Algebraic Topology of Smooth Manifolds – Problem Set 7

due Monday, June 8

(1) Let S be a connected graph (i. e. a 1-dimensional simplicial complex) with v vertices and e edges. By an embedding of S into a topological space X we mean a continuous injective map $f\colon |S| \to X$.

(a) For any embedding of S into the sphere S^2, show that $b_0(S^2 \backslash f(|S|)) = e - v + 2$.

(b) Now let $S = \{\{i, j\} : i, j \in \{1, 2, 3, 4, 5\}\}$ be the graph with five vertices and every possible edge between them. Show that f cannot be embedded in S^2, but that it can be embedded in the torus $S^1 \times S^1$.

(2) Prove for every $n \in \mathbb{N}_{>0}$:

(a) If $f\colon S^n \to S^n$ is continuous with $n \in \mathbb{N}$ even, there is a point $x \in S^n$ with $f(x) = x$ or $f(x) = -x$.

(b) Every real matrix $A \in \mathbb{R}^{n\times n}$ with positive entries has an eigenvector with positive entries.

(3) In this exercise we want to prove a decomposition statement for short exact sequences of chain complexes analogous to the interval decomposition of sequences of vector spaces in Proposition 9.11 (see also Remark 9.13 which will be discussed in the lecture on June 3). To keep the number of cases to consider reasonable, let us assume that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & 0 \end{array}$$

is a short exact sequence of chain complexes with only two non-zero rows, with all its vector spaces finite-dimensional, and such that the middle vertical map $B_1 \to B_2$ is an isomorphism.

Show that the sequence is then isomorphic to a product of sequences of the form

$$\begin{array}{ccccc} 0 \to K \to K \to 0 \to 0 & \quad & 0 \to 0 \to K \to K \to 0 & \quad & 0 \to 0 \to K \to K \to 0 \\ \downarrow \quad \downarrow \quad \downarrow & & \downarrow \quad \downarrow \quad \downarrow & & \downarrow \quad \downarrow \quad \downarrow \\ 0 \to K \to K \to 0 \to 0 & & 0 \to 0 \to K \to K \to 0 & & 0 \to K \to K \to 0 \to 0 \end{array}$$

where all maps $K \to K$ are the identity, and the number of times each of these three building blocks occurs in the product is determined uniquely by the original sequence.

(4) Generalize the proof of the Jordan Curve Theorem of Proposition 9.7 to show by induction on $n \in \mathbb{N}_{\geq 2}$ that $b_0(S^n \backslash f(S^{n-1})) = 2$ for any injective continuous map $f\colon S^{n-1} \to S^n$.