

Algebraic Topology of Smooth Manifolds – Problem Set 3

due Monday, May 11

- (1) Let S be a simplicial complex, and let $*$ and \circ be new vertices not contained in S . We construct the new simplicial complex

$$\Sigma(S) := \{\{*\}, \{\circ\}\} \cup \{\alpha, \alpha \cup \{*\}, \alpha \cup \{\circ\} : \alpha \in S\}.$$

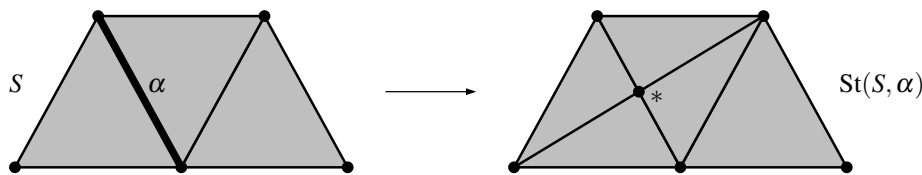
- (a) Compute the Betti numbers of $\Sigma(S)$ in terms of those of S .
 (b) Starting with the 0-dimensional simplicial complex $S = \{\{0\}, \{1\}\}$, which well-known topological space is the topological realization of the n -fold construction $\Sigma^n(S)$ of (a) for $n \in \mathbb{N}$?

- (2) Let $f, g: S \rightarrow T$ be two morphisms of simplicial complexes such that $f(\alpha) \cup g(\alpha) \in T$ for any $\alpha \in S$. Show that then the pushforwards f_* and g_* agree as maps in homology $H_p(S) \rightarrow H_p(T)$ for all $p \in \mathbb{Z}$.

- (3) Let $\alpha = \{i_0, \dots, i_p\}$ be a fixed p -simplex in a simplicial complex S for some $p \in \mathbb{N}$. We define the *stellar subdivision* $\text{St}(S, \alpha)$ of S at α to be the simplicial complex

$$\text{St}(S, \alpha) := \{\beta \in S : \alpha \not\subset \beta\} \cup \{\beta \cup \{*\} : \alpha \not\subset \beta \text{ and } \alpha \cup \beta \in S\},$$

with vertex set $V(\text{St}(S, \alpha)) = V(S) \cup \{*\}$, where $*$ denotes a new vertex.



Show:

- (a) The map f given on vertices by $f(*) = i_0$ and $f(i) = i$ for all $i \in V(S)$ is a morphism of simplicial complexes $f: \text{St}(S, \alpha) \rightarrow S$.
 (b) The linear map $g: C(S) \rightarrow C(\text{St}(S, \alpha))$ defined by

$$g([i_0, \dots, i_p, j_1, \dots, j_q]) = \sum_{k=0}^p (-1)^k [* , i_0, \dots, \widehat{i_k}, \dots, i_p, j_1, \dots, j_q]$$

for all oriented $(p + q)$ -simplices containing α , and $g(\beta) = \beta$ for all oriented simplices not containing α , is a morphism of chain complexes.

- (c) The map g of (b) is a homotopy inverse of the pushforward f_* by the morphism of (a). In particular, the simplicial homology groups of $\text{St}(S, \alpha)$ and S are isomorphic.
 (4) Let α and β be two simplices, for which we fix independently an order of its elements. We set

$$S := \{(i_0, j_0), \dots, (i_p, j_p)\} : p \in \mathbb{N}, i_0 \leq \dots \leq i_p \text{ in } \alpha, \text{ and } j_0 \leq \dots \leq j_p \text{ in } \beta\}.$$

- (a) Show that S is a simplicial complex.
 (b) Prove that $|S|$ is homeomorphic to $|D^\alpha| \times |D^\beta|$.

(In particular, for $\beta = \{0, 1\}$ this recovers the prism construction of the lecture.)