Singular Homology

6. The Construction of Singular Homology

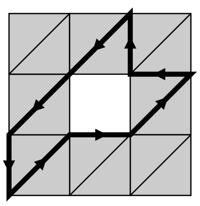
In the last chapters we have studied simplicial complexes in detail from a topological point of view. It is certainly not obvious however that the properties of simplicial homology that we obtained are actually topological in the sense that two simplicial complexes with homeomorphic topological realizations have isomorphic homology groups. In fact, although this statement is true it would be very hard to prove it using simplicial methods since the combinatorics of such simplicial complexes can be completely different.

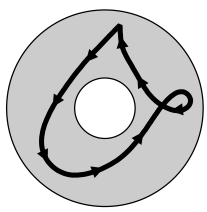
Instead, it turns out to be easier to prove this statement by constructing an entirely new homology theory called *singular homology* that is visibly topological, and then showing that singular homology actually coincides with simplicial homology when both are defined. This has the additional advantage that it also generalizes our old theory substantially since it is applicable to *all* topological spaces, and not just to topological spaces obtained from realizations of simplicial complexes. Most notably, this will then also construct homology groups of topological spaces that are not necessarily compact. It should be said clearly however that the spirit of singular homology is still the study of topological spaces that can be glued from nice pieces of $\mathbb{R}^n$ with the standard topology – so while you are able to apply singular homology to completely arbitrary topological spaces with weird topologies, this is not likely to give you anything useful.

It is the goal of the second part of these notes up to Chapter 9 to construct singular homology, show its properties, compare it to simplicial homology, and study several of its applications.

The idea to construct the singular homology of a topological space X is to replace abstract simplices by geometric simplices that are mapped continuously to X. For example, consider the "loop around an annulus" from the introduction to Chapter 2 whose simplicial version is shown again in the picture below on the left. In the singular setting in the right picture, the space X is now an actual annulus without a simplicial structure, and the 1-cycle in the picture is now a continuous map from an interval to X (of which we have only drawn its image in X). Note that in this 1-dimensional case this is the same approach as for the construction of the fundamental group [G3, Chapter 7]. In particular, the map to X really does not have to satisfy any other condition besides continuity, so it need not be injective or "smooth" in any way – which by the way is the reason for the name "singular homology". In contrast to the construction of the fundamental group however, as in the simplicial case the equivalence relation on singular 1-cycles will now be given by boundaries of 2-chains instead of homotopy, and we do not require the loops to be anchored at a given base point of X.

simplicial homology
(of a simplicial complex)





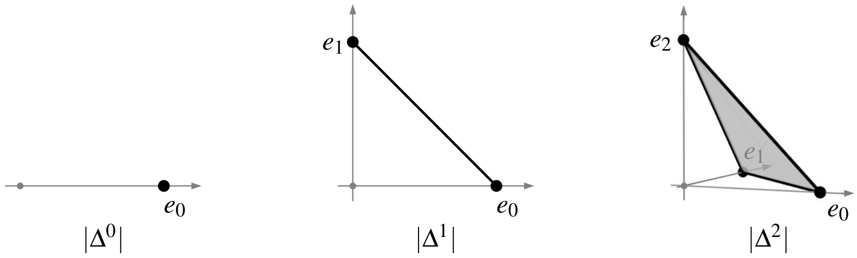
singular homology
(of a topological space)

To construct p-dimensional singular chains, we will of course just consider continuous maps from a geometric p-simplex to X. As these maps are arbitrary, we can choose a fixed p-simplex here in the same way as one considers continuous maps from the unit interval when constructing the fundamental group. With the background of Proposition 4.25, it is customary to work with ordered instead of oriented simplices here as they are simpler from an algebraic point of view. 09

Notation 6.1 (Standard simplices)**.** For $p \in \mathbb{N}$, we call $\Delta^p := (0,\dots,p)$ the **(ordered) standard p-simplex** with standard geometric realization

$$|\Delta^p| = \langle e_0,\dots,e_p\rangle = \{(x_0,\dots,x_p) : x_0,\dots,x_p \in \mathbb{R}_{\geq 0} \text{ with } x_0+\cdots+x_p = 1\} \quad \subset \mathbb{R}^{\{0,\dots,p\}}$$

as in Construction 1.12 (a).



Definition 6.2 (Singular simplices and chains)**.** Let X be a topological space, and let $p \in \mathbb{N}$.

(a) A **singular p-simplex** in X is a continuous map $\sigma\colon |\Delta^p| \to X$.

(b) We denote by $C_p(X)$ the vector space with basis given by all singular p-simplices in X. The elements of $C_p(X)$ are called **singular p-chains** in X.

As in the simplicial case, we set $C_p(X) = 0$ for $p \in \mathbb{Z}_{<0}$, and write $C_p(X,K)$ for $C_p(X)$ if we want to indicate the dependence on the ground field K.

Remark 6.3.

(a) To keep the notations concise, we have chosen to denote the groups $C_p(X)$ of singular chains of a topological space X by the same letter as the groups $C_p(S)$ of simplicial chains of a simplicial complex S in Definition 2.2. This should not lead to any confusion as the type of space in the brackets always indicates which chains are meant.

(b) Note that $C_p(X)$ is almost always highly infinite-dimensional, so it is next to impossible to do any explicit computations with it. In order to work with these spaces, it is therefore much more important than in the simplicial case to establish their general properties such as a homotopy invariance and the Mayer-Vietoris sequence. We will do this in the current and the next chapter.

(c) Although we will usually visualize a singular simplex $\sigma\colon |\Delta^p| \to X$ by drawing its image $\sigma(|\Delta^p|) \subset X$, it is important to realize that the singular simplex is actually the map itself and not just its image as a subset of X. In particular, a reparametrization of σ by precomposing it with a non-trivial homeomorphism of $|\Delta^p|$ will give rise to a different singular simplex (see however Exercise 6.17). As we are considering ordered instead of oriented simplices, this also applies if the homeomorphism is only a relabeling of the vertices. Also, note that even if $\sigma\colon |\Delta^p| \to X$ is a constant map it is a perfectly valid and non-trivial singular p-simplex that cannot be neglected in computations.

It is often useful however to allow singular simplices whose source is an ordered simplex other than Δ^p:

Construction 6.4 (Singular simplices with arbitrary domains)**.** Let X be a topological space, and let $\alpha = (i_0,\dots,i_p)$ be an abstract ordered p-simplex. Then we will consider any continuous map $\sigma\colon |\alpha| \to X$ with $|\alpha| := |\{i_0,\dots,i_p\}|$ as a singular p-simplex in X by precomposing it with the unique affine-linear map $|\Delta^p| \to |\alpha|$ that sends e_k to e_{i_k} for all $k = 0,\dots,p$. Note that this affine-linear map is a homeomorphism if all vertices of α are distinct. Otherwise, the dimension of the geometric simplex $|\alpha|$ is strictly less than p, but by precomposing σ with the map $|\Delta^p| \to |\alpha|$ we still get a singular p-simplex.

By Remark 6.3 (c), this construction depends on the ordering (and not just the orientation) of α. For simplicity, we will not indicate this dependence in the notation.

In fact, especially with our strong background in simplicial geometry that we have by now it is much more natural to not fix the domain of singular simplices to be $|\Delta^p|$, but rather allow continuous maps $\sigma\colon |\alpha| \to X$ for an arbitrary ordered simplex α as above. Consequently, in what follows we will always write singular simplices in this form, using the construction above. The only reason why we did not do so in the first place in Definition 6.2 is a very subtle set-theoretic one: Allowing maps $\sigma\colon |\alpha| \to X$ for an arbitrary ordered simplex α (hence in particular for an arbitrary underlying set for α) as a basis for $C_p(X)$ would require us to talk about the set of all sets, which unfortunately does not exist due to Russell's paradox [G1, Remark 1.13].

Having settled this issue, it is now completely straightforward to construct boundary maps for singular simplices, and thus singular homology as in Chapter 2:

Construction 6.5 (Boundary map for singular chains)**.** Let X be a topological space.

(a) For any singular simplex $\sigma\colon |\alpha| \to X$ as in Construction 6.4 and any ordered simplex β in D^α, i. e. with $\beta \subset \alpha$ as sets and thus $|\beta| \subset |\alpha|$, we set $\sigma|_\beta := \sigma|_{|\beta|}\colon |\beta| \to X$. (In our applications, β will always be obtained from α by leaving out some vertices and keeping the order of the remaining vertices, but we do not require this to be the case.) By linearity, we extend this construction to obtain a restriction map $C_p^{\text{ord}}(D^\alpha) \to C_p(X)$, $c \mapsto \sigma|_c$.

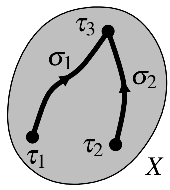
(b) For $p > 0$, the **boundary** of a singular p-simplex $\sigma\colon |\alpha| \to X$ for an ordered p-simplex $\alpha = (i_0, \dots, i_p)$ as in Construction 6.4 is the singular $(p-1)$-chain

$$\partial\sigma := \sigma|_{\partial\alpha} \overset{2.5\,(a)}{=} \sum_{k=0}^{p} (-1)^k \cdot \sigma|_{(i_0,\dots,\widehat{i_k},\dots,i_p)}.$$

Again, we extend this to linear **boundary maps** $\partial\colon C_p(X) \to C_{p-1}(X)$, and set this map to zero if $p \le 0$. Note that for this extension we need Remark 5.12 (b) stating that specifying a linear map on a basis also works for infinite-dimensional vector spaces.

(c) Following Definition 2.5 (b), a singular chain in the kernel of the boundary map is called **closed** or a **cycle**, and a singular chain in its image is called a **boundary**.

Example 6.6. The picture on the right shows the singular analogue of Examples 2.4 and 2.7: The paths σ_1 and σ_2 are (the images of) two singular 1-simplices in a topological space X, where the arrows indicate the order of their vertices. If we form from this the singular 1-chain $c = \sigma_1 + \sigma_2 \in C_1(X)$, its boundary is $\partial c = 2\tau_3 - \tau_1 - \tau_2 \in C_0(X)$, where τ_i for $i \in \{1,2,3\}$ denotes the map that sends the only point of a geometric 0-simplex to the corresponding marked point in the picture.



Remark 6.7. Let $\sigma\colon |\alpha| \to X$ be a singular simplex in a topological space X.

(a) It is obvious that $(\sigma|_\beta)|_\gamma = \sigma|_\gamma$ whenever $\gamma \subset \beta \subset \alpha$.

(b) By construction, it follows immediately from Lemma 2.12 (in its ordered version, see Construction 4.23) that

$$\partial^2\sigma = \partial\sigma|_{\partial\alpha} \overset{(a)}{=} \sigma|_{\partial^2\alpha} = 0.$$

Hence, the singular chain groups of Definition 6.2 with the boundary maps of Construction 6.5 (b) form a chain complex $C(X)$. We call it the **singular chain complex** of X.

Definition 6.8 (Singular homology)**.** Let X be a topological space. For any $p \in \mathbb{Z}$, the p-th **singular homology group** $H_p(X)$ is defined to be the p-th homology group $H_p(C(X)) := H_p(C(X,K))$ of the singular chain complex $C(X)$ of X. The corresponding Betti numbers are called the **Betti numbers** of X and denoted by

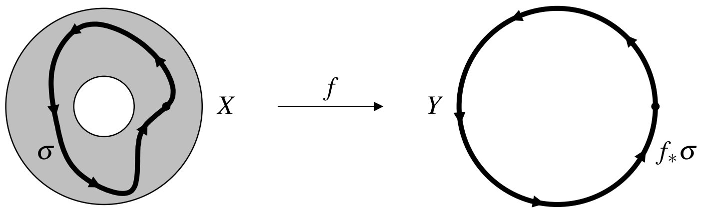
$$b_p(X) := b_p(X,K) = \dim H_p(X,K) \quad \in \mathbb{N} \cup \{\infty\}.$$

Construction 6.9 (Pushforwards and the singular chain functor)**.** In the same way as for simplicial homology in Construction 2.9 and Notation 2.19, we can make the assignment of the singular chain complex $C(X)$ to a topological space X into a functor by constructing a singular **pushforward map** f_* to any continuous map $f\colon X \to Y$ between two topological spaces X and Y. In fact, the idea for its construction is the same as well, namely just a composition with f: For a singular simplex $\sigma\colon |\alpha| \to X$ with α an ordered p-simplex, we set

$$f_*\sigma := f \circ \sigma\colon |\alpha| \to Y. \tag{$*$}$$

It is clear that this can be extended to linear maps $f_*\colon C_p(X) \to C_p(Y)$ for all $p \in \mathbb{Z}$ by linearity, giving a morphism of chain complexes $f_*\colon C(X) \to C(Y)$, and that this makes the construction of singular chains into a functor $\mathsf{Top} \to \mathsf{Ch}$. Composing with the homology functor of Lemma 2.17, we thus see that taking the p-th singular homology group $H_p(X)$ of a topological space X is a functor $\mathsf{Top} \to \mathsf{Vect}$ for all $p \in \mathbb{Z}$ as well, i. e. that the pushforward is also well-defined as a map $f_*\colon H_p(X) \to H_p(X)$ on homology. In particular, this means that homeomorphic topological spaces indeed have naturally isomorphic singular homology groups – which was one of the goals of our new homology theory.

Intuitively, the formula $(*)$ just means that the pushforward is obtained by taking images under f. Consider for example the radial projection f from the annulus X to its outer circle Y in the picture below. In this case, the pushforward of the singular simplex $\sigma \in H_1(X)$ describing a path around the annulus is just a path $f_*\sigma \in H_1(Y)$ running once around the circle.



Remark 6.10 (First properties of singular homology)**.** Let X be a topological space. The following properties of simplicial homology carry over with essentially the same proof to the singular setting:

(a) If X has finitely many path-connected components $X_1,\dots,X_n$, we have as in Lemma 2.21

$$H_p(X) \cong H_p(X_1) \times \cdots \times H_p(X_n)$$

since every singular simplex must lie in one component, and its boundary will then be in the same component.

(b) A singular 0-simplex in X is just a map from a geometric 0-simplex (i. e. a point) to X, which we can identify with its image point. As in Lemma 2.22, for $X \neq \emptyset$ there is thus a well-defined surjective *degree map* $\deg\colon H_0(X) \to K$ sending each such point in X to 1, and this is an isomorphism if X is path-connected since then the difference of any two points is the boundary of a path (i. e. of a singular 1-simplex), and hence zero in homology.

(c) Consequently, if X has finitely many path-connected components, their number is equal to $b_0(X)$ as in Corollary 2.23. However, if X has infinitely many path-connected components (such as e. g. for $X = \mathbb{Z}$) then $b_0(X)$ will be infinite – showing in contrast to the simplicial situation that Betti numbers no longer need to be finite. This also means that we cannot immediately write down Euler characteristic computations unless we know beforehand that the numbers involved in it are finite.

(d) If X is the topological space with exactly one point, there is exactly one singular p-simplex in X for any $p \in \mathbb{N}$, namely the constant map. Hence, the singular chain complex $C(X)$ is exactly the same as in Example 4.24, showing that the singular homology of X is trivial: We have

$$b_p(X) = \begin{cases} 1 & \text{if } p = 0, \\ 0 & \text{if } p \neq 0. \end{cases}$$

As already mentioned, one of our main next goals is to show that, for any simplicial complex S, the singular homology groups of $|S|$ are indeed naturally isomorphic to the simplicial homology groups for of S – as one would hope (see Proposition 8.1). We will need several preparations before we can actually show this, but it is already obvious now that there is a natural map between the two theories by reinterpreting an ordered simplex in a simplicial complex as a singular simplex in its topological realization:

Construction 6.11 (Singular from simplicial homology)**.** Let S be a simplicial complex with standard geometric realization $|S|$. Note that any ordered p-simplex α in S gives rise to a singular p-simplex

$$\rho(\alpha)\colon |\alpha| \to |S|$$

in $|S|$, which is just the inclusion map. Extending by linearity, we thus obtain a linear map $\rho\colon C_p^{\text{ord}}(S) \to C_p(|S|)$, which commutes with the boundary maps since Definition 2.5 and Construction 6.5 are clearly compatible. Hence, $\rho\colon C^{\text{ord}}(S) \to C(|S|)$ is a morphism of chain complexes; we will call it the **topological realization morphism**. By Lemma 2.17 it gives rise to linear maps $H_p^{\text{ord}}(S) \to H_p(|S|)$ between the homology groups. As we have seen already in Proposition 4.25 that ordered simplicial homology agrees with ordinary simplicial homology, we can also interpret this as natural linear maps $H_p(S) \to H_p(|S|)$.

Moreover, it is checked immediately that this construction commutes with morphisms, i. e. that a morphism $f\colon S \to T$ of simplicial complexes gives rise to a commutative diagram

$$\begin{tikzcd} H_p^{\text{ord}}(S) \arrow[r] \arrow[d] & H_p(|S|) \arrow[d] \\ H_p^{\text{ord}}(T) \arrow[r] & H_p(|T|), \end{tikzcd}$$

in which the vertical maps are the (ordered) simplicial and singular pushforwards by f and $|f|$, respectively.

Note that the map ρ only uses the structure of $|S|$ as a topological space, so that we can in fact use any geometric realization $r\colon V(S) \to W$ as in Construction 1.13 to define it. Actually, the construction even works for an arbitrary map r to give a morphism of chain complexes from $C(S)$ to $C(r(S))$, or to $C(X)$ for any other topological space X that contains $r(S)$. We will use this in the following for prisms $P(S)$ as in Definition 3.7 to consider ρ as a morphism from $C(P(S))$ to $C(|S| \times [0,1])$ instead of to $C(|P(S)|)$, choosing the map r that sends a vertex i to $(e_i, 0)$ and i' to $(e_i, 1)$ for $i \in V(S)$. As we had mentioned in Construction 3.6, this is in fact a geometric realization of the prism $P(S)$, but we did not bother to prove this since as remarked above the definition of ρ also works without this assertion.

Construction 6.11 allows us already to carry over results from the simplicial to the singular world. Let us do this for the homotopy theory of Chapter 3 to derive that homotopic continuous maps between topological spaces induce the same maps on singular homology – using the corresponding statement in the simplicial setting of Lemma 3.8.

A visualization of this situation is given in the picture below, where $X = S^1$ is a circle, embedded as the inner and outer boundary of an annulus Y by the two maps $f, g\colon X \to Y$, respectively. Of course, f and g are homotopic maps given by moving radially away from the center. Now let us consider the singular 1-chain $c = \sigma_1 + \sigma_2 + \sigma_3$ running once around the circle, made up from three singular 1-simplices defined on a geometric 1-simplex $|\alpha|$.

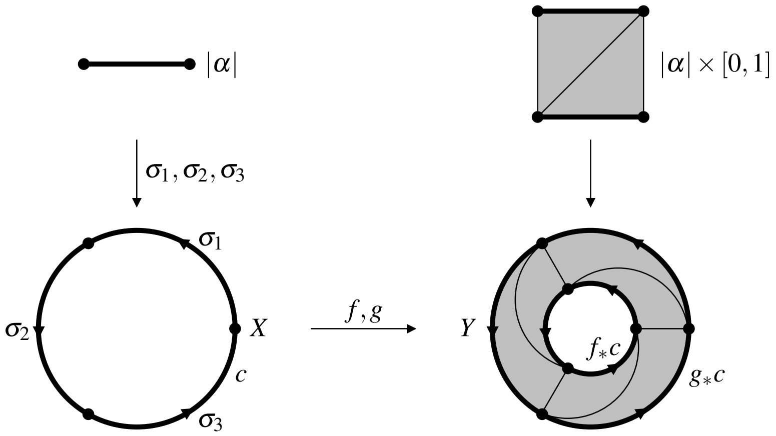
Of course, due to f and g being homotopic the pushforwards f_*c and g_*c should now define the same singular homology class in $H_1(Y)$. To prove this using our results from simplicial geometry, we can proceed for each singular simplex $\sigma_i\colon |\alpha| \to X$ in three steps:

- Consider the (simplicial) prism $P(D^\alpha)$ over D^α (resp. over α) as in Construction 3.6 and Definition 3.7. In the following picture, it is shown on the upper right.

- Pass from the simplicial to the singular world by considering this prism as a topological space rather than a simplicial complex, i. e. apply the map ρ from Construction 6.11.
- Map the prism (as a topological space) by the given homotopy between f and g continuously to the space X, shown as the "curved prisms" on the bottom right of the picture.

You can see from the picture that the three curved prisms together form a singular 2-chain whose boundary is $g_*c - f_*c$, showing that the two pushforwards agree in homology.

In the following proof, the three steps above are clearly visible when constructing the required chain homotopy between the singular homology groups.



Proposition 6.12 (Singular homotopies)**.** *Let $f, g \colon X \to Y$ be two homotopic maps between the topological spaces X and Y. Then the pushforwards f_* and g_* from $C(X)$ to $C(Y)$ are chain homotopic.*

In particular, by Proposition 3.3 these pushforwards induce the same map from $H_p(X)$ to $H_p(Y)$ for all $p \in \mathbb{Z}$.

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Proof. Let $R \colon X \times [0,1] \to Y$ be a homotopy between f and g, so that $R(\cdot\,,0) = f$ and $R(\cdot\,,1) = g$. We will construct from this a chain homotopy $h \colon C_p(X) \to C_{p+1}(Y)$ between f_* and g_*.

To do so, consider first for a fixed singular simplex $\sigma \colon |\alpha| \to X$ and $p \in \mathbb{Z}$ the composed linear map

$$h_\sigma \colon C_p^{\text{ord}}(D^\alpha) \xrightarrow{\tilde{h}} C_{p+1}^{\text{ord}}(P(D^\alpha)) \xrightarrow{\rho} C_{p+1}(|\alpha| \times [0,1]) \xrightarrow{R_*^\sigma} C_{p+1}(Y), \tag{1}$$

where $\tilde{h}$ is the simplicial prism map of Definition 3.7, ρ is the topological realization morphism of the prism in $|\alpha| \times [0,1]$ as in Construction 6.11, and R_*^σ is the singular pushforward by the continuous map $R^\sigma \colon |\alpha| \times [0,1] \to Y$, $(x,t) \mapsto R(\sigma(x),t)$ as in Construction 6.9. We set $h(\sigma) := h_\sigma(\alpha)$, and extend this by linearity to a linear map $h \colon C_p(X) \to C_{p+1}(Y)$.

Note that, if β is an ordered simplex with $\beta \subset \alpha$, to construct $h(\sigma|_\beta)$ we can still use σ in the last map of (1), i. e. we have $h(\sigma|_\beta) = h_\sigma(\beta)$. This means that

$$h(\partial\sigma) = h(\sigma|_{\partial\alpha}) = h_\sigma(\partial\alpha). \tag{2}$$

Now we know by (the ordered version of) Lemma 3.8 that $\tilde{h}$ is a chain homotopy between $\tilde{f}_*$ and $\tilde{g}_*$, where $\tilde{f}, \tilde{g} \colon D^\alpha \to P(D^\alpha)$ are the inclusions of the bottom and top face, respectively, i. e. we have

$$\partial \circ \tilde{h} + \tilde{h} \circ \partial = \tilde{g}_* - \tilde{f}_* \quad \text{as morphisms } C^{\text{ord}}(D^\alpha) \to C^{\text{ord}}(P(D^\alpha)).$$

Composing this with the morphisms ρ and R_*^σ (that commute with ∂) thus gives

$$\partial \circ h_\sigma + h_\sigma \circ \partial = R_*^\sigma \circ \rho \circ \tilde{g}_* - R_*^\sigma \circ \rho \circ \tilde{f}_*,$$

and applying this to α yields by (2) the desired relation

$$\partial h(\sigma) + h(\partial\sigma) = g_*\sigma - f_*\sigma. \qquad \square$$

Corollary 6.13 (Singular homology is a homotopy invariant)**.** *If $f\colon X \to Y$ is a homotopy equivalence between two topological spaces X and Y, the pushforward $f_*\colon H_p(X) \to H_p(Y)$ is an isomorphism for all $p \in \mathbb{Z}$.*

Proof. Let $g\colon Y \to X$ be a homotopy inverse of f. Then $g \circ f\colon X \to X$ and $f \circ g\colon Y \to Y$ are homotopic to the identity, so $(g \circ f)_* = g_* \circ f_*\colon C(X) \to C(X)$ and $(f \circ g)_* = f_* \circ g_*\colon C(Y) \to C(Y)$ are chain homotopic to the identity by 6.12. Hence, $f_*\colon C(X) \to C(Y)$ is a chain homotopy equivalence, and thus by Corollary 3.5 induces an isomorphism on homology. □

Example 6.14 (Singular homology of contractible spaces)**.** By Corollary 6.13, any contractible topological space X has the same Betti numbers as a one-pointed space, i. e. $b_0(X) = 1$ with all other Betti numbers being zero by Remark 6.10 (d).

For example, this applies to all star-shaped subsets X of $\mathbb{R}^n$ for $n \in \mathbb{N}$, in particular to all geometric simplices or to $X = \mathbb{R}^n$ itself.

Exercise 6.15. Compute the singular homology of the following topological spaces:

(a) the rational numbers $\mathbb{Q}$ (with the subspace topology of $\mathbb{R}$);

(b) the space X_n of all positive definite symmetric real matrices of a given size $n \in \mathbb{N}_{>0}$ (with the subspace topology of the space $\mathbb{R}^{n\times n}$ of all $n \times n$-matrices).

Exercise 6.16. Let $X \subset \mathbb{R}^n$ be a non-empty convex subspace with $n \in \mathbb{N}$. We have seen that there is a linear degree map $\deg\colon C_0(X) \to K$ mapping any singular 0-simplex to $1 \in K$. Regarding K as chain complex, where K is the only non-zero term and sits in degree 0, construct an explicit chain homotopy showing that deg defines a chain homotopy equivalence $C(X) \to K$.

Exercise 6.17 (Invariance of singular homology under reparametrizations)**.** Let X be a topological space. For all $p \in \mathbb{N}$ and every singular simplex $\sigma\colon |\Delta^p| \to X$ we choose a homeomorphism $f_\sigma\colon |\Delta^p| \to |\Delta^p|$ compatible with taking boundaries, i. e. such that the restriction of f_σ to $|\beta|$ for any ordered simplex β in $D^{\{0,\dots,p\}}$ is $f_{(\sigma|_\beta)}$, and set $\tilde{\sigma} := \sigma \circ f_\sigma$.

Show that the linear maps $C_p(X) \to C_p(X)$ for $p \in \mathbb{Z}$ obtained by mapping any p-simplex σ to $\tilde{\sigma}$ induce isomorphisms in homology. This justifies that we usually only draw the image of singular simplices in X when visualizing homology classes.

Exercise 6.18. In this exercise we will see why singular homology is constructed with simplices and not with cubes.

A definition with cubes would be: For any topological space X and $p \in \mathbb{N}$, let $C_p^\square(X)$ be the vector space with basis given by all continuous maps $\sigma\colon [0,1]^p \to X$. We define a boundary map by

$$\partial\colon C_p^\square(X) \to C_{p-1}^\square(X),\ \sigma \mapsto \sum_{k=1}^{p} (-1)^k \left(\sigma|_{x_k=1} - \sigma|_{x_k=0}\right) \quad \text{for } p > 0,$$

where each face $\{x \in [0,1]^p : x_k = a\}$ for $a \in \{0,1\}$ is identified with $[0,1]^{p-1}$, keeping the order of the remaining coordinates. Moreover, we construct a subdivision map

$$\mathrm{sd}\colon C_p^\square(X) \to C_p^\square(X),\ \sigma \mapsto \sum_Q \sigma|_Q,$$

where the sum is taken over all 2^p subcubes $Q \subset [0,1]^p$ of side length $\frac{1}{2}$ obtained by subdividing each interval $[0,1]$ at its midpoint, and all these subcubes are rescaled to $[0,1]^p$, so that $\sigma|_Q \in C_p^\square(X)$.

It is easy to check that this makes $C^\square(X)$ into a chain complex and $\mathrm{sd}\colon C^\square(X) \to C^\square(X)$ into a morphism of chain complexes.

(a) Compute the "cubical homology of a point", i. e. the Betti numbers of the chain complex $C^\square(X)$ for the topological space X with only one point.

(b) For an arbitrary non-empty topological space X and any ground field K, show that sd is never chain homotopic to the identity.