

Algebraic Topology of Smooth Manifolds – Problem Set 5

due Monday, May 25

- (1) Compute the singular homology of the following topological spaces:
- (a) the rational numbers \mathbb{Q} (with the subspace topology of \mathbb{R});
 - (b) the space X_n of all positive definite symmetric real matrices of a given size $n \in \mathbb{N}_{>0}$ (with the subspace topology of the space $\mathbb{R}^{n \times n}$ of all $n \times n$ -matrices).
- (2) Let X be a topological space. For all $p \in \mathbb{N}$ and every singular simplex $\sigma: |\Delta^p| \rightarrow X$ we choose a homeomorphism $f_\sigma: |\Delta^p| \rightarrow |\Delta^p|$ compatible with taking boundaries, i. e. such that the restriction of f_σ to $|\beta|$ for any ordered simplex β in $D^{\{0, \dots, p\}}$ is $f_{(\sigma|_\beta)}$, and set $\tilde{\sigma} := \sigma \circ f_\sigma$. Show that the linear maps $C_p(X) \rightarrow C_p(X)$ for $p \in \mathbb{Z}$ obtained by mapping any p -simplex σ to $\tilde{\sigma}$ induce isomorphisms in homology. (This justifies that we usually only draw the image of singular simplices in X when visualizing homology classes.)
- (3) Let $X \subset \mathbb{R}^n$ be a non-empty convex subspace with $n \in \mathbb{N}$. We have seen that there is a linear degree map $\text{deg}: C_0(X) \rightarrow K$ mapping any singular 0-simplex to $1 \in K$. Regarding K as chain complex, where K is the only non-zero term and sits in degree 0, construct an explicit chain homotopy showing that deg defines a chain homotopy equivalence $C(X) \rightarrow K$.
- (4) Let S be a simplicial complex, and let $C_{\text{ord}}^\vee(S)$ be the dual cochain complex of the chain complex $C^{\text{ord}}(S)$ of ordered simplicial chains, with vector spaces $C_{\text{ord}}^p(S) := (C_p^{\text{ord}}(S))^\vee$. As dualizing commutes with homology by Proposition 5.14, we can dualize Proposition 4.22 to see that the cohomology $H_{\text{ord}}^p(S) := H^p(C_{\text{ord}}^\vee(S))$ of this cochain complex is naturally isomorphic to $H^p(S)$ for all $p \in \mathbb{Z}$.
- (a) Determine $C^p(S)$ as a subspace of $C_{\text{ord}}^p(S)$ for $p \in \mathbb{N}$.
 - (b) For $\alpha \in C_{\text{ord}}^p(S)$ and $\beta \in C_{\text{ord}}^q(S)$ with $p, q \in \mathbb{N}$ we define $\alpha \smile \beta \in C_{\text{ord}}^{p+q}(S)$ by

$$(\alpha \smile \beta)(i_0, \dots, i_{p+q}) := \alpha(i_0, \dots, i_p) \cdot \beta(i_{p+1}, \dots, i_{p+q}).$$
- Show that this gives a well-defined (not necessarily commutative) ring structure on the ordered cohomology of S , and thus also on the oriented cohomology of S .
- (c) Let $\varphi_1, \varphi_2 \in C^1(S) \subset C_{\text{ord}}^1(S)$ be the 1-cochains on the simplicial torus S as in Example 5.16. Determine whether $\varphi_1 \smile \varphi_2 \in C^2(S)$, and whether the class $\varphi_1 \smile \varphi_2 \in H_{\text{ord}}^2(S) \cong H^2(S)$ is zero.
- (We will see this product structure on cohomology again later in class when discussing the analytic setting, where it plays a major role and occurs much more naturally.)