Andreas Gathmann and Diego Antonio Robayo Bargans Summer Semester 2026

Algebraic Topology of Smooth Manifolds – Problem Set 12

due Monday, July 13

(1) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a submersion of manifolds, and let $X := f^{-1}(\{0\})$ be the submanifold given as the vanishing set of f. For all $i = 1,\dots,n$ we set

$$\omega_i := (-1)^{i-1} \left(\frac{\partial f}{\partial x_i}\right)^{-1} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n,$$

where $\widehat{\cdot}$ means that the corresponding term has been omitted.

(a) For all $i, j = 1,\dots,n$, show that $\omega_i = \omega_j$ at the points of X where both forms are defined.

(b) Use (a) to check directly that there is a nowhere-zero $(n-1)$-form on X.

(2) We say that a smooth function

$$f\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n,\ (x,y) \mapsto (u(x,y), v(x,y))$$

satisfies the *Cauchy-Riemann equations* if

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial v_i}{\partial y_j} \quad \text{and} \quad \frac{\partial u_i}{\partial y_j} = -\frac{\partial v_i}{\partial x_j}$$

for all $i, j = 1,\dots,n$. (One can show that this is equivalent to $f\colon \mathbb{C}^n \to \mathbb{C}^n$ being infinitely complex differentiable if x and y denote the real and imaginary parts of the variables in the source, and u and v denote the real and imaginary parts of the variables in the target.)

Show that any $2n$-dimensional manifold whose transition maps satisfy the Cauchy-Riemann equations (i. e. an n-dimensional complex manifold) is orientable.

(3) Show that the tangent bundle T_X on any manifold X is an orientable manifold.

(4) (a) Show that an n-dimensional manifold X is orientable if and only if there is a nowhere-vanishing n-form on X.

(b) Now let X be a compact n-dimensional orientable manifold with boundary. Use (a) to show that ∂X is not a smooth deformation retract of X, and give a counterexample of this statement when X is non-compact.

You can already use the Theorem of Stokes that will be proven on Friday: For any $(n-1)$-form ω with compact support on an n-dimensional oriented manifold X with boundary we have $\int_{\partial X} \omega = \int_X d\omega$.