5. Simplicial Cohomology

In this last chapter on simplicial complexes we want to briefly discuss *simplicial cohomology*, which is obtained from simplicial homology by dualizing – i. e. by replacing all vector spaces and linear maps in our sequences by their duals. The goal of this is twofold: First of all, we want to set up the general algebraic framework of dualizing and study its properties since we will need this several times in the future. Secondly, and probably more importantly, from a geometric point of view this will give us a strong hint how to interpret our previous results in terms of analysis: Simplicial chains can be thought of as objects mapping *into* a simplicial complex S (in a topological realization, any simplex in S gives a subset of $|S|$ after all), so by dualizing we will study maps *from* a simplicial complex, i. e. (K-valued) functions on S. But functions on some geometric space are some of the main objects studied in analysis; and thus when we introduce algebraic topology in analytic terms in Chapter 13 it will actually be much more natural to do this in terms of cohomology instead of homology. Moreover, we will also see in this chapter that dualizing our simplicial boundary maps makes them into operations that can be interpreted as a discrete version of differentiation on S. Using this correspondence, we will then be able to establish from the properties of these boundary maps a simplicial version of the multivariate integral theorems that you might know from vector analysis, such as the divergence theorem of Gauss and the curl theorem of Stokes.

This might sound like a big project, but actually it is not since really nothing much happens when dualizing vector spaces. In case you know some more algebra, you might realize that this would be completely different for groups or modules (see e. g. [H, Chapter 3.1]) – and in fact this is one of the main reasons why we restrict to homology over *fields* in these notes. For vector spaces however we will see e. g. in Remark 5.5 already that the simplicial version of the integral theorems from vector analysis mentioned above is basically just a reformulation of the definitions.

But let us start by recalling the basic constructions for dualizing vector spaces that we will use from linear algebra.

Remark 5.1 (Dualizing vector spaces)**.** Let V be a K-vector space. From linear algebra we know [G1, Chapter 21.D]:

(a) The vector space $V^\vee := \operatorname{Hom}(V,K)$ of all linear functions from V to K is called its *dual vector space*.

(b) There is a natural bilinear pairing

$$V^\vee \times V \to K,\ (\varphi, v) \mapsto \varphi(v)$$

given by just applying the linear function $\varphi\colon V \to K$ to $v \in V$.

(c) For a linear map $f\colon V \to W$ to another vector space W, there is an associated *dual map*

$$f^\vee\colon W^\vee \to V^\vee,\ \varphi \mapsto \varphi \circ f.$$

This makes dualizing into a contravariant functor ${}^\vee\colon \mathsf{Vect} \to \mathsf{Vect}$ in the sense of Notation 1.32, i. e. we have $(g\circ f)^\vee = f^\vee \circ g^\vee$ and $\mathrm{id}_V^\vee = \mathrm{id}_{V^\vee}$ for any two composable linear maps f and g.

(d) If V is finite-dimensional with $n := \dim V$, then $\dim V^\vee = n$ as well, so $V^\vee$ is isomorphic to V (although there is no *natural* isomorphism between those spaces, i. e. you have to pick bases to construct one). However, given a basis $(v_1,\dots,v_n)$ of V there is an associated *dual basis* $(v_1^\vee,\dots,v_n^\vee)$ of $V^\vee$ defined in terms of the pairing (b) by

$$v_k^\vee(v_l) = \begin{cases} 1 & \text{if } k = l, \\ 0 & \text{if } k \neq l. \end{cases}$$

If $f\colon V \to W$ is a linear map between finite-dimensional vector spaces, its matrix with respect to fixed bases for V and W is just the transpose of the matrix of $f^\vee\colon W^\vee \to V^\vee$ with respect to the dual bases.

Let us now apply this dualization process to chain complexes. By its contravariant nature, this will reverse the directions of all linear maps, and thus the order of the indices in sequences.

Construction 5.2 (Cochain complexes)**.**

(a) For the purposes of dualizing, we will often allow a sequence of vector spaces C as in Definition 2.8 (a) to have increasing instead of decreasing indices. In this case it is customary to write its vector spaces as C^p with an upper index and its linear maps by $d_p\colon C^p \to C^{p+1}$, so that the sequence $C = ((C^p)_{p\in\mathbb{Z}}, (d_p)_{p\in\mathbb{Z}})$ reads

$$\cdots \longrightarrow C^{p-1} \xrightarrow{d_{p-1}} C^p \xrightarrow{d_p} C^{p+1} \longrightarrow \cdots.$$

As before, we will often drop the index p at the map d_p if it is clear from the context on which space it acts.

If $d^2 := d \circ d = 0$, the sequence C is called a **cochain complex** in accordance with Definition 2.13, with the prefix "co" just reminding us of the ascending indices. Correspondingly, the elements of C^p are called **p-cochains**, and the maps d_p are referred to as **coboundary maps**. Cochains in the kernel of d are called **cocycles** or **closed**, and cochains in the image of d are called **coboundaries**.

A morphism $f\colon C \to D$ of cochain complexes is still a commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C^{p-1} & \longrightarrow & C^p & \longrightarrow & C^{p+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{p-1}} & & \downarrow{\scriptstyle f_p} & & \downarrow{\scriptstyle f_{p+1}} & & \\
\cdots & \longrightarrow & D^{p-1} & \longrightarrow & D^p & \longrightarrow & D^{p+1} & \longrightarrow & \cdots
\end{array}$$

as in Definition 2.8 (b). This makes cochain complexes into a category, which we denote by $\mathsf{Ch}^\vee$.

(b) For any chain complex $C = ((C_p)_{p\in\mathbb{Z}}, (\partial_p)_{p\in\mathbb{Z}})$ as in Definition 2.13, Remark 5.1 (c) tells us that we obtain a **dual cochain complex** $C^\vee := ((C^p)_{p\in\mathbb{Z}}, (d_p)_{p\in\mathbb{Z}})$ by setting $C^p = C_p^\vee$ and $d_p = \partial_{p+1}^\vee\colon C_p^\vee \to C_{p+1}^\vee$, i. e.

$$\cdots \longrightarrow C_{p-1}^\vee \xrightarrow{\partial_p^\vee} C_p^\vee \xrightarrow{\partial_{p+1}^\vee} C_{p+1}^\vee \longrightarrow \cdots.$$

Note that this is actually a cochain complex (since $d \circ d = \partial^\vee \circ \partial^\vee = (\partial \circ \partial)^\vee = 0$) and gives rise to a contravariant dualization functor ${}^\vee\colon \mathsf{Ch} \to \mathsf{Ch}^\vee$. Hence, we can also dualize a sequence of chain complexes as in Definition 4.1 to obtain a reversed sequence of dual cochain complexes.

Of course, our most important example for the moment will come from dualizing simplicial chain complexes.

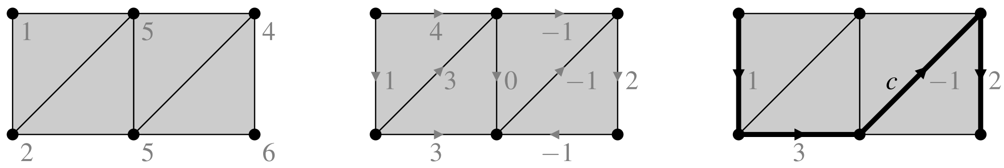
Construction 5.3 (Simplicial cochain complexes)**.** For any simplicial complex S we define the **simplicial cochain complex** as the dual cochain complex $C^\vee(S)$ of the simplicial chain complex $C(S)$ of Construction 2.9 and Example 2.14, i. e. by $C^p(S) := (C_p(S))^\vee$ and the corresponding dual boundary maps $d_p\colon C^p(S) \to C^{p+1}(S)$ according to Remark 5.1 (c) given by $d\varphi(c) = \varphi(\partial c)$ for all $\varphi \in C^p(S)$ and $c \in C_{p+1}(S)$, which by Definition 2.5 (a) means

$$d\varphi([i_0,\dots,i_{p+1}]) = \sum_{k=0}^{p+1} (-1)^k \cdot \varphi([i_0,\dots,\widehat{i_k},\dots,i_{p+1}]) \quad \in K$$

for any oriented $(p+1)$-simplex $[i_0,\dots,i_{p+1}]$ of S.

According to Construction 5.2 (a), the elements of $C^p(S)$ for $p \in \mathbb{Z}$ are called **simplicial p-cochains** of S, and the coboundary maps $d_p \colon C^p(S) \to C^{p+1}(S)$ are called the **simplicial coboundary maps** of S. Elements of $C^p(S)$ are called **simplicial p-cocycles** or **simplicial p-coboundaries** of S if they are in the kernel or image of the simplicial coboundary map, respectively.

Example 5.4 (Discrete version of differentiation and integration)**.** The picture below on the left shows a simplicial 0-cochain φ on a 2-dimensional simplicial complex S, and the middle picture shows a simplicial 1-cochain ψ on S. The gray numbers are the values of the cochains on the various simplices. Due to cochains being defined on *oriented* simplices, we had to choose an arbitrary orientation of each edge (indicated by the gray arrows) to specify ψ; reversing the orientation of an edge would replace the value of ψ on that edge by its negative.



$\varphi \in C^0(S)$ $\qquad$ $\psi = d\varphi \in C^1(S)$ $\qquad$ $\int_c \psi = 1+3-1+2 = 5$

It might seem that simplicial cochains are absolutely the same as simplicial chains, just assigning a number to every oriented simplex of the given dimension. But the geometric interpretation of this number is entirely different: For example, the picture for $\varphi \in C^0(S)$ above represents a function on the set of vertices of S (which could roughly be thought of as a discrete approximation of a function on the topological realization $|S|$), whereas the same numbers interpreted as a chain in $C_0(S)$ would correspond to a collection of points on S, having the gray number of points at the corresponding vertices. Similarly, the 1-cochain ψ is not to be thought of as an oriented path through S with some multiplicities (as it would be the case for a 1-chain), but as a function on on the oriented edges of S. For example, if we think of the simplicial complex S as a geographic map, a 0-cochain might describe the elevation at the various points, whereas a 1-cochain might describe the elevation gain when traveling along an edge.

Algebraically, the difference between chains and cochains is that their (co-)boundary maps are completely different. For example, whereas the boundary of a 0-chain is always zero by definition, by Construction 5.3 the coboundary $d\varphi \in C^1(S)$ of a cochain $\varphi \in C^0(S)$ is given by

$$d\varphi([i_0,i_1]) = \varphi([i_1]) - \varphi([i_0]) \quad \text{for all } \{i_0,i_1\} \in S,$$

i. e. it takes the difference of a 0-cocycle at adjacent vertices and can thus be thought of as a *discrete version of differentiation*. In fact, in the picture above the 1-cochain ψ is just equal to $d\varphi$, i. e. for a geographic map with elevation profile φ the 1-cocycle ψ describes the elevation gain along the edges.

On the other hand, the pairing $\varphi(c)$ of Remark 5.1 (b) by definition evaluates a p-cochain φ on every simplex in a p-chain c and takes the weighted sum of all these values, which can be interpreted geometrically as a *discrete version of integration* of a function φ on a space c, which we might write suggestively as

$$\int_c \varphi := \varphi(c) \quad \in K.$$

In fact, in the analytic analogue of our current setting this pairing will be an actual integration (see Construction 14.11). The picture above on the right shows a concrete example of a 1-chain c going from the upper left to the lower right corner of S. Note that in this case we had chosen the orientation of the gray arrows for ψ so that they match with the orientation of the edges of c, so for the integral we only have to sum up the values of ψ on the edges of c. In the geometric interpretation in terms of elevation profiles above, the integral $\psi(c)$ just computes the sum of the elevation differences when traveling along the path, so the result is if course the difference between the elevation at the end (which is 6) and at the start (which is 1). As $\psi = d\varphi$ in the example, this also follows from the defining equation $d\varphi(c) = \varphi(\partial c)$ of the coboundary map in Construction 5.3, which in integral notation becomes

$$\int_c d\varphi = \int_{\partial c} \varphi, \qquad \text{i. e.} \int_{[i_0,i_1]} d\varphi = \varphi([i_1]) - \varphi([i_0]) \quad \text{for } \varphi \in C_0(S).$$

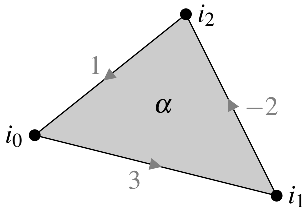
Note that this formula for 0-cochains can be understood as a discrete version of the fundamental theorem of calculus.

Remark 5.5 (Integral theorems in higher dimensions)**.** To obtain intuitive interpretations of the equation $\int_c d\varphi = \int_{\partial c} \varphi$ for $\varphi \in C^p(S)$ and $c \in C_{p+1}(S)$ with $p > 0$ on a simplicial complex S, we have to understand the geometric meaning of the coboundary operator in these cases. Let us quickly do this in two simple cases, leading to discrete versions of integral theorems that you might know from vector analysis and that we will prove later in Proposition 14.13 and Example 14.14.

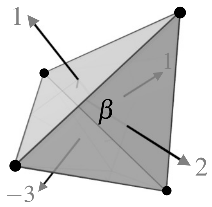
(a) For a first interpretation specifically for $p = 1$, note that by Construction 5.3 we obtain for a 1-cochain $\varphi \in C^1(S)$ and a 2-simplex $[i_0, i_i, i_2] \in S$

$$d\varphi([i_0,i_1,i_2]) = \varphi([i_0,i_1]) + \varphi([i_1,i_2]) + \varphi([i_2,i_0]).$$

Hence, e. g. in the picture below on the left we have $d\varphi(\alpha) = 3-2+1 = 2$ for $\alpha = [i_0,i_1,i_2]$. To interpret this picture geometrically, imagine that α is a triangle floating on a surface of water, and that the 1-cochain φ describes the direction and strength of the flow of water around the triangle. The number $d\varphi(\alpha)$ then determines whether the triangle starts rotating: In our example, there is a counterclockwise net flow of water of strength 2 around α, so the triangle will start rotating counterclockwise. Interpreting φ as a vector field, this number $d\varphi(\alpha)$ corresponds to what is called the *curl* of the vector field in analysis. Adding this up for an object c made up from several simplices, the equation $\int_c d\varphi = \int_{\partial c} \varphi$ thus says that the rotation speed picked up by the object is obtained by summing up the vector field φ around the boundary of the object. This is a discrete version of what is commonly called the *curl theorem of Stokes*.



$\varphi \in C^1(S)$



$\psi \in C^2(S)$

(b) Another interpretation that works well if $p = \dim S - 1$ is shown in the picture above on the right in the case $p = 2$. We can then describe the orientation of a p-simplex by an outward-pointing normal vector as in the picture, and similarly to (a) the expression $d\psi(\beta)$ for $\psi \in C^p(S)$ and a $(p+1)$-simplex β adds up the outward-pointing values of ψ at the boundary of β – so in the picture we have $d\psi(\beta) = -3+2+1+1 = 1$. To interpret this number geometrically, imagine again that ψ describes the direction and strength of a water flow (in this case three-dimensional) in these normal directions. Then the number $d\psi(\beta)$ determines if there is a water source or sink inside β: In this case, we have a net flow of water of strength 1 coming out of β, so there must be a source of water inside β. Interpreting ψ as a vector field again, this number $d\psi(\beta)$ corresponds to what is called in analysis the *divergence* of the vector field. Adding this up again over an object c made up from simplices, the equation $\int_c d\psi = \int_{\partial c} \psi$ can therefore be understood as saying that to check if there is a source of water within c we have to compute the net water flow through its boundary. This can be interpreted as a discrete version of the *divergence theorem of Gauss*.

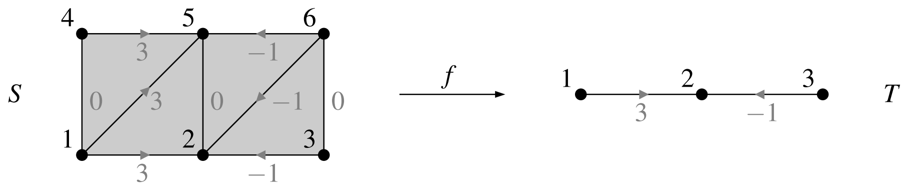
Construction 5.6 (Pullbacks and the simplicial cochain functor)**.** Note that the assignment of the simplicial cochain complex $C^\vee(S)$ to a simplicial complex S is actually a (contravariant) functor since it is the composition of the simplicial chain functor of Construction 2.9 with the dualization functor of Construction 5.2 (b). Hence, for a morphism $f\colon S \to T$ of simplicial complexes there is an associated morphism of cochain complexes from $C^\vee(T)$ to $C^\vee(S)$. In analogy to the pushforward $f_*\colon C(S) \to C(T)$ it is called the **pullback** by f and written as $f^*\colon C^\vee(T) \to C^\vee(S)$. According to the construction of the simplicial chain and dualization functors it is given by $f^*\varphi = \varphi \circ f_* \in C^p(S)$ for all $\varphi \in C^p(T)$, i. e. on p-simplices $[i_0,\dots,i_p] \in S$ by

$$(f^*\varphi)[i_0,\dots,i_p] = \varphi([f(i_0),\dots,f(i_p)]) \quad \in K.$$

Hence, intuitively it is just given by composition with f. Let us give two examples:

(a) If S is a subcomplex of a simplicial complex T, the pullback by the inclusion morphism is just the restriction $C^p(T) \to C^p(S)$, $\varphi \mapsto \varphi|_S := \varphi|_{C_p(S)}$ of linear maps $C_p(T) \to K$ to linear maps $C_p(S) \to K$.

(b) As a concrete example, the picture below shows the pullback $f^*\varphi$ of a 1-cocycle φ for the "vertical projection morphism" f of simplicial complexes given by $f(3i+j) = j$ for $i \in \{0,1\}$ and $j \in \{1,2,3\}$ (where the black numbers are the labels of the vertices and the gray numbers are the values of the cocycles on the edges).



$f^*\varphi \in C^1(S)$ $\qquad$ $\varphi \in C^1(T)$

After having understood the geometric meaning of simplicial cochain complexes in detail, let us now come back to cochain complexes in general. Of course, we also want to consider their homology, which in accordance with the language of Construction 5.2 will be called *cohomology*.

Construction 5.7 (Cohomology of cochain complexes).

(a) For a cochain complex $C = ((C^p)_{p\in\mathbb{Z}}, (d_p)_{p\in\mathbb{Z}})$ as in Construction 5.2 (a), its p-th **cohomology group** is defined as the vector space

$$H^p(C) := \ker d_p / \operatorname{im} d_{p-1}$$

as in Definition 2.15, and its dimension $b^p(C) := \dim H^p(C)$ will be called the p-th **Betti number** of C. In the same way as in Definition 4.4, the cochain complex C is called **exact** if all these cohomology groups are trivial, i. e. if $\ker d_p = \operatorname{im} d_{p-1}$ for all $p \in \mathbb{Z}$.

(b) For a simplicial complex S, the **simplicial cohomology** of S is defined to be the cohomology of its simplicial cochain complex $C^\vee(S)$ of Construction 5.3, i. e. we set $H^p(S) := H^p(C^\vee(S))$ for all $p \in \mathbb{Z}$. As usual, if we want to indicate the ground field in the notation we will write it as $H^p(S,K) = H^p(C^\vee(S,K))$.

Remark 5.8. Note that apart from the labeling of its entries a cochain complex is really just the same as a chain complex, and cohomology is just the same as homology. In particular, all results that we have obtained for chain complexes and their homology hold in our new setting as well:

(a) Cohomology H^p is a functor $\mathsf{Ch}^\vee \to \mathsf{Vect}$ for all $p \in \mathbb{Z}$ as in Lemma 2.17. In particular, the simplicial cohomology groups are contravariant functors $\mathsf{Simp} \to \mathsf{Vect}$, so for a morphism $f\colon S \to T$ the pullback maps $f^*\colon C^p(T) \to C^p(S)$ of Construction 5.6 are also well-defined on simplicial cohomology as maps $f^*\colon H^p(T) \to H^p(S)$.

(b) If all cohomology groups of a cochain complex C are finite-dimensional with almost all of them being 0, there is an *Euler characteristic* $\chi(C) = \sum_{p\in\mathbb{Z}} (-1)^p\, b^p(C)$ as in Definition 2.24, and if the same holds for C^p as well we can compute it by $\chi(C) = \sum_{p\in\mathbb{Z}} (-1)^p \dim C^p$ as in Lemma 2.25.

(c) *Cochain homotopies* can be defined in the same way as in Definition 3.1 (now lowering the index by 1 instead of raising it), and the existence of such a cochain homotopy between two morphisms of cochain complexes implies that they induce the same map on cohomology as in Proposition 3.3.

(d) Any short exact sequence of cochain complexes gives rise to a long exact sequence in cohomology as in Proposition 4.8, and this assignment is functorial as in Construction 4.10.

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In fact, at least in the simplicial case it is also immediate to see that all Betti numbers remain invariant under dualization:

Lemma 5.9 (Invariance of Betti numbers under dualization)**.** *For any chain complex C consisting only of finite-dimensional vector spaces we have $b^p(C^\vee) = b_p(C)$ for all $p \in \mathbb{Z}$.*

Proof. For a given $p \in \mathbb{Z}$ the relevant parts of the (co-)chain complexes C and $C^\vee$ are

$$\cdots \longrightarrow C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \longrightarrow \cdots \qquad \text{and} \qquad \cdots \longrightarrow C_{p-1}^\vee \xrightarrow{\partial_p^\vee} C_p^\vee \xrightarrow{\partial_{p+1}^\vee} C_{p+1}^\vee \longrightarrow \cdots.$$

Now by Remark 5.1 (d) we have $\dim C_p^\vee = \dim C_p$, and $\operatorname{rk} \partial_p^\vee = \operatorname{rk} \partial_p$ as well as $\operatorname{rk} \partial_{p+1}^\vee = \operatorname{rk} \partial_{p+1}$ since transposed matrices have the same rank. Hence by the dimension formula for linear maps we obtain

$$\begin{aligned}
\dim H^p(C^\vee) &= \dim \ker \partial_{p+1}^\vee - \operatorname{rk} \partial_p^\vee \\
&= \dim C_p^\vee - \operatorname{rk} \partial_{p+1}^\vee - \operatorname{rk} \partial_p^\vee \\
&= \dim C_p - \operatorname{rk} \partial_{p+1} - \operatorname{rk} \partial_p \\
&= \dim \ker \partial_p - \operatorname{rk} \partial_{p+1} \\
&= \dim H_p(C).
\end{aligned}$$

□

Remark 5.10.

(a) In particular, it follows from Lemma 5.9 that $\dim H^p(S) = \dim H_p(S) = b_p(S)$ for any simplicial complex S and $p \in \mathbb{Z}$.

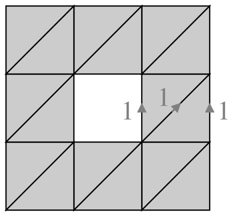
(b) Another easy consequence of Lemma 5.9 is that the dual of an exact sequence of finite-dimensional vector spaces (or chain complexes consisting of finite-dimensional vector spaces) is again exact, since exactness just means that the Betti numbers are zero. For example, for simplicial complexes S and T we obtain a *dual exact simplicial Mayer-Vietoris sequence* in cohomology

$$0 \longrightarrow C^\vee(S \cup T) \longrightarrow C^\vee(S) \times C^\vee(T) \longrightarrow C^\vee(S \cap T) \longrightarrow 0$$

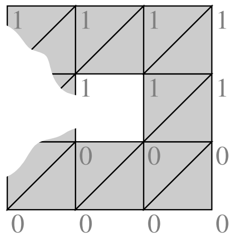
by dualizing Example 4.7.

Example 5.11. Remark 5.10 (a) means that we can detect holes in simplicial complexes using cohomology in the same way as with homology. But we have seen that the coboundary maps in cohomology look completely different from the boundary maps in homology. So let us quickly show geometrically why *cocycles* modulo *coboundaries* detect holes as well – which is probably less intuitive than considering cycles modulo boundaries. In fact, the following argument in simplicial cohomology is already a discrete version of the analytic method described in the introduction to these notes, as "functions satisfying some integrability conditions modulo functions that are derivatives".

Let S be the simplicial annulus shown below on the left, which has first Betti number $b_1(S) = 1$ by Construction 4.14 (b) due to its hole in the middle. To detect this hole using cohomology, consider the 1-cochain $\varphi \in C^1(S)$ shown in the picture (where the value of φ on all unmarked edges is zero). Note first of all that it is actually a cocycle, i. e. that $d\varphi = 0$: We could either check this by directly computing its coboundary using the formula of Construction 5.3, or – probably easier – by seeing that in the interesting right half of the annulus it is of the form $\varphi = d\psi$ for the 0-cochain ψ shown in the picture on the right, implying that $d\varphi = d^2\psi = 0$ there. In the spirit of Example 5.4 this just means that φ describes the elevation gain along the edges for the elevation function ψ.



$\varphi \in C^1(S)$



$\psi \in C^0(S)$ with $d\psi = \varphi$

However, this construction does not work globally due to the hole of S: To construct a global elevation function ψ whose elevation gain along edges is φ, the only possibility is the function ψ shown in the right picture, and this function would not be well-defined on S as the part cut out in the picture cannot be filled in consistently. In other words, the elevation gain function φ would require us to have an elevation profile in which we pick up a non-trivial elevation gain when going around the hole of the annulus, which is not possible. Hence, the cocycle φ is a non-zero element of $H^1(S)$, and thus a generator since $b_1(S) = 1$.

Hence, we can say that the cocycle φ is a coboundary locally, but not globally – which is exactly the discrete version of a function satisfying a local integrability condition $d\varphi = 0$, yet not being a global "derivative" of the form $\varphi = d\psi$.

As you might imagine, the statement of Lemma 5.9 (and also its consequences such as duals of exact sequences being exact in Remark 5.10 (b)) are absolutely crucial for homology and cohomology to interplay well, and thus for the rest of these notes. Unfortunately, there is a small technical issue here: We have used dimension arguments to prove Lemma 5.9, but starting from the next chapter most of the vector spaces in our chain complexes will actually consist of (highly) infinite-dimensional vector spaces – although usually their homology still has finite dimension. Hence, we need a replacement for the proof of this lemma that works in this more general situation as well. To prepare for this, let us first recall some properties of general (i. e. possibly infinite-dimensional) vector spaces from linear algebra to make sure that we are all on the same page.

Remark 5.12 (Infinite-dimensional linear algebra)**.** Let V be vector space, possibly infinite-dimensional. The following statements should all be well-known for finite-dimensional vector spaces; for proofs in the general case see e. g. [R]. The main guideline is that linear combinations of vectors in V are always finite by definition (after all, infinite sums would require some notion of convergence, which is not available for arbitrary ground fields). So, for example, by definition a family of vectors in V is a basis if and only if every element of V is a linear combination of *finitely many* vectors of the family in a unique way.

(a) Every linearly independent family of vectors in V can be extended to a basis of V, so in particular V always has a basis – although the proof of this statement in the infinite-dimensional case uses the Axiom of Choice and is thus not constructive. In other words, while there are some infinite-dimensional vector spaces with known bases (e. g. the vector space of polynomials in a variable x with basis $(x^k)_{k\in\mathbb{N}}$), there is in general no algorithm to compute a basis.

(b) A linear map from V to another vector space W can be constructed by picking a basis of V and specifying arbitrary image vectors in W for it; by linear extension there will then always be a unique linear map $V \to W$ realizing these values on the basis. This implies that every linear map $U \to W$ on a subspace U of V to another vector space W can be extended to a linear map $V \to W$, by extending a basis of U to one of V and setting the map to 0 on the additional basis vectors.

(c) The dimension $\dim V$ of V is well-defined as a cardinality, so e. g. a vector space with a countable basis cannot be isomorphic to a vector space with an uncountable basis.

Note that all statements of Remark 5.12 are of the form that infinite-dimensional vector spaces behave to a large extent just like one would expect from the finite-dimensional case. There is one big difference however regarding dualization (which is why this topic comes up in the current chapter): In contrast to Remark 5.1 (d), the dual $V^\vee$ of an infinite-dimensional vector space V is no longer isomorphic to V. In fact, it *never* is; one can show that its dimension is always larger than that of V in the sense of cardinality as in Remark 5.12 (c). (In case you have been exposed to some functional analysis and are surprised about this statement, let me clarify that we are talking about the *algebraic dual* $V^\vee$ of *all* linear functions on V here – although we are discussing topology, the vector spaces used for homology will actually never carry a norm or topology, so there is no notion of a *continuous dual* of all *continuous* linear functions.) We will not need this general statement in these notes, but it should warn us that the process of dualizing is not quite as harmless as it might have seemed in this chapter so far. To illustrate this fact let us consider one example here that will show up again in Corollary 16.5.

Example 5.13 (Dualizing infinite-dimensional vector spaces)**.** Let V be a real vector space with a countably infinite basis $(v_k)_{k\in\mathbb{N}}$. By Remark 5.12 (b) the elements of $V^\vee$ are uniquely given by specifying arbitrary values on these basis elements, so

$$V^\vee \cong \{(x_k)_{k\in\mathbb{N}} : x_k \in \mathbb{R}\}$$

is just the space of all real sequences (where x_k denotes the value of the element of $V^\vee$ on v_k for all $k\in\mathbb{N}$). We claim that this space $V^\vee$ does *not* have a countable basis, so that it cannot be isomorphic to V by Remark 5.12 (c). To see this, it suffices to prove that the uncountable family $(\varphi_z)_{z\in\mathbb{R}}$ in $V^\vee$ given by

$$\varphi_z = (z^k)_{k\in\mathbb{N}} = (z^0, z^1, z^2, \dots)$$

is linearly independent. So let $\lambda_1,\dots,\lambda_n \in \mathbb{R}$ for $n\in\mathbb{N}$ and $z_1,\dots,z_n\in\mathbb{R}$ be distinct with $\sum_{k=1}^n \lambda_k \varphi_{z_k} = 0 \in V^\vee$. Looking only at the first $n-1$ components of this equation gives

$$\sum_{k=1}^n \lambda_k z_k^l \quad \text{for } l = 0,\dots,n-1. \tag{$*$}$$

But this is an $n\times n$ linear system of equations for the variables $\lambda_1,\dots,\lambda_n$ with coefficient matrix $(z_k^l)_{1\le k\le n, 0\le l\le n-1}$. This matrix is known as a *Vandermonde matrix*, and it is a standard linear algebra result that it is invertible. Hence, the system $(*)$ has only the trivial solution $\lambda_1 = \cdots = \lambda_n = 0$, showing that $(\varphi_z)_{z\in\mathbb{R}}$ is an uncountable linearly independent family in $V^\vee$, and thus that $V^\vee$ is not isomorphic to V.

Fortunately however, dualization still behaves well with respect to homology, so that we obtain the following result analogous to Lemma 5.9.

Proposition 5.14. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be two linear maps of vector spaces, with corresponding dual maps $C^\vee \xrightarrow{g^\vee} B^\vee \xrightarrow{f^\vee} A^\vee$. Then:*

(a) $\ker(f^\vee) = \{\varphi \in B^\vee : \varphi|_{\operatorname{im} f} = 0\}$.

(b) $\operatorname{im}(g^\vee) = \{\varphi \in B^\vee : \varphi|_{\ker g} = 0\}$.

(c) *If $g\circ f = 0$ (and thus also $f^\vee \circ g^\vee = 0$) there is a natural isomorphism*

$$\ker(f^\vee)/\operatorname{im}(g^\vee) \cong (\ker g/\operatorname{im} f)^\vee.$$

Proof.

(a) This follows directly from the definition:

$$\ker(f^\vee) = \{\varphi \in B^\vee : \varphi\circ f = 0\} = \{\varphi \in B^\vee : \varphi|_{\operatorname{im} f} = 0\}.$$

(b) We need to show that

$$\operatorname{im}(g^\vee) = \{\varphi \in B^\vee : \varphi = \psi\circ g \text{ for some } \psi \in C^\vee\} \stackrel{!}{=} \{\varphi\in B^\vee : \varphi|_{\ker g} = 0\}.$$

"$\subset$" is obvious since any function of the form $\psi\circ g$ vanishes on $\ker g$.

"$\supset$": Let $\varphi \in B^\vee$ with $\varphi|_{\ker g} = 0$. We want to find $\psi \in C^\vee$ such that $\varphi = \psi \circ g$ to make the diagram on the right commutative. First define ψ on $\operatorname{im} g$: For any $y \in \operatorname{im} g$ with $y = g(x)$ set $\psi(y) := \varphi(x)$; this is actually well-defined since picking a different inverse image of y would change x by an element of $\ker g$, on which φ is zero.

$$\begin{tikzcd} B \arrow[rr, "\varphi"] \arrow[dr, "g"'] & & K \\ & C \arrow[ur, "\psi"'] & \end{tikzcd}$$

Hence we already have the required equation $\psi(g(x)) = \varphi(x)$ for all $x \in B$. By Remark 5.12 (b) we can now extend ψ – which so far is only defined on $\operatorname{im} g$ – to a linear function on all of C.

(c) Let us start with with the natural map $B^\vee \to (\ker g)^\vee$ that just restricts a linear function on B to one on $\ker g \subset B$. In the hope that it does not confuse you we will now restrict this restriction map to the subspace $\ker(f^\vee) \subset B^\vee$; by (a) the resulting linear functions in $(\ker g)^\vee$ will then vanish on $\operatorname{im} f$ and can thus be considered as well-defined functions in $(\ker g/\operatorname{im} f)^\vee$. We have thus obtained a natural linear map

$$\Phi\colon \ker(f^\vee) \to (\ker g/\operatorname{im} f)^\vee$$

and want to study its image and kernel:

- Φ is surjective: Any linear function in $(\ker g/\operatorname{im} f)^\vee$ comes (by composition with the quotient map) from a linear function in $(\ker g)^\vee$ that is zero on $\operatorname{im} f$, hence by extension as in Remark 5.12 (b) from a linear function in $B^\vee$ that is zero on $\operatorname{im} f$. By (a), this is just an element of $\ker(f^\vee)$.
- The kernel of Φ consists of all linear functions that are zero on $\ker g$, which by (b) is just $\operatorname{im}(g^\vee)$.

The statement thus follows from the homomorphism theorem applied to Φ. □

Remark 5.15.

(a) Applying Proposition 5.14 (c) to the linear maps in a chain complex C and its dual cochain complex $C^\vee$, we see that

$$H^p(C^\vee) \cong (H_p(C))^\vee,$$

i. e. that "(co-)homology commutes with dualization". This reproves Lemma 5.9 and establishes it also in the case when the vector spaces C_p are infinite-dimensional. If the homology groups are infinite-dimensional as well, note however that the statement $\dim H^p(C^\vee) = \dim H_p(C)$ is still correct in the sense that both dimensions are infinite, but that in this case $H^p(C^\vee)$ is not isomorphic to $H_p(C)$ by Example 5.13.

(b) As in Remark 5.10 (b), Proposition 5.14 (c) clearly implies that the dual of any exact sequence of vector spaces or chain complexes is still exact.

(c) Actually, even in the finite-dimensional case Proposition 5.14 (c) is stronger than Lemma 5.9 as it is not just a dimension statement, but also establishes $H^p(C^\vee)$ as canonically dual to $H^p(C)$. Hence, the pairing of Remark 5.1 (b) also works between $H^p(C^\vee)$ and $H_p(C)$ for all $p \in \mathbb{Z}$, so e. g. the "integral" $\int_c \varphi$ of Example 5.4 is well-defined for (co-)homology classes $c \in H_p(S)$ and $\psi \in H^p(S)$ of a simplicial complex S and $p \in \mathbb{Z}$ as well. Moreover, by Remark 5.1 (d) for any finite basis of $H_p(C)$ there is an associated dual basis of $H^p(C^\vee)$.

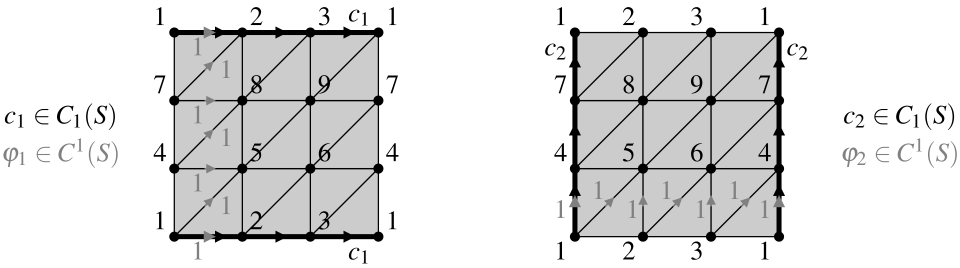
Example 5.16. Let S be a simplicial torus as in Example 1.17 (d). We have already computed its homology in Example 2.27 and found out that $b_1(S) = 2$, with a basis of $H_1(S)$ given by a horizontal circle c_1 and vertical circle c_2 as drawn in bold in the pictures below.

In the picture we have also drawn two 1-cochains φ_1 and φ_2, constructed in a similar way to Example 5.11, and with their values again given by the gray numbers and arrows. We claim that this is in fact the dual basis to c_1 and c_2 in $H^1(S)$ in the sense of Remark 5.15 (c):

- It is checked immediately that $d\varphi_1 = d\varphi_2 = 0$.
- We have $\int_{c_1} \varphi_1 = \int_{c_2} \varphi_2 = 1$: For both $i \in \{1,2\}$ the cycle c_i and the cocycle φ_i have exactly one edge in common, and they both have multiplicity 1 on it.

- In contrast, c_1 has no edge in common with φ_2, so $\int_{c_1} \varphi_2 = 0$. In the same way, we obtain $\int_{c_2} \varphi_1 = 0$.

Hence, φ_1 and φ_2 form the dual basis to c_1 and c_2 – note again as in Example 5.11 that they look completely different although they describe the same topological properties of the torus.



Exercise 5.17. Consider the simplicial sphere $S^{\underline{4}}$ for the set $\underline{4} := \{0,1,2,3,4\}$.

(a) Let
$$\varphi = [0,1,2]^\vee + [0,1,4]^\vee - [0,2,3]^\vee + [0,3,4]^\vee \quad \in C^2(S^{\underline{4}}),$$
where $[i_0,i_1,i_2]^\vee$ denotes the unique linear form $C_2(S^{\underline{4}}) \to K$ with value 1 on $[i_0,i_1,i_2]$ and 0 on all $[j_0,j_1,j_2]$ with $\{j_0,j_1,j_2\} \neq \{i_0,i_1,i_2\}$.

Show that there exists a cochain $\psi \in C^1(S^{\underline{4}})$ with $d\psi = \varphi$.

(b) For a 3-cochain $\varphi \in C^3(S^{\underline{4}})$, show that there exists a cochain $\psi \in C^2(S^{\underline{4}})$ with $d\psi = \varphi$ if and only if $\int_{\partial[0,1,2,3,4]} \varphi = 0$.