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Algebraic Topology of Smooth Manifolds – Problem Set 4

due Monday, May 18

(1) Consider the simplicial sphere $S^{\underline{4}}$ for the set $\underline{4} := \{0,1,2,3,4\}$.

(a) Let

$$\varphi = [0,1,2]^\vee + [0,1,4]^\vee - [0,2,3]^\vee + [0,3,4]^\vee \quad \in C^2(S^{\underline{4}}),$$

where $[i_0,i_1,i_2]^\vee$ denotes the unique linear form $C_2(S^{\underline{4}}) \to K$ with value 1 on $[i_0,i_1,i_2]$ and 0 on all $[j_0,j_1,j_2]$ with $\{j_0,j_1,j_2\} \neq \{i_0,i_1,i_2\}$.
Show that there exists a cochain $\psi \in C^1(S^{\underline{4}})$ with $d\psi = \varphi$.

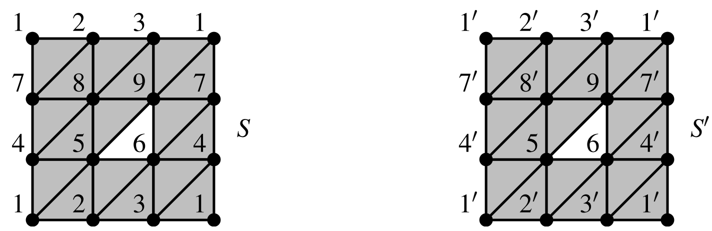
(b) For a 3-cochain $\varphi \in C^3(S^{\underline{4}})$, show that there exists a cochain $\psi \in C^2(S^{\underline{4}})$ with $d\psi = \varphi$ if and only if $\int_{\partial[0,1,2,3,4]} \varphi = 0$.

(2) Let $S_1,\dots,S_n$ with $n \in \mathbb{N}_{>0}$ be simplicial complexes, and set $S = S_1 \cup \cdots \cup S_n$. Are the following statements true for any p-cochain $\varphi \in C^p(S)$ with $p \in \mathbb{Z}$? If a statement is true, give a proof. If a statement is false, provide a counterexample.

(a) The cochain φ is closed if and only if $\varphi|_{S_i}$ is closed on S_i for every $i = 1,\dots,n$.

(b) The cochain φ is a coboundary if and only if $\varphi|_{S_i}$ is a coboundary on S_i for every $i = 1,\dots,n$.

(3) Let S and S' be the two simplicial complexes in the picture below, obtained by removing a 2-simplex from a torus as in Example 1.17 (c), and such that the boundary of this 2-simplex agrees in S and S'.



(a) Compute bases of the homology groups $H_p(S)$ for $p = 0,1,2$.

(b) Compute bases of the homology groups $H_p(S \cup S')$ for $p = 0,1,2$.

(4) Suppose that a simplicial complex S is the union of subcomplexes $S_1,\dots,S_n$ with $n \geq 2$ such that every intersection of any number of these subcomplexes is either empty or has the homology of a point.

(a) Show that $b_p(S) = 0$ for all $p \geq n-1$.

(b) Show for all n that the inequality of (a) cannot be improved.