Andreas Gathmann and Diego Antonio Robayo Bargans Summer Semester 2026

Algebraic Topology of Smooth Manifolds – Problem Set 8

due Monday, June 15

(1) Find the smallest constant $c \in \mathbb{R}_{>1}$ such that the Čech and Vietoris-Rips complexes for any finite subset M of $\mathbb{R}^2$ satisfy $\check{C}(M,r) \subset \mathrm{VR}(M,r) \subset \check{C}(M,cr)$ for all $r \in \mathbb{R}_{>0}$.

(2) Let $f\colon X \to Y$ be a continuous surjective map between two topological spaces with the following property: For every $a \in Y$ there exists a open neighborhood U_a of a such that $f^{-1}(U_a)$ is a disjoint union of open subsets each homeomorphic to U_a through f.

(a) If Y has an atlas, show that there is an atlas on X such that f is smooth.

(b) On the other hand, show by example that if X has an atlas there need not be an atlas on Y such that f is smooth.

(3) Show for any connected Hausdorff space X with an atlas:

(a) X is path-connected.

(b) For any two points $x, y \in X$ there is a homeomorphism $f\colon X \to X$ with $f(x) = y$.

(4) As usual, denote by $\mathrm{O}(n) = \{A \in \mathbb{R}^{n\times n} : A^{\mathsf{T}}A = E\}$ for $n \in \mathbb{N}_{>0}$ the group of $n \times n$ orthogonal matrices. Use the map

$$\varphi\colon U \to \mathbb{R}^{n\times n},\ A \mapsto (E-A)\cdot(E+A)^{-1} \quad \text{with} \quad U := \{A \in \mathbb{R}^{n\times n} : \det(E+A) \neq 0\}$$

to endow $\mathrm{O}(n)$ with an atlas. What is its dimension?

(Hint: A matrix $A \in U$ is orthogonal if and only if $\varphi(A)$ is antisymmetric.)