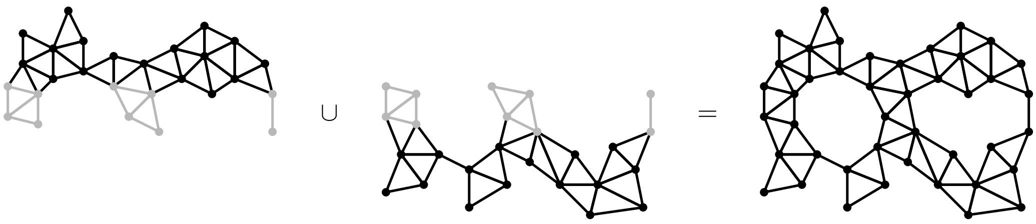
4. The Mayer-Vietoris Sequence in Simplicial Homology

In the picture below, the number of holes in the graphs S and T (i. e. their genus in the language of Example 2.26) is $b_1(S) = 24$ and $b_1(T) = 20$, respectively, and the intersection $S \cap T$ is drawn in gray. Can you figure out from this information – without explicit counting – the number of holes in $S \cup T$, i. e. the Betti number $b_1(S \cup T)$?



S T $S \cup T$

This is not too hard: As a first approximation, the holes in $S \cup T$ are the ones in S or T, so we start with the sum $b_1(S) + b_1(T)$. But there are two corrections to this number:

(a) We have counted the holes in the gray part $S \cap T$ twice, so we have to subtract $b_1(S \cap T) = 4$.

(b) In addition to the small triangular holes, we have created two big holes in the "shape ∞" of $S \cup T$ since we glued S and T in three connected components – i. e. since $b_0(S \cap T)$ is 3 and not 1.

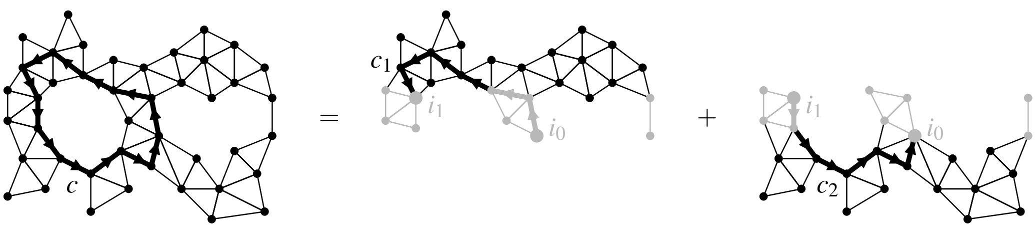
In summary, we thus obtain $b_1(S \cup T) = 24 + 20 - 4 + 2 = 42$ (an exact proof of this computation will be given in Example 4.13).

From a formal point of view, it might seem strange at first that a Betti number $b_1(S \cup T)$ in dimension 1 receives a contribution from a Betti number $b_0(S \cap T)$ in dimension 0; so let us try to understand this in more detail. Consider a hole in $S \cup T$ coming from (b) above, such as the 1-cycle c in the picture below on the left. We cannot write it as a hole in S or in T, but we can split it onto the two components in the following way (indicated in the picture on the right):

(1) Write it as a sum $c = c_1 + c_2$ for two 1-chains c_1 and c_2 in S and T, respectively.

(2) Take the boundaries of these chains; in our example case we have $\partial c_1 = [i_1] - [i_0]$ and $\partial c_2 = [i_0] - [i_1]$.

(3) Note that $\partial c_2 = \partial(c - c_1) = -\partial c_1$ as c is closed. Hence, ∂c_1 and ∂c_2 carry the same information, and they are clearly 0-cycles in $S \cap T$. Pick one of them, $\partial c_1 \in H_0(S \cap T)$ say, and assign it to c.



$S \cup T$ S T

In fact, this assignment $c \mapsto \partial c_1$ gives us a well-defined map $h\colon H_1(S \cup T) \to H_0(S \cap T)$ in homology: We might have picked a different 1-cycle representing the same hole in $S \cup T$ (i. e. the same class in $H_1(S \cup T)$) as c, and we might have chosen different splitting points (e. g. instead of i_1 in the picture above we could have taken the vertex right below it), but the result would always be the class of one vertex in the left minus one vertex in the middle component of $S \cap T$ (i. e. the same homology class by Corollary 2.23). This map $h\colon H_1(S \cup T) \to H_0(S \cap T)$ is usually called the *connecting map* since it connects homology groups in different dimensions.

Note that as in Notation 2.19 there are also natural pushforward maps $H_1(S) \to H_1(S \cup T)$ and $H_1(T) \to H_1(S \cup T)$ by the inclusions $S \to S \cup T$ and $T \to S \cup T$. Adding them, we obtain a linear map $g\colon H_1(S) \times H_1(T) \to H_1(S \cup T)$, and thus together with the connecting map we have obtained a short "sequence" of maps

$$H_1(S) \times H_1(T) \xrightarrow{g} H_1(S \cup T) \xrightarrow{h} H_0(S \cap T) \qquad (*)$$

that we can interpret as in the geometric argument above by saying that the holes in $S \cup T$ (in the middle term) are related to the holes in S or T (in the left term) and the connected components of $S \cap T$ (in the right term).

But there is one last step missing: Of course, just that these homology groups are related by linear maps does not yet mean that the middle term $H_1(S \cup T)$ is actually is made up from contributions from the two sides – after all, the maps in the sequence $(*)$ could e. g. just be the zero maps. It turns out however that this sequence has a special property:

- Coming from the left term $H_1(S) \times H_1(T)$, i. e. from holes entirely in S or T, the construction of the connecting map above is actually trivial in the sense that we do not have to introduce splitting points at all, and thus the resulting class in $H_0(S \cap T)$ will be zero. More formally, this means that $h \circ g = 0$, or that $\operatorname{im} g \subset \ker h$. In other words, we have a short version of a chain complex as in Definition 2.13.
- Conversely, if we have a cycle $c \in \ker h$, i. e. a cycle for which the connecting map does not see any splitting onto the connected components of $S \cap T$, it is intuitive that c must come from holes already present in S and T, i. e. that $c \in \operatorname{im} g$.

So altogether we see that $\operatorname{im} g = \ker h$ in $(*)$. Sequences with this property will be called *exact*, and they allow us to think about their maps as giving contributions to their vector spaces: Considering an element $c \in H_1(S \cup T)$, it does *not* give rise to anything in $H_0(S \cap T)$ (i. e. we have $c \in \ker h$) *if and only if* it comes from something in $H_1(S) \times H_1(T)$ (i. e. we have $c \in \operatorname{im} g$). This is precisely what it means to say that $H_1(S \cup T)$ "is made up from contributions from $H_1(S) \times H_1(T)$ and $H_0(S \cap T)$".

In fact, we can also interpret this in terms of the "lifespan" of elements in the sequence $(*)$ in the sense of Remark 2.11: In contrast to chain complexes, which just say that every element in the sequence survives at most two stages in the sequence, an exact sequence means that every element survives *exactly* two stages. So every element in the middle of $(*)$ comes from the left or from the right, but not both. We will pursue this perspective further in Example 9.14.

As you might already suspect, the goal of this chapter is to generalize all this and make it precise. We will see that for any two simplicial complexes S and T we can compute the homology of $S \cup T$ from that of S, T, and $S \cap T$; and to compute the homology in dimension p of $S \cup T$ for some $p \in \mathbb{Z}$ we do not only need the homology of S, T, and $S \cap T$ in dimension p, but there will also be connecting maps giving rise to contributions from dimension $p-1$.

Just as for homology and homotopy in the two preceding chapters, exact sequences in homology are a purely algebraic concept that we will encounter in many more cases than just for simplicial complexes. So let us now start the chapter again with some algebra. Generalizing sequences such as $(*)$ to arbitrary dimension of the chains will make them into sequences whose terms are actually chain complexes themselves and not just vector spaces, so let us start by introducing this concept analogously to Definition 2.8.

Definition 4.1 (Sequences of chain complexes)**.** A **sequence of chain complexes** is a collection of chain complexes $(C^{(k)}, \partial^{(k)})_{k \in \mathbb{Z}}$ and morphisms $f^{(k)}\colon C^{(k)} \to C^{(k+1)}$ between them for all $k \in \mathbb{Z}$:

$$\cdots \longrightarrow C^{(k-1)} \xrightarrow{f^{(k-1)}} C^{(k)} \xrightarrow{f^{(k)}} C^{(k+1)} \longrightarrow \cdots.$$

By Definition 2.8 we could write this as a commutative diagram of vector spaces and morphisms as follows, with the chain complexes $C^{(k)}$ running down.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & C_p^{(k-1)} \arrow[r, "f_p^{(k-1)}"] \arrow[d, "\partial_p^{(k-1)}"] & C_p^{(k)} \arrow[r, "f_p^{(k)}"] \arrow[d, "\partial_p^{(k)}"] & C_p^{(k+1)} \arrow[r] \arrow[d, "\partial_p^{(k+1)}"] & \cdots \\
\cdots \arrow[r] & C_{p-1}^{(k-1)} \arrow[r, "f_{p-1}^{(k-1)}"] \arrow[d] & C_{p-1}^{(k)} \arrow[r, "f_p^{(k)}"] \arrow[d] & C_{p-1}^{(k+1)} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Often the chain complexes $C^{(k)}$ will be zero outside a small range for k, in which case we will only write this finite part of the sequence. Moreover, as in Definition 2.8, in the notation we will often drop the lower index that indicates the position within each chain complex. However, in order to avoid confusion we will always distinguish between the individual complexes and the maps between them in the notation, either by an upper index or by using different letters. Also, we will always write the individual chain complexes vertically in the columns, and the different chain complexes horizontally in the rows.

Remark 4.2. It might seem strange that we chose the index in a sequence to count downward in Definition 2.8, but upward in Definition 4.1. The reason for this is that counting upward is probably the more natural choice, but the fact that the boundary map in homology reduces dimension made the downward counting more suggestive in Chapter 2. Of course, at the end of the day it does not matter whether we count up or down, and in fact in the next chapter when discussing cohomology we will also encounter boundary maps that increase the dimension.

Example 4.3 (The simplicial Mayer-Vietoris sequence)**.** Our main example of a sequence of chain complexes in this chapter is the following: Let S and T be simplicial complexes. Then there is a three-term sequence of simplicial chain complexes $C(S\cap T) \to C(S)\times C(T) \to C(S\cup T)$ which in each row p is just given by

$$\begin{array}{ccccc}
C_p(S\cap T) & \longrightarrow & C_p(S)\times C_p(T) & \longrightarrow & C_p(S\cup T) \\
c & \longmapsto & (c,-c) & & \\
 & & (c,c') & \longmapsto & c+c'.
\end{array}$$

We call it the **simplicial Mayer-Vietoris sequence** for S and T. The reason for the sign in the first map is that the composition of the two chain maps is zero this way – this makes the sequence into a “chain complex of chain complexes”. In fact, it is our first example of an exact sequence already mentioned in the introduction to this chapter:

Definition 4.4 (Exact sequences)**.**

(a) A sequence of vector spaces

$$\cdots \longrightarrow C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \longrightarrow \cdots$$

is called **exact** if $\ker \partial_p = \operatorname{im} \partial_{p+1}$ for all $p \in \mathbb{Z}$.

(b) A sequence of chain complexes

$$\cdots \longrightarrow C^{(k-1)} \xrightarrow{f^{(k-1)}} C^{(k)} \xrightarrow{f^{(k)}} C^{(k+1)} \longrightarrow \cdots$$

is called exact if each of its rows as in Definition 4.1

$$\cdots \longrightarrow C_p^{(k-1)} \xrightarrow{f_p^{(k-1)}} C_p^{(k)} \xrightarrow{f_p^{(k)}} C_p^{(k+1)} \longrightarrow \cdots$$

for $p \in \mathbb{Z}$ is exact in the sense of (a).

Remark 4.5.

(a) By definition, a sequence of vector spaces is exact if and only if it is a chain complex with all homology groups being zero.

(b) If $(C_p)_{p\in\mathbb{Z}}$ is an exact sequence of vector spaces, it follows directly from (a) that its Euler characteristic is $\chi(C) = 0$. Hence, if moreover all C_p are finite-dimensional and almost all of them are zero, Lemma 2.25 implies that $\sum_{p\in\mathbb{Z}}(-1)^p \dim C_p = 0$.

Example 4.6 (Short exact sequences)**.** As already mentioned in the introduction to this chapter, an exact sequence of vector spaces can be thought of as a sequence in which "every term is made up from contributions from its left and right neighbor". This is particularly evident for sequences that are short and/or contain zero vector spaces at their ends:

(a) A sequence $0 \to A \xrightarrow{f} B$ of vector spaces (with the first map of course being the zero map) is exact if and only if f is injective. Intuitively, as A cannot obtain any contribution from the left, all of A must also be contained in B. Similarly, a sequence $A \xrightarrow{f} B \to 0$ is exact if and only if f is surjective (all of B must come from A).

(b) It follows immediately from (a) that an exact sequence of vector spaces $0 \to A \to 0$ means that $A = 0$, and that an exact sequence $0 \to A \xrightarrow{f} B \to 0$ means that f is an isomorphism.

(c) The first non-trivial case is an exact sequence of vector spaces $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ with three arbitrary terms, bounded by 0; they are usually referred to as **short exact sequences**. In this case, Remark 4.5 (b) implies that $\dim B = \dim A + \dim C$ if all vector spaces are finite-dimensional, which again confirms the idea that B is made up from contributions from A and C. More naturally, f being injective means that A can be regarded as a subspace of B, and by the homomorphism theorem we have $C = \operatorname{im} g \cong B/\ker g = B/\operatorname{im} f = B/A$.

In general, not all exact sequences that we will encounter are of these types; in fact, many of them are rather long. However, most of the terms in these sequences will then be 0, and if there are at most three terms between two zero entries we can then still apply the reasoning of (b) or (c) to this part of the sequence.

Example 4.7 (The simplicial Mayer-Vietoris sequence is exact)**.** The simplicial Mayer-Vietoris sequence of Example 4.3 is in fact a short exact sequence of chain complexes

$$0 \longrightarrow C(S\cap T) \longrightarrow C(S)\times C(T) \longrightarrow C(S\cup T) \longrightarrow 0$$

for any two simplicial complexes S and T. This follows directly from the definitions: Clearly, any p-chain in $C_p(S\cup T)$ can be written as $c + c'$ for $c \in C_p(S)$ and $c' \in C_p(T)$, and such an expression $c + c'$ is zero if and only if $c' = -c$, which is only possible for $c = -c' \in C_p(S\cap T)$.

Of course, we now want to use this sequence of simplicial chain complexes to compute their homology. This is again a purely algebraic process that we will often see in these notes:

Proposition 4.8 (The long exact homology sequence)**.** *For any short exact sequence*

$$0 \longrightarrow A \xrightarrow{\;f\;} B \xrightarrow{\;g\;} C \longrightarrow 0$$

of chain complexes there are linear connecting maps $h\colon H_p(C) \to H_{p-1}(A)$ for all $p \in \mathbb{Z}$ giving rise to an induced long exact sequence in homology

$$\cdots \longrightarrow H_p(A) \xrightarrow{f} H_p(B) \xrightarrow{g} H_p(C) \xrightarrow{h} H_{p-1}(A) \xrightarrow{f} H_{p-1}(B) \xrightarrow{g} H_{p-1}(C) \xrightarrow{h} H_{p-2}(A) \longrightarrow \cdots.$$

Proof. The central part of the proof is the construction of the connecting maps $h\colon H_p(C) \to H_{p-1}(A)$ for $p \in \mathbb{Z}$. For their definition, we proceed analogously to the steps (1), (2), (3) in the introduction to this chapter for the Mayer-Vietoris sequence of Example 4.3, following the dashed arrow in the diagram below. Elements in the vector spaces not on this dashed path are marked with the numbers in the gray circles whenever they appear. Start with a closed chain $c_p \in C_p$.

(1) The map $g\colon B_p \to C_p$ is surjective, so we can choose $b_p \in B_p$ with $g(b_p) = c_p$.

(2) Take the boundary map of B to obtain $b_{p-1} := \partial b_p \in B_{p-1}$.

(3) Note that $g(b_{p-1}) = g(\partial b_p) = \partial g(b_p) = \partial c_p = 0 \in C_{p-1}$ since c_p is closed. ① As the row $p-1$ is exact, we therefore have $b_{p-1} = f(a_{p-1})$ for a unique $a_{p-1} \in A_{p-1}$.

Set $h(c_p) := a_{p-1}$.

$$
\begin{tikzcd}
 & & B_{p+1}^{⑤} \arrow[r, "g"] \arrow[d, "\partial"] & C_{p+1}^{④} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_p^{⑥} \arrow[r, "f"] \arrow[d, "\partial"] & B_p \arrow[r, "g"] \arrow[d, "\partial"] & C_p \arrow[r] \arrow[d, "\partial"] \arrow[dll, dashed, bend right] & 0 \\
0 \arrow[r] & A_{p-1} \arrow[r, "f"'] \arrow[d, "\partial"] & B_{p-1} \arrow[r, "g"'] \arrow[d, "\partial"] & C_{p-1}^{①} \arrow[r] & 0 \\
0 \arrow[r] & A_{p-2}^{③} \arrow[r, "f"'] & B_{p-2}^{②} & &
\end{tikzcd}
$$

Let us show that this map $h\colon H_p(C) \to H_{p-1}(A)$ is actually well-defined:

- a_{p-1} is closed: We have $f(\partial a_{p-1}) = \partial f(a_{p-1}) = \partial b_{p-1} = \partial^2 b_p = 0 \in B_{p-2}$, ② which means that $\partial a_{p-1} = 0 \in A_{p-2}$ since f is injective. ③
- $a_{p-1} \in H_{p-1}(A)$ is independent of choices: Let $c'_p \in C_p$ be another representative of the same class as c_p, and $b'_p \in B_p$ with $g(b'_p) = c'_p$. Correspondingly, set $b'_{p-1} = \partial b'_p$ and $a'_{p-1} \in A_{p-1}$ with $f(a'_{p-1}) = b'_{p-1}$.

  Then $c_p - c'_p = \partial c$ for some $c \in C_{p+1}$. ④ As g is surjective we can choose $b \in B_{p+1}$ with $g(b) = c$. ⑤ We obtain
  $$g(b_p - b'_p - \partial b) = c_p - c'_p - \partial g(b) = c_p - c'_p - \partial c = 0,$$
  so $b_p - b'_p - \partial b = f(a)$ for some $a \in A_p$ since row p is exact. ⑥ Hence,
  $$f(a_{p-1} - a'_{p-1}) = b_{p-1} - b'_{p-1} = \partial(b_p - b'_p) = \partial(f(a) + \partial b) = \partial f(a) + \underbrace{\partial^2 b}_{=0} = f(\partial a),$$
  which by injectivity of f means that $a_{p-1} - a'_{p-1} = \partial a$ is a boundary (i. e. zero in homology).

For obvious reasons, the technique that we applied here is usually called *diagram chasing*.

It now remains to prove that the connecting maps h give rise to an exact homology sequence as stated in the proposition. In fact, it is obvious from the constructions that this homology sequence is at least a chain complex:

- $g \circ f$ is zero on homology since this already holds for the initial chain complexes.
- $h \circ g$ is zero on homology: Applying the construction of h above to $g(b_p)$ for a *cycle* b_p means that $\partial b_p = 0$, so we will get 0 in step (2).
- $f \circ h$ is zero on homology: Applying f to the cycle a_{p-1} in the construction above gives b_{p-1}, which is a boundary by (2), and hence zero in homology.

The proof that the long homology sequence is actually exact, i. e. that also the kernel of any of its maps is included in the image of the previous map, is somewhat long and tedious (it is a long sequence after all). But it is just straightforward diagram-chasing, and we will also see a visual explanation for this statement later in Remark 9.15, so we will omit the proof here and leave it as an exercise. □

Exercise 4.9. In the situation of Proposition 4.8, show that in the long homology sequence the kernel of any map is included in the image of the previous map, thus completing the proof of the proposition.

06

Construction 4.10 (The long exact homology sequence is functorial)**.** As one would probably expect, the long exact homology sequence of Proposition 4.8 is functorial in the following sense: Consider a *morphism of short exact sequences of chain complexes*, by which of course we mean a commutative diagram of chain complexes

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "k^{(1)}"] & B \arrow[r, "g"] \arrow[d, "k^{(2)}"] & C \arrow[r] \arrow[d, "k^{(3)}"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0 \end{tikzcd}$$

with exact rows (note that this means that the vertical maps $k^{(i)}$ for $i \in \{1,2,3\}$ commute both with f and g as well as with the boundary maps of the chain complexes). Then the rows of this diagram induce a long exact homology sequence as in Proposition 4.8, and the columns pass to maps in homology by Lemma 2.17. We claim that these maps fit into a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & H_p(A) \arrow[r, "f"] \arrow[d, "k^{(1)}"] & H_p(B) \arrow[r, "g"] \arrow[d, "k^{(2)}"] & H_p(C) \arrow[r, "h"] \arrow[d, "k^{(3)}"] & H_{p-1}(A) \arrow[r] \arrow[d, "k^{(1)}"] & \cdots \\ \cdots \arrow[r] & H_p(A') \arrow[r, "f'"] & H_p(B') \arrow[r, "g'"] & H_p(C') \arrow[r, "h'"] & H_{p-1}(A') \arrow[r] & \cdots \end{tikzcd}$$

with exact rows, where h and h' are the connecting maps (i. e. into a morphism of exact sequences of vector spaces). In fact, the commutativity in the squares involving f, f', g, and g' is just the usual functoriality of homology in Lemma 2.17, so it only remains to prove that $h'(k^{(3)}(c_p)) = k^{(1)}(h(c_p))$ for all cycles $c_p \in C_p$. But this can easily be seen by following the steps (1), (2), (3) in the proof of Proposition 4.8 to construct the connecting map h' for the element $k^{(3)}(c_p) \in C'_p$, using the notations as in the proof:

(1) We can choose $k^{(2)}(b_p) \in B'_p$ since $g'(k^{(2)}(b_p)) = k^{(3)}(g(b_p)) = k^{(3)}(c_p)$.

(2) Applying the boundary map gives $\partial k^{(2)}(b_p) = k^{(2)}(\partial b_p) = k^{(2)}(b_{p-1}) \in B'_{p-1}$.

(3) We have $k^{(2)}(b_{p-1}) = k^{(2)}(f(a_{p-1})) = f'(k^{(1)}(a_{p-1}))$, so the construction gives that h' maps $k^{(3)}(c_p)$ to $k^{(1)}(a_{p-1}) = k^{(1)}(h(c_p)) \in A'_{p-1}$ as required.

Corollary 4.11. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of chain complexes for which all homology groups are finite-dimensional, and almost all of them are zero. Then $\chi(B) = \chi(A) + \chi(C)$.*

Proof. This follows from Remark 4.5 (b) applied to the long exact homology sequence of Proposition 4.8. □

Corollary 4.12 (The simplicial Mayer-Vietoris homology sequence)**.** *For any two simplicial complexes S and T, the sequence*

$$\cdots \longrightarrow H_p(S \cap T) \longrightarrow H_p(S) \times H_p(T) \longrightarrow H_p(S \cup T) \longrightarrow H_{p-1}(S \cap T) \longrightarrow \cdots$$

(with morphisms as in Example 4.3) is exact. In particular, we have

$$\chi(S \cup T) + \chi(S \cap T) = \chi(S) + \chi(T).$$

Proof. The long exact sequence follows from Proposition 4.8 applied to the Mayer-Vietoris sequence of Example 4.7, the statement about the Euler numbers then from Corollary 4.11. □

Example 4.13 (Gluing graphs)**.** We are now ready to give a proof for the intuitive computation at the beginning of this chapter: Let S and T be two graphs, and assume for simplicity that S, T, and $S \cup T$ are non-empty and connected (so that $b_0(S) = b_0(T) = b_0(S \cup T) = 1$ by Corollary 2.23). Then Corollary 4.12 implies that the first Betti number of $S \cup T$ is

$$\begin{aligned} b_1(S \cup T) &= 1 - \chi(S \cup T) = 1 - \chi(S) - \chi(T) + \chi(S \cap T) \\ &= b_1(S) + b_1(T) - b_1(S \cap T) + b_0(S \cap T) - 1. \end{aligned}$$

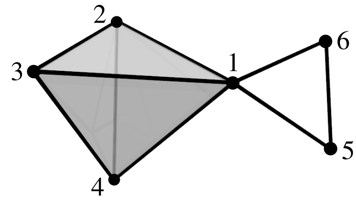
For the graphs in the introduction to this chapter we had $b_1(S \cup T) = 24 + 20 - 4 + (3 - 1) = 42$.

We could also obtain this result from the exact Mayer-Vietoris homology sequence directly: In our case of a simplicial complex of dimension 1, this sequence reads

$$0 \to H_1(S\cap T) \to H_1(S)\times H_1(T) \to H_1(S\cup T) \to H_0(S\cap T) \to H_0(S)\times H_0(T) \to H_0(S\cup T) \to 0.$$

Using Remark 4.5 (b), it leads to the same dimension computation for $b_1(S\cup T)$ as before, and the geometric interpretation of the connecting map $H_1(S\cup T) \to H_0(S\cap T)$ has already been given in the introduction.

Construction 4.14 (Wedge sums)**.** Let S and T be two simplicial complexes that share exactly one vertex, such as e. g. $S = S^{\{1,2,3,4\}}$ and $T = S^{\{1,5,6\}}$ in the picture on the right. Their union $S\cup T$ is then usually called a *wedge sum* of the two simplicial complexes. Let us assume for simplicity that S and T are connected, so that $S\cup T$ is clearly connected as well, and we are only interested in the Betti numbers $b_p(S\cup T)$ for $p > 0$.



As we have $H_p(S\cap T) = 0$ for $p > 0$, the long exact Mayer-Vietoris homology sequene of Corollary 4.12 contains exact parts

$$0 \longrightarrow H_p(S)\times H_p(T) \longrightarrow H_p(S\cup T) \longrightarrow 0 \tag{1}$$

for $p > 1$. In contrast, for $p = 1$ this exact sequence continues as

$$0 \longrightarrow H_1(S)\times H_1(T) \longrightarrow H_1(S\cup T) \longrightarrow H_0(S\cap T).$$

But the last map here is injective by Corollary 2.23, so the previous map must be the zero map since the sequence is exact. This means that we can equally well end this exact sequence with $\cdots \to H_1(S\cup T) \to 0$, so that (1) holds for $p = 1$ as well. By Example 4.6 (b) this means that $H_p(S)\times H_p(T) \cong H_p(S\cup T)$, and thus

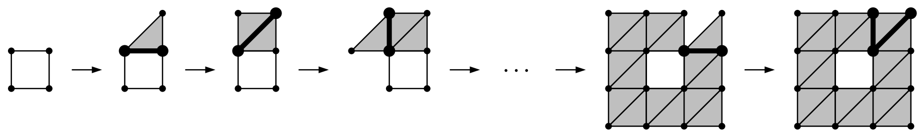
$$b_p(S\cup T) = b_p(S) + b_p(T) \quad \text{for all } p > 0. \tag{2}$$

So, in the example of the picture above, we would have $b_2(S\cup T) = b_1(S\cup T) = b_0(S\cup T) = 1$ by Example 3.13, with all other Betti numbers being zero.

Let us discuss two interesting consequences of this result:

(a) For any given collection of numbers $b_p \in \mathbb{N}$ for $p \in \mathbb{N}$, with $b_0 > 0$ and only finitely many being non-zero, there is a simplicial complex S that has these Betti numbers for any ground field: By Example 3.13, we can just take the wedge sum of b_p copies of a simplicial p-sphere for $p > 0$, and form the disjoint union with $b_0 - 1$ isolated vertices to reach b_0 as well.

(b) Obviously, the result (2) holds with the same argument also if $S\cap T$ is not just a single vertex, but has the same trivial homology as a single vertex. Specializing to the case when T has trivial homology as well, we see that passing from S to $S\cup T$ if both T and $S\cap T$ have trivial homology (e. g. for star-shaped simplicial complexes as in Proposition 3.11) does not change homology. For example, in the following picture every simplicial complex is obtained from the previous one by attaching a simplicial 2-disk along a star-shaped subcomplex (drawn in bold), which gives a rigorous proof (without even looking at the actual simplicial chain complexes) that all these spaces have isomorphic homology. In particular, the simplicial annulus on the right has the same Betti numbers as the graph on the left, namely zeroth and first Betti number 1 and all others zero. Of course, this fits well with the geometric intuition all the spaces in the picture just have one hole.



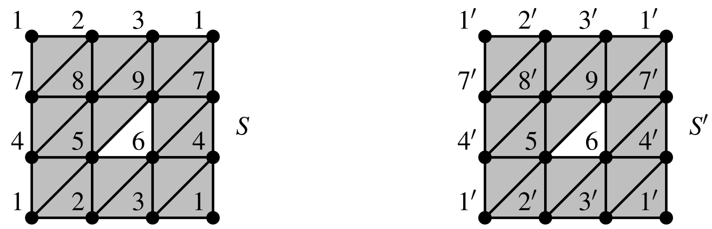
Exercise 4.15. Let S be a simplicial complex, and let $*$ and $\circ$ be new vertices not contained in S. We construct the new simplicial complex

$$\Sigma(S) := \{\{*\},\{\circ\}\} \cup \{\alpha, \alpha \cup \{*\}, \alpha \cup \{\circ\} : \alpha \in S\}.$$

(a) Compute the Betti numbers of $\Sigma(S)$ in terms of those of S.

(b) Starting with the 0-dimensional simplicial complex $S = \{\{0\},\{1\}\}$, which well-known topological space is the topological realization of the n-fold construction $\Sigma^n(S)$ of (a) for $n \in \mathbb{N}$?

Exercise 4.16. Let S and S' be the two simplicial complexes in the picture below, obtained by removing a 2-simplex from a torus as in Example 1.17 (c), and such that the boundary of this 2-simplex agrees in S and S'.



(a) Compute bases of the homology groups $H_p(S)$ for $p = 0,1,2$.

(b) Compute bases of the homology groups $H_p(S \cup S')$ for $p = 0,1,2$.

Exercise 4.17. Suppose that a simplicial complex S is the union of subcomplexes $S_1,\dots,S_n$ with $n \geq 2$ such that every intersection of any number of these subcomplexes is either empty or has the homology of a point.

(a) Show that $b_p(S) = 0$ for all $p \geq n-1$.

(b) Show for all n that the inequality of (a) cannot be improved.

There is one more important thing that we can see from Construction 4.14 (b): The Mayer-Vietoris sequence cannot only be used to simplify computations, but also opens up the path for inductive proofs of statements about the homology of simplicial complexes by writing them as unions of smaller subcomplexes. The way in which we usually use this kind of argument is the following.

Lemma 4.18 (Simplicial Mayer-Vietoris induction)**.** *Suppose that we want to prove an assertion about all simplicial complexes. Then it suffices to show:*

(a) *(Induction start) The assertion holds for the empty simplicial complex and any simplicial disk D^α.*

(b) *(Induction step) If S and T are simplicial complexes such that the assertion holds for S, T, and $S \cap T$, then it also holds for $S \cup T$.*

Proof. We show that the assertion holds for any simplicial complex S by induction on the number of its simplices. For $S = \emptyset$ we know that the assertion holds by (a), so let us assume that S is non-empty.

Let α be a simplex of maximum dimension in S. Then D^α is a subcomplex of S by Definition 1.2 (a), and $S\backslash\{\alpha\}$ is a subcomplex as well. If $S = D^\alpha$ the assertion again holds by (a), so we may assume that $D^\alpha \subsetneq S$. But then $S = (S\backslash\{\alpha\}) \cup D^\alpha$ is a union of subcomplexes with fewer simplices, so by induction the assertion holds for $S\backslash\{\alpha\}$, D^α, and $(S\backslash\{\alpha\}) \cap D^\alpha$. Hence the assertion holds for the union S as well by (b). □

Of course, the reason why this is called Mayer-Vietoris induction is that the induction step (b) in Lemma 4.18 will usually be proved using the long exact Mayer-Vietoris sequence. But there is one last problem before we can actually use this strategy in concrete cases: The Mayer-Vietoris sequence does not actually compute the homology of $S \cup T$ directly from that of S, T, and $S \cap T$, but only includes it in a long exact sequence in which every third term is a homology group of $S \cup T$, and the other terms are known. How can we use this information to extract statements about the homology of $S\cup T$? In fact, most of our statements will be that the homology of a simplicial complex is isomorphic to something else, and for such statements the following lemma solves this problem when we apply it to long exact Mayer-Vietoris homology sequences.

Lemma 4.19 (5-Lemma)**.** *Consider a commutative diagram*

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & A_3 \arrow[r, "f_3"] \arrow[d, "h_3"] & A_4 \arrow[r, "f_4"] \arrow[d, "h_4"] & A_5 \arrow[d, "h_5"] \\ B_1 \arrow[r, "g_1"] & B_2 \arrow[r, "g_2"] & B_3 \arrow[r, "g_3"] & B_4 \arrow[r, "g_4"] & B_5 \end{tikzcd}$$

of vector spaces with exact rows. If h_1, h_2, h_4, and h_5 are isomorphisms, then so is h_3.

Proof. We will show that h_3 is injective and surjective. Again, this is just tedious but straightforward diagram chasing. In both parts of the proof, the gray element in the diagram is the element to be chased, and the numbers in the gray circles denote the order of the chase.

(a) h_3 is injective: Let $a_3\in A_3$ 1 with $h_3(a_3)=0$. 2 Then $g_3(h_3(a_3))=0$, 3 so $h_4(f_3(a_3))=0$, and since h_4 is injective we have $f_3(a_3)=0$. 4 As the top row is exact, this means $a_3=f_2(a_2)$ for some $a_2\in A_2$. 5 Let $b_2=h_2(a_2)$, 6 then $g_2(b_2)=g_2(h_2(a_2))=h_3(f_2(a_2))=h_3(a_3)=0$. As the bottom row is exact, it follows that $b_2=g_1(b_1)$ for some $b_1\in B_1$. 7 Now h_1 is surjective, so $b_1=h_1(a_1)$ for some $a_1\in A_1$. 8 This element satisfies $f_1(a_1)=a_2$, since $h_2(f_1(a_1))=g_1(h_1(a_1))=g_1(b_1)=b_2=h_2(a_2)$ and h_2 is injective. But $f_2\circ f_1=0$, so $a_3=f_2(f_1(a_1))=0$.

$$\begin{tikzcd} {}^{a_1\;8}A_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & {}^{a_2\;5}A_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & {}^{a_3\;1}A_3 \arrow[r, "f_3"] \arrow[d, "h_3"] & {}^{0\;4}A_4 \arrow[r, "f_4"] \arrow[d, "h_4"] & A_5 \arrow[d, "h_5"] \\ {}^{b_1\;7}B_1 \arrow[r, "g_1"] & {}^{b_2\;6}B_2 \arrow[r, "g_2"] & {}^{0\;2}B_3 \arrow[r, "g_3"] & {}^{0\;3}B_4 \arrow[r, "g_4"] & B_5 \end{tikzcd}$$

(b) h_3 is surjective: Let $b_3\in B_3$, 1 and set $b_4=g_3(b_3)$. 2 Then $g_4(b_4)=g_4(g_3(b_3))=0$ since $g_4\circ g_3=0$. 3 As h_4 is surjective, we have $b_4=h_4(a_4)$ for some $a_4\in A_4$. 4 Then $h_5(f_4(a_4))=g_4(h_4(a_4))=g_4(b_4)=0$, hence $f_4(a_4)=0$ as h_5 is injective. 5 As the top row is exact, we have $a_4=f_3(a_3)$ for some $a_3\in A_3$. 6 Be do not necessarily have $b_3=h_3(a_3)$, but $g_3(b_3-h_3(a_3))=g_3(b_3)-g_3(h_3(a_3))=b_4-h_4(f_3(a_3))=b_4-h_4(a_4)=b_4-b_4=0$, so as the bottom row is exact we can write $b_3-h_3(a_3)=g_2(b_2)$ for some element $b_2\in B_2$. 7 Now we have $b_2=h_2(a_2)$ for some $a_2\in A_2$ since h_2 is surjective. 8 It follows that $h_3(a_3+f_2(a_2))=h_3(a_3)+h_3(f_2(a_2))=h_3(a_3)+g_2(h_2(a_2))=h_3(a_3)+g_2(b_2)=b_3$, i. e. h_3 is surjective.

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & {}^{8}A_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & {}^{a_3\;6}A_3 \arrow[r, "f_3"] \arrow[d, "h_3"] & {}^{a_4\;4}A_4 \arrow[r, "f_4"] \arrow[d, "h_4"] & {}^{0\;5}A_5 \arrow[d, "h_5"] \\ B_1 \arrow[r, "g_1"] & {}^{7}B_2 \arrow[r, "g_2"] & {}^{b_3\;1}B_3 \arrow[r, "g_3"] & {}^{b_4\;2}B_4 \arrow[r, "g_4"] & {}^{0\;3}B_5 \end{tikzcd}$$

□

Remark 4.20. The 5-Lemma is clearly an important statement that will make our Mayer-Vietoris induction arguments work. But as it is easy to misread this statement it is equally important to understand what the 5-Lemma *does not* say:

(a) The 5-Lemma does not say that a diagram as in Lemma 4.19 in which the middle vertical map f_3 is initially missing (commutative, with exact rows, and the other vertical maps being isomorphisms) can be filled in uniquely to the complete commutative with an isomorphism f_3. Instead, it is important that we must have the map f_3 already beforehand; the 5-Lemma then just ensures that it is an isomorphism. For example, the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & \mathbb{R} \arrow[r, "f_2"] \arrow[d, equal] & \mathbb{R}^2 \arrow[r, "f_3"] & \mathbb{R} \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\ 0 \arrow[r] & \mathbb{R} \arrow[r, "f_2"] & \mathbb{R}^2 \arrow[r, "f_3"] & \mathbb{R} \arrow[r] & 0 \end{tikzcd}$$

with $f_2 \colon x_1 \mapsto \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ and $f_3 \mapsto \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_2$ can be filled in with a middle vertical isomorphism $\mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + \lambda x_2 \\ x_2 \end{pmatrix}$ for any $\lambda \in \mathbb{R}$ to form a commutative diagram.

(b) Without the vertical maps being given, it is not even true that the dimensions of the outer four vector spaces in a five-term exact sequence determine the dimension of the middle space: For example, the sequences

$$\mathbb{R} \xrightarrow{\mathrm{id}} \mathbb{R} \xrightarrow{0} \mathbb{R} \xrightarrow{\mathrm{id}} \mathbb{R} \xrightarrow{0} \mathbb{R}$$
$$\text{and} \quad \mathbb{R} \xrightarrow{\mathrm{id}} \mathbb{R} \xrightarrow{0} 0 \xrightarrow{0} \mathbb{R} \xrightarrow{\mathrm{id}} \mathbb{R}$$

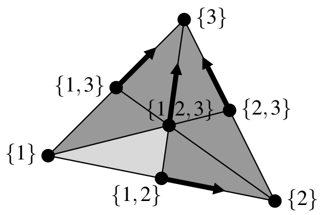
are both exact (but by the 5-Lemma there is no way to map one to the other so that the resulting diagram is commutative and the outer four maps are isomorphisms – which is also easy to see directly).

In short, it is vital for the application of the 5-Lemma that all vertical maps are known to exist and form a commutative diagram beforehand.

After this preparation, we want to finish this chapter with two interesting and clearly non-trivial statements that we can now prove by Mayer-Vietoris induction. The first is the statement that barycentric subdivision does not change the homology of a simplicial complex – which is clearly expected since by Proposition 1.21 the topological realization does not change. We have just learned in Remark 4.20 that for the Mayer-Vietoris argument to work it will be important to have a morphism between the two spaces to start with, so let us construct this first.

Construction 4.21. Let S be a simplicial complex. For any simplex $\alpha \in S$, pick an arbitrary element $f(\alpha) \in \alpha$. This defines a morphism of simplicial complexes $f \colon \mathrm{Sd}(S) \to S$: For any p-simplex $\{\alpha_0, \dots, \alpha_p\} \in \mathrm{Sd}(S)$ with $\alpha_0 \subsetneq \cdots \subsetneq \alpha_p \in S$ we have $f(\alpha_k) \in \alpha_k \subset \alpha_p$ for all $k \in \{0, \dots, p\}$, hence $f(\{\alpha_0, \dots, \alpha_p\})$ is a subset of α_p, and thus a simplex in S.

The picture on the right shows this map by arrows in the example of the 2-disk $S = D^{\{1,2,3\}}$ and the morphism $f \colon \mathrm{Sd}(S) \to S$ given on vertices of $\mathrm{Sd}(S)$ by $f(\alpha) := \max \alpha$. The light-colored 2-simplex $\{\{1\}, \{1,2\}, \{1,2,3\}\} \in \mathrm{Sd}(S)$ is the only one that maps to a 2-simplex $\{1,2,3\} \in S$; all other 2-simplices of $\mathrm{Sd}(S)$ are contracted to the boundary of the triangle.



Proposition 4.22 (Invariance of simplicial homology under barycentric subdivision)**.** *For every simplicial complex S, any morphism $f \colon \mathrm{Sd}(S) \to S$ as in Construction 4.21 induces an isomorphism $f_* \colon H_p(\mathrm{Sd}(S)) \to H_p(S)$ for all $p \in \mathbb{Z}$.*

Proof. We will prove the assertion using Mayer-Vietoris induction as in Lemma 4.18. For the induction start, note that the statement is obvious for $S = \emptyset$. For a simplicial disk, we know by Example 3.12 that $b_p(\mathrm{Sd}(S)) = b_p(S)$ for all $p \in \mathbb{Z}$, with this Betti number being 1 for $p = 0$ and 0 otherwise. Hence, the statement of the proposition holds in this case as well (for $p = 0$ note that f_* maps any vertex to a vertex, hence it is an isomorphism by Corollary 2.23).

For the induction step let S and T be simplicial complexes such that the proposition holds for S, T, and $S \cap T$. Noting that $\mathrm{Sd}(S \cap T) = \mathrm{Sd}(S) \cap \mathrm{Sd}(T)$ and $\mathrm{Sd}(S \cup T) = \mathrm{Sd}(S) \cup \mathrm{Sd}(T)$ by Remark

1.22 (b), we then obtain a commutative diagram of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & C(\mathrm{Sd}(S\cap T)) \arrow[r] \arrow[d] & C(\mathrm{Sd}(S))\times C(\mathrm{Sd}(T)) \arrow[r] \arrow[d] & C(\mathrm{Sd}(S\cup T)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C(S\cap T) \arrow[r] & C(S)\times C(T) \arrow[r] & C(S\cup T) \arrow[r] & 0,
\end{tikzcd}$$

with exact rows as in Example 4.7, and the the vertical maps being pushforwards by f. Hence, Construction 4.10 gives us commutative diagrams of vector spaces

$$\begin{tikzcd}
H_p(\mathrm{Sd}(S\cap T)) \arrow[r] \arrow[d] & H_p(\mathrm{Sd}(S))\times H_p(\mathrm{Sd}(T)) \arrow[r] \arrow[d] & H_p(\mathrm{Sd}(S\cup T)) \arrow[r] \arrow[d] & H_{p-1}(\mathrm{Sd}(S\cap T)) \arrow[r] \arrow[d] & H_{p-1}(\mathrm{Sd}(S))\times H_{p-1}(\mathrm{Sd}(T)) \arrow[d] \\
H_p(S\cap T) \arrow[r] & H_p(S)\times H_p(T) \arrow[r] & H_p(S\cup T) \arrow[r] & H_{p-1}(S\cap T) \arrow[r] & H_{p-1}(S)\times H_{p-1}(T)
\end{tikzcd}$$

for all $p\in\mathbb{Z}$ with exact rows, with the vertical maps again being pushforwards by f. As the outer four vertical maps are isomorphisms by the induction assumption, we can conclude by the 5-Lemma 4.19 that the middle vertical map is an isomorphism as well. □ 07

Our second application of Mayer-Vietoris induction is probably more surprising and goes back to the very definition of the homology groups of a simplicial complex S. Why exactly did we construct it using *oriented* simplicial chains $C_p(S)$ in Definition 2.2, and not just with *ordered* ones $C_p^{\mathrm{ord}}(S)$ as in Definition 2.1? While the notion of orientation simplified computations and seemed geometrically intuitive in our pictures, it adds an algebraic level of complexity since it makes the simplicial chain groups into quotient spaces. So let us study what would happen if we set everything up using ordered simplices instead – which is in fact what we will do for the singular homology of topological spaces introduced in Chapter 6. Fortunately, our algebraic approach admitting arbitrary chain complexes allows us to figure this out without much effort.

Construction 4.23 (Ordered simplicial homology)**.** Let S be a simplicial complex. For any $p\in\mathbb{Z}$ we define the boundary maps $\partial_p\colon C_p^{\mathrm{ord}}(S)\to C_{p-1}^{\mathrm{ord}}(S)$ for ordered chains by the same formula as in Definition 2.5, just replacing oriented simplices $[i_0,\dots,i_p]$ by ordered ones $(i_0,\dots,i_p)$. Note that the proof of Lemma 2.12 that $\partial^2=0$ never actually permutes the vertices in a simplex or uses that a simplex with repeated vertices vanishes, so it still holds in our new ordered setting with literally the same argument. Hence, we obtain an *ordered simplicial chain complex* $C^{\mathrm{ord}}(S)$. Let us call its homology groups $H_p(C^{\mathrm{ord}}(S))$ the *ordered simplicial homology* of S, denoted by $H_p^{\mathrm{ord}}(S)$.

Example 4.24 (Ordered simplicial homology of a point)**.** Already for the 0-dimensional simplicial complex $S=\{\{i_0\}\}$ consisting of a single vertex i_0 the ordered simplicial chain groups are completely different from the oriented ones: As repeated vertices in simplices are now allowed, every $C_p^{\mathrm{ord}}(S)$ for $p\geq 0$ has dimension 1, generated by the p-simplex $(i_0,\dots,i_0)$. In particular, there are now infinitely many non-zero terms in the complex $C^{\mathrm{ord}}(S)$. Due to the alternating sign in its definition, the boundary map on $(i_0,\dots,i_0)\in C_p^{\mathrm{ord}}(S)$ becomes

$$\partial_p(i_0,\dots,i_0)=\sum_{k=0}^{p}(-1)^k\cdot(i_0,\dots,i_0)=\begin{cases}(i_0,\dots,i_0)\in C_{p-1}^{\mathrm{ord}}(S) & \text{if } p \text{ is even,}\\ 0 & \text{if } p \text{ is odd.}\end{cases}$$

Hence, using the isomorphism $C_p^{\mathrm{ord}}(S)\to K,\ 1\mapsto(i_0,\dots,i_0)$ for $p\geq 0$ the ordered simplicial chain complex $C^{\mathrm{ord}}(S)$ is

$$\cdots\longrightarrow\underbrace{K}_{p=4}\xrightarrow{\mathrm{id}}\underbrace{K}_{p=3}\xrightarrow{0}\underbrace{K}_{p=2}\xrightarrow{\mathrm{id}}\underbrace{K}_{p=1}\xrightarrow{0}\underbrace{K}_{p=0}\longrightarrow\underbrace{0}_{p=-1}\longrightarrow\underbrace{0}_{p=-2}\longrightarrow\cdots,$$

and in fact, the homology $H_p^{\mathrm{ord}}(S)$ of this complex is 1-dimensional for $p=0$ and zero otherwise – just as for oriented simplicial homology!

Especially given the completely different form of $C^{\mathrm{ord}}(S)$, this result is clearly surprising. It does not really reflect a geometric idea but rather an algebraic coincidence, namely that the boundary map has some sign canceling that works out in our favor. Nevertheless, our algebraic machinery is robust enough so that this coincidence almost automatically carries over to all simplicial complexes:

Proposition 4.25 (Ordered $=$ oriented simplicial homology)**.** *For any simplicial complex S, the natural maps $C_p^{\text{ord}}(S) \to C_p(S)$, $(i_0,\dots,i_p) \mapsto [i_0,\dots,i_p]$ induce isomorphisms $H_p^{\text{ord}}(S) \to H_p(S)$ in homology for all $p \in \mathbb{Z}$.*

Proof. We will prove the statement using the Mayer-Vietoris induction of Lemma 4.18. Of course, the statement is obvious for $S = \emptyset$.

Now let S be a simplicial disk. Note that the proof of Lemma 3.8 on simplicial homotopies using the prism $P(S)$ never permutes vertices in simplices or uses that simplices with repeated vertices vanish, so it holds in the ordered setting as well. Hence, the proof of Proposition 3.11 on Betti numbers of star-shaped simplicial complexes also carries over and shows that S has the ordered homology of a point – which is trivial by Example 4.24, just as in the oriented case. This finishes the start of the induction.

Finally, for the induction step let S and T be simplicial complexes such that the proposition holds for S, T, and $S \cap T$. As in the proof of Proposition 4.22 we have a commutative diagram of chain complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^{\text{ord}}(S\cap T) & \longrightarrow & C^{\text{ord}}(S)\times C^{\text{ord}}(T) & \longrightarrow & C^{\text{ord}}(S\cup T) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C(S\cap T) & \longrightarrow & C(S)\times C(T) & \longrightarrow & C(S\cup T) & \longrightarrow & 0,
\end{array}$$

with exact rows as in Example 4.7 (and its corresponding ordered version), where the vertical maps are given by mapping each ordered simplex to the corresponding oriented one. Construction 4.10 thus gives us commutative diagrams of vector spaces

$$\begin{array}{ccccccccc}
H_p^{\text{ord}}(S\cap T) & \longrightarrow & H_p^{\text{ord}}(S)\times H_p^{\text{ord}}(T) & \longrightarrow & H_p^{\text{ord}}(S\cup T) & \longrightarrow & H_{p-1}^{\text{ord}}(S\cap T) & \longrightarrow & H_{p-1}^{\text{ord}}(S)\times H_{p-1}^{\text{ord}}(T) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_p(S\cap T) & \longrightarrow & H_p(S)\times H_p(T) & \longrightarrow & H_p(S\cup T) & \longrightarrow & H_{p-1}(S\cap T) & \longrightarrow & H_{p-1}(S)\times H_{p-1}(T)
\end{array}$$

for all $p \in \mathbb{Z}$ with exact rows. As the four outer vertical maps are isomorphisms by the induction assumption, it follows from the 5-Lemma 4.19 that the middle vertical map is an isomorphism as well. □