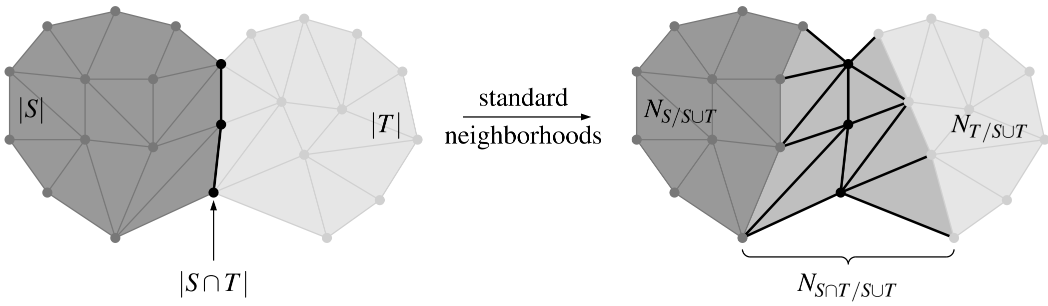
8. Comparison of Simplicial and Singular Homology

Having introduced and studied both simplicial and singular homology, we are now ready to compare these two theories. In this chapter, we will thus construct a way to pass from simplicial to singular homology, and vice versa from singular to simplicial homology.

Going from the simplicial to the singular setting seems to be straightforward: Every simplicial complex S has a topological realization $|S|$ for which singular homology is defined, and we have already seen in Construction 6.11 that there are natural topological realization morphisms $H_p(S) \to H_p(|S|)$ for all $p \in \mathbb{Z}$. We are going to show now that these maps are isomorphisms, i. e. that singular homology agrees with simplicial homology in this sense. In fact, the idea behind this statement is quite simple: We know already that both theories are trivial on simplicial disks and satisfy Mayer-Vietoris, so they should just be the same by Mayer-Vietoris induction.

Unfortunately, as it happens quite often in topology, there are some technical complications with this argument. Namely, the singular Mayer-Vietoris statement of Corollary 7.10 requires the two subspaces to be open, but (the topological realizations of) simplicial subcomplexes are not. For example, consider the picture below on the left, which is completely analogous to the situation in the introduction to Chapter 7, but in which U and V are replaced by the topological realizations of two simplicial complexes S and T. In this case the topological realization of $S \cap T$ is just a line, and thus it is clear that no subdivision of a singular simplex covering both spaces can help us to make each part of it lie in $|S|$ or $|T|$.



From the picture, the obvious way around this problem seems to be to replace $|S|$ and $|T|$ by open neighborhoods, such as the standard neighborhoods $N_{S/S\cup T}$ and $N_{T/S\cup T}$ (see Proposition 1.24) in the picture above on the right. However, we have to make sure of course that this does not change the homology of these spaces, i. e. that $|S|$ and $|T|$ are deformation retracts of these neighborhoods. By Proposition 1.24 (b) this is guaranteed if all subcomplexes are full, which we can achieve by a barycentric subdivision due to Lemma 1.23. Luckily, this barycentric subdivision changes neither the simplicial nor the singular homology of our space, so that our argument finally works out. Let us formulate this proof rigorously:

Proposition 8.1 (Singular homology of simplicial complexes)**.** *For any simplicial complex S, the topological realization morphism of Construction 6.11 induces isomorphisms between the simplicial homology $H_p(S)$ and the singular homology $H_p(|S|)$ for all $p \in \mathbb{Z}$.*

Proof. We will prove the statement by Mayer-Vietoris induction as in Lemma 4.18. For a simplicial disk D^α, both homologies are trivial: the simplicial homology of D^α by Example 3.12, and the singular homology of $|D^\alpha|$ by Example 6.14.

For the induction step, let S and T be two simplicial complexes such that the statement of the proposition holds for S, T, and $S \cap T$. By replacing these simplicial complexes by their barycentric subdivisions (which changes neither their simplicial homology nor their topological realizations by

Proposition 4.22 and Proposition 1.21), we can assume by Lemma 1.23 that $S \cap T$ is a full subcomplex of S and T, and that S and T are full subcomplexes of $S \cup T$. On the singular homology side, we can then replace $|S|$, $|T|$, and $|S \cap T|$ by their standard neighborhoods in $|S \cup T|$ as in Proposition 1.24 as they are deformation retracts of these neighborhoods and thus have the same homology by Corollary 6.13. Now consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C^{\mathrm{ord}}(S \cap T) \arrow[r] \arrow[d] & C^{\mathrm{ord}}(S) \times C^{\mathrm{ord}}(T) \arrow[r] \arrow[d] & C^{\mathrm{ord}}(S \cup T) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C(N_{S \cap T/S \cup T}) \arrow[r] & C(N_{S/S \cup T}) \times C(N_{T/S \cup T}) \arrow[r] & C(N_{S/S \cup T} | N_{T/S \cup T}) \arrow[r] & 0,
\end{tikzcd}$$

where the maps are as follows:

- The top row is just the simplicial short exact Mayer-Vietoris sequence of Example 4.7.
- The bottom row is the singular short exact Mayer-Vietoris sequence of Construction 7.1, noting that $N_{S \cap T/S \cup T} = N_{S/S \cup T} \cap N_{T/S \cup T}$ by Exercise 1.26 (b).
- The columns are the topological realization morphisms of Construction 6.11, followed by the pushforwards by the inclusions of $|S|$, $|T|$, and $|S \cap T|$ in their standard neighborhoods. On the right side of the diagram, note that any ordered simplex in $S \cup T$ actually maps to $N_{S/S \cup T}$ or $N_{T/S \cup T}$, so we can consider the vertical map as going to $C(N_{S/S \cup T} | N_{T/S \cup T})$ instead of $C(N_{S/S \cup T} \cup N_{T/S \cup T})$.

Let us now take the homology in this diagram as in Construction 4.10. In the top row we just get the ordinary simplicial homology (strictly speaking in the ordered version, but this does not make a difference by Proposition 4.25). In the bottom row we get the singular homology of the topological realizations (again, strictly speaking, in the right term we get $H_p(N_{S/S \cup T} | N_{T/S \cup T})$ instead of $H_p(N_{S/S \cup T} \cup N_{T/S \cup T}) \overset{1.26\,(a)}{=} H_p(|S \cup T|)$, but this does not make a difference by Corollary 7.9). Hence, we obtain a commutative diagram with exact rows for all $p \in \mathbb{Z}$

$$\begin{tikzcd}
H_p(S \cap T) \arrow[r] \arrow[d] & H_p(S) \times H_p(T) \arrow[r] \arrow[d] & H_p(S \cup T) \arrow[r] \arrow[d] & H_{p-1}(S \cap T) \arrow[r] \arrow[d] & H_{p-1}(S) \times H_{p-1}(T) \arrow[d] \\
H_p(|S \cap T|) \arrow[r] & H_p(|S|) \times H_p(|T|) \arrow[r] & H_p(|S \cup T|) \arrow[r] & H_{p-1}(|S \cap T|) \arrow[r] & H_{p-1}(|S|) \times H_{p-1}(|T|).
\end{tikzcd}$$

As the outer four columns are isomorphisms are isomorphisms by the induction assumption, we therefore conclude by the 5-Lemma 4.19 that the middle vertical map $H_p(S \cup T) \to H_p(|S \cup T|)$ is an isomorphism as well. This finishes the induction step for the Mayer-Vietoris induction. □

Example 8.2.

(a) If S and T are simplicial complexes with homeomorphic topological realizations $|S|$ and $|T|$, it follows from Proposition 8.1 that the simplicial homology groups $H_p(S)$ and $H_p(T)$ are isomorphic for all $p \in \mathbb{Z}$. Note however that it is in general hard to write down an explicit isomorphism without leaving the simplicial world.

(b) For an $(n+1)$-simplex α, the topological realization of the simplicial sphere S^α is just the topological n-sphere S^n. Our computation of the simplicial homology of S^α in Example 3.13 thus tells us by Proposition 8.1 that the singular Betti numbers of the sphere S^n are

$$b_p(S^n) = \begin{cases} 1 & \text{for } p = 0 \text{ or } p = n, \\ 0 & \text{otherwise.} \end{cases}$$

Geometrically, this just means that S^n has "one n-dimensional hole".

(c) The complement $\mathbb{R}^{n+1} \backslash \{0\}$ of a point in $\mathbb{R}^{n+1}$ is homotopy equivalent to S^n for all $n \in \mathbb{N}$, hence by (b) and Corollary 6.13 we also have

$$b_p(\mathbb{R}^{n+1} \backslash \{0\}) = \begin{cases} 1 & \text{for } p = 0 \text{ or } p = n, \\ 0 & \text{otherwise.} \end{cases}$$

(d) By the calculation of Example 2.27, the non-zero singular Betti numbers of the torus $S^1 \times S^1$ are $b_0(S^1 \times S^1) = b_2(S^1 \times S^1) = 1$ and $b_1(S^1 \times S^1) = 2$, with a basis of $H_1(S^1 \times S^1)$ given by circles in each of the two factors. Similarly, Example 2.28 gives the singular Betti numbers of the real projective plane and shows that they depend on the chosen ground field.

Actually, to study some concrete examples with a little more interesting homology groups, let us consider general projective spaces in more detail and start by quickly recalling their definition from [G3, Construction 5.24] or [G5, Definition 3.2]. Initially, this is a pure linear algebra construction that works over any ground field, but to make projective spaces into topological spaces interesting for our purposes we will consider them only over the real or complex numbers.

Construction 8.3 (Projective spaces)**.** Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and let $n \in \mathbb{N}$. We define the **projective** ***n*-space** over $\mathbb{K}$ to be the set $\mathbb{P}^n_{\mathbb{K}}$ of all 1-dimensional linear subspaces of $\mathbb{K}^{\{0,\dots,n\}} \cong \mathbb{K}^{n+1}$.

Clearly, such a 1-dimensional linear subspace is uniquely determined by a spanning non-zero vector in $\mathbb{K}^{\{0,\dots,n\}}$, with two such vectors giving the same linear subspace if and only if they are scalar multiples of each other. In other words, we have

$$\mathbb{P}^n_{\mathbb{K}} = (\mathbb{K}^{\{0,\dots,n\}} \backslash \{0\}) / \sim$$

with the equivalence relation

$$(x_0,\dots,x_n) \sim (x_0',\dots,x_n') \quad :\Leftrightarrow \quad x_i = \lambda x_i' \text{ for some } \lambda \in \mathbb{K}^* \text{ and all } i.$$

The equivalence class of $(x_0,\dots,x_n)$ in $\mathbb{P}^n_{\mathbb{K}}$ is usually denoted by $(x_0 : \dots : x_n)$. We make $\mathbb{P}^n_{\mathbb{K}}$ into a topological space by giving it the quotient topology inherited from $\mathbb{K}^{\{0,\dots,n\}} \backslash \{0\}$. Its main properties are:

(a) The subspaces $U_i := \{(x_0 : \dots : x_n) \in \mathbb{P}^n_{\mathbb{K}} : x_i \neq 0\}$ for $i = 0,\dots,n$ are homeomorphic to $\mathbb{K}^n$ by the map

$$U_i \to \mathbb{K}^n,\ (x_0 : \dots : x_n) \mapsto \left(\frac{x_0}{x_i},\dots,\widehat{\frac{x_i}{x_i}},\dots,\frac{x_n}{x_i}\right),$$

giving rise to an open cover $\mathbb{P}^n_{\mathbb{K}} = U_0 \cup \dots \cup U_n$ by n-dimensional $\mathbb{K}$-vector spaces.

(b) $\mathbb{P}^n_{\mathbb{K}}$ is compact, as it is the image of a compact sphere under the continuous map

$$\{x \in K^{\{0,\dots,n\}} : \|x\| = 1\} \to \mathbb{P}^n,\ (x_0,\dots,x_n) \mapsto (x_0 : \dots : x_n).$$

(c) For $n = 1$, we have $\mathbb{P}^1_{\mathbb{K}} = U_0 \cup \{(0:1)\}$, so by (a) and (b) it is a compact space obtained from $\mathbb{K}^n$ by adding a single point. We can therefore think of $\mathbb{P}^1_{\mathbb{K}}$ as the one-point compactification of $\mathbb{K}$, adding a "point at infinity". This makes $\mathbb{P}^1_{\mathbb{R}}$ homeomorphic to $\mathbb{R} \cup \{\infty\} \cong S^1$ and $\mathbb{P}^1_{\mathbb{C}}$ homeomorphic to $\mathbb{C} \cup \{\infty\} \cong S^2$. It can be shown that $\mathbb{P}^2_{\mathbb{R}}$ is just the real projective plane of Example 1.17 (d) [G5, Remark 3.6 (a)]; the other projective spaces for higher values of n are new spaces that we have not met so far in this course. They do not come with a natural embedding in a real vector space, but will be manifolds in the sense of Example 10.17 (a). To compute their homology, they could be written as topological realizations of simplicial complexes, but this is not the easiest way to proceed. Instead, let us now use singular Mayer-Vietoris for this purpose.

Example 8.4 (Homology of complex projective spaces)**.** For projective spaces, the computation of their singular homology is actually easier in the complex case than in the real case: Let us show inductively that

$$b_p(\mathbb{P}^n_{\mathbb{C}}) = \begin{cases} 1 & \text{for } p = 0,2,4,\dots,2n, \\ 0 & \text{otherwise} \end{cases}$$

for all $n \in \mathbb{N}$, with the case $n = 0$ being trivial since $\mathbb{P}^0_{\mathbb{C}}$ is just a point. For the other cases, we will use Mayer-Vietoris for the open subsets

$$U = \{(x_0 : \dots : x_n) \in \mathbb{P}^n_{\mathbb{C}} : x_0 \neq 0\} \quad \text{and} \quad V = \{(x_0 : \dots : x_n) \in \mathbb{P}^n_{\mathbb{C}} : (x_1,\dots,x_n) \neq (0,\dots,0)\},$$

which clearly cover $\mathbb{P}^n_{\mathbb{C}}$. Note first that U is just $\mathbb{C}^n$ by Construction 8.3 (a). In particular, $U \cap V$ is the complement of the origin in $\mathbb{C}^n = \mathbb{R}^{2n}$, which deformation retracts to the sphere S^{2n-1}. Finally, by moving the 0-th coordinate to zero V deformation retracts to

$$\{(0:x_1:\cdots:x_n) \in \mathbb{P}^n_{\mathbb{C}} : (x_1,\dots,x_n) \neq (0,\dots,0)\} \cong \mathbb{P}^{n-1}_{\mathbb{C}},$$

for which we can assume the homology to be known by induction. In particular, all odd homology of U and V vanishes, so the sequence of Corollary 7.10 splits into the parts

$$0 \longrightarrow H_{2k+1}(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow H_{2k}(S^{2n-1}) \longrightarrow H_{2k}(\mathbb{C}^n) \times H_{2k}(\mathbb{P}^{n-1}_{\mathbb{C}}) \longrightarrow H_{2k}(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow H_{2k-1}(S^{2n-1}) \longrightarrow 0$$

for all $k \in \mathbb{N}$. From this we can get all the information we need:

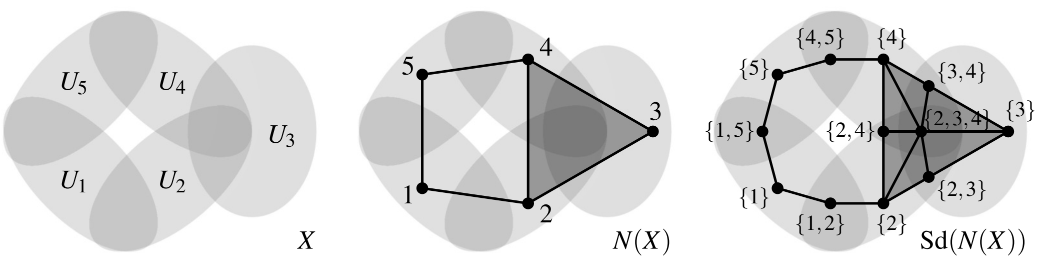
- For $k = 0$, the map $H_{2k}(S^{2n-1}) \to H_{2k}(\mathbb{C}^n) \times H_{2k}(\mathbb{P}^{n-1}_{\mathbb{C}})$ is injective since S^{2n-1} is path-connected. Hence, we must have $b_1(\mathbb{P}^n_{\mathbb{C}}) = 0$, and thus $b_0(\mathbb{P}^n_{\mathbb{C}}) = 1$ by Remark 4.5 (b).
- For $k > 0$ we have $H_{2k}(S^{2n-1}) = 0$ by Example 8.2 (b), so again looking at the beginning of our exact sequence we see that in fact all odd homology of $\mathbb{P}^n_{\mathbb{C}}$ vanishes. Looking at the end, we obtain the exact sequence
$$0 \longrightarrow H_{2k}(\mathbb{P}^{n-1}_{\mathbb{C}}) \longrightarrow H_{2k}(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow H_{2k-1}(S^{2n-1}) \longrightarrow 0.$$
Now for $k < n$ the first term has Betti number 1 by induction, for $k = n$ the last term has Betti number 1 by Example 8.2 (b), and all other outer Betti numbers are 0.

As we have claimed, this yields $b_p(\mathbb{P}^n_{\mathbb{C}}) = 1$ for $p = 0,2,4,\dots,2n$, and $b_p(\mathbb{P}^n_{\mathbb{C}}) = 0$ in all other cases. 12

Having seen how the simplicial homology of a simplicial complex can be reinterpreted as singular homology, let us now go the other way around: start with the singular homology of an arbitrary topological space and see if we can express it in terms of simplicial homology. This would e. g. be very useful for calculations, as simplicial homology is algorithmically computable. However, due to the notion of a topological space being extremely general, it should be clear that this will not work without some assumptions. Let us start with an example showing the idea of these assumptions.

Example 8.5. Let X be a topological space with a given finite open cover $(U_i)_{i \in I}$ such that all these open subsets, as well as all their possible intersections, are contractible. An example of this situation is shown in the picture below on the left, where $X = U_1 \cup \cdots \cup U_5 \subset \mathbb{R}^2$ is a union of ellipses. As in the picture, note that the contractibility assumption is clearly always satisfied if all U_i are convex sets in some vector space, since convex sets are contractible and intersections of convex sets are again convex.

In this situation, it should be geometrically clear that the homology of X – e. g. the hole in X in the picture below – does not come from the individual subsets U_i themselves, but only from the way they are glued together globally. But this information can be encoded neatly in a simplicial complex: Let us consider the simplicial complex $N(X)$ (shown in the middle picture below) whose vertex set is I and in which a subset $\{i_0,\dots,i_p\} \subset I$ forms a simplex if the intersection $U_{i_0} \cap \cdots \cap U_{i_p}$ is non-empty. It is called the *nerve* of X, and in the picture we can see that it can be viewed as some kind of simplicial approximation of X, recovering e. g. the hole in X.



We want to show now that this is a general statement: Under the given assumptions, the simplicial homology of the nerve $N(X)$ is always isomorphic to the singular homology of X. To prove this, we will first need a map in one direction, and the easiest way to set this up is to subdivide the nerve as in the picture above on the right. We can then place each vertex $\{i_0,\dots,i_p\} \in V(\mathrm{Sd}(N(X))) = N(X)$ at any point in the intersection $U_{i_0} \cap \dots \cap U_{i_p}$, and fill in the higher-dimensional simplices by the contractibility assumption. Let us now carry out this construction rigorously.

Definition 8.6 (Nerves)**.** Let X be a topological space with a fixed finite open cover $(U_i)_{i\in I}$. For any non-empty $\alpha = \{i_0,\dots,i_p\} \subset I$ we set $U_\alpha := U_{i_0} \cap \dots \cap U_{i_p}$ and assume that all non-empty U_α are contractible – we will call this a **contractible open cover**. Then the abstract simplicial complex

$$N(X) := \{\alpha \subset I : \alpha \neq \emptyset \text{ and } U_\alpha \neq \emptyset\}$$

is called the **nerve** of X (with respect to the given open cover).

Construction 8.7 (Mapping a nerve to its topological space)**.** Let $N(X)$ be the nerve of a topological space with respect to some finite contractible open cover $(U_i)_{i\in I}$ as in Definition 8.6. We want to construct a corresponding continuous map $f\colon |N(X)| \to X$. To do so, we can replace $|N(X)|$ by $|\mathrm{Sd}(N(X))|$ since these two spaces are homeomorphic by Proposition 1.21. We proceed by induction on the dimension of the simplices in $\mathrm{Sd}(N(X))$, mapping the geometric realization of every p-simplex $\{\alpha_0,\dots,\alpha_p\} \in \mathrm{Sd}(N(X))$ with $\alpha_0 \subsetneq \dots \subsetneq \alpha_p \in N(X)$ to $U_{\alpha_0} \subset X$:

(a) For $p = 0$, we pick an arbitrary point in U_{α_0} for the image of f on the geometric 0-simplex $|\{\alpha_0\}|$ (which is just a point).

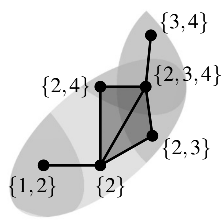
(b) For $p > 0$, by induction we have already constructed f on the boundary of $|\{\alpha_0,\dots,\alpha_p\}|$ (which is homeomorphic to S^{p-1}), with image in U_{α_0}. But U_{α_0} is contractible by assumption, so every continuous map to U_{α_0} from a sphere S^{p-1} can be extended to a continuous map from the disk D^p by contracting to the center. Hence, we can extend f to all of $|\{\alpha_0,\dots,\alpha_p\}|$, again with image in U_{α_0}; we do so in an arbitrary way.

Actually, part (b) of the construction simplifies if all U_i are convex sets in some vector space: In this case we can always pick the unique affine-linear extension; the convexity assumption then ensures that this extension really lies in U_{α_0}. The map f will then be affine-linear on each simplex of $\mathrm{Sd}(N(X))$, as e. g. in the right picture in Example 8.5. Note however that, in contrast to that picture, the map f need not be a geometric realization of $\mathrm{Sd}(N(X))$ in general (see e. g. Remark 8.9 (b)).

Proposition 8.8 (**Nerve Theorem**)**.** *Let X be a topological space with a finite contractible open cover as in Definition 8.6. Then the simplicial homology of the nerve $N(X)$ is isomorphic to the singular homology of X (with an isomorphism given by combining the topological realization morphism of Construction 6.11 with the pushforward by the map $|N(X)| \to X$ of Construction 8.7).*

Proof. For a fixed contractible open cover $(U_i)_{i\in I}$ of a given space X, we will prove the statement by a Mayer-Vietoris type induction argument on subspaces of X.

Namely, for all subsets $A = \{\alpha_1,\dots,\alpha_n\} \subset N(X)$ with $n \in \mathbb{N}$ we set, as shown in the picture for $A = \{\{2\},\{3,4\}\}$ in Example 8.5:



- Let $X(A) \subset X$ be the union of all U_α with $\alpha \in A$. Equivalently, this is the union of all U_β with $\alpha \subset \beta \in N(X)$ for some $\alpha \in A$.
- Let $S(A) \subset \mathrm{Sd}(N(X))$ be the full subcomplex with vertices given by all β with $\alpha \subset \beta \in N(X)$ for some $\alpha \in A$.

We will show by induction on n that the topological realization morphism of Construction 6.11 with the pushforward of Construction 8.7 induces isomorphisms $H_p(S(A)) \to H_p(|S(A)|) \to H_p(X(A))$ for all $p \in \mathbb{Z}$ and A as above. As $X(A) = X$ and $S(A) = \mathrm{Sd}(N(X))$ for $A = \{\{i\} : i \in I\}$, this shows that $H_p(\mathrm{Sd}(N(X))) \cong H_p(X)$ in this way, thus proving the proposition since $H_p(N(X)) \cong H_p(\mathrm{Sd}(N(X)))$ by Proposition 4.22.

The induction start for $n = 1$ is easy: For $A = \{\alpha\}$ we know that $X(A) = U_\alpha$ is contractible by assumption, and the simplicial complex $S(A)$ is star-shaped with respect to α. Both homologies are therefore trivial (see Example 6.14 and Proposition 3.11).

For the induction step let $A = \{\alpha_1, \dots, \alpha_n\}$ be a set with $n > 1$ elements. Note that by construction we have for $A_1 := \{\alpha_1, \dots, \alpha_{n-1}\}$ and $A_2 := \{\alpha_1 \cup \alpha_n, \dots, \alpha_{n-1} \cup \alpha_n\}$

$$X(A_1) \cup X(\{\alpha_n\}) = X(A) \quad \text{and} \quad S(A_1) \cup S(\{\alpha_n\}) = S(A),$$

as well as

$$X(A_1) \cap X(\{\alpha_n\}) = X(A_2) \quad \text{and} \quad S(A_1) \cap S(\{\alpha_n\}) = S(A_2).$$

The simplicial and singular Mayer-Vietoris sequences of Example 4.7 and Construction 7.1 thus give a morphism of short exact sequences of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & C^{\text{ord}}(S(A_2)) \arrow[r] \arrow[d] & C^{\text{ord}}(S(A_1)) \times C^{\text{ord}}(S(\{\alpha_n\})) \arrow[r] \arrow[d] & C^{\text{ord}}(S(A)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C(X(A_2)) \arrow[r] & C(X(A_1)) \times C(X(\{\alpha_n\})) \arrow[r] & C(X(A_1) \,|\, X(\{\alpha_n\})) \arrow[r] & 0,
\end{tikzcd}$$

with the vertical maps given by topological realization and Construction 8.7. The proof now continues in the same way as in Proposition 8.1: With Proposition 4.25 and Corollary 7.9, this diagram gives rise by Construction 4.10 to a commutative homology diagram with exact rows

$$\begin{tikzcd}
H_p(S(A_2)) \arrow[r] \arrow[d] & H_p(S(A_1)) \times H_p(S(\{\alpha_n\})) \arrow[r] \arrow[d] & H_p(S(A)) \arrow[r] \arrow[d] & H_{p-1}(S(A_2)) \arrow[r] \arrow[d] & H_{p-1}(S(A_1)) \times H_{p-1}(S(\{\alpha_n\})) \arrow[d] \\
H_p(X(A_2)) \arrow[r] & H_p(X(A_1)) \times H_p(X(\{\alpha_n\})) \arrow[r] & H_p(X(A)) \arrow[r] & H_{p-1}(X(A_2)) \arrow[r] & H_{p-1}(X(A_1)) \times H_{p-1}(X(\{\alpha_n\})).
\end{tikzcd}$$

As the outer four columns are isomorphisms by induction, the isomorphism $H_p(S(A)) \cong H_p(X(A))$ thus follows from the 5-Lemma 4.19. □

Remark 8.9.

(a) As we have already remarked in Example 8.5 and Construction 8.7, a common case of the Nerve Theorem occurs if a topological space X is a union of finitely many convex open sets in a real vector space: In this case, the contractibility condition is automatic, and the map from $|N(X)| \cong |\text{Sd}(N(X))|$ to X inducing an isomorphism in homology can be obtained by choosing arbitrary points for the vertices of $\text{Sd}(N(X))$ in their respective intersections and affine-linear interpolation.

This situation occurs e. g. in topological data analysis in Remark 9.19 (a).

(b) Under some mild additional assumptions, it can be shown that the map of Construction 8.7 does not only preserve homology, but is actually also a homotopy equivalence [H, Corollary 4G.3]. Note however that it is clearly not a homeomorphism, as we can already see from Example 8.5. In fact, the nerve $N(X)$ can easily acquire higher dimension as expected or desired: If we add another open subset U_6 to Example 8.5 which is just equal to U_3, this would not change the topological space X at all, but it would make the nerve 3-dimensional due to the intersection $U_2 \cap U_3 \cap U_4 \cap U_6$ now being non-empty – although X was given as a subset of the plane.

To finish this chapter, let us quickly also introduce singular cohomology and compare it to the simplicial setting. As you would probably expect from Chapter 5, not much is happening here as we are just talking about dualizing all our vector spaces. We have already discussed the geometric meaning of cohomology when discussing its simplicial version, so let us restrict ourselves now to its construction and properties.

Construction 8.10 (Singular cohomology)**.** Let X be a topological space. Analogously to Construction 5.3, the **singular cochain complex** of X is defined to be the dual cochain complex $C^\vee(X)$ of the singular chain complex $C(X)$ of Remark 6.7 (b) (see Construction 5.2), consisting of the **singular cochain** groups $C^p(X) = (C_p(X))^\vee$ and **singular coboundary maps** $d_p \colon C^p(X) \to C^{p+1}(X)$ of X. Elements of $C^p(X)$ are called **singular p-cocycles** or **singular p-coboundaries** of X if they are in the kernel or image of the singular coboundary map, respectively.

As in Construction 5.6, assigning to a topological space X its singular cochain complex $C^\vee(X)$ is a contravariant functor, with the morphism $C^\vee(Y) \to C^\vee(X)$ associated to a continuous map $f\colon X \to Y$ called the **pullback** f^*.

Applying the cohomology functor to $C^\vee(X)$ as in Construction 5.7, we obtain the **singular cohomology** $H^p(X,K) := H_p(X) = \ker d_p / \operatorname{im} d_{p-1}$ of X for all $p \in \mathbb{Z}$. In particular, the pullback maps for a continuous map $f\colon X \to Y$ pass to cohomology $f^*\colon H^p(Y) \to H^p(X)$.

By Proposition 5.14, homology commutes with dualization (also for infinite-dimensional spaces, which is important here!), so that we have natural isomorphisms $H^p(X) \cong (H_p(X))^\vee$ – telling us that singular cohomology does not contain any new information compared to homology, and that the dimensions of $H^p(X)$ are just the usual Betti numbers $b_p(X)$. As for our results on singular homology, this means:

(a) Homotopic continuous maps induce with their pullbacks the same map on cohomology (Proposition 6.12), and homotopy equivalent topological spaces have isomorphic singular cohomology (Corollary 6.13).

(b) For any two open subsets U and V of X there is a long exact Mayer-Vietoris sequence

$$\cdots \longrightarrow H^p(U \cup V) \longrightarrow H^p(U) \times H^p(V) \longrightarrow H^p(U \cap V) \longrightarrow H^{p+1}(U \cup V) \longrightarrow \cdots$$

obtained by dualizing Corollary 7.10.

(c) For any simplicial complex S, the simplicial cohomology $H^p(S)$ is naturally isomorphic to the singular homology $H^p(|S|)$ for all $p \in \mathbb{Z}$ by Proposition 8.1. For any contractible finite open cover of a topological space X, the Nerve Theorem of Proposition 8.8 implies that the simplicial cohomology $H^p(N(X))$ is naturally isomorphic to the singular cohomology $H^p(X)$.