7. The Mayer-Vietoris Sequence in Singular Homology

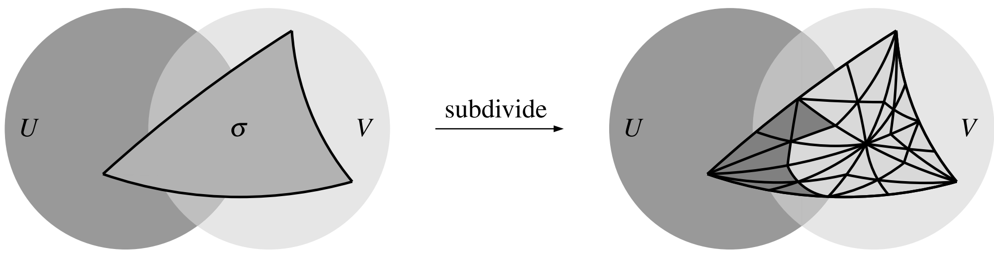
In simplicial geometry, one of the main tools both for computing and for proving general statements on homology was clearly the Mayer-Vietoris sequence of Corollary 4.12. We therefore want to investigate now if a corresponding result holds in singular homology as well, i. e. if for two subspaces U and V of a topological space we have a long exact sequence

$$\cdots \longrightarrow H_p(U\cap V) \longrightarrow H_p(U)\times H_p(V) \longrightarrow H_p(U\cup V) \longrightarrow H_{p-1}(U\cap V) \longrightarrow \cdots. \qquad (1)$$

As in simplicial geometry, we would expect this sequence to come from a short exact sequence of singular chain complexes

$$0 \longrightarrow C(U\cap V) \longrightarrow C(U)\times C(V) \longrightarrow C(U\cup V) \longrightarrow 0, \qquad (2)$$

with morphisms constructed from the pushforwards by the various inclusions. However, there is a big difference compared to the simplicial setting here: Whereas an abstract simplex in the union of two simplicial complexes must always lie in one of these simplicial complexes, the corresponding statement is no longer true in the singular setting as there will be singular simplices in $U\cup V$ that do not lie in U or V alone, such as σ in the picture below on the left. This means that the last map $C(U)\times C(V)\to C(U\cup V)$ in (2) is no longer surjective. In other words, the sequence (2) fails to be exact, so that we cannot use it to derive the long exact sequence (1) from it.



Fortunately, it turns out however that the long exact sequence (1) still exists – we just have to adapt our proof. The idea is that it should be possible – at least for open subsets U and V that overlap a bit – to subdivide σ fine enough so that each singular simplex in the subdivision lies in U or V, restoring the exactness of (2). We have shown this situation on the right side of the picture, where we took a double barycentric subdivision of σ and shaded the singular subsimplices of σ in dark or light color depending on whether they lie in U or V.

It is the main goal of this chapter to prove these statements. Let us start by formalizing the observation that (2) would be exact – and hence we could derive the long exact sequence (1) from it – if we could restrict our attention to singular simplices in $U\cup V$ that lie completely in U or V.

Construction 7.1 (Singular chains on two open subsets)**.** Let U and V be subsets of a topological space X. For all $p\in\mathbb{Z}$ we denote by $C_p(U\,|\,V)$ the vector subspace generated by all singular simplices $\sigma\colon |\alpha|\to U\cup V$ whose image lies completely in U or in V. Note that the singular boundary map sends $C_p(U\,|\,V)$ to $C_{p-1}(U\,|\,V)$, and hence we obtain a chain complex $C(U\,|\,V)$ whose homology groups we denote by $H_p(U\,|\,V)$. By construction, the sequence of chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(U\cap V) & \longrightarrow & C(U)\times C(V) & \longrightarrow & C(U\,|\,V) & \longrightarrow & 0 \\ & & c & \longmapsto & (c,-c) & & & & \\ & & & & (c,c') & \longmapsto & c+c' & & \end{array}$$

is then exact just as in Example 4.7, so that we obtain a long exact homology sequence

$$\cdots \longrightarrow H_p(U\cap V) \longrightarrow H_p(U)\times H_p(V) \longrightarrow H_p(U\,|\,V) \longrightarrow H_{p-1}(U\cap V) \longrightarrow \cdots$$

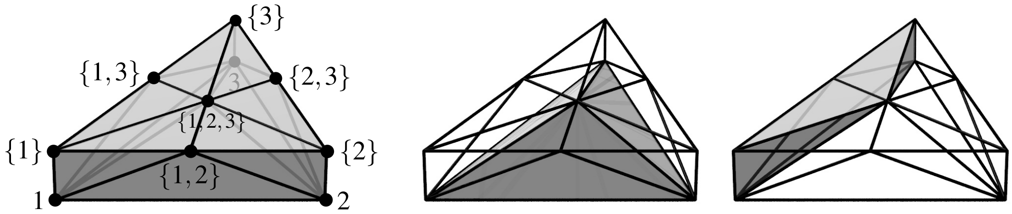
by Proposition 4.8. Hence, to obtain the desired result we only have to show that the inclusion morphism $C(U\,|\,V)\to C(U\cup V)$ induces an isomorphism in homology $H_p(U\,|\,V)\to H_p(U\cup V)$ for all $p\in\mathbb{Z}$ if U and V are open. As explained above, it is the main task of this chapter to prove this, with an inverse morphism given by sufficiently fine subdivisions.

To study these singular subdivisions and show that they give rise to an isomorphism in homology, let us first go back to simplicial geometry and construct a homotopy between a simplicial complex and its barycentric subdivision. For this we need an alternative version of the prism of Construction 3.6.

Construction 7.2 (Subdivision prism construction)**.** Let S be a simplicial complex. We want to construct a simplicial complex $Q(S)$ realizing the prism $|S|\times[0,1]$ by taking S for the bottom face and its barycentric subdivision $\mathrm{Sd}(S)$ for the top face. To do so, we connect every vertex $\alpha\in V(\mathrm{Sd}(S))$ to all subsets of α (considered as vertices of $\mathrm{Sd}(S)$ in the top face) and all elements of α (considered as vertices of S in the bottom face), i. e. we set

$$Q(S) := \big\{\{i_0,\dots,i_{k-1},\alpha_k,\dots,\alpha_p\} : k\le p \text{ and } \{i_0,\dots,i_{k-1}\}\subset\alpha_k\subset\cdots\subset\alpha_p\in S\big\}.$$

The following picture on the left shows this construction for $S=D^{\{1,2,3\}}$.



As in our original prism construction, we do not need an exact proof that $Q(S)$ realizes $|S|\times[0,1]$ as we will just define the corresponding homotopy algebraically and thus also give an algebraic proof that it works. For this we will proceed by induction on the dimension rather than writing down an explicit formula as in Definition 3.7 (b) because $Q(S)$ is a little more complicated than $P(S)$ in Construction 3.6. First define a "connecting to α map" b_α for any oriented simplex of $Q(D^\alpha)$ by

$$b_\alpha[i_0,\dots,i_{k-1},\alpha_k,\dots,\alpha_p] := [\alpha,i_0,\dots,i_{k-1},\alpha_k,\dots,\alpha_p],$$

extended to a linear map $b_\alpha\colon C_p(Q(D^\alpha))\to C_{p+1}(Q(D^\alpha))$ for all p. Note that then

$$\partial b_\alpha[i_0,\dots,i_{k-1},\alpha_k,\dots,\alpha_p] = [i_0,\dots,i_{k-1},\alpha_k,\dots,\alpha_p] - b_\alpha(\partial[i_0,\dots,i_{k-1},\alpha_k,\dots,\alpha_p])$$

by Definition 2.5 (a), so we have

$$\partial b_\alpha + b_\alpha\partial = \mathrm{id} \tag{1}$$

on $C(Q(D^\alpha))$. We can use this to define the *subdivision map* $\mathrm{sd}\colon C(S)\to C(\mathrm{Sd}(S))$ (that intuitively sends a simplex at the bottom of $Q(S)$ to its subdivision at the top) inductively by $\mathrm{sd}([i]) := [\{i\}]$ for a 0-simplex, and

$$\mathrm{sd}(\alpha) := b_\alpha(\mathrm{sd}(\partial\alpha)) \tag{2}$$

otherwise: To construct the subdivision of α, we subdivide its boundary and connect every simplex in this subdivision of $\partial\alpha$ to the barycenter. Let us also check inductively that sd is a morphism of chain complexes, i. e. commutes with the boundary map: This is obvious in dimension 0, and for $\alpha\in C_p(S)$ with $p>0$ we have

$$\partial\,\mathrm{sd}(\alpha) \overset{(2)}{=} \partial b_\alpha(\mathrm{sd}(\partial\alpha)) \overset{(1)}{=} \mathrm{sd}(\partial\alpha) - b_\alpha(\partial\,\mathrm{sd}(\partial\alpha)) \overset{\text{ind.}}{=} \mathrm{sd}(\partial\alpha) - b_\alpha(\mathrm{sd}(\partial^2\alpha)) = \mathrm{sd}(\partial\alpha). \tag{3}$$

Finally, we have to construct our homotopy $h\colon C_p(S)\to C_{p+1}(Q(S))$ that assigns to every simplex $\alpha\in C(S)$ the prism over it. If you believe in the left picture above you may want to directly accept the following Lemma 7.3 – which is just the algebraic version of the picture – without going through the following construction and computations. Otherwise, as shown in the other pictures above in the case $\alpha = \{1,2,3\}$, this prism can be assembled recursively from two components: a pyramid over α with apex the vertex $\alpha \in V(\mathrm{Sd}(S))$ (as in the middle picture), and for every face of α the prism over it, connected again to $\alpha \in V(\mathrm{Sd}(S))$ (as in the right picture). In formulas (and getting the signs right) this means that we set $h([i]) := [i,\{i\}]$ in dimension 0, and

$$h(\alpha) := -b_\alpha(\alpha + h(\partial\alpha)) \tag{4}$$

otherwise. Indeed, we can then check that h is a homotopy resp. prism map between the top and bottom face, i.e. that $\partial h(\alpha) + h(\partial\alpha) = \mathrm{sd}(\alpha) - \alpha$ for all $\alpha \in C_p(S)$: Again this is obvious for $p = 0$, and for $p > 0$ we have

$$\begin{aligned}
\partial h(\alpha) + h(\partial\alpha) &\overset{(4)}{=} -\partial b_\alpha(\alpha + h(\partial\alpha)) + h(\partial\alpha) \\
&\overset{(1)}{=} -\alpha - h(\partial\alpha) + b_\alpha(\partial\alpha + \partial h(\partial\alpha)) + h(\partial\alpha) \\
&= -\alpha + b_\alpha(\partial\alpha + \mathrm{sd}(\partial\alpha) - \partial\alpha - h(\partial^2\alpha)) \qquad \text{(induction applied to } \partial h(\partial\alpha)) \\
&\overset{(2)}{=} \mathrm{sd}(\alpha) - \alpha.
\end{aligned}$$

Summarizing, we have thus shown the following statement analogous to Lemma 3.8:

Lemma 7.3 (Subdivision homotopy)**.** *Let S be a simplicial complex. With notations as in Construction 7.2, the prism map h is a homotopy between the top and bottom face maps of $Q(S)$, i.e. we have*

$$\partial \circ h + h \circ \partial = g - f$$

as morphisms $C(S) \to C(Q(S))$ of chain complexes, where $f\colon C(S) \to C(Q(S))$, $\alpha \mapsto \alpha$ is the push-forward by the bottom face inclusion and $g\colon C(S) \to C(Q(S))$, $\alpha \mapsto \mathrm{sd}(\alpha)$ subdivides a simplex and embeds it in the top face.

As in Proposition 6.12, let us now push this homotopy to the singular setting by the topological realization morphism, giving a homotopy between singular simplices and their barycentric subdivisions. In fact, the proof of Proposition 7.5 below is almost identical to that of Proposition 6.12, just using our new prism construction and forgetting about the given topological homotopy.

Construction 7.4 (Barycentric subdivision of singular simplices)**.** Let $\sigma\colon |\alpha| \to X$ be a singular simplex in a topological space X, with α an ordered simplex as in Construction 6.4.

(a) As in Construction 6.5 (a), for every ordered simplex β in the barycentric subdivision of α we have $r(\beta) \subset |\alpha|$ for the barycentric realization r, and set $\sigma|_\beta := \sigma|_{r(\beta)}\colon r(\beta) \to X$. We extend this by linearity to a restriction map $C_p^{\mathrm{ord}}(\mathrm{Sd}(D^\alpha)) \to C_p(X)$, $c \mapsto \sigma|_c$ for all $p \in \mathbb{Z}$.

(b) We define the **barycentric subdivision** of σ to be $\mathrm{sd}(\sigma) := \sigma|_{\mathrm{sd}(\alpha)}$ as in (a), with $\mathrm{sd}(\alpha)$ defined in Construction 7.2. Again, by linearity we extend this to a singular *subdivision map* $\mathrm{sd}\colon C_p(X) \to C_p(X)$ for all $p \in \mathbb{Z}$. Note that

$$\partial\,\mathrm{sd}(\sigma) \overset{6.5\,(b)}{=} \sigma|_{\partial\,\mathrm{sd}(\alpha)} \overset{7.2}{=} \sigma|_{\mathrm{sd}(\partial\alpha)} = \mathrm{sd}(\partial\sigma),$$

so $\mathrm{sd}\colon C(X) \to C(X)$ is a morphism of chain complexes, and thus induces maps on homology $H_p(X) \to H_p(X)$ by Lemma 2.17.

Proposition 7.5 (Singular subdivisions)**.** *For any topological space X, the subdivision morphism $\mathrm{sd}\colon C(X) \to C(X)$ of Construction 7.4 (b) is chain homotopic to the identity. In particular, by Proposition 3.3 the corresponding maps $\mathrm{sd}\colon H_p(X) \to H_p(X)$ on homology are the identity for all $p \in \mathbb{Z}$.*

Proof. To construct a chain homotopy between sd and the identity, consider first for a fixed singular simplex $\sigma\colon |\alpha| \to X$ and $p \in \mathbb{Z}$ the composed linear map

$$h_\sigma\colon C_p^{\mathrm{ord}}(D^\alpha) \xrightarrow{\tilde h} C_{p+1}^{\mathrm{ord}}(Q(D^\alpha)) \xrightarrow{\rho} C_{p+1}(|\alpha| \times [0,1]) \xrightarrow{R_*^\sigma} C_{p+1}(Y), \tag{1}$$

where $\tilde{h}$ is the ordered version of the simplicial homotopy of Construction 7.2, ρ is the topological realization morphism of the prism in $|\alpha| \times [0,1]$ as in Construction 6.11, and R_*^σ is the singular pushforward by the continuous map $R^\sigma \colon |\alpha| \times [0,1] \to X$, $(x,t) \mapsto \sigma(x)$ as in Construction 6.9. We set $h(\sigma) := h_\sigma(\alpha)$ and extend this by linearity to a linear map $h \colon C_p(X) \to C_{p+1}(X)$.

As in the proof of Proposition 6.12, if β is an ordered simplex with $\beta \subset \alpha$, to construct $h(\sigma|_\beta)$ we can still use σ in the last map of (1), i. e. we have $h(\sigma|_\beta) = h_\sigma(\beta)$. Applying this to the boundary faces of α, we thus see that

$$h(\partial\sigma) = h(\sigma|_{\partial\alpha}) = h_\sigma(\partial\alpha). \tag{2}$$

Now we know by (the ordered version of) Lemma 7.3 that

$$\partial \circ \tilde{h} + \tilde{h} \circ \partial = \tilde{g} - \tilde{f} \quad \text{as morphisms } C^{\text{ord}}(D^\alpha) \to C^{\text{ord}}(Q(D^\alpha)),$$

where $\tilde{f}$ is the bottom face inclusion and $\tilde{g}$ the top face subdivision map. Composing this with the morphisms ρ and R_*^σ thus gives

$$\partial \circ h_\sigma + h_\sigma \circ \partial = R_*^\sigma \circ \rho \circ \tilde{g} - R_*^\sigma \circ \rho \circ \tilde{f},$$

and applying this to α yields by (2) the desired equation

$$\partial h(\sigma) + h(\partial\sigma) = g(\sigma) - f(\sigma). \qquad \square$$

Remark 7.6. Assume that the topological space X in Proposition 7.5 is the union of two subspaces U and V. Then the subdivision map sd as well as all other maps involved in (the proof of) the proposition clearly map simplices in U or V again to simplices in U or V, respectively. Hence, literally the same proof shows that sd is also chain homotopic to the identity as morphisms of chain complexes $C(U\,|\,V) \to C(U\,|\,V)$, and thus induces the identity $H_p(U\,|\,V) \to H_p(U\,|\,V)$ for all $p \in \mathbb{Z}$ as well.

We have thus seen now that subdivisions of singular simplices do not affect homology. To carry out our idea to establish the singular Mayer-Vietoris sequence, it therefore only remains to show that sufficiently fine subdivisions of singular simplices in a union $U \cup V$ of two open subsets U and V of a topological space consist only of simplices in U or V. Again, let us start with a simplicial preparation proving the geometrically obvious fact that repeated barycentric subdivisions of a geometric simplex will lead to arbitrarily small subsimplices.

Lemma 7.7. *Let S be a simplicial complex. For any $\varepsilon \in \mathbb{R}_{>0}$ there is a number $n \in \mathbb{N}$ such that every edge of the n-fold barycentric subdivision $\mathrm{Sd}^n(S)$ has length smaller than ε (in its corresponding barycentric subdivision, and with respect to any fixed norm).*

Proof. For $n \in \mathbb{N}$ and any vertex $i \in V(\mathrm{Sd}^n(S))$ let us denote by $r(i) \in \mathbb{R}^{V(S)}$ the position of this vertex in its barycentric realization. Moreover, let

$$a_n := \max\{\|r(i_0) - r(i_1)\| : \{i_0, i_1\} \in \mathrm{Sd}^n(S)\}$$

the maximum length of an edge in $\mathrm{Sd}^n(S)$ in this realization. To see what happens to this number when we pass from $\mathrm{Sd}^n(S)$ to $\mathrm{Sd}^{n+1}(S)$, consider any edge between two vertices of $\mathrm{Sd}^{n+1}(S)$, which by Definition 1.20 we can write as $\{i_0, \dots, i_l\}$ and $\{i_0, \dots, i_p\}$ with $0 \le l < p$ for a p-simplex $\{i_0, \dots, i_p\} \in \mathrm{Sd}^n(S)$. By Proposition 1.21 we then have

$$r(\{i_0,\dots,i_p\}) - r(\{i_0,\dots,i_l\}) = \frac{r(i_0)+\cdots+r(i_p)}{p+1} - \frac{r(i_0)+\cdots+r(i_l)}{l+1} = \sum_{j=0}^{l} \sum_{k=l+1}^{p} \frac{r(i_k) - r(i_j)}{(l+1)(p+1)},$$

and thus by taking norms

$$\|r(\{i_0,\dots,i_p\}) - r(\{i_0,\dots,i_l\})\| \le \sum_{j=0}^{l} \sum_{k=l+1}^{p} \frac{a_n}{(l+1)(p+1)} = \frac{p-l}{p+1}\, a_n \le \frac{p}{p+1}\, a_n \le \frac{\dim S}{\dim S + 1}\, a_n.$$

Hence, we obtain

$$a_{n+1} \le \frac{\dim S}{\dim S+1}\, a_n, \quad \text{and thus by induction} \quad a_n \le \left(\frac{\dim S}{\dim S+1}\right)^n a_0.$$

11 As this bound tends to 0 for $n \to \infty$, the result follows. $\square$

Corollary 7.8. *Let $c \in C_p(U \cup V)$ be a singular p-chain for some $p \in \mathbb{N}$ in the union of two open subsets U and V of a topological space. Then for some $n \in \mathbb{N}$ we have $\operatorname{sd}^n(c) \in C_p(U\,|\,V)$, i. e. the n-fold barycentric subdivision of c contains only singular simplices lying entirely in U or V.*

Proof. It suffices to consider a single singular simplex $\sigma\colon |\alpha| \to U \cup V$, since c contains only finitely many of such simplices, and we can take the maximum n needed for one of them.

Note that then the geometric simplex $|\alpha|$ is compact and covered by the two open subsets $\sigma^{-1}(U)$ and $\sigma^{-1}(V)$. It is a result from topology that in this situation there is an $\varepsilon \in \mathbb{R}_{>0}$ (a so-called *Lebesgue number*) such that any open ε-ball in $|\alpha|$ (i. e. with any center point) lies entirely in $\sigma^{-1}(U)$ or $\sigma^{-1}(V)$ [G3, Lemma 8.10]. By Lemma 7.7 applied to D^α there is now a number $n \in \mathbb{N}$ such that the barycentric realization $r(\beta) \subset |\alpha|$ of any simplex β in the n-fold subdivision of α has all edge lengths smaller than ε. Hence $r(\beta)$ lies within a ball of radius less than ε, and must therefore be contained in $\sigma^{-1}(U)$ or $\sigma^{-1}(V)$; which means that $\sigma|_\beta$ lies in U or V. $\square$

Corollary 7.9. *Let U and V be open subsets of a topological space. Then for all $p \in \mathbb{Z}$ the map $H_p(U\,|\,V) \to H_p(U \cup V)$ in homology induced by the inclusion morphism of chain complexes $C(U\,|\,V) \to C(U \cup V)$ is an isomorphism.*

Proof. We will show that the map is injective and surjective.

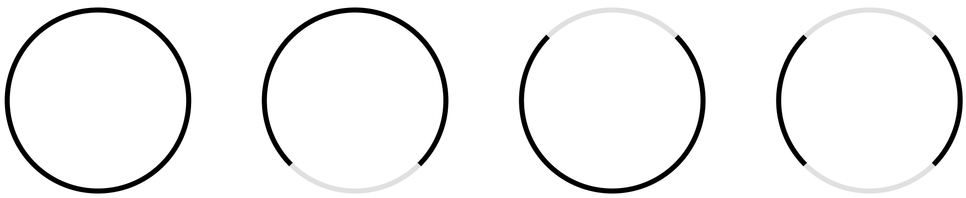
- Injectivity: Let $c \in H_p(U\,|\,V)$ with $c = 0 \in H_p(U \cup V)$, i. e. we have $c = \partial c'$ for a chain $c' \in C_{p+1}(U \cup V)$. By Corollary 7.8 there is some $n \in \mathbb{N}$ with $\operatorname{sd}^n(c') \in C_{p+1}(U\,|\,V)$. Then $\partial \operatorname{sd}^n(c') = \operatorname{sd}^n(\partial c') = \operatorname{sd}^n(c)$, so $\operatorname{sd}^n(c) = 0 \in H_p(U\,|\,V)$. By Remark 7.6 this means that $c = 0 \in H_p(U\,|\,V)$.
- Surjectivity: Let $c \in H_p(U \cup V)$, i. e. $\partial c = 0$. By Corollary 7.8 we have $\operatorname{sd}^n(c) \in H_p(U\,|\,V)$. It is mapped to $\operatorname{sd}^n(c) \in H_p(U \cup V)$, which agrees with c by Proposition 7.5. $\square$

Corollary 7.10 (The singular Mayer-Vietoris homology sequence)**.** *For any two open subsets U and V of a topological space, there is an exact **Mayer-Vietoris sequence***

$$\cdots \longrightarrow H_p(U \cap V) \longrightarrow H_p(U) \times H_p(V) \longrightarrow H_p(U \cup V) \longrightarrow H_{p-1}(U \cap V) \longrightarrow \cdots.$$

Proof. This follows immediately by combining Construction 7.1 with Corollary 7.9. $\square$

Example 7.11 (Homology of a circle)**.** As a simple concrete example, let us compute the singular homology of a circle S^1 by writing it as $S^1 = U \cup V$ as in the picture below, where U, V, and $U \cap V$ are drawn in black.



S^1 $\qquad U \qquad V \qquad U \cap V$

As U and V are contractible, their Betti numbers are

$$b_p(U) = b_p(V) = \begin{cases} 1 & \text{if } p = 0, \\ 0 & \text{if } p \neq 0 \end{cases}$$

by Example 6.14. Moreover, $U \cap V$ is a disjoint union of two two contractible spaces, hence by Remark 6.10 (a) we have $b_0(U \cap V) = 2$ and $b_p(U \cap V) = 0$ for all $p \neq 0$. The Mayer-Vietoris sequence of Corollary 7.10 thus splits for $p > 1$ as

$$\underbrace{H_p(U) \times H_p(V)}_{=0} \longrightarrow H_p(S^1) \longrightarrow \underbrace{H_{p-1}(U \cap V)}_{=0},$$

showing that in this case $b_p(S^1) = 0$, and in its last part as

$$\underbrace{H_1(U) \times H_1(V)}_{=0} \longrightarrow H_1(S^1) \longrightarrow \underbrace{H_0(U \cap V)}_{\cong K^2} \longrightarrow \underbrace{H_0(U) \times H_0(V)}_{\cong K^2} \longrightarrow H_0(S^1) \longrightarrow 0.$$

But S^1 is path-connected, so we have $b_0(S^1) = 1$ by Remark 6.10 (b), and thus it also follows from this sequence that $b_1(S^1) = 1$ by Remark 4.5 (b).

Note that this is clearly what we would have expected from geometry: firstly, since S^1 is just the topological realization of the simplicial 1-sphere $S^{\{1,2,3\}}$, which has the same Betti numbers by Example 2.26; and secondly, since S^1 has just one "one-dimensional hole".

Remark 7.12. Due to the need for subdivisions to pass from $C(U\,|\,V)$ to $C(U \cup V)$ in Construction 7.1, the connecting map $H_p(U \cup V) \to H_{p-1}(U \cap V)$ for $p \in \mathbb{Z}$ in the singular Mayer-Vietoris homology sequence of Corollary 7.10 is a little more complicated than in the simplicial case: As in (1) in the introduction to Chapter 4, the first step in this map is to write a closed chain $c \in H_p(U \cup V)$ as a sum $c = c_1 + c_2$ of two chains in U and V, respectively – but for singular chains, we will in general need to apply a suitable subdivision first to make this splitting possible.

Clearly, we could go on now and use the Mayer-Vietoris sequence of Corollary 7.10 to compute the singular homology of many other topological spaces. For example, it would be just as easy to determine the singular homology of the other spheres S^n in the same way as well – to obtain the same result as in the simplicial case in Example 3.13. However, our very next step will be to prove that the simplicial homology of a simplicial complex always agrees with the singular homology of its topological realization anyway. So let us not waste any time now with concrete examples and go straight to the proof of this general statement in the next chapter, which will then immediately give us the singular homology of most spaces we are interested in.

Exercise 7.13 (Relative homology)**.** For any subset Y of a topological space X and $p \in \mathbb{N}$, let $C_p(X;Y) := C_p(X)/C_p(Y)$. The boundary maps of $C(X)$ descend to these quotients and make $C(X;Y)$ into a chain complex. The homology groups of this complex are called the *relative homology groups* of X and Y and denoted by $H_p(X;Y)$. Of course, by definition there is then a short exact sequence of chain complexes

$$0 \longrightarrow C(Y) \longrightarrow C(X) \longrightarrow C(X;Y) \longrightarrow 0,$$

which induces a corresponding long exact sequence in homology.

Now let $A \subset Y \subset X$ such that A is closed and Y is open in X. Show that the inclusion maps induce isomorphisms on relative homology groups $H_p(X\backslash A; Y\backslash A) \to H_p(X;Y)$.

(Hint: It is useful to express $C(X\backslash A; Y\backslash A)$ in terms of the complex $C(U\,|\,V)$ for suitable U and V.)

Exercise 7.14 (Local homology)**.** Let X be a topological space and $x \in X$ such that $\{x\} \subset X$ is closed.

(a) Show that the relative homology groups $H_p(U; U\backslash\{x\})$ of Exercise 7.13, where U is any open neighborhood of x, are independent of U. They are called the *local homology groups* of X at x.

(b) Compute these local homology groups for the half-space $X = \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_n \geq 0\}$ with $n \in \mathbb{N}_{>0}$, and any $x \in X$.