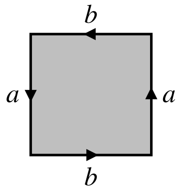
Andreas Gathmann and Diego Antonio Robayo Bargans Summer Semester 2026

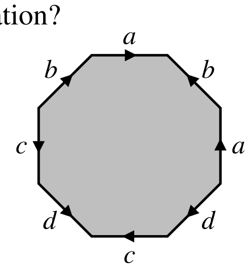
Algebraic Topology of Smooth Manifolds – Problem Set 1

due Monday, April 27

(1) (a) Recall that the real projective plane can be obtained from a square by identifying its sides as in the picture below on the left.
Draw an abstract simplicial complex whose topological realization is the real projective plane and that has fewer than 10 vertices.

(b) Draw an abstract simplicial complex whose topological realization is the topological space obtained from an octagon by identifying its sides as in the picture below on the right. What does this toplogical space look like after this identification?



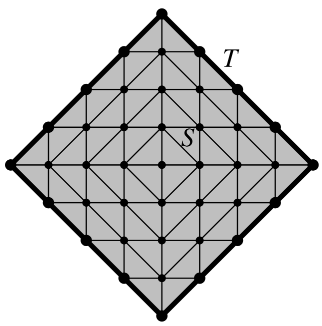


(2) Let $T_1, \dots, T_n$ be the connected components of a simplicial complex S. Show that $|T_1|, \dots, |T_n|$ are the path-connected components of the topological space $|S|$.

(3) Let S be the 2-dimensional simplicial complex in the picture below, with vertices labeled (i, j) for $i, j \in \mathbb{Z}$ and $|i| + |j| \leq 4$ according to their position in the plane. We denote its "boundary" by T, i. e. the full 1-dimensional subcomplex drawn in bold containing the vertices (i, j) with $|i| + |j| = 4$. Obviously, the topological realizations of S and T are the 2-dimensional disk D^2 and its boundary circle S^1, respectively.

The following two facts are known from topology:

- There is no continuous map $f\colon D^2 \to S^1$ with $f|_{S^1} = \mathrm{id}$.
- Every continuous map $f\colon D^2 \to D^2$ has a fixed point.



Are the following simplicial versions of these statements true or false? If a statement is true, give a purely simplicial proof as well as a proof deriving it from the corresponding topological fact above. If a statement is false, provide a counterexample.

(a) There is no morphism $f\colon S \to T$ with $f|_T = \mathrm{id}$.

(b) Every morphism $f\colon S \to S$ has a fixed vertex.

(4) Show that every simplicial complex S has a geometric realization in $\mathbb{R}^n$ with $n = 2\dim S + 1$.

(Hint: Start with the standard geometric realization and try to project it linearly to smaller-dimensional subspaces as long as possible. It might be useful that the complement of a finite union of proper affine subspaces of a real vector space is a dense open subset.)

Please put your solutions (in groups of up to 3 people) in Diego's mailbox next to room 48-210, or submit them in the OLAT course as a PDF file.