mathbb{Z}$ with $p < 0$ or $p > n$.

We want to consider the formal expression $dx_{i_1} \wedge \dots \wedge dx_{i_p}$ as alternating in the coordinates, i. e. we set

- $dx_{i_{\pi(1)}} \wedge \dots \wedge dx_{i_{\pi(p)}} := \operatorname{sign}(\pi) \cdot dx_{i_1} \wedge \dots \wedge dx_{i_p} \in \Omega^p(U)$ for all $i_1 < \dots < i_p$ and permutations π of $\{1, \dots, p\}$, and
- $dx_{i_1} \wedge \dots \wedge dx_{i_p} := 0 \in \Omega^p(U)$ if $i_1, \dots, i_p$ are not pairwise distinct.

In this way, expressions of the form $f \, dx_{i_1} \wedge \dots \wedge dx_{i_p} \in \Omega^p(U)$ are defined for all smooth functions $f : U \to \mathbb{R}$ and $i_1, \dots, i_p \in \{1, \dots, n\}$, and they generate $\Omega^p(U)$ as a real vector space.

(b) As a vector space, note that $\Omega^p(U)$ is just the $\binom{n}{p}$-fold product of $\Omega^0(U)$ with itself. However, differential forms also admit a natural bilinear product called the **wedge product**

$$\Omega^p(U) \times \Omega^q(U) \to \Omega^{p+q}(U), \ (\omega, \eta) \mapsto \omega \wedge \eta$$

for all $q \in \mathbb{N}$ with $q \le n$, defined on generators by

$$(f\,dx_{i_1} \wedge \cdots \wedge dx_{i_p}) \wedge (g\,dx_{j_1} \wedge \cdots \wedge dx_{j_q}) := fg\,dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q}$$

for all $f, g \in \Omega^0(U)$ and $i_1, \dots, i_p, j_1, \dots, j_q \in \{1, \dots, n\}$.

It is obvious that this product is well-defined in the sense that it is compatible with the definition of $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ and $dx_{j_1} \wedge \cdots \wedge dx_{j_q}$ in the case of indices that are not strictly increasing.

Remark 13.2.

(a) Clearly, the wedge product of Construction 13.1 (b) is associative. Moreover, the formal symbols $dx_{i_1} \wedge \cdots \wedge dx_{i_p} \in \Omega^p(U)$ of Construction 13.1 (a) can actually be interpreted as the wedge products of $dx_{i_1}, \dots, dx_{i_p} \in \Omega^1(U)$.

If you know this construction from commutative algebra, you will probably realize that $\Omega^p(U)$ is the so-called *alternating tensor product* of a free rank-n module $\Omega^1(U)$ over the ring $\Omega^0(U)$, where the basis elements of this module are written as $dx_1, \dots, dx_n$. The wedge product then makes the direct sum of all $\Omega^p(U)$ into an associative algebra over $\Omega^0(U)$, which is called the *exterior algebra* over $\Omega^1(U)$. In what follows, we will not use this terminology however.

(b) The wedge product is commutative up to sign in the following sense: Note that

$$(dx_{i_1} \wedge \cdots \wedge dx_{i_p}) \wedge (dx_{j_1} \wedge \cdots \wedge dx_{j_q}) = (-1)^{pq} (dx_{j_1} \wedge \cdots \wedge dx_{j_q}) \wedge (dx_{i_1} \wedge \cdots \wedge dx_{i_p})$$

since we can transform the ordering on the left to the ordering on the right by moving each $dx_{i_p}, \dots, dx_{i_1}$ by q positions to the right, leading to pq transpositions and thus to a sign of $(-1)^{pq}$. Hence, we have

$$\omega \wedge \eta = (-1)^{pq}\, \eta \wedge \omega \quad \text{for all } \omega \in \Omega^p(U) \text{ and } \eta \in \Omega^q(U)$$

with $U \subset \mathbb{R}^n$ open.

(c) The formal symbols dx_i for $i = 1, \dots, n$ in Construction 13.1 are usually called *differentials*. They can be interpreted geometrically as an "infinitesimally small change in the variable x_i", similarly to standard analysis when you write a one-dimensional integral as $\int f(x)\,dx$ rather than $\int f$ (in fact we will see in Chapter 14 that well-defined integrals over one-dimensional objects require 1-forms and not functions), or when you just plainly write the derivative of a function $f\colon \mathbb{R} \to \mathbb{R}$ as $f' = \frac{df}{dx}$. In fact, let us now define the derivative of differential forms in this spirit. It is usually called the *exterior derivative* since its values lie in an exterior algebra in the sense of (a).

Definition 13.3 (Exterior derivative)**.** Let $U \subset \mathbb{R}^n$ be an open subset with $n \in \mathbb{N}$.

(a) For a smooth function $f \in \Omega^0(U)$ we define

$$df := \sum_{i=1}^{n} \partial_i f \cdot dx_i \quad \in \Omega^1(U).$$

(b) For a p-form $f\,dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ with $f \in \Omega^0(U)$ we set

$$d(f\,dx_{i_1} \wedge \cdots \wedge dx_{i_p}) := df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} \quad \in \Omega^{p+1}(U)$$

with df as in (a), and extend this by linearity to a map $d\colon \Omega^p(U) \to \Omega^{p+1}(U)$.

22 For all $p \in \mathbb{N}$, the map $d\colon \Omega^p(U) \to \Omega^{p+1}(U)$ is called the **(exterior) derivative**.

Lemma 13.4 (Properties of the exterior derivative)**.** *For any open subset $U \subset \mathbb{R}^n$, $\omega \in \Omega^p(U)$, and $\eta \in \Omega^q(U)$ we have*

(a) $d^2\omega = 0$;

(b) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p\, \omega \wedge d\eta$.

Proof. By linearity, it suffices to check both equations on generators $\omega = f\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ and $\eta = g\, dx_{j_1} \wedge \cdots \wedge dx_{j_q}$.

(a) We have

$$d^2\omega = d\left(\sum_{i=1}^n \partial_i f \cdot dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}\right) = \sum_{i,j=1}^n \partial_j \partial_i f \cdot dx_j \wedge dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

which vanishes since $\partial_j \partial_i f = \partial_i \partial_j f$ [G1, Proposition 26.6] and $dx_j \wedge dx_i = -dx_i \wedge dx_j$.

(b) By the usual product rule for (multivariate) derivatives [G1, Proposition 25.29 (c)] we obtain

$$\begin{aligned} d(\omega \wedge \eta) &= \left(\sum_{i=1}^n \partial_i f \cdot g + f \cdot \partial_i g\right) dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q} \\ &= \sum_{i=1}^n \partial_i f \cdot dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge g\, dx_{j_1} \wedge \cdots \wedge dx_{j_q} \\ &\qquad + (-1)^p \sum_{i=1}^n f\, dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge \partial_i g \cdot dx_i \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q} \\ &= d\omega \wedge \eta + (-1)^p\, \omega \wedge d\eta, \end{aligned}$$

where the factor $(-1)^p$ arises from shifting dx_i by p positions to the right in the second sum. □

Example 13.5.

(a) Note that the name dx_i for the formal symbols in Construction 13.1 plays well with the notation d for the exterior derivative, since dx_i is actually the exterior derivative of the i-th coordinate function $x \mapsto x_i$ for $i = 1, \dots, n$.

(b) For the smooth function $f \colon \mathbb{R}^2 \to \mathbb{R}$, $x \mapsto x_1 x_2$ we have $df = x_2\, dx_1 + x_1\, dx_2 \in \Omega^1(\mathbb{R}^2)$, and consequently

$$d^2 f = d(x_2\, dx_1 + x_1\, dx_2) = dx_2 \wedge dx_1 + dx_1 \wedge dx_2 = 0 \quad \in \Omega^2(\mathbb{R}^2)$$

in accordance with Lemma 13.4 (a).

(c) On an open subset U of $\mathbb{R}^3$, the exterior derivative reproduces the standard differential operators of vector analysis:

- For a smooth function $f \in \Omega^0(U)$ we have

$$df = \partial_1 f \cdot dx_1 + \partial_2 f \cdot dx_2 + \partial_3 f \cdot dx_3 \quad \in \Omega^1(U),$$

so written as a triple as in Construction 13.1 (a) this is just $(\partial_1 f, \partial_2 f, \partial_3 f)$, i. e. the *gradient* $\operatorname{grad} f$ of f.

- If we consider a given vector field $f = (f_1, f_2, f_3) \colon U \to \mathbb{R}^3$ in the same way as a 1-form $f = f_1\, dx_1 + f_2\, dx_2 + f_3\, dx_3 \in \Omega^1(U)$, we have

$$df = (\partial_1 f_2 - \partial_2 f_1)\, dx_1 \wedge dx_2 + (\partial_2 f_3 - \partial_3 f_2)\, dx_2 \wedge dx_3 + (\partial_3 f_1 - \partial_1 f_3)\, dx_3 \wedge dx_1$$

in $\Omega^2(U)$. Note that 2-forms in $\mathbb{R}^3$ have three component functions as well. Written again as a triple $df = g_1\, dx_2 \wedge dx_3 + g_2\, dx_3 \wedge dx_1 + g_3\, dx_1 \wedge dx_2$, the vector field

$$g = (\partial_2 f_3 - \partial_3 f_2, \partial_3 f_1 - \partial_1 f_3, \partial_1 f_2 - \partial_2 f_1) \colon U \to \mathbb{R}^3$$

is called the *curl* of f as in Remark 5.5 (a), denoted by $\operatorname{curl} f$.

- Finally, if we write a given vector field $f = (f_1, f_2, f_3) \colon U \to \mathbb{R}^3$ as above as 2-form $f = f_1\, dx_2 \wedge dx_3 + f_2\, dx_3 \wedge dx_1 + f_3\, dx_1 \wedge dx_2$, its exterior derivative is

$$df = (\partial_1 f_1 + \partial_2 f_2 + \partial_3 f_3)\, dx_1 \wedge dx_2 \wedge dx_3 \quad \in \Omega^3(U),$$

which is a 3-form whose only component $\partial_1 f_1 + \partial_2 f_2 + \partial_3 f_3$ is called the *divergence* $\operatorname{div} f$ of f as in Remark 5.5 (b).

Note that the equation $d^2\omega = 0$ of Lemma 13.4 (a) thus means that $\operatorname{curl}\operatorname{grad} = 0$ (when applied to 0-forms) and $\operatorname{div}\operatorname{curl} = 0$ (when applied to 1-forms), which are well-known relations in vector analysis.

Of course, this relation $d^2\omega = 0$ for $\omega \in \Omega^p(U)$ with $U \subset \mathbb{R}^n$ would already allow us to construct analytic cohomology groups for such open subsets U of $\mathbb{R}^n$. But let us first transfer our definitions to general manifolds, where differential forms will become sections of suitable vector bundles. We will set up these vector bundles using transition maps as in Construction 12.23, so we need to study how differential forms behave under coordinate transformations. In fact, differential forms will be a contravariant functor, with their pullback being given by the following definition.

Definition 13.6 (Pullback of differential forms)**.** Let $f\colon U \to V$ be a smooth map between open subsets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ with component functions $f_1, \dots, f_m$. Then we define a **pullback** map $f^*\colon \Omega^p(V) \to \Omega^p(U)$ for all p by linear extension of

$$f^*(g\,dx_{i_1} \wedge \cdots \wedge dx_{i_p}) := (g \circ f) \cdot df_{i_1} \wedge \cdots \wedge df_{i_p}$$

for all $g \in \Omega^0(V)$ and $i_1, \dots, i_p \in \{1, \dots, m\}$.

Lemma 13.7 (Properties of pullbacks)**.** *Let $f\colon U \to V$ be a smooth map between open subsets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$, and let $\omega \in \Omega^p(V)$ be a p-form on V.*

(a) *For any q-form $\eta \in \Omega^q(V)$ we have $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$.*

(b) *$df^*\omega = f^*(d\omega)$.*

(c) *If $g\colon W \to U$ is another smooth map from an open subset $W \subset \mathbb{R}^k$ then $g^*(f^*\omega) = (f \circ g)^*\omega$.*

(d) *If $m = n$ then $f^*(dx_1 \wedge \cdots \wedge dx_n) = \det f' \cdot dx_1 \wedge \cdots \wedge dx_n$.*

Proof. By linearity, it suffices to take $\omega = h\,dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ for a smooth function $h \in \Omega^0(V)$ and $i_1, \dots, i_p \in \{1, \dots, m\}$.

(a) This follows directly from the definition.

(b) First, in the case $p = 0$, i. e. for $\omega = h \in \Omega^0(V)$ a smooth function, we have by the multivariate chain rule [G1, Proposition 25.30]

$$\begin{aligned} df^*h = d(h \circ f) &= \sum_{i=1}^n \partial_i(h \circ f)\,dx_i = \sum_{i=1}^n \sum_{j=1}^m (\partial_j h \circ f) \cdot \partial_i f_j\,dx_i = \sum_{j=1}^m (\partial_j h \circ f) \cdot df_j \\ &= f^*\left(\sum_{j=1}^m \partial_j h \cdot dx_j\right) = f^*(dh). \end{aligned}$$

Now for arbitrary p, when computing $df^*\omega = d((h \circ f) \cdot df_{i_1} \wedge \cdots \wedge df_{i_p})$ we see by Lemma 13.4 that we can split the exterior derivative onto the factors, and leave out $d^2 f_{i_1}, \dots, d^2 f_{i_p}$. Hence, we obtain

$$df^*\omega = d(h \circ f) \wedge df_{i_1} \wedge \cdots \wedge df_{i_p} = f^*(dh) \wedge f^*dx_{i_1} \wedge \cdots \wedge f^*dx_{i_p} \overset{\text{(a)}}{=} f^*(d\omega).$$

(c) By (a) and (b), the pullbacks f^* and g^* commute with the wedge product and the exterior derivative. Hence, to check the statement for any $\omega = h\,dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ it suffices to check it for the functions $h, x_{i_1}, \dots, x_{i_p} \in \Omega^0(V)$, where it is obvious.

(d) By (a), we have

$$f^*(dx_1 \wedge \cdots \wedge dx_n) = df_1 \wedge \cdots \wedge df_n = \sum_{i_1, \dots, i_n = 1}^n \partial_{i_1} f_1 \cdot \cdots \cdot \partial_{i_n} f_n\,dx_{i_1} \wedge \cdots \wedge dx_{i_n}.$$

But $dx_{i_1} \wedge \cdots \wedge dx_{i_n}$ is $\operatorname{sign}(\pi) \cdot dx_1 \wedge \cdots \wedge dx_n$ if $(i_1, \dots, i_n)$ is a permutation π of $(1, \dots, n)$ and zero otherwise, so we obtain

$$f^*(dx_1 \wedge \cdots \wedge dx_n) = \sum_{\pi} \operatorname{sign}(\pi) \cdot \partial_{\pi(1)} f_1 \cdot \cdots \cdot \partial_{\pi(n)} f_n \, dx_1 \wedge \cdots \wedge dx_n,$$

with the sum taken over all these permutations. By [G1, Remark 18.14], this is just the same as $\det f' \cdot dx_1 \wedge \cdots \wedge dx_n$. □

Example 13.8. Consider the map

$$f \colon \mathbb{R} \to \mathbb{R}^2, \, x \mapsto y := \left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1} \right)$$

with image $S^1 \backslash \{(0,1)\}$ that we saw already in Example 10.4 (b) as the inverse of a chart for S^1. Let us denote the coordinates in the target $\mathbb{R}^2$ by $y = (y_1, y_2)$.

(a) The pullback by f of the 1-form $\omega = y_1 \, dy_2$ on $\mathbb{R}^2$ is obtained as expected by the usual rules of substitution:

$$f^* \omega = \frac{2x}{x^2+1} \cdot d\left(\frac{x^2-1}{x^2+1} \right) = \frac{2x}{x^2+1} \cdot \frac{4x}{(x^2+1)^2} \, dx = \frac{8x^2}{(x^2+1)^3} \, dx.$$

(b) The pullback by f of any 2-form on $\mathbb{R}^2$ is clearly zero since there are non non-trivial 2-forms on $\mathbb{R}^1$.

We are now ready to carry over differential forms to corresponding vector bundles on manifolds.

Construction 13.9 (Differential forms on manifolds)**.** Let $(X, (\varphi_i)_{i \in I})$ be an n-dimensional manifold.

(a) For simplicity, let us first construct 1-forms on X. The idea is that on any chart $\varphi_i \colon U_i \to V_i$ with $i \in I$ we will just see a 1-form on $V_i \subset \mathbb{R}^n$ in the sense of Construction 13.1 (a), i. e. an expression of the form

$$\omega_i = (f_{i,1} \circ \varphi_i^{-1}) \, dx_1 + \cdots + (f_{i,n} \circ \varphi_i^{-1}) \, dx_n \quad \in \Omega^1(V_i)$$

for a smooth vector-valued map $f_i = (f_{i,1}, \dots, f_{i,n}) \colon U_i \to \mathbb{R}^n$, i. e. a section of the trivial bundle $U_i \times \mathbb{R}^n$ over U_i. Now if $\varphi_j \colon U_j \to V_j$ is another chart with $j \in I$, we have a similar representation of $\omega_j \in \Omega^1(V_j)$ by $f_j \colon U_j \to \mathbb{R}^n$, and as ω_i and ω_j are supposed to be different coordinate representations of the same 1-form on $U_i \cap U_j \subset X$, they should be related by the transition map $\varphi_i \circ \varphi_j^{-1}$ using the pullback of Definition 13.6. We thus require that $\omega_j = (\varphi_i \circ \varphi_j^{-1})^* \omega_i$ on the common intersection, which means in terms of f_i and f_j that

$$\begin{aligned}
\sum_{k=1}^{n} (f_{j,k} \circ \varphi_j^{-1}) \, dx_k &= (\varphi_i \circ \varphi_j^{-1})^* \left(\sum_{l=1}^{n} (f_{i,l} \circ \varphi_i^{-1}) \, dx_l \right) \\
&= \sum_{l=1}^{n} (f_{i,l} \circ \varphi_j^{-1}) \, d(\varphi_i \circ \varphi_j^{-1})_l \\
&= \sum_{k,l=1}^{n} (f_{i,l} \circ \varphi_j^{-1}) \, \partial_k((\varphi_i \circ \varphi_j^{-1})_l) \, dx_k.
\end{aligned}$$

Comparing the dx_k-coefficients and precomposing with φ_j, this means that $f_j = A_{i,j} \cdot f_i$ for the matrix-valued function

$$A_{i,j} \colon U_i \cap U_j \to \mathrm{GL}(n, \mathbb{R}), \, x \mapsto \big(\partial_k((\varphi_i \circ \varphi_j^{-1})_l)(\varphi_j(x)) \big)_{k,l} = ((\varphi_i \circ \varphi_j^{-1})')^{\mathsf{T}}(\varphi_j(x)). \qquad (*)$$

It is checked immediately that these functions satisfy the compatibility conditions $A_{i,i} = E$ and $A_{i,k} = A_{j,k} \cdot A_{i,j}$ of Construction 12.23 where defined for all $i, j, k \in I$, so as in Example 12.25 they define a vector bundle Ω^1_X of rank n on X that has these transition maps. Note that the matrices $A_{i,j}(x)$ for $x \in U_i \cap U_j$ appearing here are just the transposed inverse matrices of the ones for the tangent bundle T_X in Example 12.25 (b). One can show that this means that

Ω^1_X is the *dual bundle* of T_X, obtained by making the dual vector spaces $T^\vee_{X,x}$ of the tangent spaces $T_{X,x}$ for $x \in X$ into a new vector bundle. We will not need this notion of dual bundles and therefore not go into details here, but just use this observation as a motivation why Ω^1_X is usually called the **cotangent bundle** of X.

(b) The construction of general p-forms for $p \in \mathbb{N}$ with $p \leq n$ is completely analogous. Again, a p-form on X will be given by a collection of p-forms $\omega_i \in \Omega^p(V_i)$ for all $i \in I$, satisfying the compatibility condition that $\omega_j = (\varphi_i \circ \varphi_j^{-1})^* \omega_i$ where defined for all $i, j \in I$. The only difference is that we now need $\binom{n}{p}$ instead of n coordinate functions to describe these forms, and hence that the transition maps will be more complicated when expressed in terms of matrix functions. Fortunately however, we do not need to know what they look like – the compatibility of pullbacks with wedge products of Lemma 13.7 (a) ensures that everything works out, and for actual computations we will just use the equation $\omega_j = (\varphi_i \circ \varphi_j^{-1})^* \omega_i$.

In this way we obtain **vector bundles Ω^p_X of p-forms** on X, of rank $\binom{n}{p}$. Of course, for $p = 0$ we just get the trivial bundle $\Omega^0_X = X \times \mathbb{R}$ of smooth functions on X, and for $p < 0$ or $p > n$ we set Ω^p_X to be the trivial vector bundle of rank 0.

(c) Having constructed the vector bundles Ω^p_X for all $p \in \mathbb{Z}$, a *differential form* of degree p (or *p-form*) on X is just defined to be a section of Ω^p_X over X. For simplicity, we will denote the vector space of all such p-forms on X by $\Omega^p(X)$ instead of $\Omega^p_X(X)$.

By (b), elements of $\Omega^p(X)$ can be given by choosing an atlas $(\varphi_i)_{i \in I}$ for X and specifying differential forms $\omega_i \in \Omega^p(V_i)$ for all charts $\varphi_i \colon U_i \to V_i \subset \mathbb{R}^n$ such that the compatibility condition $\omega_j = (\varphi_i \circ \varphi_j^{-1})^* \omega_i$ holds for all $i, j \in I$. In particular, if X is an open subset of $\mathbb{R}^n$ (for which only one chart is needed) our new definition of $\Omega^p(X)$ coincides with the old one from Construction 13.1 (a).

(d) Finally, note that wedge products and exterior derivatives commute with pullbacks by Lemma 13.7, and thus with the compatibility condition $\omega_j = (\varphi_i \circ \varphi_j^{-1})^* \omega_i$. Hence, we obtain well-defined bilinear *wedge products* $\Omega^p(X) \times \Omega^q(X) \to \Omega^{p+q}(X)$ and *exterior derivatives* $d \colon \Omega^p(X) \to \Omega^{p+1}(X)$ for all $p, q \in \mathbb{Z}$ on any manifold X, given on coordinate charts by Construction 13.1 (b) and Definition 13.3.

Remark 13.10.

(a) Clearly, the properties of the exterior derivative in Lemma 13.4 carry over to differential forms on manifolds since they can be checked locally on coordinate charts.

(b) Similarly to Remark 13.2 (a), there is actually a general construction of *alternating tensor products* for vector bundles, and in this sense Ω^p_X is just the p-th alternating tensor product of the cotangent bundle Ω^1_X. But again, we will not use this general construction or terminology in these notes.

(c) Note that the exterior derivative is a map on sections $d \colon \Omega^p(X) \to \Omega^{p+1}(X)$ for any manifold X, but clearly not a morphism of vector bundles $\Omega^p_X \to \Omega^{p+1}_X$: By definition, such a morphism would have to be a linear map in each fiber, but differentiation *is not even defined in fibers* as the value of a derivative $f'(x)$ at a point $x \in X$ does not only depend on $f(x)$, but on the values of f in a neighborhood of x.

However, as a map on sections the exterior derivative is even a contravariant functor:

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Construction 13.11 (Differential forms are functorial)**.** Let $f \colon (X, (\varphi_i)_{i \in I}) \to (Y, (\psi_j)_{j \in J})$ be a morphism of manifolds. Then for any differential form $\omega \in \Omega^p(Y)$ we can construct a *pullback* $f^*\omega \in \Omega^p(X)$ using Definition 13.6 on coordinate charts: For any charts $\varphi_i \colon U_i \to V_i$ and $\psi_j \colon U'_j \to V'_j$ with $f(U_i) \subset U'_j$ the form ω is given in coordinates on V'_j by some $\omega_j \in \Omega^p(V'_j)$, and we assign to it the pullback $(\psi_j \circ f \circ \varphi_i^{-1})^* \omega_j \in \Omega^p(V_j)$ in the sense of Definition 13.6. It is checked immediately using Lemma 13.7 (c) that this is compatible with coordinate changes in the source and target, hence gives rise to a well-defined differential form $f^*\omega \in \Omega^p(X)$.

Moreover, Lemma 13.7 (c) confirms that this assignment $X \mapsto \Omega^p(X)$ is functorial for morphisms of manifolds for all p, making it into a contravariant functor $\Omega^p\colon \mathsf{Man} \to \mathsf{Vect}$. By Lemma 13.7 (a) and (b), it commutes with wedge products and the exterior derivative.

Note that the pullback $f^*\colon \Omega^p(Y) \to \Omega^p(X)$ is particularly easy to construct for embeddings $X \to Y$ of submanifolds: In this case, it is just obtained by restricting sections of the vector bundle Ω^p_Y to X as in Construction 12.4 (a). Hence, in this case we will often write a pullback $f^*\omega$ as $\omega|_X$, or even just ω if it is clear that we are considering the form only on X.

Example 13.12. Consider the manifold $X = S^1 \subset \mathbb{R}^2$ with the charts $\varphi_1\colon U_1 = X\backslash\{(0,1)\} \to V_1 = \mathbb{R}$ and $\varphi_2\colon U_2 = X\backslash\{(0,-1)\} \to V_2 = \mathbb{R}$ as in Example 10.4 (b). Moreover, let S^1 be embedded as usual in the plane $\mathbb{R}^2$, whose coordinates we denote by $y = (y_1, y_2)$.

(a) The restriction $\omega := y_1\,dy_2|_{S^1}$ of the form $y_1\,dy_2 \in \Omega^1(\mathbb{R}^2)$ is a well-defined 1-form on S^1. In the local coordinate x_1 on $V_1 = \mathbb{R}$ it is given by

$$\omega_1 := (\varphi_1^{-1})^*(y_1\,dy_2) \overset{13.8}{=} \frac{8x_1^2}{(x_1^2+1)^3}\,dx_1.$$

Similarly, one can compute its representation in the local coordinate x_2 on $V_2 = \mathbb{R}$ to be $\omega_2 = -\frac{8x_2^2}{(x_2^2+1)^3}\,dx_2$.

(b) Conversely, we could define the same form $\omega \in \Omega^1(S^1)$ without referring to the ambient $\mathbb{R}^2$ by only giving the two explicit formulas above for ω_1 and ω_2 on V_1 and V_2, respectively, noting that these two local representations are compatible since the transition map is given by $\varphi_2 \circ \varphi_1^{-1}\colon V_1\backslash\{0\} \to V_2\backslash\{0\}$, $x_1 \mapsto x_2 = \frac{1}{x_1}$ by Example 10.4 (b), and

$$(\varphi_2 \circ \varphi_1^{-1})^*\omega_2 = -\frac{8(\frac{1}{x_1})^2}{((\frac{1}{x_1})^2+1)^3}\,d(\tfrac{1}{x_1}) = -\frac{8(\frac{1}{x_1})^2}{((\frac{1}{x_1})^2+1)^3} \cdot \frac{-1}{{x_1}^2}\,dx_1 = \omega_1.$$

Exercise 13.13. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a submersion of manifolds, and let $X := f^{-1}(\{0\})$ be the submanifold given as the vanishing set of f. For all $i = 1,\dots,n$ we set

$$\omega_i := (-1)^{i-1}\left(\frac{\partial f}{\partial x_i}\right)^{-1} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n,$$

where $\widehat{\cdot}$ means that the corresponding term has been omitted.

(a) For all $i, j = 1,\dots,n$, show that $\omega_i = \omega_j$ at the points of X where both forms are defined.

(b) Use (a) to check directly that there is a nowhere-zero $(n-1)$-form on X.

As announced, we can now use Construction 13.9 to set up an analytic version of cohomology.

Construction 13.14 (De Rham cohomology)**.** Let X be a manifold.

(a) By Lemma 13.4 (a) and Remark 13.10 (a) there is a cochain complex

$$\cdots \longrightarrow \Omega^{p-1}(X) \xrightarrow{d} \Omega^p(X) \xrightarrow{d} \Omega^{p+1}(X) \longrightarrow \cdots$$

of differential forms on X. It is called the **de Rham cochain complex** of X and denoted by $\Omega(X)$. By Construction 13.11, the assignment $X \mapsto \Omega(X)$ is a contravariant functor from Man to $\mathsf{Ch}^\vee$, hence we call it the **de Rham cochain functor**.

(b) As for general cochain complexes, differential forms in the kernel of d will be called **closed** or **cocycles**, and forms in the image of d will be called **coboundaries**. In de Rham theory, it is also common to call such coboundaries **exact** differential forms.

(c) The cohomology groups of the de Rham cochain complex

$$H^p_{\mathrm{dR}}(X) := H^p(\Omega(X))$$

are called the **de Rham cohomology groups** of X; hence in the wording of (b) the elements of $H^p_{\mathrm{dR}}(X)$ are closed p-forms modulo exact p-forms on X. As usual, the dimensions $b^p_{\mathrm{dR}}(X) = \dim H^p_{\mathrm{dR}}(X) \in \mathbb{N} \cup \{\infty\}$ are called the **de Rham Betti numbers** of X.

As the composition of the de Rham cochain functor with the cohomology functor, the assignment $X \mapsto H^p_{\mathrm{dR}}(X)$ is clearly a contravariant functor $\mathsf{Man} \to \mathsf{Vect}$. We will see in Proposition 15.10 that $H^p_{\mathrm{dR}}(X)$ is naturally isomorphic to the singular cohomology group $H^p(X)$ for the ground field $\mathbb{R}$ (so that in particular we obtain the same Betti numbers $b^p_{\mathrm{dR}}(X) = b_p(X)$ as in the singular setting) – but as long as we have not shown this we will use the index "dR" to distinguish between the two groups and Betti numbers.

Example 13.15.

(a) If a manifold X has finitely many connected components $X_1, \dots, X_n$, we clearly have $\Omega^p(X) = \Omega^p(X_1) \times \cdots \times \Omega^p(X_n)$ for all $p \in \mathbb{Z}$ as giving a p-form on X is the same as giving a p-form independently on each component. Also, the exterior differential $d\colon \Omega^p(X) \to \Omega^{p+1}(X)$ is just the product of the exterior differentials on the components, and thus we see that $H^p_{\mathrm{dR}}(X) \cong H^p_{\mathrm{dR}}(X_1) \times \cdots \times H^p_{\mathrm{dR}}(X_n)$.

(b) A smooth function $f \in \Omega^0(X)$ on a manifold X satisfies $df = 0 \in \Omega^1(X)$ if and only if it is locally constant: As this is a local statement we may assume that $X \subset \mathbb{R}^n$ is an open subset, in which case

$$df = \partial_1 f\, dx_1 + \cdots + \partial_n f\, dx_n \stackrel{!}{=} 0$$

is equivalent to $f' = 0$, which exactly means that f is locally constant. In particular, if X is connected then the closed 0-forms are just the constant functions. As 0-forms can never be exact, we conclude that $H^0_{\mathrm{dR}}(X) \cong \mathbb{R}$ in this case, i. e. that $b^0_{\mathrm{dR}}(X) = 1$. If X is not necessarily connected, then by (a) the Betti number $b^0_{\mathrm{dR}}(X)$ is the number of connected components of X.

Of course, this just corresponds to the analogous statement for singular (co-)homology in Remark 6.10. In case you are surprised about *connected* instead of *path-connected* components showing up here: Connected components of X are the natural setting for functions from X to $\mathbb{R}$ (which we have here – such a function is locally constant if and only if it is constant on each *connected* component of X), whereas path-connected components are the natural setting for functions from $\mathbb{R}$ to X (which are needed in singular homology). For manifolds however, this does not make a difference by Exercise 10.12 (a).

(c) Let $X = (a,b) \subset \mathbb{R}$ be an open interval in $\mathbb{R}$. Clearly, we have $b^p_{\mathrm{dR}}(X) = 0$ for $p < 0$ or $p > 1$, and $b^0_{\mathrm{dR}}(X) = 1$ by (b). To compute $b^1_{\mathrm{dR}}(X)$, note that every $\omega \in \Omega^1(X)$ is of the form $f\,dx$ for a smooth function f, and by integrating f we can clearly find a smooth function $F \in \Omega^0(X)$ such that $dF = F'\,dx = f\,dx$. Hence, every 1-form on X is exact, which means that $b^1_{\mathrm{dR}}(X) = 0$ (as expected from topology).

(d) Recalling the initial motivating example from the introduction to these notes, the "derivative of the angle in polar coordinates"

$$\omega = d \arctan \frac{x_2}{x_1} = -\frac{x_2}{x_1^2 + x_2^2}\, dx_1 + \frac{x_1}{x_1^2 + x_2^2}\, dx_2 \quad \in \Omega^1(\mathbb{R}^2 \backslash \{0\})$$

is closed since $d^2 = 0$, but not exact: Assume for a contradiction that there is a smooth function $f \in \Omega^0(\mathbb{R}^2 \backslash \{0\})$ with $df = \omega$. Then on $(\mathbb{R} \backslash \{0\}) \times \mathbb{R}$, where the angle function $g\colon x \mapsto \arctan \frac{x_2}{x_1}$ is well-defined, we have $dg = \omega$ as well, and thus $d(f-g) = 0$. As in (b) this means that $f - g$ is locally constant, i. e. up to an additive constant f must be the angle function on $\mathbb{R}_{>0} \times \mathbb{R}$ and $\mathbb{R}_{<0} \times \mathbb{R}$ – but there is no continuous way to make this into a function on all of $\mathbb{R}^2 \backslash \{0\}$.

Hence, we see that ω is a non-zero element in $H^1_{\mathrm{dR}}(\mathbb{R}^2 \backslash \{0\})$, showing that $b^1_{\mathrm{dR}}(\mathbb{R}^2 \backslash \{0\}) \geq 1$. Of course, this is just the differential form corresponding to the non-zero simplicial cocycle in the annulus in Example 5.11, and once we know that de Rham cohomology coincides with singular cohomology it will be clear that $b^1_{\text{dR}}(\mathbb{R}^2\backslash\{0\}) = 1$, detecting the one hole in the punctured plane.

Remark 13.16 (Ring structure on de Rham cohomology)**.** Note that the wedge product of Construction 13.1 (b) passes to a well-defined product

$$H^p_{\text{dR}}(X) \times H^q_{\text{dR}}(X) \to H^{p+q}_{\text{dR}}(X),\ (\omega,\eta) \mapsto \omega \wedge \eta$$

on any manifold X: If $\omega \in \Omega^p(X)$ and $\eta \in \Omega^q(X)$ are closed then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p\, \omega \wedge d\eta = 0$$

is closed as well by Lemma 13.4 (b), and if in addition e. g. $\omega = d\varphi$ is exact then by the same lemma

$$\omega \wedge \eta = d\varphi \wedge \eta = d\varphi \wedge \eta + (-1)^{p-1}\, \varphi \wedge d\eta = d(\varphi \wedge \eta)$$

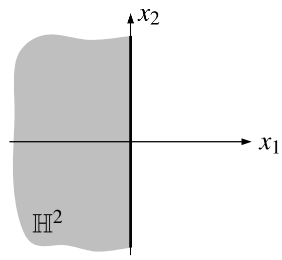
is also exact. This makes the product of all $H^p_{\text{dR}}(X)$ for $p \in \mathbb{Z}$ into a (graded, associative, but non-commutative) ring. Note that this product resp. ring structure on de Rham cohomology appears very naturally, in contrast to simplicial cohomology (where a corresponding structure could also be defined, but with much more effort). We will see the geometric meaning of this ring structure in Proposition 16.11.

14. Integration of Differential Forms

In the last chapter, we have introduced the concept of the de Rham cohomology groups $H^p(X)$ of a smooth manifold X. Recall that $H^p(X)$ describes differential forms $\omega \in \Omega^p(X)$ that are closed but not exact, i. e. that satisfy $d\omega = 0$ but cannot be written as $\omega = d\eta$ for some $\eta \in \Omega^{p-1}(X)$. Hence, we are facing an "inverse differentiation problem": Given a (closed) p-form ω, is there a $(p-1)$-form η whose exterior derivative is $\omega = d\eta$? From standard analysis, we would clearly expect that such a question is related to integration, so let us study in this chapter how to define and compute integrals of differential forms.

Just a side remark to start with: From standard analysis you would probably also expect that integrals are related to computing volumes under graphs of functions. But manifolds do not carry any kind of natural metric – for example, any two spheres of the same dimension but different radii are isomorphic – so it might seem strange that there is a well-defined notion of integration on such spaces. Actually, it is for exactly that reason why it is not possible to integrate *functions* on a manifold: The integral e. g. over the constant function 1 would have to give the "volume" of the manifold, which is ill-defined. Such integrals over functions can only be constructed for Riemannian manifolds as in Remark 12.29, which do carry a natural metric, but are not in the scope of these notes. However, it turns out that integrals over *differential forms* can be defined on manifolds without referring to a metric. For example, already the standard notation $\int f(x)\,dx$ for one-dimensional integrals over (a subset of) the real line $\mathbb{R}$ suggests that we rather do not integrate a function f but a 1-form $f(x)\,dx$. In fact, a coordinate change to a new variable y with $x = \varphi(y)$ will transform this 1-form into $f(\varphi(y))\,d\varphi(y) = f(\varphi(y))\,\varphi'(y)\,dy$ by Definition 13.3, which then has the same integral as $f(x)\,dx$ due to the integration rule of substitution. For several variables, the transformation rule for multivariate integrals will do the same trick.

Before we can define such integrals on manifolds however, we have to quickly discuss two straightforward extensions of the notion of manifolds. The first one is motivated by the observation that the spaces we want to integrate over are often not quite manifolds due to "boundary points", such as closed intervals in $\mathbb{R}$ as in Example 10.4 (e). To allow this, let us introduce so-called *manifolds with boundary*. Their definition is entirely analogous to that of ordinary manifolds, except that they do not locally look like $\mathbb{R}^n$, but like a closed half-space in $\mathbb{R}^n$ in the sense of the following notation, and as shown in the picture on the right.



Notation 14.1 (Half-spaces)**.** For $n \in \mathbb{N}_{>0}$, we call

$$\mathbb{H}^n := \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_1 \le 0\}$$

the **(standard closed) half-space** in $\mathbb{R}^n$.

Construction 14.2 (Smooth functions on half-spaces)**.** A priori, as in Notation 10.1 smooth maps and diffeomorphisms are only defined in analysis for a map $\varphi\colon U \to V$ between open subsets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ for $m,n \in \mathbb{N}$, in order to be able to define limits for $x \to a \in U$ with x coming from any direction. To be able to apply these notions to half-spaces (which are not of this form) let us define a map $\varphi\colon A \to B$ for arbitrary subsets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ to be smooth if it can be extended to a smooth map from an open neighborhood of A to $\mathbb{R}^m$. As before, we call φ a diffeomorphism if it is bijective, and both φ and φ^{-1} are smooth.

Note that such maps do not have well-defined derivatives in general since the limits needed to construct them may depend on the chosen extensions: For example, if A is a single point in $\mathbb{R}^n$ with $n > 0$ there is clearly no meaningful way to define the derivative of a map $\varphi\colon A \to \mathbb{R}^m$. However, we will compute derivatives of smooth maps in our new sense only on open subsets of $\mathbb{H}^n$, where they can be defined without using extensions at all, just with limits $x \to a \in \mathbb{H}^n$ being one-sided for $x_1 \leq 0$ if $a_1 = 0$.

Construction 14.3 (Manifolds with boundary)**.** Let X be a topological space, and let $n \in \mathbb{N}_{>0}$.

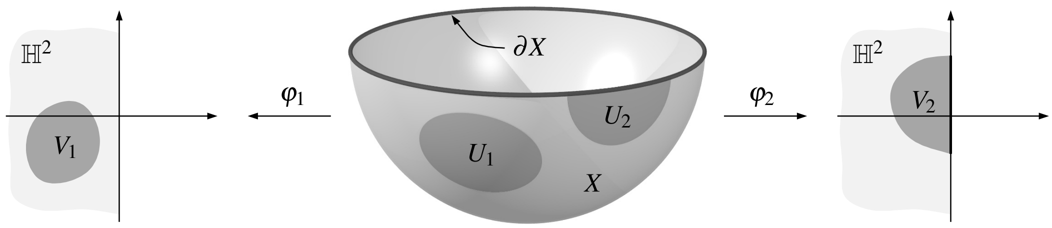
(a) As in Definition 10.2, an (n-dimensional coordinate) *chart* for manifolds with boundary on X is a homeomorphism $\varphi\colon U \to V$ from an open subset $U \subset X$ to an open subset $V \subset \mathbb{H}^n$. Two such coordinate charts $\varphi_1\colon U_1 \to V_1$ and $\varphi_2\colon U_2 \to V_2$ are called *compatible* if their *transition map* $\varphi_2 \circ \varphi_1^{-1}\colon \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is a diffeomorphism in the sense of Construction 14.2. An (n-dimensional) *atlas* for manifolds with boundary on X is a family of pairwise compatible n-dimensional charts for manifolds with boundary covering X. As in Construction 10.9, an n-dimensional *smooth structure* for a manifold with boundary on X is a maximal atlas for manifolds with boundary on X.

(b) An (n-dimensional) **manifold with boundary** is a second countable Hausdorff space $X \neq \emptyset$ together with an n-dimensional smooth structure in the sense of (a). A morphism of manifolds with boundary is just a smooth map as in Construction 10.5, with smoothness defined as in Construction 14.2.

(c) Note that a transition map $\varphi_2 \circ \varphi_1^{-1}$ as in (a) necessarily sends any point $x \in \mathbb{H}^n$ with $x_1 < 0$ to a point $\varphi_2(\varphi_1^{-1}(x))$ with negative first coordinate as well since $\varphi_2 \circ \varphi_1^{-1}$ is then defined on an open neighborhood of x in $\mathbb{R}^n$, which must be mapped to a neighborhood of $\varphi_2(\varphi_1^{-1}(x)) \subset \mathbb{R}^n$ by the Inverse Function Theorem of Remark 11.4 – but this image neighborhood also has to be contained in $\mathbb{H}^n$, which is only possible if $\varphi_2(\varphi_1^{-1}(x))$ is not on the boundary of $\mathbb{H}^n$.

Hence, there is a well-defined subset $\partial X \subset X$ of all points whose first coordinate in an arbitrary chart is zero; we call it the **boundary** of X. Restricting the charts for X to these points, making them into charts to open subsets of $\mathbb{R}^{n-1}$ by dropping the first coordinate, the boundary ∂X is an $(n-1)$-dimensional manifold (without boundary) contained in X as a closed subset if it is non-empty. Clearly, the complement $X \backslash \partial X$ is an open subset of X that is an n-dimensional manifold without boundary, with charts given by restricting the charts for X to the open subsets with negative first coordinate.

As a visual example, the picture below shows a half-sphere X as a 2-dimensional manifold with boundary; its boundary ∂X is a 1-dimensional circle. We will consider this space rigorously in Example 14.5 (c). Note that a chart $\varphi\colon U \to V$ may or may not touch the boundary ∂X of X, resp. the boundary $\{x \in \mathbb{R}^n : x_1 = 0\}$ of $\mathbb{H}^n$.



Remark 14.4. Most properties of manifolds that we have seen so far carry over to manifolds with boundary in the expected way. For example, submanifolds, vector bundles, and differential forms can be introduced and studied for manifolds with boundary in the same way as for regular manifolds. Most of these extensions are straightforward and only more complicated notationally, and we will not need them for what follows. Hence, let us not go into details here, but only list a few example constructions that will be enough for these notes. The only property of manifolds with boundary that we will actually use is a partition of unity, but it is easy to see that the preparing results of Lemma 11.12 and Proposition 11.13, as well as the final statements of Proposition 11.16 and Corollary 11.17 hold literally in the same way, with (almost) the same proofs.

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Example 14.5.

(a) Clearly, any manifold X with $\dim X > 0$ in the usual sense of Definition 10.16 is also a manifold with boundary, just with $\partial X = \emptyset$.

(b) A general example that covers most use cases is the following: Let $f: X \to \mathbb{R}$ be a smooth function on an n-dimensional manifold X with $n > 0$. Assume that f is a submersion, or – to obtain a more general case – that the rank of f is 1 on $f^{-1}(\{0\})$. We then have seen already in Proposition 11.9 (a) and Remark 11.10 (a) that the inverse image $Z := f^{-1}(\{0\})$ is an $(n-1)$-dimensional submanifold of X if it is non-empty. In fact, the very same proof shows that $Y := f^{-1}(\mathbb{R}_{\leq 0}) \subset X$ is an n-dimensional manifold with boundary Z: By the local normal form of Lemma 11.5, the map f is just given by $(x_1,\dots,x_n) \mapsto x_1$ in suitable local coordinates around a point in the boundary, hence restricting these coordinates to $x_1 \leq 0$ will precisely give charts for Y as an n-dimensional manifold with boundary Z. On the other hand, the points not on the boundary just form the open subset $f^{-1}(\mathbb{R}_{<0})$ of X, and thus have charts as well by Example 10.17 (b).

(c) As a concrete example of (b), consider for $n \in \mathbb{N}_{>0}$ the half-sphere

$$Y := \{x \in S^n \subset \mathbb{R}^{n+1} : x_1 \leq 0\}$$

as in the picture of Construction 14.3. By the multivariate version of Example 10.7 (b), the rank of $f: S^n \to \mathbb{R}$, $(x_1,\dots,x_{n+1}) \mapsto x_1$ is 1 at all points except $(\pm 1,0,\dots,0) \notin f^{-1}(\{0\})$. Hence, we conclude by (b) again that $Y = f^{-1}(\mathbb{R}_{\leq 0})$ is an n-dimensional manifold with boundary $\partial Y = \{x \in S^n : x_1 = 0\} \cong S^{n-1}$.

(d) In the standard smooth structure of $\mathbb{R}^2$, the quadrant $\mathbb{R}^2_{\leq 0}$ is not a manifold with boundary since its boundary $(\mathbb{R}_{\leq 0} \times \{0\}) \cup (\{0\} \times \mathbb{R}_{\leq 0})$ is not a submanifold of $\mathbb{R}^2$. Instead, $\mathbb{R}^2_{\leq 0}$ is a so-called *manifold with corners* – which is defined analogously to a manifold with boundary, except that it does not look locally like $\mathbb{R}^n$ or $\mathbb{H}^n$, but like $\{x \in \mathbb{R}^n : x_i \leq 0 \text{ for all } i\}$.

The second extension of the notion of manifolds that we will need for integration stems from orientation issues. Recall from the introduction to this chapter that the integration rule of substitution [G1, Proposition 12.31], resp. the transformation rule for multivariate integrals [G1, Proposition 30.8] will eventually make our construction of integrals over differential forms coordinate-independent, and thus well-defined on manifolds. However, both these rules are sensitive to a change of orientation, i. e. to space reflections: A one-dimensional integral changes its sign on swapping its lower and upper bound, and the transformation rule for multivariate integrals visibly contains *the absolute value of* the determinant of the Jacobian matrix of the coordinate change in the integral, whereas differential forms are multiplied just with this determinant regardless of its sign by Lemma 13.7 (d). Hence, in order to obtain full coordinate independence we have to remember whether we think of a coordinate chart as containing a space reflection or not, and adjust the sign of the integral accordingly. This leads to the following notation of *oriented manifolds*.

Construction 14.6 (Oriented manifolds)**.** Let $(X,(\varphi_i)_{i\in I})$ be an n-dimensional manifold (with or without boundary), with charts $\varphi_i: U_i \to V_i$ for $i \in I$. Recall that the transition maps

$$\varphi_{i,j} := \varphi_j \circ \varphi_i^{-1}: \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

are diffeomorphisms for all $i, j \in I$ by Definition 10.2, so the determinants of their Jacobian matrices are nowhere zero. An **orientation** on X is a collection $(\sigma_i)_{i\in I}$ of smooth (i. e. locally constant) **sign functions** $\sigma_i: V_i \to \{\pm 1\}$ such that

$$\sigma_i(x) \cdot \sigma_j(\varphi_{i,j}(x)) \cdot \det(\varphi'_{i,j}(x)) > 0 \quad \text{for all } x \in \varphi_i(U_i \cap U_j),$$

i. e.

$$\sigma_i(x) \cdot \sigma_j(\varphi_{i,j}(x)) \cdot \det(\varphi'_{i,j}(x)) = |\det(\varphi'_{i,j}(x))| \quad \text{for all } x \in \varphi_i(U_i \cap U_j).$$

We say that the manifold is **orientable** if it admits such an orientation; in this case the manifold together with an orientation is called an **oriented manifold**.

Remark 14.7. There are several equivalent definitions of an oriented manifold X, which are all a little more technical than desired. The one of Construction 14.6 will be the most convenient for us, so let us look at its meaning and implications in a bit more detail.

(a) Construction 14.6 corresponds to the intuitive notion of being able to define a consistent orientation on X, where similarly to all our simplicial pictures in Chapter 2 an orientation can be visualized e. g. on a 1-dimensional manifold by specifying a forward or backward direction, and on a 2-dimensional manifold by a circular arrow choosing a clockwise or counterclockwise direction of rotation. Consequently, for example, spheres and the torus are orientable, whereas the Möbius strip and the real projective plane are not (which we will show in Example 16.6 (b)). In the same way as for manifolds with boundary, we do not need to go into details here, but we will consider a few rigorous constructions in Example 14.8 that will be sufficient for our purposes.

(b) It is easy to verify that it suffices to specify sign functions on an arbitrary atlas of X; the condition of Construction 14.6 then ensures that there is always a unique way to extend these functions to all charts of the maximal atlas.

(c) Geometrically, the sign function $\sigma_i\colon V_i \to \{\pm 1\}$ for a chart $\varphi_i\colon U_i \to V_i$ specifies whether the chart transfers the orientation on X to the standard one on $\mathbb{R}^n$ (if the sign is 1) or the opposite one (if the sign is -1). If the chart is connected (which clearly almost all charts are that we use to construct manifolds), this is just a constant, but if U_i and hence also V_i consists of several components it might be that φ_i is orientation-preserving on some components and orientation-reversing on others.

(d) Although the full maximal atlas of an oriented manifold will usually contain charts with both signs, another consequence of (b) is that we can almost always construct oriented manifolds by using charts for which all sign functions are 1: Instead of using a chart with sign function -1 we could equally well just replace one of its coordinates by its negative (i. e. do a space reflection) and use the sign function 1 instead. We can then construct an oriented manifold without specifying any sign functions at all (since they are all 1 anyway), only by Construction 14.6 requiring that all determinants of the Jacobian matrices of the transition maps are positive.

   In fact, this would clearly be an easier method of construction resp. definition of oriented manifolds – except that this fails in *exactly two cases*, unfortunately:

   - A point (as a 0-dimensional manifold) has *two* orientations, given by the two possible signs ± 1, but this cannot be absorbed into replacing a coordinate by its negative since there is no coordinate.
   - Similarly, for a 1-dimensional manifold with boundary such as e. g. the interval $[0,1]$ any chart around the left endpoint would have to map it to the right endpoint 0 of $\mathbb{H}^1$, hence it is necessarily decreasing and cannot reflect the natural orientation of the interval unless we set its sign function to -1.

Example 14.8.

(a) Every open subset $X \subset \mathbb{R}^n$ (or $X \subset \mathbb{H}^n$) is naturally an n-dimensional oriented manifold (with boundary) with the one trivial identity chart and sign function 1, since there are no transition maps to check. We will take this as the standard orientation in this case.

(b) Expanding Example 14.5 (b), let now $f\colon X \to \mathbb{R}$ be a smooth function on an n-dimensional oriented manifold X. Assume again that the rank of f is 1 on $f^{-1}(\{0\})$, so that $Y := f^{-1}(\mathbb{R}_{\leq 0}) \subset X$ is an n-dimensional manifold with boundary. Then Y has a natural orientation as well, obtained by taking only charts which lie in the interior of Y or for which f is in local normal form, and just keeping the sign function for these charts.

(c) Let X be an n-dimensional oriented manifold with boundary; we claim that ∂X has a natural induced orientation. To see this, consider as in Construction 14.3 (c) a chart $\varphi\colon U \to V \subset \mathbb{H}^n$ for X around a point $a \in \partial X$, and make it into a chart $\tilde{\varphi}\colon U \cap \partial X \to \{y \in V : y_1 = 0\} \subset \mathbb{R}^{n-1}$

for ∂X around a by leaving out the first coordinate. The sign function for $\tilde{\varphi}$ will just be the restriction of the sign function for φ.

For $x \in U \cap \partial X$, the transition map between two such charts φ_1 and φ_2 then has Jacobian matrix

$$(\varphi_2 \circ \varphi_1^{-1})'(x) = \left(\begin{array}{c|c} \lambda(x) & 0 \\ \hline * & (\tilde{\varphi}_2 \circ \tilde{\varphi}_1^{-1})'(x) \end{array} \right)$$

for some smooth positive function λ, since it sends points with first coordinate zero to points with first coordinate zero, and points with negative first coordinate to points with negative first coordinate. This means that the determinants of $(\tilde{\varphi}_2 \circ \tilde{\varphi}_1^{-1})'$ and $(\varphi_2 \circ \varphi_1^{-1})'$ have the same sign, and thus that the condition of Construction 14.6 will hold for ∂X if it holds for X, i. e. that ∂X is an oriented manifold as well.

Note however that we have made one (global) choice in this construction: The orientation for $\partial X \subset X$ is determined by first taking a coordinate pointing outwards from ∂X, and then the coordinates for ∂X. We could have made other choices here, but the one above turns out to be most convenient for the theorem of Stokes in Proposition 14.13.

(d) As a concrete example, the closed n-dimensional disk

$$D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\} = f^{-1}(\mathbb{R}_{\leq 0}) \quad \text{with } f\colon \mathbb{R}^n \to \mathbb{R},\ x \mapsto x_1^2 + \cdots + x_n^2 - 1$$

is an n-dimensional manifold with boundary that has a natural orientation by (b) (since the rank of f is only zero at the origin, which is not in $f^{-1}(0)$), and thus its boundary sphere S^{n-1} also has a natural orientation by (c).

Exercise 14.9. We say that a smooth function

$$f\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n,\ (x,y) \mapsto (u(x,y), v(x,y))$$

satisfies the *Cauchy-Riemann equations* if

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial v_i}{\partial y_j} \quad \text{and} \quad \frac{\partial u_i}{\partial y_j} = -\frac{\partial v_i}{\partial x_j}$$

for all $i, j = 1, \dots, n$. (One can show that this is equivalent to $f\colon \mathbb{C}^n \to \mathbb{C}^n$ being infinitely complex differentiable if x and y denote the real and imaginary parts of the variables in the source, and u and v denote the real and imaginary parts of the variables in the target.)

Show that any $2n$-dimensional manifold whose transition maps satisfy the Cauchy-Riemann equations (i. e. an n-dimensional complex manifold) is orientable.

Exercise 14.10. Show that the tangent bundle T_X on any manifold X is an orientable manifold.

We are now ready to construct well-defined integrals over n-forms on n-dimensional oriented manifolds (with or without boundary) by transferring them to ordinary integrals on $\mathbb{R}^n$ using coordinate charts. To avoid complications with improper integrals, we will always assume that our differential forms have compact support, so that we only have to integrate smooth and bounded functions. Note that this condition of compact support is automatic if we integrate over a compact manifold, since the support is then always a closed subset of a compact space.

Construction 14.11 (Integration of differential forms with compact support)**.**

(a) Let $\omega \in \Omega^n(V)$ be an n-form with compact support on an open subset V of $\mathbb{R}^n$. Writing it as $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$ for a smooth function f with compact support on V, we define its integral by

$$\int_V \omega := \int_V f(x)\, d(x_1, \dots, x_n) \quad \in \mathbb{R}$$

in the usual (Riemann or Lebesgue) sense of multivariate analysis as e. g. in [G1, Definition 29.5]. Note that this is clearly well-defined: As in Construction 11.15, the function f can be extended smoothly by zero to all of $\mathbb{R}^n$, so we can consider its integral over a sufficiently large box containing its support.

(b) The main property of (a) is that it is invariant under signed coordinate transformations in the following sense: For any diffeomorphism $\varphi \colon V \to \tilde{V}$ between open subsets of $\mathbb{R}^n$, sign functions $\sigma \colon V \to \{\pm 1\}$ and $\tilde{\sigma} \colon \tilde{V} \to \{\pm 1\}$ with

$$\sigma(x) \cdot \tilde{\sigma}(\varphi(x)) \cdot \det(\varphi'(x)) = |\det(\varphi'(x))| \quad \text{for all } x \in V \tag{$*$}$$

as in Construction 14.6, and an n-form $\omega = f\,dx_1 \wedge \cdots \wedge dx_n$ with compact support on $\tilde{V}$ we have

$$\begin{aligned}
\int_V \sigma \cdot \varphi^* \omega &= \int_V \sigma \cdot \varphi^*(f\,dx_1 \wedge \cdots \wedge dx_n) \\
&\overset{(1)}{=} \int_V \sigma(x) \cdot f(\varphi(x)) \cdot \det \varphi'(x)\, d(x_1, \ldots, x_n) \\
&\overset{(*)}{=} \int_V \tilde{\sigma}(\varphi(x)) \cdot f(\varphi(x)) \cdot |\det \varphi'(x)|\, d(x_1, \ldots, x_n) \\
&\overset{(2)}{=} \int_{\tilde{V}} \tilde{\sigma}(x) \cdot f(x)\, d(x_1, \ldots, x_n) \\
&= \int_{\tilde{V}} \tilde{\sigma} \cdot \omega,
\end{aligned}$$

where (1) follows from Lemma 13.7, and (2) is the transformation rule for multivariate integrals [G1, Proposition 30.8].

(c) Now let $\omega \in \Omega^n(X)$ be an n-form with compact support on an n-dimensional oriented manifold X, and assume for the moment that its support is contained in the open subset $U \subset X$ of a coordinate chart $\varphi \colon U \to V \subset \mathbb{R}^n$ with sign function $\sigma \colon V \to \{\pm 1\}$. We then define its integral as

$$\int_X \omega := \int_V \sigma \cdot (\varphi^{-1})^* \omega,$$

where the integral over V on the right hand side is defined by (a). By our constructions, this is now independent of the chosen chart: If $\tilde{\varphi} \colon \tilde{U} \to \tilde{V}$ is another chart with sign function $\tilde{\sigma} \colon V \to \{\pm 1\}$ and containing the support of ω, we have

$$\int_V \sigma \cdot (\varphi^{-1})^* \omega \overset{13.7\,(c)}{=} \int_V \sigma \cdot (\tilde{\varphi} \circ \varphi^{-1})^* (\tilde{\varphi}^{-1})^* \omega \overset{(b)}{=} \int_{\tilde{V}} \tilde{\sigma} \cdot (\tilde{\varphi}^{-1})^* \omega$$

since the sign functions σ and $\tilde{\sigma}$ satisfy the condition $(*)$ of (b) for the transition map $\tilde{\varphi} \circ \varphi^{-1}$ by Construction 14.6.

(d) Finally, let $\omega \in \Omega^n(X)$ be a general n-form with compact support on an n-dimensional oriented manifold X. By compactness, we can find charts $\varphi_i \colon U_i \to V_i$ with $i \in I$ covering X so that only finitely many of them intersect the support of ω. Choose a partition of unity $(f_i)_{i \in I}$ of X subordinate to this cover in the sense of Corollary 11.17, and set

$$\int_X \omega := \sum_{i \in I} \int_{U_i} f_i \cdot \omega,$$

with the integrals on the right defined by (c), and only finitely many of them being non-zero due to our finiteness assumption. Again, let us check that this is independent of choices: If we pick different charts $\psi_j \colon U'_j \to V'_j$ for $j \in J$ and another partition of unity $(g_j)_{j \in J}$, we obtain

$$\begin{aligned}
\sum_{j \in J} \int_{U'_j} g_j \cdot \omega &= \sum_{i \in I} \sum_{j \in J} \int_{U'_j} f_i g_j \cdot \omega && ((f_i)_{i \in I} \text{ is a partition of unity}) \\
&= \sum_{i \in I} \sum_{j \in J} \int_{U_i \cap U'_j} f_i g_j \cdot \omega && (\text{the support of } f_i g_j \text{ is contained in } U_i \cap U'_j) \\
&= \sum_{i \in J} \int_{U_i} f_i \cdot \omega && (\text{same calculation backwards}).
\end{aligned}$$

Clearly, the same constructions also work for differential forms with compact support on manifolds with boundary, replacing open subsets of $\mathbb{H}^n$ instead of $\mathbb{R}^n$ (and using partitions of unity as in Remark 14.4).

Example 14.12. Construction 14.11 shows rigorously that there is a well-defined notion of integrals over n-dimensional forms with compact support on an n-dimensional manifold, without problems regarding divergent or non-existing integrals. However, it can be shown that for actual computations the formulas above may also be applied without the compact support assumption if one knows beforehand that the integrals actually exist. Let us consider two examples of this in the plane $\mathbb{R}^2$, whose coordinates we will denote by $y = (y_1, y_2)$.

(a) If we want to compute the integral of the 2-form $dy_1 \wedge dy_2$ over the 2-disk $D^2 \subset \mathbb{R}^2$ (with the standard orientation as in Example 14.8 (d)) directly by definition, we would have to take an open cover of the manifold with boundary D^2 by charts whose images lie in $\mathbb{H}^2$, and then use partitions of unity as in Construction 14.11 (d) for this cover. However, it is much easier to use the well-known fact that null sets do not matter for integration, so that we can equally well integrate $dy_1 \wedge dy_2$ over the open 2-disk $\{y \in \mathbb{R}^2 : \|y\| < 1\}$ (although the support of this form is no longer compact), for which we only need the standard chart. As expected, the
25 integral $\int_{D^2} dy_1 \wedge dy_2 = \int_{\|y\|<1} d(y_1, y_2) = \pi$ then just gives the area of the disk.

(b) Let us compute the integral of (the restriction of) the 1-form $y_1\, dy_2$ over the circle $S^1 \subset \mathbb{R}^2$, again with the standard orientation as in Example 14.8 (d). As in (a), rather than covering S^1 with two coordinate charts and using a partition of unity, it is sufficient to consider $S^1 \backslash \{(0,1)\}$, for which we only need one chart. We want to take the chart

$$\varphi_1 : S^1 \backslash \{(0,1)\} \to \mathbb{R},\ (y_1, y_2) \mapsto \frac{y_1}{1 - y_2}$$

of Example 10.4 (b), but have to check first that it has the correct orientation as the boundary of D^2, as otherwise we would get the wrong sign in the result: Locally around e. g. the point $(0, -1) \in S^1$, the map

$$\varphi\colon (y_1, y_2) \mapsto \left(y_1^2 + y_2^2 - 1,\ \frac{y_1}{1 - y_2} \right) \quad \text{with} \quad \det \varphi'(0, -1) = \det \begin{pmatrix} 0 & -2 \\ \frac{1}{2} & 0 \end{pmatrix} = 1 > 0$$

gives a chart for D^2 to $\mathbb{H}^2$ with sign function 1, and thus its second component φ_1 is a chart for S^1 also with sign function 1 by Example 14.8 (c).

Finally, we have already computed $(\varphi_1^{-1})^*(y_1\, dy_2) = \frac{8x^2}{(x^2+1)^3}\, dx$ in Example 13.12 (a), and hence we obtain by Construction 14.11 (c)

$$\int_{S^1} y_1\, dy_2 = \int_{\mathbb{R}} \frac{8x^2}{(x^2+1)^3}\, dx = \lim_{c \to \infty} \left[\frac{x(x^2-1)}{(x^2+1)^2} + \arctan x \right]_{-c}^{c} = \pi.$$

In fact, the equality of the two integrals in Example 14.12 is not a coincidence: Note that the form $dy_1 \wedge dy_2$ in (a) is just the exterior derivative of $y_1\, dy_2$ in (b), and the sphere S^1 in (b) is the boundary of the disk D^2 in (a). The following theorem of Stokes – which is the main result of this chapter – asserts that taking exterior derivatives of forms is compatible in general with boundaries of manifolds in this sense. Given the somewhat complicated computations in the example above its proof might seem surprisingly simple, but in fact we have already done the most important part of the work by setting up a coordinate-invariant theory of differentiation and integration. This allows us to reduce the general statement to local statements on $\mathbb{R}^n$ and $\mathbb{H}^n$ which are easy to verify directly.

Proposition 14.13 (Theorem of Stokes). *For any $n \in \mathbb{N}_{>0}$ and an $(n-1)$-form ω with compact support on an n-dimensional oriented manifold X with boundary we have*

$$\int_{\partial X} \omega = \int_X d\omega.$$

Proof. Both sides of the equation are linear in ω, so using the coordinate pullbacks and partitions of unity of Construction 14.11 (c) and (d) we may assume that X is an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$, and thus in fact by extension by zero that ω is a form with compact support on $X = \mathbb{R}^n$ or $X = \mathbb{H}^n$ itself. Moreover, linearity implies that we can take the sign function to be 1 and consider a form

$$\omega = f\,dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \quad \Rightarrow \quad d\omega = (-1)^{i-1}\,\partial_i f\,dx_1 \wedge \cdots \wedge dx_n$$

for a smooth function f with compact support, where as usual $\widehat{dx_i}$ denotes that this differential is left out. Let us now compute the integrals on both sides of the theorem, noting that the order of integration is arbitrary by Fubini's theorem [G1, Chapter 28.B].

If $i > 1$, both sides of the theorem are zero: the integral $\int_{\partial X} \omega$ vanishes since $dx_1 = 0$ on ∂X (which is empty for $\mathbb{R}^n$ and given by $x_1 = 0$ on $\mathbb{H}^n$), and $\int_X d\omega$ is zero since $\int \partial_i f(x)\,dx_i$ gives the difference of f at points with $x_i \to \infty$ and $x_i \to -\infty$, which are both zero since f has compact support.

If $i = 1$ and $X = \mathbb{R}^n$, the same argument holds by symmetry of the coordinates.

Finally, if $i = 1$ and $X = \mathbb{H}^n$ we get

$$\int_X d\omega = \int_{\mathbb{H}^n} \partial_1 f(x_1,\dots,x_n)\,d(x_1,\dots,x_n) \overset{(*)}{=} \int_{\mathbb{R}^{n-1}} f(0,x_2,\dots,x_n)\,d(x_2,\dots,x_n) = \int_{\partial X} \omega,$$

where $(*)$ is obtained by integrating $\partial_1 f$ over x_1 from $-\infty$ (where f is zero due to its compact support) to 0. $\square$

Example 14.14. Note that we had already seen a discrete simplicial version of Proposition 14.13 in Example 5.4 and Remark 5.5. Correspondingly, there are similar examples now that use actual differentiation and integration instead of the simplicial coboundary operator and the pairing between simplicial homology and cohomology:

(a) A closed interval $[a,b] \subset \mathbb{R}$ is a 1-dimensional oriented manifold with boundary $\{a,b\}$, where b has sign 1 and a has sign -1. Applying Proposition 14.13 to a smooth function $f \in \Omega^0([a,b])$ thus gives

$$f(b) - f(a) = \int_{\partial[a,b]} f = \int_{[a,b]} df = \int_a^b f'(x)\,dx,$$

which is just the Fundamental Theorem of Calculus.

(b) Let $X \subset \mathbb{R}^3$ be a 2-dimensional oriented manifold with boundary, and let $\omega \in \Omega^1(X)$ be the restriction of a 1-form on $\mathbb{R}^3$, so that we can write it as $\omega = f_1\,dx_1 + f_2\,dx_2 + f_3 dx_3$ for smooth functions $f_1, f_2, f_3 \colon \mathbb{R}^3 \to \mathbb{R}$. Then the statement $\int_{\partial X} \omega = \int_X d\omega$ translates into

$$\begin{aligned}&\int_{\partial X} (f_1\,dx_1 + f_2\,dx_2 + f_3 dx_3)\\ &\quad = \int_X \big((\partial_1 f_2 - \partial_2 f_1)\,dx_1 \wedge dx_2 + (\partial_2 f_3 - \partial_3 f_2)\,dx_2 \wedge dx_3 + (\partial_3 f_1 - \partial_1 f_3)\,dx_3 \wedge dx_1\big).\end{aligned}$$

As the three functions integrated over X are just the curl of f in the sense of Example 13.5 (c), this statement is usually called the *curl theorem of Stokes*.

(c) Similarly, if $X \subset \mathbb{R}^3$ is a 3-dimensional oriented manifold with boundary and $\omega \in \Omega^2(X)$, we have $\omega = f_1\,dx_2 \wedge dx_3 + f_2\,dx_3 \wedge dx_1 + f_3\,dx_1 \wedge dx_2$ for smooth functions $f_1, f_2, f_3 \colon X \to \mathbb{R}$, and the Theorem of Stokes becomes

$$\int_{\partial X} \big(f_1\,dx_2 \wedge dx_3 + f_2\,dx_3 \wedge dx_1 + f_3\,dx_1 \wedge dx_2\big) = \int_X (\partial_1 f_1 + \partial_2 f_2 + \partial_3 f_3)\,dx_1 \wedge dx_2 \wedge dx_3.$$

This time, the function occurring in the integral over X is the divergence of f as in Example 13.5 (c), and thus this statement is named the *divergence theorem of Gauss*.

Hence, the Theorem of Stokes of Proposition 14.13 unifies several classical theorems from vector calculus, and generalizes them to arbitrary dimension.

Remark 14.15 (Integrals in cohomology)**.** From the topological point of view, the Theorem of Stokes will have many implications in de Rham cohomology since it relates the exterior derivative (i. e. the coboundary operator of the de Rham cochain complex) to the geometric boundary of manifolds – similarly to the corresponding statement in simplicial geometry in Example 5.4. For the moment, let us just note that on a compact n-dimensional oriented manifold X the integration of forms yields a well-defined linear integration map

$$\int : H^n_{\mathrm{dR}}(X) \to \mathbb{R},\ \omega \mapsto \int_X \omega,$$

since X has empty boundary, and thus the integral vanishes on exact forms. In fact, this map will turn out to be an isomorphism if X is connected; we will study this in more detail in Proposition 16.4 and Example 16.6 (a).

Exercise 14.16.

(a) Show that an n-dimensional manifold X is orientable if and only if there is a nowhere-vanishing n-form on X.

(b) Now let X be a compact n-dimensional orientable manifold with boundary. Use (a) to show that ∂X is not a smooth deformation retract of X, and give a counterexample of this statement when X is non-compact.

15. Comparison of Singular and De Rham Cohomology

In this chapter we now want to prove the long-awaited statement that the analytic de Rham cohomology of a smooth manifold is canonically isomorphic to its singular cohomology, usually called the *de Rham's Theorem* in the literature. The very rough idea of our proof will be a Mayer-Vietoris type induction similarly to Lemma 4.18 in the simplicial case, i. e. we will start with simple base cases and then glue them together to obtain the statement for arbitrary manifolds. In order do to this, we need to address the following three points:

(A) For the induction start similarly to Lemma 4.18 (a), we have to show that the de Rham cohomology of sufficiently simple spaces is trivial.

(B) For the induction step similarly to Lemma 4.18 (b), we need a Mayer-Vietoris statement for de Rham cohomology.

(C) Finally, we have to establish a manifold version of the Mayer-Vietoris induction principle (note that, in contrast to simplicial complexes, a manifold is not built up from finitely many simple pieces in an obvious way).

All three points will require a little bit of work, but fortunately we are well-prepared for it by our previous results. Let us start with (A), which in singular (co-)homology was a homotopy argument in Proposition 3.11 for star-shaped simplicial complexes. We will copy this idea and show that the de Rham cohomology of a star-shaped region $X \subset \mathbb{R}^n$ is trivial, using a homotopy $\Omega^p(X) \to \Omega^{p-1}(X)$ between the identity and a trivial map (in the algebraic sense of Definition 3.1; note that the index now goes down and not up since we are using cohomology with ascending indices). This homotopy will be constructed from integrating a p-form on X along lines from the center point of the star, eating up one of the differentials to leave us with a $(p-1)$-form. In fact, we have seen the very simplest case of this argument already in Example 13.15 (c) where we showed that the first de Rham cohomology of an open interval is trivial since every 1-form on it is the coboundary of its integral.

In order to be able to construct this map, we need a version of integration that only integrates over one of the coordinates, and not over all of them as in Construction 14.11. It will be sufficient for us to do this in $\mathbb{R}^n$, so we do not have to worry about coordinate changes or orientations.

Construction 15.1 (Integration of forms over one variable)**.** Let $U \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ with coordinates (x,t) be an open subset for $n \in \mathbb{N}$. We assume that the non-empty fibers of the projection $U \to \mathbb{R}^n$, $(x,t) \mapsto x$ are open intervals of $\mathbb{R}$ containing 0. For all $p \in \mathbb{N}$ we can then construct a linear map $h\colon \Omega^p(U) \to \Omega^{p-1}(U)$ that integrates p-forms over t, i. e. it is defined on generators by

$$h(f\,dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{p-1}})(x,t) = \left(\int_0^t f(x,u)\,du \right) dx_{i_1} \wedge \cdots \wedge dx_{i_{p-1}}$$

for any form containing dt, and $h(f\,dx_{i_1} \wedge \cdots \wedge dx_{i_p}) = 0$ for any form not containing dt.

There is a projection map $\pi\colon U \to \mathbb{R}^n$, $(x,t) \mapsto x$, and a zero section $s\colon \pi(U) \to U$, $x \mapsto (x,0)$ of π. The composition $s \circ \pi$ then just sets t to 0 on U, but note that this implies on forms that the pullback $(s \circ \pi)^* = \pi^* s^*$ also sets dt to 0.

Lemma 15.2 (Integration homotopy)**.** *In the situation of Construction 15.1, the map h that integrates over t is a homotopy between $\pi^* s^*\colon \Omega^p(U) \to \Omega^p(U)$ and the identity, i. e. we have*

$$d \circ h + h \circ d = \mathrm{id} - \pi^* s^* \quad \text{on } \Omega^p(U)$$

for all $p \in \mathbb{Z}$.

Proof. This is just a straightforward computation, slightly different depending on whether a form contains dt or not: If $\omega = f\,dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{p-1}}$ contains dt, setting $F(x,t) := \int_0^t f(x,u)\,du$ and thus $\partial_i F(x,t) = \int_0^t \partial_i f(x,u)\,du$ we have

$$
\begin{aligned}
(d\circ h + h\circ d)(\omega) &= \Bigg(dF + \sum_{i=1}^n h(\partial_i f \underbrace{dx_i \wedge dt}_{=-dt\wedge dx_i})\Bigg) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{p-1}} \\
&= \Bigg(f\,dt + \sum_{i=1}^n \partial_i F\,dx_i - \sum_{i=1}^n \partial_i F\,dx_i\Bigg) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{p-1}} \\
&= f\,dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{p-1}} \\
&= (\mathrm{id} - \pi^* s^*)(\omega)
\end{aligned}
$$

since $\pi^* s^* \omega = 0$ due to the presence of dt in ω. On the other hand, if $\omega = f\,dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ does not contain dt, we clearly have $h(\omega) = 0$, and thus

$$
\begin{aligned}
(d\circ h + h\circ d)(\omega) &= h\Big(\frac{\partial f}{\partial t}\,dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}\Big) \\
&= (f(x,t) - f(x,0))\,dx_{i_1} \wedge \cdots \wedge dx_{i_p} \\
&= (\mathrm{id} - \pi^* s^*)(\omega).
\end{aligned}
$$

□

Corollary 15.3 (Poincaré Lemma). *For every star-shaped open subset $X \subset \mathbb{R}^n$ with $n \in \mathbb{N}$ we have $H^p_{\mathrm{dR}}(X) = 0$ for all $p > 0$.*

Proof. We may assume that the center point of the star is the origin. Let $\omega \in \Omega^p(X)$ be a closed form; we need to show that it is also exact. In order to integrate it along lines starting at the origin, consider the map

$$g\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n,\ (x,t) \mapsto tx$$

on the open subset $U := g^{-1}(X)$, which satisfies the assumptions of Lemma 15.2 since X is star-shaped with center 0. Using the notations of the lemma, we thus obtain for the form $g^*\omega$ on U

$$(d\circ h + h\circ d)(g^*\omega) = g^*\omega - \pi^* s^* g^* \omega \quad \in \Omega^p(U).$$

In this equation, we first of all have $dg^*\omega = g^* d\omega = 0$ by Lemma 13.7 (b) since ω is closed. Moreover, the composition $g \circ s$ is the constant zero map, hence $\pi^* s^* g^* \omega = 0$ as well by Lemma 13.7 (c) since $p > 0$. We are thus left with

$$(d\circ h)(g^*\omega) = g^*\omega \quad \in \Omega^p(U).$$

Finally, we pull this equation back along the map $f\colon X \to U$, $x \mapsto (x,1)$ that sets t equal to 1. As $g \circ f = \mathrm{id}$, we obtain by Lemma 13.7 (b) again

$$d f^*(h(g^*\omega)) = \omega.$$

Hence, the form ω is exact, which shows that $H^p_{\mathrm{dR}}(X) = 0$. □

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Next, let us address problem (B), i. e. establish a Mayer-Vietoris statement for de Rham cohomology. In the singular setting in Chapter 7, the main tool to prove Mayer-Vietoris for a union of two open subsets U and V was the construction of a sufficiently fine subdivision of simplices in $U \cup V$ so that each subsimplex lies in U or V, and we had already remarked that the analytic analogue of this is a partition of unity that allows us to write every smooth function on $U \cup V$ as a sum of two smooth functions with support in U and V, respectively. As we have all these tools already at our disposal, the proof of Mayer-Vietoris for de Rham cohomology is in fact quite straightforward.

Proposition 15.4 (De Rham Mayer-Vietoris sequence). *For any two open subsets U and V of a manifold X the sequence of de Rham cochain complexes*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega(U\cup V) & \longrightarrow & \Omega(U)\times\Omega(V) & \longrightarrow & \Omega(U\cap V) & \longrightarrow & 0 \\ & & \omega & \longmapsto & (\omega|_U,\omega|_V) & & & & \\ & & & & (\omega,\eta) & \longmapsto & \omega|_{U\cap V}-\eta|_{U\cap V} & & \end{array}$$

is exact. In particular, by Remark 5.8 (d) we obtain a functorial long exact sequence in cohomology

$$\cdots \longrightarrow H^p_{\mathrm{dR}}(U\cup V) \longrightarrow H^p_{\mathrm{dR}}(U)\times H^p_{\mathrm{dR}}(V) \longrightarrow H^p_{\mathrm{dR}}(U\cap V) \longrightarrow H^{p+1}_{\mathrm{dR}}(U\cup V) \longrightarrow \cdots.$$

Proof. The only non-trivial part of the statement is that the last map in the sequence is surjective. So let $\omega\in\Omega^p(U\cap V)$ be any p-form on the intersection. Choose a partition of unity (f_U,f_V) subordinate to the open cover (U,V) of $U\cup V$ as in Corollary 11.17. Then we can consider $f_V\,\omega$ to be a smooth form on U which is zero on $U\backslash V$ (and in fact even on the neighborhood $U\backslash\operatorname{supp} f_V$ of any point of $U\backslash V$ in U, which guarantees smoothness). In the same way, $f_U\,\omega$ is a differential form on V, and thus $(f_V\,\omega,-f_U\,\omega)$ is an inverse image of ω under the last map in the sequence since $(f_V\,\omega)|_{U\cap V}-(-f_U\,\omega)|_{U\cap V}=\omega$. □

Example 15.5. As an example of Proposition 15.4 and the general strategy in this chapter, let us compute the de Rham cohomology groups $H^p_{\mathrm{dR}}(S^1)$ of a circle. Clearly, this group is trivial by definition for $p<0$ or $p>1$, and it is isomorphic to $\mathbb{R}$ for $p=0$ by Example 13.15 (b). Hence, it remains to determine $H^1_{\mathrm{dR}}(S^1)$, and from the viewpoint of singular cohomology we clearly expect this Betti number to be 1 as well.

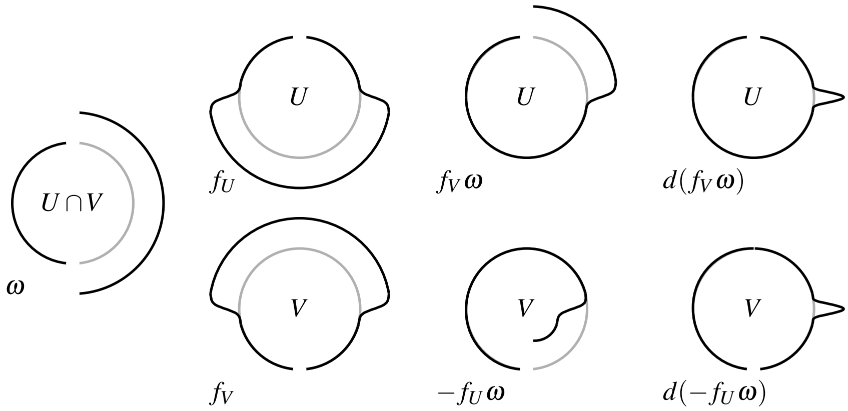
(a) Let us quickly confirm this expectation with with the Mayer-Vietoris sequence analogously to Example 7.11. We write $S^1\subset\mathbb{R}^2$ as the union of two open subsets $U=S^1\backslash\{(0,1)\}$ and $V=S^1\backslash\{(0,-1)\}$ isomorphic to $\mathbb{R}$, with the intersection $U\cap V$ consisting of two disjoint open intervals. By Example 13.15 (b) we then have $b^0_{\mathrm{dR}}(U)=b^0_{\mathrm{dR}}(V)=b^0_{\mathrm{dR}}(U\cup V)=1$ and $b^0_{\mathrm{dR}}(U\cap V)=2$, and by Example 13.15 (c) we have $b^1_{\mathrm{dR}}(U)=b^1_{\mathrm{dR}}(V)=0$. From the exact Mayer-Vietoris sequence of Proposition 15.4

$$0\to\underbrace{H^0_{\mathrm{dR}}(S^1)}_{\cong\mathbb{R}}\to\underbrace{H^0_{\mathrm{dR}}(U)\times H^0_{\mathrm{dR}}(V)}_{\cong\mathbb{R}^2}\to\underbrace{H^0_{\mathrm{dR}}(U\cap V)}_{\cong\mathbb{R}^2}\to H^1_{\mathrm{dR}}(S^1)\to\underbrace{H^1_{\mathrm{dR}}(U)\times H^1_{\mathrm{dR}}(V)}_{=0}$$

we thus conclude by Remark 4.5 (b) that $b^1_{\mathrm{dR}}(S^1)=1$, as expected.

(b) In fact, it is also instructive to explicitly find a generator for $H^1_{\mathrm{dR}}(S^1)$ by following the construction of the connecting map of the long exact cohomology sequence as in the proof of Proposition 4.8. This is shown in the picture below, where the columns from left to right correspond to the steps of the construction, the spaces U, V, and $U\cap V$ are drawn in gray, and the functions resp. forms in black with positive values pointing outwards (where "positive" for the forms in the last column means a positive coefficient function at the differential dx, with x increasing counterclockwise).

More precisely, by (a) we have to start with an element $\omega\in H^0_{\mathrm{dR}}(U\cap V)$ not in the image of $H^0_{\mathrm{dR}}(U)\times H^0_{\mathrm{dR}}(V)$, e. g. with a function which is 1 on one and 0 on the other component of $U\cap V$, as shown in the first column. To construct its image under the connecting map, we follow the proof of Proposition 15.4: To obtain an inverse image of ω in $\Omega^0(U)\times\Omega^0(V)$ we construct a partition of unity (f_U,f_V) subordinate to the cover (U,V) (shown in the second column) and form the pair $(f_V\,\omega,-f_U\,\omega)$ (in the third column). Then we apply the coboundary map d to obtain the pair $(d(f_V\,\omega),d(-f_U\,\omega))\in\Omega^1(U)\times\Omega^1(V)$ (shown in the last column), and by the general construction in the proof of Proposition 4.8 this element must come from a closed form in $\Omega^1(S^1)$ that we are looking for. In fact, one can see in the picture that the two 1-forms $d(f_V\,\omega)$ on U and $d(-f_U\,\omega)$ on V glue to a 1-form on S^1, which is just a bump 1-form giving 1 when integrated over S^1.



(c) Note that this result fits well with our earlier cohomological manifestations of $b_1(S^1) = 1$:

- In simplicial geometry, we have seen in Example 5.11 that a generator of the first cohomology of a simplicial annulus is a cocycle which is almost everywhere 0, except for a small region in which it has values so that it gives 1 on integration over this region. This is precisely the simplicial analogue of the bump 1-form obtained above.
- By Remark 14.15, the integration of 1-forms gives a well-defined map $H^1_{\mathrm{dR}}(S^1) \to \mathbb{R}$, so any 1-form on S^1 (which is necessarily closed) that has a non-zero integral must be a non-trivial de Rham cohomology class, and thus a generator of $H^1_{\mathrm{dR}}(S^1)$. The bump 1-form above clearly satisfies this.
- We had also mentioned already in Example 13.15 (d) that the derivative of the angle in polar coordinates is an example of a closed 1-form on $\mathbb{R}^2\backslash\{0\}$ that is not exact, showing that the first Betti number of this space is non-zero. In fact, this is just another 1-form whose integral around the hole in $\mathbb{R}^2\backslash\{0\}$ is non-zero (namely 2π), and the same holds after restriction to S^1 – so again we see that its integral being non-zero is what makes this cohomology class non-trivial and thus a generator of $H^1_{\mathrm{dR}}(S^1)$.

With our current techniques we could compute and compare the cohomology groups of manifolds if they have a finite open cover such that all possible intersections of the open subsets are isomorphic to star-shaped open subsets of $\mathbb{R}^n$, similarly to the contractible open covers of Definition 8.6. But not all manifolds are of this type – e. g. manifolds with infinitely many connected components, to name a trivial example. Let us therefore now address problem (C) from the introduction to this chapter: In order to establish a Mayer-Vietoris type induction as in Lemma 4.18 for general manifolds we have to include the possibility of infinite unions in some way. Luckily, it turns out that their onion structure ensures that *disjoint* infinite unions suffice, which are then again easy enough to handle.

Lemma 15.6 (Mayer-Vietoris induction for manifolds)**.** *To prove an assertion about all manifolds, it suffices to show:*

(a) *(Induction start) The assertion holds for any manifold isomorphic to a convex open subset of $\mathbb{R}^n$ for any $n \in \mathbb{N}$.*

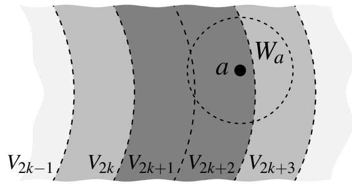
(b) *(Finite induction step) If U and V are open subsets of a manifold X such that the assertion holds for U, V, and $U \cap V$, then it also holds for $U \cup V$.*

(c) *(Infinite induction step) If a manifold X is the disjoint union of countably many submanifolds X_k for $k \in \mathbb{N}$ such that the assertion holds for every X_k, it also holds for X.*

Proof. Starting from convex open subsets of $\mathbb{R}^n$ as in (a), we will show the assertion for progressively larger classes of manifolds in four steps, where steps 1 and 3 use (b), and steps 2 and 4 use (c).

Step 1: The assertion holds for finite unions of convex open subsets of $\mathbb{R}^n$. To see this, write any union $U_1 \cup \cdots \cup U_k \subset \mathbb{R}^n$ of convex open subsets as $U \cup V$ with $U = U_1 \cup \cdots \cup U_{k-1}$ and $V = U_k$. As the intersection of convex subsets of $\mathbb{R}^n$ is convex, the intersection $U \cap V = (U_1 \cap U_k) \cup \cdots \cup (U_{k-1} \cup U_k)$ is a union of $k-1$ convex open subsets of $\mathbb{R}^n$, so the statement follows from (b) by induction on k.

Step 2: The assertion holds for any open subset X of $\mathbb{R}^n$. To prove this, we proceed similarly to the proof of partitions of unity in Proposition 11.16. As X is an n-dimensional manifold, we can pick an onion structure $(V_k)_{k\in\mathbb{N}}$ for X by Proposition 11.13, setting $V_k = \emptyset$ for $k < 0$. For any $k \in \mathbb{N}$ and $a \in \overline{V_{2k+2}} \backslash V_{2k}$ (shown in dark gray in the picture), pick a (convex) open ball W_a around a contained in the open set $V_{2k+3} \backslash \overline{V_{2k-1}}$ (shown in dark and light gray).



As $\overline{V_{2k+2}} \backslash V_{2k}$ is compact, we can pick a finite subset $J \subset \overline{V_{2k+2}} \backslash V_{2k}$ such that $U_k := \bigcup_{a \in J_k} W_a$ covers $\overline{V_{2k+2}} \backslash V_{2k}$. By construction, we then have $X = \bigcup_{k\in\mathbb{N}} U_k$, and every U_k intersects at most the neighboring open subsets U_{k-1} and U_{k+1}.

Now the assertion holds for any U_k by Step 1, and for $U := \bigcup_{k\in\mathbb{N}} U_{2k}$ and $V := \bigcup_{k\in\mathbb{N}} U_{2k+1}$ by (c) since these unions are disjoint. Similarly, it also holds for each intersection $U_k \cap U_{k+1}$ by Step 1, and thus for $U \cap V = \bigcup_{k\in\mathbb{N}} (U_k \cap U_{k+1})$ by (c) as these unions are also disjoint. By (b), the assertion thus holds for $U \cup V = \bigcup_{k\in\mathbb{N}} U_k = X$ as well.

Step 3: The assertion holds for any manifold that can be covered by finitely many charts. To see this, note that the assertion holds for a manifold with a single chart by Step 2, and that the intersection of charts is a chart. Hence, the argument is the same as in Step 1, replacing "convex open subset of $\mathbb{R}^n$" by "chart".

Step 4: The assertion holds for any manifold. Again, this is the same argument as in Step 2, taking for W_a arbitrary charts instead of (convex) open balls, and replacing references to Step 1 by Step 3. $\square$

Remark 15.7. The infinite induction step (c) of Lemma 15.6 is usually easy. Let us quickly check the behavior of our (co-)homology theories for a manifold X that is a disjoint union of countably many components X_n for $n \in \mathbb{N}$:

(a) For singular homology, every singular simplex lies in exactly one component X_n, so every singular chain lies in finitely many components, and the boundary map acts on each component separately. Hence, as in Remark 6.10 (a) the singular homology $H_p(X)$ for $p \in \mathbb{Z}$ is the so-called *direct sum* of all $H_p(X_n)$, i. e. the vector space of all families $(c_n)_{n\in\mathbb{N}}$ with $c_n \in H_p(X_n)$ such that only finitely many c_n are non-zero.

(b) When dualizing, direct sums become ordinary products as in Example 5.13. Hence, the singular cohomology $H^p(X)$ will be the ordinary product of all $H^p(X_n)$ for $n \in \mathbb{N}$ (without any condition that only finitely many components are non-zero).

(c) In de Rham theory, the vector spaces $\Omega^p(X)$ are also visibly the products of $\Omega^p(X_n)$ for all $n \in \mathbb{N}$ since differential forms can be specified on each component separately. The coboundary operator also acts independently on any component, and thus again $H^p_{\mathrm{dR}}(X)$ will be the product of all $H^p_{\mathrm{dR}}(X_n)$ for $n \in \mathbb{N}$.

We are now almost ready to compare de Rham to singular cohomology on manifolds. There is only one piece missing: Recall from Remark 4.20 that to prove the equivalence of two (co-)homology theories using Mayer-Vietoris (and thus the 5-Lemma) we first need a map between the two theories for which we can then prove that it is an isomorphism! As the definitions of de Rham and singular cohomology theories is quite different, this is a bit tricky here – in particular since the singular setting works with *continuous* maps, whereas de Rham cohomology requires *smooth* maps. To deal with this difference, let us construct a smooth version of singular (co-)homology for manifolds.

Construction 15.8 (Smooth singular homology)**.** For a manifold, we can set up a *smooth singular homology theory* in the same way as ordinary singular homology, just requiring the singular simplices to be smooth in the sense of Construction 14.2:

- For $p \in \mathbb{N}$, as in Definition 6.2 a **smooth singular p-simplex** in a manifold X is a smooth map $\sigma\colon |\Delta^p| \to X$. We denote by $C_p^{\mathrm{sm}}(X)$ the K-vector space having them as a basis, and call its elements **smooth singular p-chains** in X. Clearly, $C_p^{\mathrm{sm}}(X)$ is a vector subspace of $C_p(X)$.
- Note that restrictions of smooth maps are smooth, hence the boundary map of Construction 6.5 maps $C_p^{\mathrm{sm}}(X)$ to $C_{p-1}^{\mathrm{sm}}(X)$. We thus obtain a **smooth singular chain complex** $C^{\mathrm{sm}}(X)$ as in Remark 6.7, and corresponding **smooth singular homology groups** $H_p^{\mathrm{sm}}(X) := H_p(C^{\mathrm{sm}}(X))$. Of course, the inclusion $C^{\mathrm{sm}}(X) \to C(X)$ induces natural maps $H_p^{\mathrm{sm}}(X) \to H_p(X)$ between homology groups by Lemma 2.17.

It is now easy to check by the Mayer-Vietoris induction argument of Lemma 15.6 that this natural map is an isomorphism for any manifold X, i. e. that smooth singular homology agrees with ordinary singular homology:

(a) Both singular homologies (smooth and ordinary) are trivial for convex open subsets $X \subset \mathbb{R}^n$: The linear homotopy sending X to any of its points smoothly contracts X to a one-pointed space, so the argument of Example 6.14 (i. e. the proof of Proposition 6.12 and Corollary 6.13) carries over to the smooth case.

(b) The singular Mayer-Vietoris sequence of Corollary 7.10 holds in the same way for smooth singular homology since its main input (namely the subdivision of simplices) does not leave the smooth category. Hence, for any two open subsets U and V of a manifold X we obtain a commutative diagram with exact rows and vertical maps induced by inclusions

$$\begin{array}{ccccccccc}
H_p^{\mathrm{sm}}(U\cap V) & \to & H_p^{\mathrm{sm}}(U)\times H_p^{\mathrm{sm}}(V) & \to & H_p^{\mathrm{sm}}(U\cup V) & \to & H_{p-1}^{\mathrm{sm}}(U\cap V) & \to & H_{p-1}^{\mathrm{sm}}(U)\times H_{p-1}^{\mathrm{sm}}(V) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_p(U\cap V) & \longrightarrow & H_p(U)\times H_p(V) & \longrightarrow & H_p(U\cup V) & \to & H_{p-1}(U\cap V) & \to & H_{p-1}(U)\times H_{p-1}(V).
\end{array}$$

By the 5-Lemma 4.19 this means that the smooth singular homology of $U \cup V$ agrees with the ordinary one if this holds for U, V, and $U \cap V$.

(c) For disjoint unions, the statement of Remark 15.7 (a) holds for smooth as well as for ordinary singular homology.

Consequently, by Remark 5.15 (a) singular cohomology can also be defined using smooth simplices, i. e. the singular cohomology groups $H^p(X)$ can also be constructed as the cohomology groups of the cochain complex $(C^{\mathrm{sm}})^\vee(X)$ instead of $C^\vee(X)$.

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Construction 15.9 (Integration over smooth singular chains)**.** We can now construct a natural map from the de Rham cohomology to the singular cohomology of a manifold X: First of all, given a p-form $\omega \in \Omega^p(X)$ and a smooth singular p-simplex $\sigma\colon |\Delta^p| \to X$ as in Construction 15.8, we can form the integral

$$\int_\sigma \omega := \int_{|\Delta^p|} \sigma^*\omega \qquad (*)$$

as in Construction 14.11. Note that this does not need an orientation on X, but only on $|\Delta^p|$ (for which we take the standard one). However, $|\Delta^p|$ is strictly speaking not a manifold with boundary (which would be necessary to apply Construction 14.11), but only a manifold with corners in the sense of Remark 14.5 (d). There are two possible ways to solve this technical problem: Either one sets up all of Chapter 14 for manifolds with corners, or one uses null set arguments to reduce the integral $(*)$ to an integral over the subset of $|\Delta^p|$ on which at most one of the coordinates is zero – which is a manifold with boundary again, with the boundary given by the interiors of the $(p-1)$-dimensional faces of $|\Delta^p|$. Either way, one can make the definition $(*)$ and also the Theorem of Stokes work in this slightly more general situation, and in order to avoid overly technical constructions we will just assume this in what follows.

Extending $(*)$ by linearity, we can define the integral $\int_c \omega$ also for any p-chain $c \in C_p^{\mathrm{sm}}(X)$ with real coefficients, and thus consider integration over ω as an element of $(C_p^{\mathrm{sm}}(X))^\vee$. As this map is linear in ω as well, we obtain a linear integration map

$$\int : \Omega^p(X) \to (C_p^{\mathrm{sm}}(X))^\vee.$$

In fact, by the Theorem of Stokes as in Proposition 14.13 this gives us a morphism of cochain complexes $\Omega(X) \to (C^{\mathrm{sm}})^\vee(X)$, i. e. the diagram

$$\begin{tikzcd}
\Omega^p(X) \arrow[r, "d"] \arrow[d, "\int"'] & \Omega^{p+1}(X) \arrow[d, "\int"] \\
(C_p^{\mathrm{sm}}(X))^\vee \arrow[r, "d"] & (C_{p+1}^{\mathrm{sm}}(X))^\vee
\end{tikzcd}$$

commutes, where the top row is the exterior derivative and the bottom row the dual of the singular boundary map: Applying first $\int$ and then d to a p-form ω we obtain the linear map $\sigma \mapsto \int_{\partial\sigma} \omega$ in $(C_{p+1}^{\mathrm{sm}}(X))^\vee$, whereas first applying d and then $\int$ gives $\sigma \mapsto \int_\sigma d\omega$.

By Lemma 2.17 and Construction 15.8, the morphism $\Omega(X) \to (C^{\mathrm{sm}})^\vee(X)$ of cochain complexes passes to cohomology to give linear integration maps $H^p_{\mathrm{dR}}(X) \to H^p(X,\mathbb{R})$ for all $p \in \mathbb{Z}$. With all our preparations, it is now in fact easy to prove that this map is an isomorphism.

Proposition 15.10 (De Rham's Theorem). *For any manifold X, the integration over chains map $H^p_{\mathrm{dR}}(X) \to H^p(X)$ of Construction 15.9 is an isomorphism between the de Rham cohomology and the real (smooth) singular cohomology $H^p(X) := H^p(X,\mathbb{R})$ for all $p \in \mathbb{Z}$.*

Proof. We prove the statement by the Mayer-Vietoris induction principle of Lemma 15.6, noting that we already know everything that is necessary:

(a) For an open convex subset of $\mathbb{R}^n$, the de Rham cohomology is trivial by the Poincaré Lemma of Corollary 15.3, and the singular cohomology is trivial by Example 6.14 (and Construction 8.10).

(b) For open subsets U and V of a manifold X, there is a commutative diagram with exact rows

$$\begin{tikzcd}
H^{p-1}_{\mathrm{dR}}(U) \times H^{p-1}_{\mathrm{dR}}(V) \arrow[r] \arrow[d] & H^{p-1}_{\mathrm{dR}}(U \cap V) \arrow[r] \arrow[d] & H^p_{\mathrm{dR}}(U \cup V) \arrow[r] \arrow[d] & H^p_{\mathrm{dR}}(U) \times H^p_{\mathrm{dR}}(V) \arrow[r] \arrow[d] & H^p_{\mathrm{dR}}(U \cap V) \arrow[d] \\
H^{p-1}(U) \times H^{p-1}(V) \arrow[r] & H^{p-1}(U \cap V) \arrow[r] & H^p(U \cup V) \arrow[r] & H^p(U) \times H^p(V) \arrow[r] & H^p(U \cap V),
\end{tikzcd}$$

where the vertical maps are given by Construction 15.9: The top row is the Mayer-Vietoris sequence of Proposition 15.4, the bottom row the Mayer-Vietoris sequence of Construction 8.10 (b), and the diagram commutes since all maps of the corresponding cochain complexes are just given by restrictions. Hence, by the 5-Lemma 4.19, de Rham's Theorem holds for $U \cup V$ if it holds for U, V, and $U \cap V$.

(c) Finally, by Remark 15.7 (b) and (c), de Rham's Theorem holds for countable disjoint unions if it holds for the individual components. □

Example 15.11.

(a) By Proposition 15.10, it follows from Example 8.2 (b) that the only non-trivial de Rham Betti numbers of the sphere S^n with $n \in \mathbb{N}_{>0}$ are $b^0_{\mathrm{dR}}(S^n) = b^n_{\mathrm{dR}}(S^n) = 1$. In particular, for $n = 1$ this reproduces the results of Example 15.5.

(b) Similarly, the only non-trivial de Rham Betti numbers of the punctured plane $X = \mathbb{R}^2 \backslash \{0\}$ are $b^0_{\mathrm{dR}}(X) = b^1_{\mathrm{dR}}(X) = 1$, as this space is homotopy equivalent to S^1. Note that we have not shown the homotopy invariance of de Rham cohomology, but it clearly follows now from Proposition 15.10 as we have shown it for singular cohomology in Construction 8.10 (a). One could also prove its homotopy invariance directly by an argument similar to that in the Poincaré Lemma of Corollary 15.3.

In particular, the statement $b^1_{\mathrm{dR}}(X) = 1$ also shows that the non-zero element of $H^1_{\mathrm{dR}}(X)$ that we had found in Example 13.15 (d) is actually a generator.

16. Poincaré Duality

In the previous chapters, we have constructed the de Rham cohomology of manifolds and saw that it is naturally isomorphic to singular cohomology. To conclude these notes, we now want to show in this final chapter that our results actually also place some interesting restrictions on the possible values of the Betti numbers of compact oriented manifolds. Due to time constraints, we will only sketch the proofs in this chapter. To simplify notation, we will denote the (de Rham or singular) cohomology groups of a manifold X just by $H^p(X)$ for $p \in \mathbb{Z}$ as in our singular treatment, but we will think of its elements as de Rham cohomology classes, i. e. classes of closed differential forms.

Our starting point is the observation that, for any compact oriented n-dimensional manifold X, combining the wedge product of de Rham cohomology classes of Remark 13.16 with the integral of Remark 14.15 we obtain a well-defined linear map

$$D\colon H^p(X) \to (H^{n-p}(X))^\vee,\ \omega \mapsto \int \omega \wedge \cdot\,. \qquad (*)$$

It is our goal to show that this is in fact an isomorphism. In particular, this implies the probably surprising statement that the Betti numbers in degree p and $n-p$ of such a manifold must be the same! Moreover, applying this isomorphism twice shows that every cohomology group $H^p(X)$ is isomorphic to its double dual, which is only possible if $H^p(X)$ is finite-dimensional.

The idea to prove this isomorphism will again be a Mayer-Vietoris induction as in Lemma 15.6. This induction principle works for oriented manifolds in the same way as for ordinary manifolds, but clearly not for compact ones, since it relies on building up manifolds from open convex subsets of $\mathbb{R}^n$. Hence, we have to extend our isomorphism statement and thus the map D to non-compact manifolds first – however, this does not work out of the box since then the integral in $(*)$ might be divergent, making D ill-defined. The solution to this problem is to let $D(\omega)$ for $\omega \in H^p(X)$ act only on forms with compact support, i. e. to set up a modified version of de Rham cohomology first that works exclusively with differential forms with compact support.

Construction 16.1 (De Rham cohomology with compact support)**.** For any n-dimensional manifold X and $p \in \{0,\dots,n\}$, we denote by $\Omega_c^p(X) \subset \Omega^p(X)$ the vector subspace of all p-forms on X with compact support. Clearly, the exterior derivative maps forms with compact support to forms with compact support, so we obtain a cochain complex $\Omega_c(X)$. Its cohomology groups $H_c^p(X) := H^p(\Omega_c(X))$ are called the **de Rham cohomology groups with compact support**.

Of course, for a compact manifold X we just have $H_c^p(X) = H^p(X)$ since then the support of any differential form is a closed subset of a compact space, and thus itself compact. For non-compact manifolds however, the compact support condition can change the cohomology groups drastically, as the following simple example shows.

Example 16.2. Let $X = \mathbb{R}$; let us compute $H_c^p(X)$ for $p \in \{0,1\}$.

(a) For $p = 0$, a function $f \in \Omega_c^0(\mathbb{R})$ is closed if and only if $df = f'\,dx$ is zero, which requires f to be constant. But a constant function only has compact support if it is zero, hence $H_c^0(\mathbb{R}) = 0$.

(b) For $p = 1$, every 1-form on $\mathbb{R}$ is closed. Moreover, for 1-forms with compact support there is a linear integration map

$$\Phi\colon \Omega_c^1(\mathbb{R}) \to \mathbb{R},\ f\,dx \mapsto \int f(x)\,dx$$

whose value is also given by $F(\infty)$ for the unique function $F \in \Omega^0(\mathbb{R})$ with $F' = f$ and $F(-\infty) = 0$. Now $f\,dx$ is exact in $\Omega_c(\mathbb{R})$ if and only if F has compact support as well, i. e. if $\Phi(f\,dx) = F(\infty) = 0$. Hence, the cohomology group $H_c^1(\mathbb{R})$ is the quotient $\Omega_c^1(\mathbb{R})/\ker\Phi$, which by the homomorphism theorem is $\operatorname{im}\Phi = \mathbb{R}$. In other words, we have $\dim H^1_c(\mathbb{R}) = 1$, with a generator given by any 1-form $f\,dx$ where f is a function with compact support and non-zero integral, e. g. any bump function as in Construction 11.15. We will call this a *bump 1-form*.

In particular, the dimensions of the de Rham cohomology groups of $\mathbb{R}$ with compact support are exactly opposite to the ordinary ones in Example 13.15. Note that this shows another peculiarity of compactly supported de Rham cohomology, namely that it is not homotopy invariant: The real line is homotopy equivalent to a point, but does not have the same compactly supported de Rham cohomology groups.

Remark 16.3. More surprisingly, the compact support condition also changes the functoriality properties of the cohomology groups: For simplicity, let us only consider the case of inclusion maps $f\colon U \to X$ for an open subset U of a manifold X. In contrast to ordinary de Rham cohomology, there is now no longer a pullback (i. e. restriction) $f^*\colon \Omega^p_c(X) \to \Omega^p_c(U)$ since intersecting the compact support of a form on X with U is in general not compact. However, a form with compact support in U can be extended by zero to a form on X as in Construction 11.15, leading to a *pushforward map* $f_*\colon \Omega^p_c(U) \to \Omega^p_c(X)$! One consequence of this is that reverses the order of the cochain complexes in the Mayer-Vietoris sequence of Proposition 15.4: There is now a sequence

$$0 \longrightarrow \Omega_c(U\cap V) \longrightarrow \Omega_c(U)\times\Omega_c(V) \longrightarrow \Omega_c(U\cup V) \longrightarrow 0$$

whose exactness can be shown using partitions of unity similarly to Proposition 15.4 [BT, Proposition 2.7]. Note that the order of the terms in this sequence is the same as for homology, but it is still a sequence of cochain complexes, with the coboundary operator increasing the degree of the forms. Hence, the corresponding long exact cohomology sequence is

$$\cdots \longrightarrow H^p_c(U\cap V) \longrightarrow H^p_c(U)\times H^p_c(V) \longrightarrow H^p_c(U\cup V) \longrightarrow H^{p+1}_c(U\cap V) \longrightarrow \cdots.$$

Proposition 16.4 (Poincaré duality). *For any n-dimensional oriented manifold X, the map*

$$D\colon H^p(X) \to (H^{n-p}_c(X))^\vee,\ \omega \mapsto \int \omega\wedge\cdot$$

is an isomorphism for all $p = 0,\dots,n$.

Sketch proof. The proof works by Mayer-Vietoris induction as in Lemma 15.6, following the three steps.

As for (a), for a non-empty convex open subset $X\subset\mathbb{R}^n$ we know by the Poincaré Lemma of Corollary 15.3 that $H^p(X)$ is zero for $p\neq 0$ and one-dimensional for $p=0$. Using an adapted integration homotopy, one can prove a similar Poincaré Lemma for compactly supported de Rham cohomology, showing that

$$H^p_c(X) \cong \begin{cases} \mathbb{R} & \text{for } p=n,\\ 0 & \text{otherwise,}\end{cases}$$

where $H^n_c(X)$ is generated by a bump n-form as in Example 16.2. This shows that D is an isomorphism in this case.

The finite induction step (b) follows as usual from a diagram of Mayer-Vietoris sequences. For open subsets U and V of an oriented manifold X, we have a commutative diagram with exact rows

$$\begin{tikzcd}
H^{p-1}(U)\times H^{p-1}(V) \arrow[r] \arrow[d] & H^{p-1}(U\cap V) \arrow[r] \arrow[d] & H^p(U\cup V) \arrow[r] \arrow[d] & H^p(U)\times H^p(V) \arrow[r] \arrow[d] & H^p(U\cap V) \arrow[d] \\
H^{n-p+1}_c(U)^\vee\times H^{n-p+1}_c(V)^\vee \arrow[r] & H^{n-p+1}_c(U\cap V)^\vee \arrow[r] & H^{n-p}_c(U\cup V)^\vee \arrow[r] & H^{n-p}_c(U)^\vee\times H^{n-p}_c(V)^\vee \arrow[r] & H^{n-p}_c(U\cap V)^\vee,
\end{tikzcd}$$

where the top row is Proposition 15.4, the bottom row is the dual sequence of Remark 16.3, and the vertical maps are given by D.

The infinite induction step (c) follows since disjoint unions of manifolds lead to products of cohomology groups $H^p(X)$ by Remark 15.7 (c), whereas they yield direct sums of compactly supported cohomology groups $H_c^{n-p}(X)$ as in Remark 15.7 (a) (since only finite unions of compact spaces are compact), which are then transformed into products when dualizing by Example 5.13. □

Corollary 16.5. *Let X be an n-dimensional compact oriented manifold, and let $p \in \mathbb{Z}$.*

(a) *The Poincaré duality map gives an isomorphism $H^p(X) \cong (H^{n-p}(X))^\vee$. In particular, we have $b_p(X) = b_{n-p}(X)$.*

(b) *All Betti numbers $b_p(X)$ are finite.*

(c) *For singular homology there are natural isomorphisms $H_p(X) \cong (H^p(X))^\vee \cong H^{n-p}(X)$.*

Proof.

(a) This follows directly from Proposition 16.4 since $H^{n-p}(X) = H_c^{n-p}(X)$ for a compact manifold X.

(b) By Mayer-Vietoris induction it follows that the compactly supported de Rham cohomology of any manifold has at most countable dimension. Now if the dimension of $H^p(X)$ was countably infinite, it would follow by (a) that $H^{n-p}(X) \cong (H^p(X))^\vee$ has uncountable dimension by Example 5.13, which is a contradiction. Hence, $H^p(X)$ must be finite-dimensional.

(c) By Construction 8.10 we have $H^p(X) \cong (H_p(X))^\vee$. Taking duals, we obtain the first isomorphism by (b) since the double dual of a finite-dimensional vector space is isomorphic to the original space. The second isomorphism is just (a). □

Example 16.6.

(a) If X is an n-dimensional connected oriented manifold, the statement $H^0(X) \cong \mathbb{R}$ of Example 13.15 (b) translates into $H_c^n(X) \cong \mathbb{R}$. In particular, if X is also compact then $H^n(X) \cong \mathbb{R}$, i. e. $b_n(X) = 1$.

(b) The sphere S^n is a compact oriented manifold for all n by Example 14.8 (d), and in fact its Betti numbers (which are $b_0(S^n) = b_n(S^n) = 1$ and zero otherwise by Example 8.2 (b)) satisfy the symmetry property of Corollary 16.5 (a). Similarly, this holds for the complex projective spaces $\mathbb{P}^n_\mathbb{C}$ in Example 8.4.

(c) In contrast, the real projective plane $\mathbb{P}^2_\mathbb{R}$ has $b_0(\mathbb{P}^2_\mathbb{R}) = 1 \neq 0 = b_2(\mathbb{P}^2_\mathbb{R})$ over the ground field $\mathbb{R}$ by Example 8.2 (d). As $\mathbb{P}^2_\mathbb{R}$ is a compact 2-dimensional manifold, Corollary 16.5 (a) proves that $\mathbb{P}^2_\mathbb{R}$ is not orientable.

Remark 16.7.

(a) In fact, Poincaré duality can also be established purely in the world of singular homology and cohomology, without referring to de Rham cohomology or differential forms – and thus over any ground field. The general idea to construct the duality map by using a product in cohomology and integrating over an orientation on a manifold is still the same as in our setting, but one needs different (and less intuitive) replacement constructions for this product and the orientations. This generalized version of Poincaré duality does not need smooth structures, but still (topological) manifolds; just topological spaces do not suffice. For details, see e. g. [H, Theorem 3.30].

(b) On the other hand, Corollary 16.5 (c) would allow us to construct *de Rham homology* groups of a compact oriented manifold X e. g. as the duals of $H^p_{\mathrm{dR}}(X)$, so that they are isomorphic to singular homology, but only the analytic methods of de Rham theory are needed. Thinking of $H_p(X)$ in this way, one method to produce an element of it is integration over submanifolds: Given a p-dimensional compact oriented submanifold Y of X, there is a linear map

$$\int_Y : H^p(X) \to \mathbb{R},\ \omega \mapsto \int_Y \omega,$$

defining an element of $(H^p(X))^\vee$, and thus of $H_p(X)$. Hence, such submanifolds determine elements in homology, although homology is not defined in terms of them.

Combining this with Poincaré duality, a submanifold even defines a cohomology class in the following sense.

Construction 16.8 (Poincaré dual of a submanifold)**.** Let X be an n-dimensional compact oriented manifold, and let $Y \subset X$ be a p-dimensional compact oriented submanifold. Then integration over Y defines an element of $(H^p(X))^\vee$ as in Remark 16.7 (b), and thus by Poincaré duality as in Corollary 16.5 (a) an element η_Y of $H^{n-p}(X)$. By the definition of the Poincaré duality map, it is determined by the equation

$$\int_Y \omega = \int_X \eta_Y \wedge \omega \quad \text{for all } \omega \in H^p(X)$$

(i. e. for any closed p-form ω on X). We call η_Y the **Poincaré dual** of Y.

Example 16.9 (Poincaré dual of a point in S^1)**.** Let Y be a point in $X = S^1$. Then by definition the Poincaré dual $\eta_Y \in H^1(S^1)$ satisfies

$$\int_Y f = \int_{S^1} \eta_Y f \tag{$*$}$$

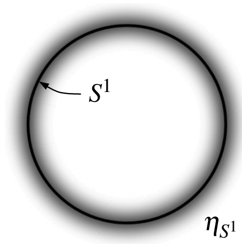
for any closed $f \in \Omega^0(S^1)$. If we forget the closedness condition for a moment this looks as if η_Y is some kind of *Dirac delta function* (resp. functional) [G1, Remark 21.51]: As the left side is just $f(Y)$ and thus does not depend on the values of f at other points, these other values should be zeroed out by η_Y on the right side as well, i. e. η_Y should be zero at all points except Y. On the other hand, for the integrals to match η_Y would have to be infinite at the point Y in such a way that the integral over it is 1 (and thus the integral over $\eta_Y f$ is $1 \cdot f(Y)$).

Of course, such a smooth form η_Y does not exist; it can only be approximated by taking for η_Y a bump 1-form in a small neighborhood of Y with integral 1 as in Example 15.5 (b). Clearly, for arbitrary functions f, the equation $(*)$ would then only be approximately true. But for closed functions $(*)$ becomes an equality again: Such functions must be (locally) constant, hence the right side is just this constant times the integral of η_Y, i. e. again $f(Y)$. This should not really come as a surprise as de Rham's Theorem of Proposition 15.10 tells us that de Rham cohomology classes are topological, i. e. $(*)$ is a topological statement that should not change if we approximate the Dirac delta function by a smooth function.

Remark 16.10 (Localization of Poincaré duals)**.** In fact, our observation of Example 16.9 generalizes: Let Y be a p-dimensional compact oriented submanifold of an n-dimensional compact oriented manifold X. By the Whitney embedding of Proposition 11.21 we may assume that X is embedded in some $\mathbb{R}^m$, and thus that the normal bundle $N_{Y/X}$ is defined as a subbundle of $T_X|_Y$ as in Construction 12.16.

Intuitively speaking, the defining equation of Construction 16.8 says that $\eta_Y \in H^{n-p}(X)$ should be some kind of "Dirac delta form" that is non-zero only on Y, and such that its integral over the $n-p$ normal directions of Y in X is 1. Again, such a form does not exist, but it can be shown that η_Y can be represented by a bump $(n-p)$-form with compact support in an arbitrarily small neighborhood of Y in X [BT, end of §5]. By the Tubular Neighborhood Theorem of 12.21 we may take this neighborhood to be isomorphic to the normal bundle $N_{Y/X}$, and η_Y is then characterized by the condition that its integral over any fiber of this bundle is 1 [BT, Proposition 6.18].

As a concrete visual example, the picture on the right shows a circle $Y = S^1$ in an ambient 2-dimensional compact oriented manifold X that contains a neighborhood of S^1 in $\mathbb{R}^2$. It has a small tubular neighborhood $N_{S^1/X}$ as in Example 12.17 which supports the Poincaré dual $\eta_{S^1} \in H^1(X)$. Concretely, in polar coordinates with radius r we can take $\eta_{S^1} = f(r)\,dr$ where f is a bump function for the radius 1 and with integral 1, so that η_{S^1} integrates to 1 along any radial line intersecting S^1 transversely. In the picture, the value of η_{S^1} resp. f is shown by the shading, with darker colors corresponding to higher values.

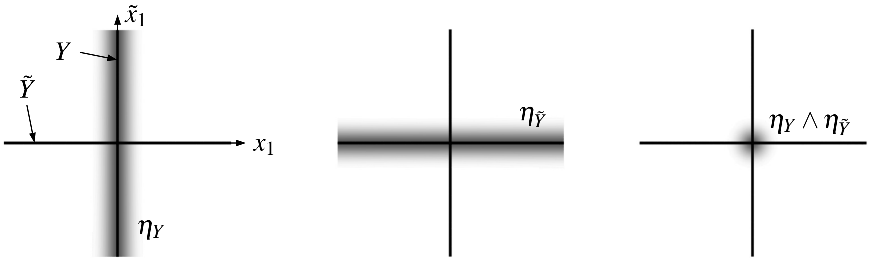


This relation between submanifolds and de Rham cohomology by Poincaré duals can in fact be used to give a geometric interpretation of the ring structure in cohomology, with which we want to end these notes: Under Poincaré duality, taking wedge products of de Rham cohomology classes simply corresponds to intersecting submanifolds.

Proposition 16.11 (Intersection products). *Let Y and $\tilde{Y}$ be compact oriented submanifolds of dimensions p resp. $\tilde{p}$ in an n-dimensional compact oriented manifold X. If Y and $\tilde{Y}$ intersect transversely, i. e. at every intersection point $a \in Y \cap \tilde{Y}$ their normal spaces $N_{Y/X,a}$ and $N_{\tilde{Y}/X,a}$ (considered as subspaces of $T_{X,a}$ as in Remark 16.10) form a direct sum, then $Y \cap \tilde{Y}$ is naturally a compact oriented submanifold of dimension $p+\tilde{p}-n$ in X, and we have*

$$\eta_Y \wedge \eta_{\tilde{Y}} = \eta_{Y\cap\tilde{Y}} \quad \in H^{2n-p-\tilde{p}}(X).$$

Sketch proof. The situation is shown in the picture below: As Y and $\tilde{Y}$ intersect transversely, similarly to Construction 11.2 we can find an adapted chart for X locally around any point $a \in Y \cap \tilde{Y}$ in which Y is given by the vanishing of some coordinates $x_1,\dots,x_{n-p}$, and $\tilde{Y}$ is given by the vanishing of other coordinates $\tilde{x}_1,\dots,\tilde{x}_{n-\tilde{p}}$. By Remark 16.10, the Poincaré duals η_Y and $\eta_{\tilde{Y}}$ can then be localized in small tubular neighborhoods of Y and $\tilde{Y}$, respectively, with integrals 1 along the normal directions at each point. As in the picture below, the wedge product $\eta_Y \wedge \eta_{\tilde{Y}}$ is thus localized in a small neighborhood of $Y \cap \tilde{Y}$, and by Fubini's Theorem its integral over the normal directions of $Y \cap \tilde{Y}$ (which are just the normal directions of Y together with the ones of $\tilde{Y}$) is also 1. Hence, $\eta_Y \wedge \eta_{\tilde{Y}}$ is the Poincaré dual of $Y \cap \tilde{Y}$.



□

Example 16.12 (Bézout's Theorem). As a concluding interesting example of intersection products from the area of algebraic geometry, let us consider the complex projective space $\mathbb{P}^n_{\mathbb{C}}$ of Construction 8.3, which is a compact oriented $2n$-dimensional manifold. Any homogeneous complex polynomial f of degree d in the coordinates $(x_0:\dots:x_n)$ of $\mathbb{P}^n_{\mathbb{C}}$ defines a subset

$$Y := \{(x_0:\dots:x_n) \in \mathbb{P}^n_{\mathbb{C}} : f(x_0,\dots,x_n) = 0\}$$

which is a submanifold of $\mathbb{P}^n_{\mathbb{C}}$ of dimension $2n-2$ by Proposition 11.9 (a) if f satisfies the corresponding rank condition (which is the case for a sufficiently general choice of f). Expressed over $\mathbb{C}$, this means that Y has complex dimension $n-1$ in the complex n-dimensional projective space; it is called a *hypersurface* in $\mathbb{P}^n_{\mathbb{C}}$. The degree of f is also called the *degree* $\deg Y$ of Y.

Hypersurfaces in projective spaces are among most studied objects in algebraic geometry. One important question about them is the following: Given n hypersurfaces $Y_1,\dots,Y_n$ in $\mathbb{P}^n_{\mathbb{C}}$, a naive dimension count shows that one expects them to intersect in finitely many points – how many intersection points are there? In fact, intersection products can answer this question, by considering the Poincaré duals of various submanifolds of $\mathbb{P}^n_{\mathbb{C}}$. For this, note that any complex p-dimensional submanifold Y of $\mathbb{P}^n_{\mathbb{C}}$ has a natural orientation by Exercise 14.9, so that its Poincaré dual $\eta_Y \in H^{2n-2p}(\mathbb{P}^n_{\mathbb{C}})$ is well-defined, and recall from Example 8.4 that these cohomology groups $H^{2n-2p}(\mathbb{P}^n_{\mathbb{C}})$ are all one-dimensional.

(a) By Example 13.15 (b), any closed 0-form ω on $\mathbb{P}^n_{\mathbb{C}}$ is constant. Hence its integral over any point $P \in \mathbb{P}^n_{\mathbb{C}}$ is the same, which means by Construction 16.8 that all points have the same Poincaré dual $\eta_P \in H^0(\mathbb{P}^n_{\mathbb{C}})$, which is clearly non-zero.

(b) If $Y \subset \mathbb{P}^n_{\mathbb{C}}$ is a hypersurface and $L \subset \mathbb{P}^n_{\mathbb{C}}$ a complex line, then $Y \cap L$ is obtained by solving a polynomial equation of degree $\deg Y$ in one complex variable, leading to $\deg Y$ intersection points if L is general enough so that the intersection is transverse. Hence, by Proposition 16.11 we obtain $\eta_Y \wedge \eta_L = \deg Y \cdot \eta_P$ with $\eta_Y \in H^2(\mathbb{P}^n_{\mathbb{C}})$ and $\eta_L \in H^{2n-2}(\mathbb{P}^n_{\mathbb{C}})$. Note that η_L actually does not depend on the chosen line since this space is one-dimensional and different lines lead to the same result $\det Y \cdot \eta_P$. The same then also holds for Y as long as $\deg Y$ is fixed.

(c) As a special case, a hypersurface of degree 1 (i. e. a linear subspace of $\mathbb{P}^n_{\mathbb{C}}$ of complex dimension $n-1$) is called a *hyperplane*; we denote its Poincaré dual by $\eta_H \in H^2(\mathbb{P}^n_{\mathbb{C}})$. Note that by (b) we have for any hypersurface Y

$$\eta_Y \wedge \eta_L = \deg Y \cdot \eta_P = \deg Y \cdot \eta_H \wedge \eta_L,$$

so as $H^2(\mathbb{P}^n_{\mathbb{C}})$ is one-dimensional we see that $\eta_Y = \deg Y \cdot \eta_H \in H^2(\mathbb{P}^n_{\mathbb{C}})$.

(d) Finally, n (sufficiently general) hyperplanes clearly intersect transversely in one point, i. e. by Proposition 16.11 we have $\eta_H \wedge \cdots \wedge \eta_H = \eta_P$ for the n-fold wedge product of η_H with itself. Hence, we obtain by (c) for n hypersurfaces $Y_1, \dots, Y_n$

$$\begin{aligned} \eta_{Y_1} \wedge \cdots \wedge \eta_{Y_n} &= \deg Y_1 \cdot \cdots \cdot \deg Y_n \cdot \eta_H \wedge \cdots \wedge \eta_H \\ &= \deg Y_1 \cdot \cdots \cdot \deg Y_n \cdot \eta_P, \end{aligned}$$

so if the hypersurfaces intersect transversely, their number of intersection points is the product $\deg Y_1 \cdot \cdots \cdot \deg Y_n$ of their degrees. This statement is known as *Bézout's Theorem* and arguably the most important intersection-theoretic result in algebraic geometry. 28

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