

## Algebraic Topology of Smooth Manifolds – Problem Set 9

due Monday, June 22

The support of a smooth function  $f: X \rightarrow \mathbb{R}$  on a manifold  $X$  is defined to be the closure of  $\{x \in X : f(x) \neq 0\}$  in  $X$ .

- (1) Let  $U_i := \{(x_1, x_2) \in \mathbb{R}^2 : x_i > 0\} \subset \mathbb{R}^2$  for  $i \in \{1, 2\}$ , and set  $X := U_1 \cup U_2 \subset \mathbb{R}^2$ .

Find explicit smooth functions  $f_1, f_2: X \rightarrow \mathbb{R}_{\geq 0}$  such that the support of  $f_i$  is contained in  $U_i$  for  $i \in \{1, 2\}$ , and  $f_1(x) + f_2(x) = 1$  for all  $x \in X$ .

(The pair  $(f_1, f_2)$  is called a partition of unity on  $X$  subordinate to the open cover  $(U_1, U_2)$ . We will prove its existence in Corollary 11.17, but to find one explicitly in this specific example it is actually easier to construct it directly rather than to follow the proof of the general statement).

- (2) Let  $f: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  for  $n \in \mathbb{N}$  be a homogeneous polynomial of degree  $d \in \mathbb{N}$ , i. e. a function of the form

$$f(x_0, \dots, x_n) = \sum_{i_0 + \dots + i_n = d} a_{i_0, \dots, i_n} x_0^{i_0} \cdots x_n^{i_n} \quad \text{with } a_{i_0, \dots, i_n} \in \mathbb{R}.$$

Show:

- (a) The subset  $X := \{(x_0 : \dots : x_n) \in \mathbb{P}_{\mathbb{R}}^n : f(x_0, \dots, x_n) = 0\}$  of  $\mathbb{P}_{\mathbb{R}}^n$  is well-defined.  
 (b) If the partial derivatives  $\frac{\partial f}{\partial x_i}$  for  $i = 0, \dots, n$  are nowhere simultaneously zero on  $X$ , then  $X$  is a submanifold of  $\mathbb{P}_{\mathbb{R}}^n$  of dimension  $n - 1$ .
- (3) Let  $U \subset \mathbb{R}^n$  and  $V \subset \mathbb{R}^m$  be open neighborhoods of the origin, and let  $f: U \rightarrow V$  be a smooth map with  $f(0) = 0$  such that  $r := \text{rk}_a f$  is constant for  $a \in U$ .

Show that after possibly shrinking  $U$  and  $V$  there are diffeomorphisms  $\varphi: U \rightarrow U'$  and  $\psi: V \rightarrow V'$  to open neighborhoods  $U'$  and  $V'$  of the origin in  $\mathbb{R}^n$  and  $\mathbb{R}^m$ , respectively, such that  $\psi \circ f \circ \varphi^{-1}: U' \rightarrow V'$  is the map  $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_r, 0, \dots, 0)$ .

- (4) (a) Let  $(f_n)_{n \in \mathbb{N}}$  be a family of smooth functions  $f_n: \mathbb{R} \rightarrow \mathbb{R}$  with compact support. Show that there are constants  $c_n \in \mathbb{R}_{>0}$  for  $n \in \mathbb{N}$  such that the sum  $f := \sum_{n=0}^{\infty} c_n f_n: \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$  converges and is a smooth function as well.

(Hint: It is known from standard analysis that a sequence  $g_n: \mathbb{R} \rightarrow \mathbb{R}$  of differentiable functions converges to a differentiable function with derivative  $\lim_{n \rightarrow \infty} g'_n$  if both  $(g_n)_{n \in \mathbb{N}}$  and  $(g'_n)_{n \in \mathbb{N}}$  converge uniformly.)

- (b) Show that every closed subset  $A \subset \mathbb{R}$  is the zero locus of a smooth function, i. e. that there is a smooth function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with  $A = \{x \in \mathbb{R} : f(x) = 0\}$ .

(Hint: Use second countability and suitable bump functions for the complement  $\mathbb{R} \setminus A$ .)