Algebraic Topology of Smooth Manifolds – Problem Set 11

due Monday, July 6

(1) Let $f\colon X \to Y$ be a morphism of manifolds such that X is connected and $f'(x)\colon T_{X,x} \to T_{Y,f(x)}$ is the zero map for all $x \in X$. Show that f is a constant map.

(2) Let $X \subset \mathbb{R}^n$ be a submanifold.

(a) Show that $\Delta := \{(x,x) : x \in X\}$ is a submanifold of $X \times X$, and that

$$\delta\colon X \to \Delta,\ x \mapsto (x,x)$$

is an isomorphism.

(b) Show that the pullback bundle $\delta^* N_{\Delta/X\times X}$ as in Exercise 3 of Problem Set 10 is isomorphic to the tangent bundle T_X.

(3) Let $(U_i)_{i\in I}$ be an open cover of a manifold X, and let $A_{i,j}, B_{i,j}\colon U_i \cap U_j \to \mathrm{GL}(r,\mathbb{R})$ for $i,j \in I$ be two sets of transition maps for vector bundles, i. e. such that

$$A_{i,i} = B_{i,i} = E, \quad A_{i,k} = A_{j,k} \cdot A_{i,j}, \quad B_{i,k} = B_{j,k} \cdot B_{i,j}$$

(where defined) for all $i,j,k \in I$. Show that the vector bundle with transition maps $A_{i,j}$ is isomorphic to the vector bundle with transition maps $B_{i,j}$ if and only if there exist smooth maps $\Lambda_i\colon U_i \to \mathrm{GL}(r,\mathbb{R})$ with

$$A_{i,j} = \Lambda_j \cdot B_{i,j} \cdot \Lambda_i^{-1} \quad \text{on } U_i \cap U_j$$

for all $i,j \in I$.

(4) In this exercise we want to see an alternative construction of tangent spaces and tangent bundles that does not use embeddings in $\mathbb{R}^N$. Let X be an n-dimensional manifold, and let $\varphi\colon U \to V \subset \mathbb{R}^n$ be a chart. Recall that $\Omega^0(U)$ denotes the vector space of smooth real-valued functions on U.

(a) Show that the maps

$$\frac{\partial}{\partial \varphi_i}\colon \Omega^0(U) \to \Omega^0(U),\ f \mapsto \frac{\partial f}{\partial \varphi_i} := \frac{\partial (f \circ \varphi^{-1})}{\partial x_i} \circ \varphi$$

for $1 \leq i \leq n$ define linear homomorphisms verifying the Leibniz rule, i. e. such that

$$\frac{\partial (fg)}{\partial \varphi_i} = \frac{\partial f}{\partial \varphi_i} \cdot g + f \cdot \frac{\partial g}{\partial \varphi_i} \quad \text{for all } f,g \in \Omega^0(U).$$

(b) For all $x \in U$ and $i \in \{1,\dots,n\}$, consider the linear map

$$\left.\frac{\partial}{\partial \varphi_i}\right|_x\colon \Omega^0(X) \to \mathbb{R},\ f \mapsto \frac{\partial f|_U}{\partial \varphi_i}(x).$$

Show that $\left(\left.\frac{\partial}{\partial \varphi_i}\right|_x\right)_{1\leq i\leq n}$ is a basis of the vector space D_x of all linear maps $v\colon \Omega^0(X) \to \mathbb{R}$ satisfying $v(fg) = v(f)\cdot g(x) + f(x)\cdot v(g)$ for all $f,g \in \Omega^0(X)$.

(c) Show that the vector spaces D_x for $x \in X$ form a vector bundle over X isomorphic to the tangent bundle T_X.