Andreas Gathmann and Diego Antonio Robayo Bargans Summer Semester 2026

Algebraic Topology of Smooth Manifolds – Problem Set 10

due Monday, June 29

(1) (a) Show that the tangent bundle of the sphere S^1 is trivial.

(b) Note that the tangent bundle of S^2 is not trivial by the Hairy Ball Theorem.
Is the tangent bundle of S^3 trivial?

(2) Let $f\colon E \to F$ be a morphism between two vector bundles $\pi\colon E \to X$ and $\psi\colon F \to X$ over a manifold X. We denote by $\ker f := \bigcup_{x\in X} \ker(f|_{E_x}) \subset E$ the union of the kernels of the linear maps in the fibers given by f.

(a) Show that $\ker f$ is a subbundle of E if and only if the dimension of the vector spaces $\ker(f|_{E_x})$ for $x \in X$ is constant.

(b) Give an example of a morphism f for which $\ker f$ is not a subbundle of E.

(3) Let $f\colon X \to Y$ be a morphism of manifolds, and let $\pi\colon E \to Y$ a vector bundle on Y of rank r. We set

$$f^*(E) := \{(x,v) \in X \times E : f(x) = \pi(v)\} \quad \subset X \times E.$$

(a) Show that $f^*(E)$ is a submanifold of $X \times E$, and that the natural projection $\pi\colon f^*(E) \to X$ onto the first factor is a smooth map with $\pi^{-1}(\{x\}) = E_{f(x)}$ for all $x \in X$.

(b) Show that $\pi\colon f^*(E) \to X$ is a rank-r vector bundle. It is called the *pullback bundle* of E by f.

(4) Show that every vector bundle on a compact manifold is a subbundle of a trivial bundle.