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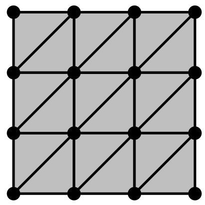
Algebraic Topology of Smooth Manifolds – Problem Set 2

due Monday, May 4

(1) The Klein bottle is obtained as a topological space by taking the unit square and identifying the top with the bottom side in the same direction, and the left with the right side in opposite directions. Starting from a subdivision of the unit square into 18 triangles as in the figure below, construct a simplicial complex realizing the Klein bottle, and compute its Betti numbers. Do they depend on the chosen ground field?



(2) Sometimes it is more natural to slightly modify our very first definition of simplicial complexes by dropping the condition that a simplex has to be non-empty. The empty set $\emptyset$ is then a valid (-1)-simplex that is a proper face of every other simplex. For a simplicial complex S we can form the *augmented simplicial complex* $\tilde{S} := S \cup \{\emptyset\}$, satisfying again our definition of a simplicial complex with the new notion of faces. As for the constructions on singular homology in class, every augmented simplicial complex has an oriented (-1)-simplex $[\,]$, and the formula for the boundary map ∂_p is extended to $p = 0$ in the natural way by setting $\partial[i_0] = [\,]$ for all 0-simplices $\{i_0\}$.

If we leave all other constructions regarding simplicial homology unchanged, the resulting simplicial homology of the augmented simplicial complex $\tilde{S}$ is called the *reduced simplicial homology* of S and denoted $\tilde{H}_p(S) := H_p(\tilde{S})$ for all $p \in \mathbb{Z}$. How it is related to $H_p(S)$?

(3) (a) Let C be a complex of finite-dimensional vector spaces, with only finitely many C_p non-zero, and let $f : C \to C$ be a morphism. Show that

$$\sum_{p\in\mathbb{Z}} (-1)^p \operatorname{tr}(f : C_p \to C_p) = \sum_{p\in\mathbb{Z}} (-1)^p \operatorname{tr}(f : H_p(C) \to H_p(C)), \qquad (*)$$

where tr denotes the trace of the corresponding linear map. If f is the pushforward $C(S) \to C(S)$ by a morphism $g : S \to S$ of simplicial complexes, we call $(*)$ the *Lefschetz number* of g.

(b) Show that any morphism $g : S \to S$ of simplicial complexes with non-zero Lefschetz number has a fixed simplex.

(c) Now suppose that S is a connected simplicial complex with Betti numbers $b_p(S) = 0$ for all $p > 0$. Show that any simplicial morphism $g : S \to S$ must fix at least one simplex. Use this to show that any simplicial morphism from a tree (i. e. a connected graph of genus 0) to itself must have a fixed simplex.

(4) Let S be a 2-dimensional simplicial complex with barycentric subdivision $\operatorname{Sd}(S)$. Show that the linear maps

$$\begin{aligned}
\operatorname{sd}_0 &: C_0(S) \to C_0(\operatorname{Sd}(S)),\ [i_0] \mapsto [\{i_0\}],\\
\operatorname{sd}_1 &: C_1(S) \to C_1(\operatorname{Sd}(S)),\ [i_0,i_1] \mapsto [\{i_0\},\{i_0,i_1\}] - [\{i_1\},\{i_0,i_1\}],\\
\operatorname{sd}_2 &: C_2(S) \to C_2(\operatorname{Sd}(S)),\ [i_0,i_1,i_2] \mapsto \sum_\pi \operatorname{sign}\pi \cdot [\{i_{\pi(0)}\},\{i_{\pi(0)},i_{\pi(1)}\},\{i_{\pi(0)},i_{\pi(1)},i_{\pi(2)}\}]
\end{aligned}$$

(where the sum runs over all permutations π of $\{0,1,2\}$) define a morphism of chain complexes $\operatorname{sd} : C(S) \to C(\operatorname{Sd}(S))$ that induces an isomorphism in homology $H_p(S) \cong H_p(\operatorname{Sd}(S))$ for all p.