Algebraic Topology of Smooth Manifolds – Problem Set 6

due Monday, June 1

(1) Let K be a field with $2 \neq 0 \in K$. Show for all $n \in \mathbb{N}_{>0}$ that the Betti numbers of the real projective n-space are

$$b_p(\mathbb{P}^n_{\mathbb{R}}, K) = \begin{cases} 1 & \text{if } p = 0, \text{ or } p = n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

(2) In this exercise we will see why singular homology is constructed with simplices and not with cubes. A definition with cubes would be: For any topological space X and $p \in \mathbb{N}$, let $C^{\square}_p(X)$ be the vector space with basis given by all continuous maps $\sigma\colon [0,1]^p \to X$. We define a boundary map by

$$\partial\colon C^{\square}_p(X) \to C^{\square}_{p-1}(X),\ \sigma \mapsto \sum_{k=1}^{p} (-1)^k \left(\sigma|_{x_k=1} - \sigma|_{x_k=0}\right) \quad \text{for } p > 0,$$

where each face $\{x \in [0,1]^p : x_k = a\}$ for $a \in \{0,1\}$ is identified with $[0,1]^{p-1}$, keeping the order of the remaining coordinates. Moreover, we construct a subdivision map

$$\mathrm{sd}\colon C^{\square}_p(X) \to C^{\square}_p(X),\ \sigma \mapsto \sum_{Q} \sigma|_Q,$$

where the sum is taken over all 2^p subcubes $Q \subset [0,1]^p$ of side length $\frac{1}{2}$ obtained by subdividing each interval $[0,1]$ at its midpoint, and all these subcubes are rescaled to $[0,1]^p$, so that $\sigma|_Q \in C^{\square}_p(X)$.

It is easy to check that this makes $C^{\square}(X)$ into a chain complex and $\mathrm{sd}\colon C^{\square}(X) \to C^{\square}(X)$ into a morphism of chain complexes.

(a) Compute the "cubical homology of a point", i. e. the Betti numbers of the chain complex $C^{\square}(X)$ for the topological space X with only one point.

(b) For an arbitrary non-empty topological space X and any ground field K, show that sd is never chain homotopic to the identity.

(3) For any subset Y of a topological space X and $p \in \mathbb{N}$, let $C_p(X;Y) := C_p(X)/C_p(Y)$. The boundary maps of $C(X)$ descend to these quotients and make $C(X;Y)$ into a chain complex. The homology groups of this complex are called the *relative homology groups* of X and Y and denoted by $H_p(X;Y)$. Of course, by definition there is then a short exact sequence of chain complexes

$$0 \longrightarrow C(Y) \longrightarrow C(X) \longrightarrow C(X;Y) \longrightarrow 0,$$

which induces a corresponding long exact sequence in homology.

Now let $A \subset Y \subset X$ such that A is closed and Y is open in X. Show that the inclusion maps induce isomorphisms on relative homology groups $H_p(X\backslash A; Y\backslash A) \to H_p(X;Y)$.

(Hint: It is useful to express $C(X\backslash A; Y\backslash A)$ in terms of the complex $C(U|V)$ from class for suitable U and V.)

(4) Let X be a topological space and $x \in X$ such that $\{x\} \subset X$ is closed.

(a) Show that the relative homology groups $H_p(U; U\backslash\{x\})$ of Exercise 3, where U is any open neighborhood of x, are independent of U. They are called the *local homology groups* of X at x.

(b) Compute these local homology groups for the half-space $X = \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_n \geq 0\}$ with $n \in \mathbb{N}_{>0}$, and any $x \in X$.